SLE$_\kappa$: correlation functions in coefficient problem

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Abstract

We apply method of correlation functions to coefficient problem in stochastic geometry. In particular, we give a proof for some universal patterns conjectured by M. Zinsmeister [1] for second moments of coefficients of conformal mappings for special values of κ in the whole-plane Schramm-Loewner evolution (SLE$_\kappa$). We propose to use multi-point correlation functions for study of higher moments in coefficient problem. Generalizations related to the Levy-type processes are also considered.

1 Coefficient problem for radial and whole plane SLE$_\kappa$: main results

Radial stochastic Schramm-Loewner Evolution (SLE$_\kappa$, see e.g. [4]) describes dynamics of a slit domain in the z-plane that is a growth of a random planar curve $\Gamma = \Gamma(t)$ starting from a point on a unit circle $|z| = 1$ at $t = 0$. It is represented by the time dependent conformal mapping $z = F(w, t)$:

$$\frac{\partial F(w,t)}{\partial t} = w\frac{\partial F(w,t)}{\partial w}\frac{w + e^{\mathrm{i}B(t)}}{w - e^{\mathrm{i}B(t)}}, \quad t \geq 0, \quad F(w,0) = w, \tag{1}$$

where $B(t)$ is Brownian motion with "temperature" κ:

$$\langle (B(t) - B(t'))^2 \rangle = \kappa |t - t'|.$$

Everywhere through the article $\langle\rangle$ denotes expectation.

One can consider either "exterior" problem, where curve is growing in an exterior of the unit circle Ω_+ in the w-plane, which is mapped by F to $\Omega_+ \backslash \Gamma(t)$ in the z-plane

$$F(w,t) = e^t \left(w + \sum_{i=0}^{\infty} \frac{F_i(t)}{w^i} \right), \quad |w| > 1,$$

or an "interior" problem, where an interior of a unit circle Ω_- is mapped to $\Omega_- \backslash \Gamma(t)$

$$F(w,t) = e^{-t} \left(w + \sum_{i=2}^{\infty} F_i(t) w^i \right), \quad |w| < 1.$$

Since SLE$_\kappa$ is a conformally invariant stochastic process, exterior and interior problems are related by inversion $F(w,t) \to 1/F(1/w, t)$.

The whole plane SLE$_\kappa$ is the limit of an interior problem

$$\mathcal{F}(w,t) = \lim_{T\to\infty} e^T F(w, T - t) \tag{2}$$

describing the growth process in an infinite slit domain by "erasing" in time a curve/slit that starts at some point on the plane and goes to infinity. Here $\mathcal{F}$ maps an interior of a unit circle to $C\backslash\Gamma(t)$

$$\frac{\partial \mathcal{F}(w,t)}{\partial t} = -w\frac{\partial \mathcal{F}(w,t)}{\partial w}\frac{w+e^{\mathrm{i}B(t)}}{w-e^{\mathrm{i}B(t)}}, \quad \mathcal{F}(w,t) = e^t\left(w+\sum_{i=2}^{\infty}\mathcal{F}_i(t)w^i\right), \quad |w|<1 \tag{3}$$

In the work [1], the Bieberbach conjecture (see e.g. [4], [5]) has been revisited in the framework of the SLE$_\kappa$: The authors of [1] performed calculations of expectation values of squares of absolute values of several first Taylor coefficients $\langle|\mathcal{F}_i|^2\rangle := \lim_{t\to\infty}\langle|\mathcal{F}_i(t)|^2\rangle$ for stochastic Loewner evolution driven by Levy processes. They observed some universal patterns, and in particular for the SLE$_\kappa$ with $\kappa=6$ and $\kappa=2$

$$\kappa=2, \quad \langle|\mathcal{F}_n|^2\rangle = n$$
$$\kappa=6, \quad \langle|\mathcal{F}_n|^2\rangle = 1$$

In the present article we prove the above conjecture.

Our study relies on technique of correlation functions, which is introduced in the next section. The multi-point correlation functions are considered in the context of evaluation of higher moments of expectations of the Taylor coefficients. This approach is also generalized to the case of the Loewner evolution driven by Levy processes.

2 Correlation functions of SLE$_\kappa$

We rewrite Loewner equations (1) and (3) in the "moving/rotating frame", where the point $w=1$ is mapped by f to the tip of the growing/erasing curve. For the exterior radial SLE$_\kappa$ we then get

$$df(w,t) = w\frac{\partial f(w,t)}{\partial w}\left(\frac{w+1}{w-1}dt+\mathrm{i}dB\right), \quad f(w,t) = F(we^{\mathrm{i}B(t)},t) = e^{t+\mathrm{i}B(t)}\left(w+\sum_{j=0}^{\infty}f_j(t)/w^j\right)$$

while for the whole plane SLE$_\kappa$ we have

$$df(w,t) = -w\frac{\partial f(w,t)}{\partial w}\left(\frac{w+1}{w-1}dt+\mathrm{i}dB\right), \quad f(w,t) = \mathcal{F}(we^{\mathrm{i}B(t)},t) = e^{t+\mathrm{i}B(t)}\left(w+\sum_{j=2}^{\infty}f_j(t)w^j\right)$$

Remind, that in the "fixed frame" (1), (3), the rotating point $w=e^{\mathrm{i}B(t)}$ on the circle was mapped to the moving tip of the growing/erasing curve.

We now define the correlation function

$$\rho(w,\bar{w}|q;\kappa) = \lim_{t\to\infty} e^{-qt}\langle(f'(w,t)\bar{f}'(\bar{w},t))^{q/2}\rangle \tag{4}$$

where prime denotes w-derivative. We do not assume that the second argument of the above correlation function $\bar{w}$ is a complex conjugate of the first one w, so we call this correlation function the "2-point" one. Bar over a function denotes its complex conjugation i.e. if $f(w)=\sum_i f_i w^i$, then $\bar{f}(w)=\sum_i \bar{f}_i w^i$, where $\bar{f}_i$ is the complex conjugate of f_i. In the case when $\bar{w}$ is a complex conjugate of w, function ρ is the q-th moment of the harmonic measure.

Expectations of squares of absolute values of Laurent/Taylor coefficients are same in both frames $\langle f_j(t)\bar{f}_j(t)\rangle = \langle F_j(t)\bar{F}_j(t)\rangle$ and by the definition (4)

$$\lim_{t\to\infty}\langle|F_j(t)|^2\rangle = \rho_{j,j}(2,\kappa),$$

where $ij\rho_{i,j}(q,\kappa)$ are coefficients of expansion of $\rho(w,\bar{w}|q;\kappa)$:

$$\rho(w,\bar{w}|q;\kappa) = 1 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{ij\rho_{i,j}(q,\kappa)}{w^{i+1}\bar{w}^{j+1}} \tag{5}$$

for exterior radial SLE or

$$\rho(w,\bar{w}|q;\kappa) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\rho_{i,j}(q,\kappa) w^{i-1}\bar{w}^{j-1}, \quad \rho_{1,1} = 1, \tag{6}$$

for whole plane SLE.

Diagonal coefficients $\rho_{j,j}(q,\kappa)$ are invariant in any rotating frame.

Lemma 1: The 2-point correlation function (4) satisfies the following differential equation

$$\mathcal{L}[\rho](w,\bar{w}|q;\kappa) = \sigma q \rho(w,\bar{w}|q;\kappa) \tag{7}$$

$$\mathcal{L} = -\frac{\kappa}{2}\left(w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{w}}\right)^2 + \frac{w+1}{w-1}w\frac{\partial}{\partial w} + \frac{\bar{w}+1}{\bar{w}-1}\bar{w}\frac{\partial}{\partial \bar{w}} - q\left(\frac{1}{(w-1)^2} + \frac{1}{(\bar{w}-1)^2} - 1\right)$$

where $\sigma = 1$ for exterior radial SLE$_\kappa$, and $\sigma = -1$ for interior radial or the whole plane SLE$_\kappa$.

Proof: The proof relies on procedure introduced by Hastings [2] for derivation of multi-fractal spectrum of SLE$_\kappa$. We start with exterior radial SLE$_\kappa$.

From (4) it follows that at $t \to \infty$

$$\frac{\partial}{\partial t}\langle g(w,\bar{w},t)\rangle = q\langle g(w,\bar{w},t)\rangle, \quad g(w,\bar{w},t) = (f'(w,t)\bar{f}'(\bar{w},t))^{q/2} \tag{8}$$

Using the fact that for the Loewner evolution $f(w,t+\delta t) = f(\psi(w,\delta t),t)$, where

$$d\psi(w,\delta t) = w\frac{\partial\psi(w,\delta t)}{\partial w}\left(\frac{w+1}{w-1}d\delta t + \mathrm{i}dB(t+\delta t)\right), \quad \psi(w,\delta t = 0) = w$$

we find that in the first order in δt

$$f(w,t+\delta t) = f\left(w + \delta\phi(w), t\right), \quad \delta\phi(w) = w\frac{w+1}{w-1}\delta t + \mathrm{i}w\delta B - \frac{w}{2}(\delta B)^2, \quad \delta B = B(t+\delta t) - B(t)$$

and

$$g(w,\bar{w},t+\delta t) = g\left(w + \delta\phi(w), \bar{w} + \delta\bar{\phi}(\bar{w}), t\right)\left(1 + \delta\phi'(w)\right)^{q/2}\left(1 + \delta\bar{\phi}'(\bar{w})\right)^{q/2}$$

Equating expectations of RHS and LHS of the above and taking into account that $< \delta B >= 0$, $< (\delta B)^2 >= \kappa\delta t$ we obtain

$$\frac{\partial}{\partial t}\langle g(w,\bar{w},t)\rangle = \mathcal{L}\left[\langle g(w,\bar{w},t)\rangle\right],$$

Finally, with the help of (4) and (8) we arrive at differential equation (7) with $\sigma = 1$ for the exterior radial SLE$_\kappa$.

For the whole plane SLE$_\kappa$, equation for correlation functions coincides, due to (2), with that for an interior radial SLE$_\kappa$. As a consequence derivation for the interior case coincides with that of the exterior, except the factor e^{qt} must replace e^{-qt} in (4), so that

$$\frac{\partial}{\partial t}\langle g(w,\bar{w},t)\rangle = -q\langle g(w,\bar{w},t)\rangle,$$

and we arrive at equation (7) with $\sigma = -1$.

3 Whole plane SLE$_\kappa$

Consider the whole plane or interior radial SLE$_\kappa$ first. Substituting (6) into (7) we get recurrence relation for $\rho_{i,j}$:

$$\sum_{n=0}^{2}\sum_{k=0}^{2} C_{i,j}^{n,k}\theta_{i-n,j-k} = 0, \quad \theta_{i,j} = ij\rho_{i,j}, \quad \theta_{1,1} = 1, \quad \theta_{i\le 0,j} = \theta_{i,j\le 0} = 0 \tag{9}$$

$$C_{i,j}^{0,0} = -\left(\frac{\kappa}{2}(i-j)^2 + i + j - 2\right)$$

$$C_{i,j}^{1,1} = -4\left(\frac{\kappa}{2}(i-j)^2 - 2q\right)$$

$$C_{i,j}^{2,2} = -\left(\frac{\kappa}{2}(i-j)^2 - i - j + 6 - 2q\right)$$

$$C_{i,j}^{0,1} = 2\left(\frac{\kappa}{2}(j-i-1)^2 + i - 1 - q\right), \quad C_{i,j}^{1,0} = 2\left(\frac{\kappa}{2}(i-j-1)^2 + j - 1 - q\right)$$

$$C_{i,j}^{0,2} = -\left(\frac{\kappa}{2}(j-i-2)^2 + i - j + 2 - q\right), \quad C_{i,j}^{2,0} = -\left(\frac{\kappa}{2}(i-j-2)^2 + j - i + 2 - q\right),$$

$$C_{i,j}^{1,2} = 2\left(\frac{\kappa}{2}(i-j+1)^2 + 3 - j - 2q\right), \quad C_{i,j}^{2,1} = 2\left(\frac{\kappa}{2}(j-i+1)^2 + 3 - i - 2q\right)$$

From this recurrence relation any $\rho_{i,j}(q,\kappa)$ can be found in a consecutive manner: We express $\rho_{i,j}$ as a linear combination of 8 coefficients $\rho_{i,j-1}$, $\rho_{i,j-2}$, $\rho_{i-1,j}$, $\rho_{i-1,j-1}$, $\rho_{i-1,j-2}$, $\rho_{i-2,j}$, $\rho_{i-2,j-1}$, $\rho_{i-2,j-2}$. At the first step we find $\rho_{1,2}$, then $\rho_{2,2}$ etc up to $\rho_{1,n}$. Repeating similar procedure for the second row $\rho_{2,j}, j = 1..n$, then for the third row $\rho_{3,j}$... up to the n-th row $\rho_{n,j}, j = 1..n$, we get all $\rho_{1..n,1..n}$.

For example

$$\rho_{2,2}(q,\kappa) = \frac{2q^2}{2+\kappa}$$

$$\rho_{3,3}(q,\kappa) = \frac{q^2}{36}\frac{9\kappa^2 + 8(14q + 7 + 16q^2)\kappa + 12(4q+1)^2}{(2+\kappa)(1+\kappa)(6+\kappa)}$$

$$\rho_{4,4}(q,\kappa) = \frac{q^2}{72}\big(240(2+3q+4q^2)^2 + 16\kappa^5 + 8(744q^4 + 340 + 1152q + 2363q^2 + 1572q^3)\kappa^2$$
$$+8(701q^2 + 378q^3 + 414q + 192 + 144q^4)\kappa^3 + 32(635q^2 + 56 + 498q^3 + 272q^4 + 258q)\kappa$$
$$+(204q + 243q^2 + 284)\kappa^4\big) \,/\, \big((2+\kappa)^2(1+\kappa)(6+\kappa)(2+3\kappa)(10+\kappa)\big)$$

etc.

Several first expectations of $\lim_{t\to\infty}\langle|F_j(t)|^2\rangle = \rho_{j,j}(2,\kappa)$ are then

$$\langle|F_2|^2\rangle = \frac{8}{2+\kappa}$$

$$\langle|F_3|^2\rangle = \frac{88\kappa + 108 + \kappa^2}{(2+\kappa)(1+\kappa)(6+\kappa)}$$

$$\langle|F_4|^2\rangle = \frac{8(104\kappa^4 + 22896\kappa + 4576\kappa^3 + 18288\kappa^2 + \kappa^5 + 8640)}{9(2+\kappa)^2(1+\kappa)(6+\kappa)(2+3\kappa)(10+\kappa)}$$

etc, which coincides with results of computer experiments given in [1].

There are two particular values of κ for which the following special results hold

Theorem 1:

- For $\kappa = 6$, $\langle |F_n|^2 \rangle = 1$
- For $\kappa = 2$, $\langle |F_n|^2 \rangle = n$

Proof: Proof is by direct solution of recurrence relation (9)

- For $\kappa = 6$, matrix $\rho_{i,j}$ is tri-diagonal, i.e. $\rho_{i,j} = 0$, if $|i-j| > 1$. Nonzero matrix elements are $\rho_{i,i} = 1$, and $\rho_{i,i-1} = \rho_{i-1,i} = -1/2$. One verifies the first statement of the theorem by direct substitution of $q = 2, \kappa = 6$, and the above matrix elements into (9).
- For $\kappa = 2$, matrix $\rho_{i,j}$ is five-diagonal , i.e. $\rho_{i,j} = 0$, if $|i - j| > 2$. Nonzero matrix elements are $\rho_{i,i} = i$, $\rho_{i,i-1} = \rho_{i-1,i} = (1-2i)/3$, $\rho_{i,i-2} = \rho_{i-2,i} = (i-1)/6$. One verifies the second statement of the theorem by direct substitution of $q = 2, \kappa = 2$, and the above matrix elements into (9).

Remark: An alternative proof can be done using the explicit solutions of (7) for $q = 2$ and $\kappa = 6, \kappa = 2$

$$\rho(w, \bar{w}|q = 2; \kappa = 6) = \frac{(1-w)(1-\bar{w})}{(1-w\bar{w})^3}, \quad \rho(w, \bar{w}|q = 2; \kappa = 2) = \frac{(1-w)^2(1-\bar{w})^2}{(1-w\bar{w})^4} \tag{10}$$

It is interesting to note that

$$\rho\left(w, \bar{w}|q = \frac{(2+\kappa)(6+\kappa)}{8\kappa}; \kappa\right) = \frac{((1-w)(1-\bar{w}))^{\frac{6+\kappa}{2\kappa}}}{(1-w\bar{w})^{\frac{(6+\kappa)^2}{8\kappa}}} \tag{11}$$

which comprises two of the above special cases.

4 Exterior radial SLE$_\kappa$

In the exterior problem coefficients in the recurrence relation (9) take the following values

$$C^{0,0}_{i,j} = -\left(\frac{\kappa}{2}(i-j)^2 + i + j + 2\right)$$

$$C^{1,1}_{i,j} = -2\kappa(i-j)^2$$

$$C^{2,2}_{i,j} = -\left(\frac{\kappa}{2}(i-j)^2 - i - j + 2 + 2q\right)$$

$$C^{0,1}_{i,j} = 2\left(\frac{\kappa}{2}(j-i-1)^2 + i + 1\right), \quad C^{1,0}_{i,j} = 2\left(\frac{\kappa}{2}(i-j-1)^2 + j + 1\right)$$

$$C^{0,2}_{i,j} = -\left(\frac{\kappa}{2}(j-i-2)^2 + i - j + 2 + q\right), \quad C^{2,0}_{i,j} = -\left(\frac{\kappa}{2}(i-j-2)^2 + j - i + 2 + q\right),$$

$$C^{1,2}_{i,j} = 2\left(\frac{\kappa}{2}(i-j+1)^2 - j + 1 + q\right), \quad C^{2,1}_{i,j} = 2\left(\frac{\kappa}{2}(j-i+1)^2 - i + 1 + q\right)$$

Similarly to the whole-plane case, any $\rho_{i,j}$ can be found recursively. Here we list several first $\rho_{i,i}$

$$\rho_{1,1}(q, \kappa) = \frac{q^2}{4(\kappa+1)}$$

$$\rho_{2,2}(q,\kappa) = \frac{2q^2\kappa(6+\kappa)}{9(\kappa+1)(3\kappa+2)(\kappa+10)}$$

$$\rho_{3,3} = \frac{q^2}{576}\big(840(q-2)^2 + 4(1084-1012q+63q^2)\kappa + 4(3q^2-3q+602)\kappa^4 + 6(3876+52q+349q^2)\kappa^2$$
$$+108\kappa^5 + (532q+305q^2+13620)\kappa^3\big) \,/\, ((\kappa+1)(\kappa+3)(3\kappa+2)(2\kappa+1)(\kappa+10)(\kappa+14))$$

etc. For $q=2$ we have expectation values of $\langle |F_i|^2\rangle$:

$$\langle |F_1|^2\rangle = \frac{1}{(\kappa+1)}$$

$$\langle |F_2|^2\rangle = \frac{8\kappa(6+\kappa)}{9(\kappa+1)(3\kappa+2)(\kappa+10)}$$

$$\langle |F_3|^2\rangle = \frac{\kappa(6+\kappa)(27\kappa^3+446\kappa^2+1300\kappa+264)}{36(\kappa+1)(\kappa+3)(3\kappa+2)(2\kappa+1)(\kappa+10)(\kappa+14)}$$

etc.

5 Multi-point correlation functions

To estimate expectations of higher degrees of the Taylor coefficients, one needs to introduce multi-point correlation functions. For instance, the 4-point correlation function

$$\rho(w_1,w_2,\bar w_1,\bar w_2|q_1,q_2;\kappa) = \lim_{t\to\infty} e^{-(q_1+q_2)t/2}\langle (f'(w_1,t)\bar f'(\bar w_1,t))^{q_1/2}(f'(w_2,t)\bar f'(\bar w_2,t))^{q_2/2}\rangle$$

allows to estimate

$$\langle F_iF_j\bar F_l\bar F_n\rangle, \quad i+j=l+n$$

from the Taylor/Laurent expansion of $\rho(w_1,w_2,\bar w_1,\bar w_2|2,2;\kappa)$. Similarly to the 2-point case the Taylor/Laurent coefficients can be found by solution of a PDE for the $2n$-point correlation function. By analogy with Lemma 1 we have the following

Proposition 1: The $2n$-point correlation function

$$\rho(w_1,\dots w_n,\bar w_1,\dots,\bar w_n|q_1,q_2,\dots q_n;\kappa)$$
$$= \lim_{t\to\infty} e^{-(q_1+q_2+\dots q_n)t/2}\langle (f'(w_1,t)\bar f'(\bar w_1,t))^{q_1/2}\cdots(f'(w_n,t)\bar f'(\bar w_n,t))^{q_n/2}\rangle$$

satisfies the following second-order linear PDE

$$\mathcal{L}[\rho](w_1,\dots w_n,\bar w_1,\dots,\bar w_n|q_1,q_2,\dots q_n;\kappa) = \sigma\left(\sum_{i=1}^n q_i\right)\rho(w_1,\dots w_n,\bar w_1,\dots,\bar w_n|q_1,q_2,\dots q_n;\kappa) \tag{12}$$

$$\mathcal{L} = \frac{\kappa}{2}\left(\sum_{1\le i<j\le n}\left(w_i\frac{\partial}{\partial w_i} - w_j\frac{\partial}{\partial w_j}\right)^2 + \sum_{1\le i<j\le n}\left(\bar w_i\frac{\partial}{\partial \bar w_i} - \bar w_j\frac{\partial}{\partial \bar w_j}\right)^2 - \sum_{i=1}^n\sum_{j=1}^n\left(w_i\frac{\partial}{\partial w_i} - \bar w_j\frac{\partial}{\partial \bar w_j}\right)^2\right)$$
$$+\sum_{i=1}^n\left(\frac{w_i+1}{w_i-1}w_i\frac{\partial}{\partial w_i} + \frac{\bar w_i+1}{\bar w_i-1}\bar w_i\frac{\partial}{\partial \bar w_i}\right) - \sum_{i=1}^n q_i\left(\frac{1}{(w_i-1)^2} + \frac{1}{(\bar w_i-1)^2} - 1\right)$$

where $\sigma = 1$ for exterior radial SLE_κ and $\sigma = -1$ for interior radial/whole plane SLE_κ.

Similarly to the 2-point case (4), any coefficient $\rho_{i_1,i_2,\ldots i_n;j_1,j_2,\ldots j_n}$ of the Taylor/Laurent expansion of the $2n$-point function ρ (below we use vector notations $i := i_1,\ldots,i_n; j := j_1,\ldots,j_n; w := w_1,\ldots,w_n; \bar{w} := \bar{w}_1,\ldots,\bar{w}_n; q := q_1,\ldots,q_n$)

$$\rho(w,\bar{w}|q,\kappa)$$
$$= 1 + \sum_{i_1=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{j_n=1}^{\infty} i_1 i_2\cdots i_n j_1 j_2\cdots j_n \rho_{i,j}(q;\kappa) w_1^{-\sigma i_1-1}\cdots w_n^{-\sigma i_n-1}\bar{w}_1^{-\sigma j_1-1}\cdots\bar{w}_n^{-\sigma j_n-1},$$

can be found in a consecutive manner using the "$2n$-dimensional" recursion relation

$$\sum_{l_1=0}^{2}\cdots\sum_{l_n=0}^{2}\sum_{k_1=0}^{2}\cdots\sum_{k_n=0}^{2} C_{i,j}^{l,k}\rho_{i-l,j-k} = 0, \quad \rho_{1,1,\ldots,1} = 1, \quad \rho_{i<0,j} = \rho_{i,j<0} = 0,$$

where expressions for $C_{i,j}^{l,k}$ can be explicitly found by substitution of the above Taylor/Laurent expansion of $\rho(w,\bar{w}|q;\kappa)$ into (12).

6 Non-Brownian processes

The above considerations generalize straightforwardly to the case of stochastic Loewner evolution driven by Levy processes $L(t)$. Following Hastings' approach [2], one arrives to the analog of equation (7) for the two-point correlation function

$$-\hat{\eta}[\rho](w,\bar{w}) + \frac{w+1}{w-1}w\frac{\partial\rho(w,\bar{w})}{\partial w} + \frac{\bar{w}+1}{\bar{w}-1}\bar{w}\frac{\partial\rho(w,\bar{w})}{\partial\bar{w}} - q\left(\frac{1}{(w-1)^2} + \frac{1}{(\bar{w}-1)^2} - 1 + \sigma\right)\rho(w,\bar{w}) = 0, \tag{13}$$

where

$$\hat{\eta}[\rho](w,\bar{w}) = -\lim_{t\to 0}\frac{\langle\rho\left(e^{\mathrm{i}L(t)}w, e^{-\mathrm{i}L(t)}\bar{w}\right) - \rho(w,\bar{w})\rangle}{t}, \quad L(t=0) = 0.$$

Equations for multi-point correlation functions can be also derived in a similar manner.

As suggested in [1] one can consider the coefficient problem for Levy processes with characteristic functions of the following type:

$$\langle e^{\mathrm{i}\tau L(t)}\rangle = e^{-t\eta(\tau)}, \quad \eta(n) = \bar{\eta}(n) = \eta(-n).$$

For these processes, in the whole plane/interior radial ($\sigma = -1$, (6)) case we have

$$\hat{\eta}[\rho](w,\bar{w}) = \eta\left(w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial\bar{w}}\right)[\rho](w,\bar{w}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\rho_{i,j}\eta_{i-j}w^{i-1}w^{j-1}.$$

where $\eta_i := \eta(i)$.

By analogy with the Brownian case, the coefficient moment expectations can be obtained from recurrence relation (9) with

$$C_{i,j}^{0,0} = -\eta_{i-j} - i - j + 2,$$
$$C_{i,j}^{1,1} = -2(\eta_{i-j} - 4q),$$

$$C^{2,2}_{i,j} = -\eta_{i-j} + i + j - 6 + 2q,$$
$$C^{0,1}_{i,j} = 2\left(\eta_{i-j+1} + i - 1 - q\right), \quad C^{1,0}_{i,j} = 2\left(\eta_{i-j-1} + j - 1 - q\right),$$
$$C^{0,2}_{i,j} = -\eta_{i-j+2} + j - i - 2 + q, \quad C^{2,0}_{i,j} = -\eta_{i-j-2} + i - j - 2 + q,$$
$$C^{1,2}_{i,j} = 2\left(\eta_{i-j+1} + 3 - j - 2q\right), \quad C^{2,1}_{i,j} = 2\left(\eta_{i-j-1} + 3 - i - 2q\right).$$

As conjectured in [1] (by calculations of $\langle |F_n|^2 \rangle$ for $n \leq 20$), an analog of Theorem 1 holds for Levy processes with $\eta_1 = 1, 3$, that correspond to $\kappa = 2, 6$ in the SLE$_\kappa$ case.

The case $\eta_1 = 3$ can be easily proved either by repeating arguments of Theorem 1 that now use the above recurrence coefficients or by explicit solution of equation (13): The $\eta_1 = 3$ solution of (13) coincides with the $q = 2, \kappa = 6$ solution (10) of equation (7).

The case $\eta_1 = 1$ seems to be more involved.

7 Orthogonal polynomials, multi-fractal spectrum etc

Concluding the paper, we would like to discuss the 2-dimensional recurrence relation (9) which seems to be an interesting structure.

The first line $\rho_{1,j}$ of matrix $\rho_{i,j}$ corresponds to function $\rho(0, \bar{w}|q; \kappa)$

$$\rho(0, \bar{w}|q; \kappa) = \sum_{j=1}^{\infty} j \rho_{1,j} \bar{w}^{j-1} \tag{14}$$

that satisfies the $w = 0$ reduction of equation (7). The reduced equation is amenable, by a gauge transform, to the Gauss hypergeometric equation. Choosing an appropriate solution of hypergeometric equation (or considering three-term recurrence relation for $\rho_{1,j}$, which are related to Dual Hahn polynomials [3]), we see that expansion (14) truncates at $j = N + 1$ (i.e. $\rho_{1,j} = 0, j > N + 1$) if

$$q = \frac{Nn(2N - n + 1)}{N^2 + n^2 - n}, \quad \kappa = 2\frac{q + N^2}{N} = 2\frac{2n + N}{N^2 + n^2 - n}, \quad N, n = 0, 1, 2, 3 \ldots \tag{15}$$

Furthermore, since for the above values of parameters, $C^{0,2}_{i,i+N+2} = 0$ in the recurrence relation (9), the whole matrix $\rho_{i,j}$ becomes $2N + 1$-diagonal, i.e. $\rho_{i,j} = 0$ if $|i - j| > N$ when (15) holds.

Remark: The cases $q = 2, \kappa = 6$ and $q = 2, \kappa = 2$ of the Theorem 1 correspond to $N = 1, n = 1$ and $N = 2, n = 1$ in (15).

It is worth to note that the orthogonal polynomials also appear in asymptotic expansion of $\rho(w, \bar{w})$: In more details, in $i \to \infty$ limit

$$\rho_{i,i}(q, \kappa) \to \mathcal{C}(q, \kappa) i^{\beta(q,\kappa) - 3}$$

where $\beta(q, \kappa)$ is an integral "β-spectrum":

$$\beta = \lim_{\epsilon \to 0} \frac{\int_0^{2\pi} \rho\left(e^{-\epsilon + \imath\theta}, e^{-\epsilon - \imath\theta}|q; \kappa\right) d\theta}{-\log \epsilon}.$$

Using the anzatz

$$\rho_{i,i+l} \to f_l i^{\beta-3},$$

from the recurrence relation (9) in the leading order in i we get

$$g_{l+1} + g_{l-1} - 2g_l = 0, \quad g_l = L[f]_l - \beta f_l,$$

where L is the three-diagonal difference operator

$$L[f]_l = \frac{1}{2}\left(a_{l+1}f_{l+1} + b_l f_l + c_{l-1}f_{l-1}\right), \tag{16}$$

$$a_l = \frac{\kappa l^2}{2} + l - q, \quad c_l = a_{-l} = \frac{\kappa l^2}{2} - l - q, \quad b_l = -a_l - c_l + 2(q-3) = -\kappa l^2 + 4q$$

Since the "boundary conditions" $\rho_{i<0,j} = \rho_{i,j<0} = 0$ must hold and $\rho_{ij} = \bar{\rho}_{ij} = \rho_{ji}$,

$$f_l \to 0, \quad l \to \pm\infty, \quad f_{-l} = f_l. \tag{17}$$

This is only possible if g_l=0, and we have

$$L[f]_l = \beta f_l. \tag{18}$$

Thus, the value of β-spectrum for given κ and q is an eigenvalue of the operator L whose eigenvector has properties (17).

In the case when parameters q and κ take values (15), i.e. in the case of the $2N+1$ band truncation of $\rho_{i,j}$, the difference equation (18) corresponds to the three-term recurrence relation for the Dual Hahn polynomials [3] with the following spectrum:

$$\beta = \frac{2N(n + 6nN - 3n^2 - N) - (8nN - 2n^2 - N + 2N^2)j + (2n+N)j^2}{2(N^2 + n^2 - n)},$$

where $j \in 0, 1, 2, \ldots 2N$. To choose a proper j, corresponding to a value of β-spectrum for specified parameters, additional analysis (e.g. analysis of (7) in neighborhood of singular point $w = 1, \bar{w} = 1$) is needed.

It is interesting to understand what is particular significance of the above $2N+1$-diagonal structures and whether the whole two-dimensional recurrence relation (9) is "exactly solvable".

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