# On complex singularity analysis for linear partial $q$-difference-differential equations using nonlinear differential equations

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**Abstract**

We investigate the existence of local holomorphic solutions of linear $q$-difference-differential equations in two variables $t, z$ whose coefficients have poles or algebraic branch points singularities in the variable $t$. These solutions are shown to develop poles or algebraic branch points along half $q$-spirals. We also give bounds for the rate of growth of the solutions near the singular points. We construct these solutions with the help of functions of infinitely many variables that satisfy functional equations that involve $q$-difference, partial derivatives and shift operators. We show that these functional equations have solutions in some Banach spaces of holomorphic functions in $\mathbb{C}^{\infty}$ having sub-exponential growth.

Key words: functional differential equations, $q$-difference equations, partial differential equations with infinitely many variables, entire functions with $q$-exponential growth, entire functions with infinitely many variables, singularity analysis. 2000 MSC: 35C10, 35C20.

## 1 Introduction

In this paper, we study linear partial $q$-difference-differential equations of the form

$$\partial_z^S u(t,z) = \sum_{h=(h_1,h_2)\in\mathcal{S}} b_h(t,z)(\partial_t^{h_1}\partial_z^{h_2}u)(q^{m_{0,h}}t, zq^{-m_{1,h}}) \tag{1}$$

where $\mathcal{S}$ is a subset of $\mathbb{N}^2$, $q$ is a complex number with modulus $|q| > 1$ and $S, m_{0,h}, m_{1,h}$ are positive integers which satisfy the constraints (49). The coefficients $b_h(t,z)$ are holomorphic functions with singularities in the $t$ variable and polynomial in the $z$ variable for given initial data $(\partial_z^j u)(t,0) = \varphi_j(t)$, $0 \leq j \leq S-1$.

Our goal is the construction of local holomorphic solutions of (1) and the study of their behaviour near the singular points of the coefficients $b_h(t,z)$ and initial data $\varphi_j(t)$.

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In the framework of linear partial differential equations (i.e $m_{0,h} = m_{1,h} = 0$), there exists a huge litterature on the study of complex singularities and analytic continuation of their holomorphic solutions starting from the fundamental contributions of J. Leray in [18]. Several authors have considered Cauchy problems $a(x,D)u(x) = 0$, where $a(x,D)$ is a differential operator of some order $m \geq 1$, for initial data $\partial_{x_0}^h u_{|x_0=0} = w_h$, $0 \leq h < m$. Under specific hypotheses on the symbol $a(x,\xi)$, precise descriptions of the solutions of these problems are given near the singular locus of the initial data $w_h$. For meromorphic initial data, we may refer to [8], [23], [24] and for more general ramified multivalued initial data, we may cite [7], [13], [29], [33], [34].

In the real shrinking setting (i.e $q \in \mathbb{R}$, $|q| < 1$ and $m_{1,h} < 0$) general results concerning the construction of local solutions and their regularity have been achieved, see for instance [2], [14], [16], [17].

These recent years many authors focused on the case of mixed type $q$-difference-differential equations with $|q| > 1$. This subject is now of great interest from both algebraic and analytic point of view, see for instance [5], [9], [11], [25], [26], [27], [35], [36].

The kind of problem (1) we consider in this work enters this new trend of research and extends aspects of both of the previous situations studied in [19], [20].

In the paper [19], we considered differential equations in the variable $z$ with dilations and contractions in both variables $t$,$z$, whose coefficients are polynomial in $t$,$z$ and a function $e(t)$ satisfying some $q$-difference equation having meromorphic singularities along unions of half $q$-spirals (which are sets of the form $q^{-\mathbb{N}}a$, for some $a \in \mathbb{C}^*$). We constructed local holomorphic solutions with respect to $t$ near the points of the half $q$-spirals, entire for $z$ in $\mathbb{C}$ and showed that their growth rate is at most sub-exponential with bounds of the form $C\exp(M(\log|t-t_0|)^2)$ near the singularities $t_0 \in q^{-\mathbb{N}}a$ for some constants $C, M > 0$. In the classical situation of systems of $q$-difference equations with rational coefficients of the form $Y(qz) = A(z)Y(z)$, we refer to [6] for the construction of local meromorphic solutions near the origin and the point at infinity on the Riemann sphere but it is worthwhile saying that the construction of local solutions near the singularities of $A(z)$ outside 0 and $\infty$ remains an unsolved problem.

In the work [20], the authors investigated the construction and behaviour of local holomorphic solutions to linear partial differential equations in $\mathbb{C}^2$ near the singular locus of the initial data. These initial data are assumed to be polynomial in $t$,$z$ and a function $u(t)$ satisfying some nonlinear differential equation of first order and owning an isolated singularity $t_0$ on some domain $\mathcal{D} \subset \mathbb{C}$, which is, by a result of P. Painlevé, either a pole or an algebraic branch point. Following the principle of the classical *tanh method* introduced in [21], they have considered formal series solutions of the form

$$u(t,z) = \sum_{l \geq 0} u_l(t,z)(u(t))^l \tag{2}$$

where $u_l$ are holomorphic functions on $\mathcal{D} \times D$ where $D \subset \mathbb{C}$ is a small disc centered at 0. They have given suitable conditions for these series to converge for $t$ in a sector $S$ with vertex $t_0$ and to have at most exponential bounds estimates of the form $C\exp(M|t-t_0|^{-\mu})$ for all $t \in S$ near $t_0$ for some constants $C, M, \mu > 0$.

Like in the work [20], the coefficients $b_h(t,z)$ of (1) and the initial data are polynomials in $t$,$z$ and a solution $u(t)$ of some nonlinear differential equation of first order. We require the function $u(t)$ to be bounded at 0 and $\infty$ (see Section 3.1). For a suitable choice of $u(t)$, one can choose for instance $b_h(t,z)$ to be some rational function in $t$ and polynomial in $z$ (see Example 1 from Section 3.1).

In our setting, one cannot content oneself with formal expansions in the function $u(t)$ like (2) due to the presence of the dilation operator $t \mapsto qt$. In order to get suitable recursion formulas, it turns out that we need to deal with series expansions that take into account all the functions $u(q^j t)$, $j \geq 0$. This is the reason why the construction of the solutions will follow the one introduced in a recent work of H. Tahara and will involve Banach spaces of holomorphic functions with infinitely many variables.

In the paper [30], H. Tahara has studied a new equivalence problem between given two non-linear partial differential equations of first order in the complex domain. He showed that the equivalence maps have to satisfy so called coupling equations which are non linear partial differential equation of first order but with infinitely many variables. In a more general setting, within the framework of mathematical physics, spaces of functions of infinitely many variables play a fundamental role in the study of nonlinear integrable partial differential equations known as solitons equations as described in the theory of M. Sato, see [22] for an introduction. Important contributions have been obtained these recent years to the study of higher order Painlevé equations, see for instance [15], [31], and applications to quantum field theory, see [1], [32].

In the first part of this paper we construct Banach spaces of formal power series of infinitely many variables as sums of entire functions of finite numbers of variables having at most a polynomial growth. We show that these power series are convergent and define holomorphic functions on every polydiscs of finite radii in $\mathbb{C}^{\infty}$. Moreover, we prove that these functions have $q$-exponential growth rate (in the terminology of [28]) as one makes one radius of the polydisc increase (Proposition 1).

In the section 3, we construct holomorphic functions of the form $u(t,z) = \phi(t,z,(u(q^j t))_{j \geq 0})$ where $\phi(t,z,(u_j)_{j \geq 0})$ is a holomorphic function of infinitely many variables that belongs to the Banach spaces constructed in the previous section. Under suitable conditions on the function $u(t)$, these functions $u(t,z)$ are defined on product sets $q^{-\mathbb{N}}S \times \mathbb{C}$ where $S$ is some open sector having finite radius and with vertex $t_0$ in $\mathbb{C}^*$. Moreover, these functions $u(t,z)$ are shown to have at most sub-exponential growth with bounds of the form $C\exp(M(\log|q^k t - t_0|)^2)$ for all $t \in q^{-k}S$ near the singularity $q^{-k}t_0$, for all $k \geq 0$, for some constants $C, M > 0$.

It turns out that such a function $u(t,z)$ satisfies the equation (1) for the given initial conditions if $\phi$ satisfies a functional equation (43) which involves partial derivatives and shift operators in the variables $(u_j)_{j \geq 0}$. In the proof we use a Faà di Bruno formula in several variables obtained in [3] (see Proposition 3).

In Proposition 4, we give sufficient conditions for the linear operators involved in this functional equation (43) to be continuous maps on the Banach spaces constructed above. This is the most technical part in the proof of our main result. In Section 3.4, we show that the functional equation (43) has a unique solution $\phi$ in the Banach space introduced in the first part. Finally, we use this function $\phi$ to construct a solution $u(t,z)$ of the problem (1) with given initial conditions $\varphi_j(t)$, $0 \leq j \leq S-1$, having the upper-mentioned sub-exponential growth estimates (Theorem 1).

## 2 Weighted Banach spaces of holomorphic functions with infinitely many variables

**Definition 1** *Let $c, c_0 > 0$ be two positive real numbers. For all integers $\beta \geq 0$, we denote by $A(\mathbb{C}^{\beta+1})$ the vector space of entire functions on $\mathbb{C}^{\beta+1}$ and $\mathrm{SE}_{\beta}(\mathbb{C}^{\beta+1})$ the vector subspace of $A(\mathbb{C}^{\beta+1})$ of entire functions $f((u_j)_{0 \leq j \leq \beta}) : \mathbb{C}^{\beta+1} \rightarrow \mathbb{C}$ such that*

$$||f||_{\beta} := \sup_{(u_j)_{0 \leq j \leq \beta} \in \mathbb{C}^{\beta+1}} |f((u_j)_{0 \leq j \leq \beta})| \prod_{j=0}^{\beta} (1+|u_j|)^{-c\beta - c_0}$$

*exists.*

In the following, we will denote by $\mathbb{C}[[t,z,\underline{u}]]$ the vector space of formal series $\phi$ in the infinitely many variables $t, z, \underline{u} = (u_j)_{j\geq 0}$ with coefficients in $\mathbb{C}$ which can be written in the form

$$\phi(t,z,\underline{u}) = \sum_{l,\beta\geq 0} \phi_{l,\beta}((u_j)_{0\leq j\leq \beta})\frac{t^l}{l!}\frac{z^\beta}{\beta!}$$

where $\phi_{l,\beta}((u_j)_{0\leq j\leq\beta})$ are formal series in the variables $u_j$, $0 \leq j \leq \beta$, with coefficients in $\mathbb{C}$, for all $l, \beta \geq 0$. For general facts about formal series and holomorphic functions of infinitely many variables, we refer to the book [4].

**Definition 2** *Let $q > 1$, $T, X > 0$ be real numbers. Let $P(l,\beta)$ be the polynomial $l\beta - \beta^2 - l^2$. We denote by $\mathbb{SE}(T,X,q,c,c_0)$ the subspace of $\mathbb{C}[[t,z,\underline{u}]]$ of formal series in the variables $t, z, \underline{u} = (u_j)_{j\geq 0}$,*

$$\phi(t,z,\underline{u}) = \sum_{l,\beta\geq 0} \phi_{l,\beta}((u_j)_{0\leq j\leq \beta})\frac{t^l}{l!}\frac{z^\beta}{\beta!}$$

*where $\phi_{l,\beta} \in \mathrm{SE}_\beta(\mathbb{C}^{\beta+1})$, for all $l, \beta \geq 0$, such that*

$$||\phi(t,z,\underline{u})||_{(T,X)} := \sum_{l,\beta\geq 0} \frac{||\phi_{l,\beta}((u_j)_{0\leq j\leq\beta})||_\beta}{q^{P(l,\beta)}}\frac{T^l}{l!}\frac{X^\beta}{\beta!}$$

*converges. It is easy to see that $\mathbb{SE}(T,X,q,c,c_0)$ is a Banach space for the norm $||.||_{(T,X)}$.*

In the next proposition, we analyse the convergence of the series in the Banach space constructed above.

**Proposition 1** *We define the functions*

$$H_{1,c}(r) = \frac{1}{4\log(q^{1/2}/(1+r)^c)} \quad , \quad H_{2,c_0}(r) = 2(1+r)^{c_0}$$

*for all $r \geq 0$. Let $\phi(t,z,\underline{u}) \in \mathbb{SE}(T,X,q,c,c_0)$. Then, there exists a constant $C_\phi > 0$ such that, for all integers $\nu \geq 0$, for all $T_0, X_0 > 0$ and all $R, R_\nu > 0$ such that $(1+R)^c < q^{1/2}$ and $R_\nu \geq R$, we have*

$$(3) \quad |\phi(t,z,\underline{u})| \leq C_\phi \exp(\frac{1}{2\log(q)}(\log(2T_0/T))^2) \times (1+R_\nu)^{c_0}\exp(H_{1,c}(R)(\log(H_{2,c_0}(R)(1+R_\nu)^c\frac{X_0}{X}))^2)$$

*for all $t, z \in \mathbb{C}$, with $|t| < T_0$, $|z| < X_0$, all $\underline{u} = (u_j)_{j\geq 0}$ with $|u_j| \leq R$, for all $j \neq \nu$ and $|u_\nu| \leq R_\nu$.*

**Proof** Let $\phi(t,z,\underline{u}) \in \mathbb{SE}(T,X,q,c,c_0)$. By definition, there exists a constant $C_0 > 0$ such that

$$(4) \qquad |\phi_{l,\beta}((u_j)_{0\leq j\leq\beta})| \leq C_0(\prod_{j=0}^{\beta}(1+|u_j|)^{c\beta+c_0})q^{P(l,\beta)}(\frac{1}{T})^l(\frac{1}{X})^\beta l!\beta!$$

for all $l, \beta \geq 0$. From the fact that $P(l,\beta) \leq -l^2/2 - \beta^2/2$, for all $l,\beta \geq 0$, we deduce that there exists $C_1 > 0$ such that

$$|\phi(t,z,\underline{u})| \leq C_1 A_1(|t|) A_2(R, R_\nu, |z|) \tag{5}$$

where

$$A_1(|t|) = \sum_{l \geq 0} q^{-l^2/2} (\frac{|t|}{T})^l$$

and

$$A_2(R, R_\nu, |z|) = (1+R_\nu)^{c_0} \sum_{\beta \geq 0} (\frac{(1+R)^c}{q^{1/2}})^{\beta^2} (\frac{(1+R_\nu)^c (1+R)^{c_0} |z|}{X})^{\beta},$$

for all $t, z \in \mathbb{C}$, all $\underline{u} = (u_j)_{j \geq 0}$ with $|u_j| \leq R$, for all $j \neq \nu$ and $|u_\nu| \leq R_\nu$. Now, we give estimates for $A_1(|t|)$ and $A_2(R, R_\nu, |z|)$. We will use the next lemma which is given in [28] (Lemma 2.2).

**Lemma 1** *There exists a constant $M > 0$ such that*

$$\sum_{n \geq 0} q^{-n^2/(2s)} r^n \leq M \exp(\frac{s}{2\log(q)} (\log(2r))^2) \tag{6}$$

*for all $r > 0$, all $s > 0$.*

Using Lemma 1, for $s = 1$ and $s = \frac{\log(q)}{2\log(q^{1/2}/(1+R)^c)}$, we get a constant $M > 0$ such that

$$\begin{aligned} &A_1(|t|) \leq A_1(T_0) \leq M \exp(\frac{1}{2\log(q)} (\log(2T_0/T))^2), \\ &A_2(R, R_\nu, |z|) \leq A_2(R, R_\nu, X_0) \leq M(1+R_\nu)^{c_0} \exp(H_{1,c}(R)(\log(H_{2,c_0}(R)(1+R_\nu)^c \frac{X_0}{X}))^2) \end{aligned} \tag{7}$$

for all $t, z \in \mathbb{C}$ with $|t| < T_0$, $|z| < X_0$. The estimates (3) follow. □

# 3 Linear operators on $\mathbb{SE}(T, X, |q|, c, c_0)$ and a functional equation

## 3.1 Some nonlinear differential equations

We consider the following first order nonlinear differential equation

$$u'(t) = \sum_{j=0}^{d} p_j (u(t))^j, \tag{8}$$

where $d \geq 2$ is an integer, with constant coefficients $p_j \in \mathbb{C}$, $0 \leq j \leq d$, with $p_d \neq 0$. As a consequence of a result of P. Painlevé (see [10], Theorem 3.3.2), we know that the only singularities in $\mathbb{C}$ of the solutions $u(t)$ of (8) are poles or algebraic branch points. In the following, we denote by $D_\theta(t,r)$ an open disc $D(t,r)$ centered at $t \in \mathbb{C}$ with radius $r > 0$ minus the half line $[t, re^{i\theta})$, for $\theta \in \mathbb{R}$. We make the following assumptions.

**Assumption 1.** There exist $\theta_0 \in \mathbb{R}$, $0 < r_0 < 1$, $t_0 \in \mathbb{C}^*$ and a function $u(t)$ which is holomorphic and a solution of (8) on $D_{\theta_0}(t_0, r_0)$ which admits the following puiseux convergent expansion

$$u(t) = \sum_{l \geq -m} u_l (t - t_0)^{\mu l} \tag{9}$$

for all $t \in D_{\theta_0}(t_0, r_0)$, where $\mu > 0$ is a real number and $m \geq 1$ is an integer such that $u_{-m} \neq 0$. If $\mu$ is a positive integer, the point $t_0$ is called a pole of order $m\mu$ and in this case the function $u(t)$ is holomorphic on $D(t_0, r_0) \setminus \{t_0\}$, otherwise, the point $z_0$ is called an algebraic branch of order $m\mu$.

**Assumption 2.** There exists $q \in \mathbb{C}$ with $|q| > 1$ such that, for all $j \in \mathbb{Z}^*$, the function $t \mapsto u(q^j t)$ is well defined and holomorphic on $S_{d_0,r_0,\delta_0}(t_0)$ and moreover, there exists a constant $M > 0$ such that $|u(q^j t)| \leq M$ for all $t \in S_{d_0,r_0,\delta_0}(t_0)$, all $j \in \mathbb{Z}^*$, where

$$S_{d_0,r_0,\delta_0}(t_0) = \{t \in \mathbb{C} \setminus \{t_0\} / |\arg(t - t_0) - d_0| < \delta_0/2, |t - t_0| < r_0\} \subset D_{\theta_0}(t_0, r_0)$$

is an open sector centered at $t_0$ contained in $D_{\theta_0}(t_0, r_0)$.

**Example 1:** For all $c \in \mathbb{C}^*$, the function $u(t) = -1/(t - c)$ is a solution of $u'(t) = (u(t))^2$ on $\mathbb{C} \setminus \{c\}$. Then, one checks that the assumptions **1.** and **2.** are satisfied for $t_0 = c$, any $q \in \mathbb{C}$ with $|q| > 1$ and $r_0$ small enough.

**Example 2:** Let $S_{d_0,r_0,\delta_0}(1/2)$ be an open sector centered at $1/2$, not containing the origin. For all $j \in \mathbb{Z}$, we fix a determination of the logarithm $t \mapsto \log(1 - 2t)$ on $q^j S_{d_0,r_0,\delta_0}(1/2)$. Then, the function $u(t) = \exp(-(1/2)\log(1 - 2t))$ is a holomorphic solution of $u'(t) = (u(t))^3$ on $q^j S_{d_0,r_0,\delta_0}(1/2)$, for all $j \in \mathbb{Z}$. One checks that the assumptions **1.** and **2.** are satisfied for every $q \in \mathbb{C}$ with $|q| > 1$.

## 3.2 Composition series

In the next proposition, we construct holomorphic functions in two variables defined on open $q$-spirals using holomorphic functions with infinitely many variables constructed in the previous section and a solution of a nonlinear differential equation satisfying the assumptions **1.** and **2.**.

**Proposition 2** *Let $u(t)$ be a holomorphic solution of a nonlinear differential equation (8) satisfying the assumptions* **1.** *and* **2.** *from the section 3.1. Let $\phi(t, z, \underline{u}) \in \mathbb{SE}(T, X, |q|, c, c_0)$. We make the assumption that $(1+M)^c < |q|^{1/2}$ (for the constant $M > 0$ defined in Assumption* **2.**). *Then, the function*

$$w(t, z) = \sum_{l,\beta \geq 0} \phi_{l,\beta}((u(q^j t))_{0 \leq j \leq \beta}) \frac{t^l}{l!} \frac{z^\beta}{\beta!}$$

*is holomorphic on $q^{-\mathbb{N}} S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$. Moreover, the following estimates hold: there exist two constants $M' > 0$ (depending on $u(t)$ and $q$) and $C_\phi$ (depending on $\phi$) such that*

$$(10) \quad |w(t, z)| \leq C_\phi \exp(\frac{1}{2\log(|q|)}(\log(2(|t_0| + r_0)/T))^2) \times (1 + M'|q^k t - t_0|^{-m\mu})^{c_0} \exp(H_{1,c}(M)(\log(H_{2,c_0}(M)(1 + M'|q^k t - t_0|^{-m\mu})^c \frac{X_0}{X}))^2)$$

*for all $X_0 > 0$, for all $t \in q^{-k} S_{d_0,r_0,\delta_0}(t_0)$, for all $k \geq 0$, for all $|z| < X_0$, where the functions $H_{1,c}(r)$ and $H_{2,c_0}(r)$ are defined in the proposition 1.*

**Proof** Let $k \geq 0$, let $t = q^{-k}\tilde{t} \in q^{-k} S_{d_0,r_0,\delta_0}(t_0)$, where $\tilde{t} \in S_{d_0,r_0,\delta_0}(t_0)$. From the assumption **2.**, there exists a constant $M > 0$ such that $|u(q^{j-k}\tilde{t})| \leq M$, for all $j \neq k$, all $\tilde{t} \in S_{d_0,r_0,\delta_0}(t_0)$. From the assumption **1.**, there exists a constant $M' > M$ such that

$$(11) \qquad |u(\tilde{t})| \leq M' |\tilde{t} - t_0|^{-m\mu}$$

for all $\tilde{t} \in S_{d_0,r_0,\delta_0}(t_0)$. Using the estimates (3) for $\nu = k$, $T_0 = |t_0| + r_0$, $X_0 > 0$, $R = M$, $R_k = M'|\tilde{t} - t_0|^{-m\mu}$, we get that the function $w(t,z)$ is holomorphic on $q^{-k}S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$ and satisfies moreover the estimates (10). □

We state a lemma concerning the $n$-th derivative of the function $u(t)$, which can be easily checked by induction on $n \geq 1$, see [20].

**Lemma 2** *For all $n \geq 1$, there exists $Q_n(X) \in \mathbb{C}[X]$ with $\deg(Q_n) = n(d-1)+1$ such that*

$$u^{(n)}(t) = Q_n(u(t))$$

*for all $t \in S_{d_0,r_0,\delta_0}(t_0)$.*

In the following, we define the operator $\partial_z^{-1}$ of integration by $\partial_z^{-1}w(z) := \int_0^z w(z)dz$, for all entire functions $w(z)$ on $\mathbb{C}$. In the next proposition we compute the $n$-th derivative of the function $w(t,z)$ with respect to $t$.

**Proposition 3** *Let $w(t,z)$ be the holomorphic function constructed in Proposition 2 with the help of a function $\phi \in \mathbb{SE}(T,X,|q|,c,c_0)$. Let $k = (k_1,k_2) \in \mathbb{N}^2$, $\alpha = (\alpha_0,\alpha_1,\alpha_2) \in \mathbb{N}^3$ and $m_{0,k}, m_{1,k} \geq 0$. We have that*

$$(12) \quad (u(t))^{\alpha_0} t^{\alpha_1} z^{\alpha_2} (\partial_t^{k_1}\partial_z^{-k_2} w)(q^{m_{0,k}}t, zq^{-m_{1,k}}) =$$
$$\sum_{\kappa_1^1+\kappa_1^2=k_1,\kappa_1^1\geq 1} \frac{k_1!}{\kappa_1^1!\kappa_1^2!} \sum_{l\geq\alpha_1,\beta\geq k_2+\alpha_2} \Bigg( \sum_{(\lambda_0,\ldots,\lambda_{\beta-\alpha_2-k_2})\in A_{\beta-\alpha_2-k_2+1,\kappa_1^1}}$$
$$\{(u(t))^{\alpha_0} \partial_{u_0}^{\lambda_0} \cdots \partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}} \phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u(q^{j+m_{0,k}}t))_{0\leq j\leq\beta-\alpha_2-k_2}) \times \mathcal{A}_{l-\alpha_1,\beta-\alpha_2,u}(t)\}$$
$$\times q^{m_{0,k}(l-\alpha_1)} q^{-m_{1,k}(\beta-\alpha_2)} \frac{l!}{(l-\alpha_1)!} \frac{\beta!}{(\beta-\alpha_2)!}\Bigg) \frac{t^l}{l!}\frac{z^\beta}{\beta!}$$
$$+ \sum_{l\geq\alpha_1,\beta\geq k_2+\alpha_2} \Bigg((u(t))^{\alpha_0} \phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u(q^{j+m_{0,k}}t))_{0\leq j\leq\beta-\alpha_2-k_2})$$
$$\times q^{m_{0,k}(l-\alpha_1)} q^{-m_{1,k}(\beta-\alpha_2)} \frac{l!}{(l-\alpha_1)!}\frac{\beta!}{(\beta-\alpha_2)!}\Bigg) \frac{t^l}{l!}\frac{z^\beta}{\beta!}$$

*for all $(t,z) \in q^{-\mathbb{N}}S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$, where*

$$A_{\beta-\alpha_2-k_2+1,\kappa_1^1} = \{(\lambda_0,\ldots,\lambda_{\beta-\alpha_2-k_2}) \in \mathbb{N}^{\beta-\alpha_2-k_2+1} \;\; / \;\; 1 \leq \lambda_0 + \ldots + \lambda_{\beta-\alpha_2-k_2} \leq \kappa_1^1\}$$

*and*

$$\mathcal{A}_{l-\alpha_1,\beta-\alpha_2,u}(t) = \sum_{(k_j^h)_{1\leq j\leq\kappa_1^1,0\leq h\leq\beta-\alpha_2-k_2}\in B_{\beta-\alpha_2-k_2+1,\kappa_1^1}} \kappa_1^1! q^{(\sum_{j=1}^{\kappa_1^1}\sum_{h=0}^{\beta-\alpha_2-k_2} jhk_j^h)}$$
$$\times \prod_{j=1}^{\kappa_1^1} \frac{\prod_{h=0}^{\beta-\alpha_2-k_2}(Q_j(u(q^{h+m_{0,k}}t)))^{k_j^h}}{k_j^0!\cdots k_j^{\beta-\alpha_2-k_2}!(j!)^{\sum_{h=0}^{\beta-\alpha_2-k_2}k_j^h}}$$

*where*

$$B_{\beta-\alpha_2-k_2+1,\kappa_1^1}$$
$$= \{(k_j^h)_{1\leq j\leq\kappa_1^1,0\leq h\leq\beta-\alpha_2-k_2} \in \mathbb{N}^{\kappa_1(\beta-\alpha_2-k_2+1)} / \sum_{j=1}^{\kappa_1^1} k_j^h = \lambda_h, \sum_{j=1}^{\kappa_1^1} j(\sum_{h=0}^{\beta-\alpha_2-k_2} k_j^h) = \kappa_1^1\}$$

**Proof** From the Leibniz formula, we have that

(13)
$$\partial_t^{k_1}\partial_z^{-k_2}w(t,z) = \sum_{l\geq 0,\beta\geq k_2}\Big(\sum_{\kappa_1^1+\kappa_1^2=k_1, 0\leq \kappa_1^2\leq l}\frac{k_1!}{\kappa_1^1!\kappa_1^2!}\partial_t^{\kappa_1^1}(\phi_{l,\beta-k_2}((u(q^jt))_{0\leq j\leq \beta-k_2}))\frac{t^{l-\kappa_1^2}}{(l-\kappa_1^2)!}\frac{z^\beta}{\beta!}\Big)$$
$$= \sum_{\kappa_1^1+\kappa_1^2=k_1}\frac{k_1!}{\kappa_1^1!\kappa_1^2!}\Big(\sum_{l\geq 0,\beta\geq k_2}\partial_t^{\kappa_1^1}(\phi_{l+\kappa_1^2,\beta-k_2}((u(q^jt))_{0\leq j\leq \beta-k_2}))\frac{t^l}{l!}\frac{z^\beta}{\beta!}\Big)$$

Now, we recall the Faà di Bruno formula in several variables, obtained in [3], Corollary 2.11.

**Lemma 3** *Let $g^1(x),\ldots,g^m(x)\in\mathcal{O}(\Omega_{x_0})$ be $m\geq 1$ holomorphic functions on some neighborhood $\Omega_{x_0}$ of $x_0\in\mathbb{C}$ and let $f(y)\in\mathcal{O}(\Omega_{y_0})$ be a holomorphic function on some neighborhood $\Omega_{y_0}$ of $y_0=(g^1(x_0),\ldots,g^m(x_0))\in\mathbb{C}^m$. Then, for all $n\geq 1$, the $n$-th derivative of $h(x)=f(g^1(x),\ldots,g^m(x))$ is given by the following formula:*

$$\partial_x^n h(x) = \sum_{(\lambda_1,\ldots,\lambda_m)\in A_{m,n}}(\partial_{y_1}^{\lambda_1}\cdots\partial_{y_m}^{\lambda_m}f)(g^1(x),\ldots,g^m(x))$$
$$\times\Big(\sum_{(k_j^h)_{1\leq j\leq n,1\leq h\leq m}\in B_{m,n}} n!\prod_{j=1}^n\frac{(\partial_x^j g^1(x))^{k_j^1}\cdots(\partial_x^j g^m(x))^{k_j^m}}{k_j^1!\cdots k_j^m!(j!)^{\sum_{h=1}^m k_j^h}}\Big)$$

*where*

$$A_{m,n}=\{(\lambda_1,\ldots,\lambda_m)\in\mathbb{N}^m \ / \ 1\leq\lambda_1+\ldots+\lambda_m\leq n\},$$
$$B_{m,n}=\{(k_j^h)_{1\leq j\leq n,1\leq h\leq m}\in\mathbb{N}^{mn}/\sum_{j=1}^n k_j^h=\lambda_h,\sum_{j=1}^n j(\sum_{h=1}^m k_j^h)=n\}$$

From the Lemma 2 and 3, we deduce that

(14)
$$\partial_t^{\kappa_1^1}(\phi_{l+\kappa_1^2,\beta-k_2}((u(q^jt))_{0\leq j\leq\beta-k_2}))$$
$$= \sum_{(\lambda_0,\ldots,\lambda_{\beta-k_2})\in A_{\beta-k_2+1,\kappa_1^1}}\partial_{u_0}^{\lambda_0}\cdots\partial_{u_{\beta-k_2}}^{\lambda_{\beta-k_2}}\phi_{l+\kappa_1^2,\beta-k_2}((u(q^jt))_{0\leq j\leq\beta-k_2})$$
$$\times\Big(\sum_{(k_j^h)_{1\leq j\leq\kappa_1^1,0\leq h\leq\beta-k_2}\in B_{\beta-k_2+1,\kappa_1^1}}\kappa_1^1!q^{(\sum_{j=1}^{\kappa_1^1}\sum_{h=0}^{\beta-k_2}jhk_j^h)}\prod_{j=1}^{\kappa_1^1}\frac{\prod_{h=0}^{\beta-k_2}(Q_j(u(q^ht)))^{k_j^h}}{k_j^0!\cdots k_j^{\beta-k_2}!(j!)^{\sum_{h=0}^{\beta-k_2}k_j^h}}\Big)$$

for all $\kappa_1^1\geq 1$, for all $l\geq 0$, all $\beta\geq k_2$. Finally, from (13) and (14), multiplying by the function $(u(t))^{\alpha_0}t^{\alpha_1}z^{\alpha_2}$, we get the expansion (12). □

## 3.3 Linear operators on $\mathbb{SE}(T,X,|q|,c,c_0)$

We first define some linear operators that will be useful in the sequel.

**Definition 3** *For any tuples $k=(k_1,k_2)\in\mathbb{N}^2$, $\alpha=(\alpha_0,\alpha_1,\alpha_2)\in\mathbb{N}^3$, for non negative integers $m_{0,k},m_{1,k}\geq 1$ which satisfy the assumption that*

$$1\leq m_{0,k}\leq\alpha_2+k_2,$$

*we define the following linear operator* $\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}$ *from* $\mathbb{C}[[t,z,\underline{u}]]$ *into* $\mathbb{C}[[t,z,\underline{u}]]$ *by*

(15)

$$\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}(\phi(t,z,\underline{u})) := \sum_{\kappa_1^1+\kappa_1^2=k_1,\kappa_1^1\geq 1} \frac{k_1!}{\kappa_1^1!\kappa_1^2!} \sum_{l\geq\alpha_1,\beta\geq\alpha_2+k_2} \Bigg( \sum_{(\lambda_0,\ldots,\lambda_{\beta-\alpha_2-k_2})\in A_{\beta-\alpha_2-k_2+1,\kappa_1^1}} \{(u_0)^{\alpha_0}\partial_{u_0}^{\lambda_0}\cdots\partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}}\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u_{j+m_{0,k}})_{0\leq j\leq\beta-\alpha_2-k_2})\mathcal{A}_{l-\alpha_1,\beta-\alpha_2,\underline{u}}\} \times q^{m_{0,k}(l-\alpha_1)}q^{-m_{1,k}(\beta-\alpha_2)}\frac{l!}{(l-\alpha_1)!}\frac{\beta!}{(\beta-\alpha_2)!}\Bigg)\frac{t^l}{l!}\frac{z^\beta}{\beta!}$$
$$+ \sum_{l\geq\alpha_1,\beta\geq k_2+\alpha_2} \Big((u_0)^{\alpha_0}\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_{j+m_{0,k}})_{0\leq j\leq\beta-\alpha_2-k_2}) \times q^{m_{0,k}(l-\alpha_1)}q^{-m_{1,k}(\beta-\alpha_2)}\frac{l!}{(l-\alpha_1)!}\frac{\beta!}{(\beta-\alpha_2)!}\Big)\frac{t^l}{l!}\frac{z^\beta}{\beta!}$$

*where*

$$\mathcal{A}_{l-\alpha_1,\beta-\alpha_2,\underline{u}} = \sum_{(k_j^h)_{1\leq j\leq\kappa_1^1,0\leq h\leq\beta-\alpha_2-k_2}\in B_{\beta-\alpha_2-k_2+1,\kappa_1^1}} \kappa_1^1! q^{(\sum_{j=1}^{\kappa_1^1}\sum_{h=0}^{\beta-\alpha_2-k_2} jhk_j^h)} \times \prod_{j=1}^{\kappa_1^1} \frac{\prod_{h=0}^{\beta-\alpha_2-k_2}(Q_j(u_{h+m_{0,k}}))^{k_j^h}}{k_j^0!\cdots k_j^{\beta-\alpha_2-k_2}!(j!)^{\sum_{h=0}^{\beta-\alpha_2-k_2}k_j^h}}$$

*and where* $Q_j(X)$, $j \geq 0$, *are the polynomials introduced in the lemma 2.*

In the next proposition, we show that the linear operators constructed above are continuous on the spaces $\mathbb{SE}(T,X,|q|,c,c_0)$.

**Proposition 4** *We make the assumptions that*

$$(16)\quad \alpha_0 \leq c_0 \ , \ dk_1 \leq c(\alpha_2+k_2) \ , \ 1 \leq m_{0,k} < \alpha_2+k_2-2\alpha_1, \quad -\alpha_1+k_1+2(\alpha_2+k_2)+\frac{ck_1\log(3/2)}{\log(|q|)} < m_{1,k}$$

*Then, there exists a constant* $C_1 > 0$ *(depending on* $|q|$,$c$,$c_0$,$k$,$\alpha$,$m_{0,k}$,$m_{1,k}$ *and* $Q_j(X)$, $1 \leq j \leq k_1$*) such that*

$$(17)\qquad ||\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}(\phi(t,z,\underline{u}))||_{(T,X)} \leq C_1(1+\frac{1}{T})^{k_1}T^{\alpha_1}X^{\alpha_2+k_2}||\phi(t,z,\underline{u})||_{(T,X)}$$

*for all* $\phi \in \mathbb{SE}(T,X,|q|,c,c_0)$.

**Proof** Let

$$\phi(t,z,\underline{u}) = \sum_{l,\beta\geq 0}\phi_{l,\beta}((u_j)_{0\leq j\leq\beta})\frac{t^l}{l!}\frac{z^\beta}{\beta!}$$

be a formal series in $\mathbb{SE}(T,X,|q|,c,c_0)$. First a all, we start the proof with an important lemma.

**Lemma 4** **I)** *There exists a constant $C_0 > 0$ (depending on $c_0$,$\alpha_0$,$\kappa_1^1$ and $Q_j(X)$, $1 \leq j \leq \kappa_1^1$) such that*

$$(18)\quad ||(u_0)^{\alpha_0}(\partial_{u_0}^{\lambda_0} \cdots \partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}} \phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2})((u_{j+m_{0,k}})_{0 \leq j \leq \beta-\alpha_2-k_2}) \times \prod_{j=1}^{\kappa_1^1} \frac{\prod_{h=0}^{\beta-\alpha_2-k_2}(Q_j(u_{h+m_{0,k}}))^{k_j^h}}{k_j^0! \cdots k_j^{\beta-\alpha_2-k_2}!(j!)^{\sum_{h=0}^{\beta-\alpha_2-k_2} k_j^h}}||_\beta \leq C_0||\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u_j)_{0\leq j \leq \beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2} \times (\frac{3}{2})^{c\kappa_1^1\beta}$$

*for all $l \geq \alpha_1$, $\beta \geq \alpha_2 + k_2$, all $(\lambda_0,\ldots,\lambda_{\beta-\alpha_2-k_2}) \in A_{\beta-\alpha_2-k_2+1,\kappa_1^1}$ and all $(k_j^h)_{1 \leq j \leq \kappa_1^1, 0 \leq h \leq \beta-\alpha_2-k_2} \in B_{\beta-\alpha_2-k_2+1,\kappa_1^1}$.*
**II)** *We have*

$$(19)\quad ||(u_0)^{\alpha_0}\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_{j+m_{0,k}})_{0\leq j\leq\beta-\alpha_2-k_2})||_\beta \leq ||\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}$$

*for all $l \geq \alpha_1$, $\beta \geq \alpha_2 + k_2$.*

**Proof** We first show the part **I)**.
From the Cauchy formula in several variables, see [12], we have that

$$(20)\quad (\partial_{u_0}^{\lambda_0} \cdots \partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}} \phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2})(u_{m_{0,k}},\ldots,u_{\beta-\alpha_2-k_2+m_{0,k}}) = (\frac{1}{2i\pi})^{\beta-\alpha_2-k_2+1} \int_{C(u_{m_{0,k}},a_{m_{0,k}})} \cdots \int_{C(u_{\beta-\alpha_2-k_2+m_{0,k}},a_{m_{0,k}+\beta-\alpha_2-k_2})} \lambda_0! \ldots \lambda_{\beta-\alpha_2-k_2}! \frac{\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}(\xi_0,\ldots,\xi_{\beta-\alpha_2-k_2})}{(\xi_0 - u_{m_{0,k}})^{\lambda_0+1} \cdots (\xi_{\beta-\alpha_2-k_2} - u_{\beta-\alpha_2-k_2+m_{0,k}})^{\lambda_{\beta-\alpha_2-k_2}+1}} d\xi_0 \cdots d\xi_{\beta-\alpha_2-k_2}$$

for all $l \geq \alpha_1$, all $\beta \geq \alpha_2 + k_2$, all $(u_{m_{0,k}},\ldots,u_{\beta-\alpha_2-k_2+m_{0,k}}) \in \mathbb{C}^{\beta-\alpha_2-k_2+1}$, where $C(u,r)$ denotes the positively oriented circle centered at $u$ with radius $r > 0$ and where $a_{m_{0,k}+h}$, $0 \leq h \leq \beta - \alpha_2 - k_2$, are arbitrary positive real numbers. Introducing the term

$$\prod_{j=0}^{\beta-\alpha_2-k_2} (1 + |\xi_j|)^{-c(\beta-\alpha_2-k_2)-c_0}$$

and it's inverse under the integrals of the representation (20), we deduce that

$$(21)\quad |(\partial_{u_0}^{\lambda_0} \cdots \partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}} \phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2})(u_{m_{0,k}},\ldots,u_{\beta-\alpha_2-k_2+m_{0,k}})| \leq ||\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}(u_0,\ldots,u_{\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2} \times \frac{\lambda_0! \ldots \lambda_{\beta-\alpha_2-k_2}!}{a_{m_{0,k}}^{\lambda_0} \cdots a_{m_{0,k}+\beta-\alpha_2-k_2}^{\lambda_{\beta-\alpha_2-k_2}}} \times \prod_{j=0}^{\beta-\alpha_2-k_2} (1 + |u_{j+m_{0,k}}| + a_{j+m_{0,k}})^{c(\beta-\alpha_2-k_2)+c_0}$$

From the inequality (21), we get that

$$(22)\quad |(u_0)^{\alpha_0}(\partial_{u_0}^{\lambda_0}\cdots\partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}}\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2})(u_{m_{0,k}},\ldots,u_{\beta-\alpha_2-k_2+m_{0,k}})|$$
$$\times|\prod_{j=1}^{\kappa_1^1}\frac{\prod_{h=0}^{\beta-\alpha_2-k_2}(Q_j(u_{h+m_{0,k}}))^{k_j^h}}{k_j^0!\cdots k_j^{\beta-\alpha_2-k_2}!(j!)^{\sum_{h=0}^{\beta-\alpha_2-k_2}k_j^h}}|\times\prod_{j=0}^{\beta}(1+|u_j|)^{-c\beta-c_0}$$
$$\leq ||\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}(u_0,\ldots,u_{\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}\mathcal{E}(\beta,\underline{u})$$

where

$$(23)\quad \mathcal{E}(\beta,\underline{u})=\frac{\lambda_0!\ldots\lambda_{\beta-\alpha_2-k_2}!}{a_{m_{0,k}}^{\lambda_0}\cdots a_{m_{0,k}+\beta-\alpha_2-k_2}^{\lambda_{\beta-\alpha_2-k_2}}}|\prod_{j=1}^{\kappa_1^1}\frac{\prod_{h=0}^{\beta-\alpha_2-k_2}(Q_j(u_{h+m_{0,k}}))^{k_j^h}}{k_j^0!\cdots k_j^{\beta-\alpha_2-k_2}!(j!)^{\sum_{h=0}^{\beta-\alpha_2-k_2}k_j^h}}|$$
$$\times|u_0|^{\alpha_0}\times\prod_{j=0}^{m_{0,k}-1}(1+|u_j|)^{-c\beta-c_0}\times\prod_{j=\beta-\alpha_2-k_2+m_{0,k}+1}^{\beta}(1+|u_j|)^{-c\beta-c_0}$$
$$\times\prod_{j=m_{0,k}}^{\beta-\alpha_2-k_2+m_{0,k}}(1+|u_j|+a_j)^{c(\beta-\alpha_2-k_2)+c_0}\times\prod_{j=m_{0,k}}^{\beta-\alpha_2-k_2+m_{0,k}}(1+|u_j|)^{-c\beta-c_0}$$

for all $\beta\geq\alpha_2+k_2$, all $(u_0,\ldots,u_\beta)\in\mathbb{C}^{\beta+1}$. In the rest of the proof, we will give estimates for the expression $\mathcal{E}(\beta,\underline{u})$.

**Lemma 4.1** *We put*

$$(24)\quad A_h^\beta(u_{h+m_{0,k}},a_{h+m_{0,k}})=\frac{1}{a_{h+m_{0,k}}^{\lambda_h}}|\prod_{j=1}^{\kappa_1^1}(Q_j(u_{h+m_{0,k}}))^{k_j^h}|$$
$$\times(1+|u_{h+m_{0,k}}|+a_{h+m_{0,k}})^{c(\beta-\alpha_2-k_2)+c_0}\times(1+|u_{h+m_{0,k}}|)^{-c\beta-c_0}$$

*for all* $0\leq h\leq\beta-\alpha_2-k_2$.
**1)** *If* $\lambda_h=0$, *then we put* $a_{h+m_{0,k}}=0$, *and*

$$(25)\quad A_h^\beta(u_{h+m_{0,k}},a_{h+m_{0,k}})\leq 1$$

*for all* $u_{h+m_{0,k}}\in\mathbb{C}$, *all* $\beta\geq\alpha_2+k_2$.
**2)** *There exists a constant* $C_{0,1}>0$ *(depending on* $c_0$,$\kappa_1^1$ *and* $Q_j(X)$, $1\leq j\leq\kappa_1^1$*) with the following properties: for all* $\lambda_h\neq 0$, *there exists a radius* $a_{h+m_{0,k}}>0$ *such that*

$$(26)\quad A_h^\beta(u_{h+m_{0,k}},a_{h+m_{0,k}})\leq C_{0,1}(\frac{3}{2})^{c\beta}$$

*for all* $u_{h+m_{0,k}}\in\mathbb{C}$, *all* $\beta\geq\alpha_2+k_2$..

**Proof**
**Case 1)** If we have $\lambda_h=0$, then by construction of the set $B_{\beta-\alpha_2-k_2+1,\kappa_1^1}$ we have that $k_j^h=0$, for all $1\leq j\leq\kappa_1^1$. We can put $a_{h+m_{0,k}}=0$. Then, we get that

$$A_h^\beta(u_{h+m_{0,k}},0)=(1+|u_{h+m_{0,k}}|)^{-c(\alpha_2+k_2)}\leq 1$$

for all $u_{h+m_{0,k}} \in \mathbb{C}$, all $\beta \geq \alpha_2 + k_2$.

**Case 2)** Assume that $\lambda_h > 0$. From the lemma 2, we get two constants $E_0 > 0$ and $R_0 > 1$ such that

$$|Q_j(x)| \leq E_0 |x|^{j(d-1)+1} \tag{27}$$

for all $|x| > R_0$, for all $1 \leq j \leq \kappa_1^1$. Moreover, by construction of the set $B_{\beta-\alpha_2-k_2+1,\kappa_1^1}$, we have the following inequalities

$$\sum_{j=1}^{\kappa_1^1} k_j^h = \lambda_h \leq \kappa_1^1 \quad , \quad \sum_{j=1}^{\kappa_1^1} k_j^h j(d-1) + k_j^h \leq d\kappa_1^1. \tag{28}$$

In the following, we put $a_{h+m_{0,k}} = 1/2$. From (27) we deduce that

$$A_h^\beta(u_{h+m_{0,k}}, a_{h+m_{0,k}}) \leq \frac{1}{a_{h+m_{0,k}}^{\lambda_h}} E_0^{\sum_{j=1}^{\kappa_1^1} k_j^h} |u_{h+m_{0,k}}|^{\sum_{j=1}^{\kappa_1^1} k_j^h (j(d-1))+k_j^h} \\ \times (1 + |u_{h+m_{0,k}}| + a_{h+m_{0,k}})^{c(\beta-\alpha_2-k_2)+c_0} (1 + |u_{h+m_{0,k}}|)^{-c\beta-c_0},$$

for all $u_{h+m_{0,k}} \in \mathbb{C}$ with $|u_{h+m_{0,k}}| > R_0$. From (28), we get that

$$A_h^\beta(u_{h+m_{0,k}}, a_{h+m_{0,k}}) \leq 2^{\kappa_1^1} E_0^{\kappa_1^1} \times \frac{|u_{h+m_{0,k}}|^{d\kappa_1^1}}{(1 + |u_{h+m_{0,k}}| + 1/2)^{c(\alpha_2+k_2)}} \\ \times (\frac{1 + |u_{h+m_{0,k}}| + 1/2}{1 + |u_{h+m_{0,k}}|})^{c\beta+c_0}$$

for all $u_{h+m_{0,k}} \in \mathbb{C}$ with $|u_{h+m_{0,k}}| > R_0$. From the hypotheses (16), we have in particular that $d\kappa_1^1 \leq c(\alpha_2 + k_2)$. So that

$$A_h^\beta(u_{h+m_{0,k}}, a_{h+m_{0,k}}) \leq (2E_0)^{\kappa_1^1} (\frac{3}{2})^{c\beta+c_0} \tag{29}$$

for all $u_{h+m_{0,k}} \in \mathbb{C}$ with $|u_{h+m_{0,k}}| > R_0$. On the other hand, we have

$$A_h^\beta(u_{h+m_{0,k}}, a_{h+m_{0,k}}) \leq 2^{\kappa_1^1} \frac{\Pi_{j=1}^{\kappa_1^1} (\sup_{|z| \leq R_0} |Q_j(z)|)^{k_j^h}}{(1 + |u_{h+m_{0,k}}| + 1/2)^{c(\alpha_2+k_2)}} \times (\frac{1 + |u_{h+m_{0,k}}| + 1/2}{1 + |u_{h+m_{0,k}}|})^{c\beta+c_0}$$

for all $u_{h+m_{0,k}} \in \mathbb{C}$ with $|u_{h+m_{0,k}}| \leq R_0$. So that

$$A_h^\beta(u_{h+m_{0,k}}, a_{h+m_{0,k}}) \leq 2^{\kappa_1^1} \Pi_{j=1}^{\kappa_1^1} (\sup_{|z| \leq R_0} |Q_j(z)|)^{k_j^h} (\frac{3}{2})^{c\beta+c_0} \tag{30}$$

for all $u_{h+m_{0,k}} \in \mathbb{C}$ with $|u_{h+m_{0,k}}| \leq R_0$. Finally, from (29) and (30), we get the inequality (26). □

From the expression (23), we have that

$$\mathcal{E}(\beta, \underline{u}) \leq \lambda_0! \dots \lambda_{\beta-\alpha_2-k_2}! \prod_{j=1}^{\kappa_1^1} \frac{1}{k_j^0! \cdots k_j^{\beta-\alpha_2-k_2}! (j!)^{\sum_{h=0}^{\beta-\alpha_2-k_2} k_j^h}} \\ \times \prod_{h=0}^{\beta-\alpha_2-k_2} A_h^\beta(u_{h+m_{0,k}}, a_{h+m_{0,k}}) \times \frac{|u_0|^{\alpha_0}}{(1 + |u_0|)^{c_0}} \tag{31}$$

On the other hand, from the construction of the sets $A_{\beta-\alpha_2-k_2+1,\kappa_1^1}$, we have that

$$\lambda_0!\ldots\lambda_{\beta-\alpha_2-k_2}! \leq (\lambda_0+\ldots+\lambda_{\beta-\alpha_2-k_2})! \leq \kappa_1^1! \ , \quad \mathrm{card}\{h\in\{0,\ldots,\beta-\alpha_2-k_2\}/\lambda_h\neq 0\} \leq \kappa_1^1 \tag{32}$$

From the hypotheses (16), we have $\alpha_0 \leq c_0$, and from the inequalities (32) and the lemma 4.1, we deduce from (31) that

$$\mathcal{E}(\beta,\underline{u}) \leq \kappa_1^1!(C_{0,1})^{\kappa_1^1}(\frac{3}{2})^{c\kappa_1^1\beta} \tag{33}$$

for all $\beta \geq \alpha_2+k_2$, all $(u_0,\ldots,u_\beta)\in\mathbb{C}^{\beta+1}$. Finally, from the estimates (22) and (33), we deduce that the inequality (18) holds.

We show the part **II)**.
Introducing the term

$$\prod_{j=m_{0,k}}^{\beta+m_{0,k}-\alpha_2-k_2}(1+|u_j|)^{-c(\beta-\alpha_2-k_2)-c_0}$$

and it's inverse in the definition of the norm $||.||_\beta$, we get that

$$||(u_0)^{\alpha_0}\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_{j+m_{0,k}})_{0\leq j\leq\beta-\alpha_2-k_2})||_\beta \leq ||\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}\mathcal{F}(\beta,\underline{u}) \tag{34}$$

where

$$\mathcal{F}(\beta,\underline{u}) = \prod_{j=m_{0,k}}^{\beta+m_{0,k}-\alpha_2-k_2}(1+|u_j|)^{c(\beta-\alpha_2-k_2)+c_0-c\beta-c_0} \times |u_0|^{\alpha_0}\prod_{j=0}^{m_{0,k}-1}(1+|u_j|)^{-c\beta-c_0} \times \prod_{j=\beta+m_{0,k}-\alpha_2-k_2+1}^{\beta}(1+|u_j|)^{-c\beta-c_0} \tag{35}$$

for all $\beta \geq \alpha_2+k_2$, for all $(u_0,\ldots,u_\beta)\in\mathbb{C}^{\beta+1}$. From the hypothesis $\alpha_0 \leq c_0$ (see (16)), we get that

$$\mathcal{F}(\beta,\underline{u}) \leq 1 \tag{36}$$

for all $\beta \geq \alpha_2+k_2$, for all $(u_0,\ldots,u_\beta)\in\mathbb{C}^{\beta+1}$. From the estimates (34) and (36), we deduce that the inequality (19) holds. □

In the following, we put

$$\mathcal{P}_{1,\kappa_1^1,k_2,\alpha_2}(\beta) = \sum_{h=1}^{\kappa_1^1}\frac{(h+\beta-\alpha_2-k_2)^h}{h!} \ , \ \mathcal{P}_{2,\kappa_1^1,k_2,\alpha_2}(\beta) = \sum_{h=1}^{\kappa_1^1}\frac{(h+\kappa_1^1(\beta-\alpha_2-k_2+1)-1)^h}{h!}$$

for all $\beta \geq \alpha_2+k_2$. Using the classical fact that the number of $k$-tuples $(a_1,\ldots,a_k)\in\mathbb{N}^k$, solutions of the equation $a_1+\ldots+a_k=n$ is equal to $(n+k-1)!/((k-1)!n!)$, for all $k,n\geq 1$, we get that

$$\mathrm{Card}(A_{\beta-\alpha_2-k_2+1,\kappa_1^1}) \leq \mathcal{P}_{1,\kappa_1^1,k_2,\alpha_2}(\beta) \ , \ \mathrm{Card}(B_{\beta-\alpha_2-k_2+1,\kappa_1^1}) \leq \mathcal{P}_{2,\kappa_1^1,k_2,\alpha_2}(\beta) \tag{37}$$

for all $\beta \geq \alpha_2 + k_2$. We also have that

$$\sum_{j=1}^{\kappa_1^1} \sum_{h=0}^{\beta-\alpha_2-k_2} jhk_j^h \leq \beta\kappa_1^1 \tag{38}$$

for all $\beta \geq \alpha_2 + k_2$, all $(k_j^h)_{1 \leq j \leq \kappa_1^1, 0 \leq h \leq \beta-\alpha_2-k_2} \in B_{\beta-\alpha_2-k_2+1,\kappa_1^1}$. From the Lemma 4 I) and the estimates (37), (38), we deduce that

$$\begin{aligned}
(39)\quad & || \sum_{(\lambda_0,\ldots,\lambda_{\beta-\alpha_2-k_2}) \in A_{\beta-\alpha_2-k_2+1,\kappa_1^1}} \\
& \{(u_0)^{\alpha_0}\partial_{u_0}^{\lambda_0} \cdots \partial_{u_{\beta-\alpha_2-k_2}}^{\lambda_{\beta-\alpha_2-k_2}} \phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u_{j+m_{0,k}})_{0\leq j\leq \beta-\alpha_2-k_2})\mathcal{A}_{l-\alpha_1,\beta-\alpha_2,\underline{u}}\} \\
& \times q^{m_{0,k}(l-\alpha_1)} q^{-m_{1,k}(\beta-\alpha_2)} \times \frac{l!}{(l-\alpha_1)!}\frac{\beta!}{(\beta-\alpha_2)!}||_\beta \\
& \leq C_0 ||\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2} \times \mathcal{P}_{1,\kappa_1^1,k_2,\alpha_2}(\beta)\mathcal{P}_{2,\kappa_1^1,k_2,\alpha_2}(\beta)(\frac{3}{2})^{c\kappa_1^1\beta} \\
& \times |q|^{\kappa_1^1\beta + m_{0,k}(l-\alpha_1)-m_{1,k}(\beta-\alpha_2)} \times l^{\alpha_1}\beta^{\alpha_2}
\end{aligned}$$

for all $l \geq \alpha_1$, all $\beta \geq \alpha_2 + k_2$. From (39) and Lemma 4 II), we get that

$$\begin{aligned}
(40)\quad & ||\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}(\phi(t,z,\underline{u}))||_{(T,X)} \leq \sum_{\kappa_1^1+\kappa_1^2=k_1,\kappa_1^1\geq 1} \frac{k_1!}{\kappa_1^1!\kappa_1^2!} \\
& \times \Big( \sum_{l\geq\alpha_1,\beta\geq\alpha_2+k_2} \{\frac{||\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}}{|q|^{P(l,\beta)}} C_0 \\
& \times \mathcal{P}_{1,\kappa_1^1,\alpha_2,k_2}(\beta)\mathcal{P}_{2,\kappa_1^1,\alpha_2,k_2}(\beta)(\frac{3}{2})^{c\kappa_1^1\beta} \times |q|^{\kappa_1^1\beta+m_{0,k}(l-\alpha_1)-m_{1,k}(\beta-\alpha_2)} \times l^{\alpha_1}\beta^{\alpha_2}\}\frac{T^l}{l!}\frac{X^\beta}{\beta!}) \\
& + \sum_{l\geq\alpha_1,\beta\geq\alpha_2+k_2} \{\frac{||\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}}{|q|^{P(l,\beta)}} \\
& \times |q|^{m_{0,k}(l-\alpha_1)-m_{1,k}(\beta-\alpha_2)} \times l^{\alpha_1}\beta^{\alpha_2}\}\frac{T^l}{l!}\frac{X^\beta}{\beta!}
\end{aligned}$$

From (40), we deduce that

$$\begin{aligned}
(41)\quad & ||\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}(\phi(t,z,\underline{u}))||_{(T,X)} \leq \sum_{\kappa_1^1+\kappa_1^2=k_1,\kappa_1^1\geq 1} \frac{k_1!}{\kappa_1^1!\kappa_1^2!} \\
& \times \left( \sum_{l\geq\alpha_1,\beta\geq\alpha_2+k_2} \mathcal{M}_{l,\beta} \frac{||\phi_{l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}}{|q|^{P(l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2)}} \frac{T^{l-\alpha_1+\kappa_1^2}}{(l-\alpha_1+\kappa_1^2)!}\frac{X^{\beta-\alpha_2-k_2}}{(\beta-\alpha_2-k_2)!}\right) \\
& + \sum_{l\geq\alpha_1,\beta\geq\alpha_2+k_2} \mathcal{M}_{l,\beta}^1 \frac{||\phi_{l-\alpha_1+k_1,\beta-\alpha_2-k_2}((u_j)_{0\leq j\leq\beta-\alpha_2-k_2})||_{\beta-\alpha_2-k_2}}{|q|^{P(l-\alpha_1+k_1,\beta-\alpha_2-k_2)}} \frac{T^{l-\alpha_1+k_1}}{(l-\alpha_1+k_1)!}\frac{X^{\beta-\alpha_2-k_2}}{(\beta-\alpha_2-k_2)!}
\end{aligned}$$

where

$$\mathcal{M}_{l,\beta} = \frac{C_0 \mathcal{P}_{1,\kappa_1^1,k_2,\alpha_2}(\beta)\mathcal{P}_{2,\kappa_1^1,k_2,\alpha_2}(\beta)(\frac{3}{2})^{c\kappa_1^1\beta}}{|q|^{P(l,\beta)-P(l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2)+m_{1,k}(\beta-\alpha_2)-m_{0,k}(l-\alpha_1)-\kappa_1^1\beta}} \times l^{\alpha_1}\beta^{\alpha_2}\frac{(l-\alpha_1+\kappa_1^2)!}{l!}\frac{(\beta-\alpha_2-k_2)!}{\beta!}T^{\alpha_1-\kappa_1^2}X^{\alpha_2+k_2}$$

and

$$\mathcal{M}_{l,\beta}^1 = \frac{l^{\alpha_1}\beta^{\alpha_2}}{|q|^{P(l,\beta)-P(l-\alpha_1+k_1,\beta-\alpha_2-k_2)+m_{1,k}(\beta-\alpha_2)-m_{0,k}(l-\alpha_1)}} \times \frac{(l-\alpha_1+k_1)!}{l!}\frac{(\beta-\alpha_2-k_2)!}{\beta!}T^{\alpha_1-k_1}X^{\alpha_2+k_2}$$

for all $l \geq \alpha_1$, all $\beta \geq \alpha_2 + k_2$. From the assumptions (16), we get some constants $A, B, C > 0$ (depending on $|q|$,$c$,$\alpha$,$k$,$m_{0,k}$,$m_{1,k}$) with $0 < A < 1$, $0 < B < 1$, such that

$$\frac{(\frac{3}{2})^{c\kappa_1^1\beta}}{|q|^{P(l,\beta)-P(l-\alpha_1+\kappa_1^2,\beta-\alpha_2-k_2)+m_{1,k}\beta-m_{0,k}l-\kappa_1^1\beta}} = (|q|^{-\alpha_2-k_2-2(-\alpha_1+\kappa_1^2)+m_{0,k}})^l(|q|^{-m_{1,k}-\alpha_1+k_1+2(\alpha_2+k_2)+\frac{c\kappa_1^1\log(3/2)}{\log(|q|)}})^\beta \times |q|^{-(\alpha_2+k_2)(\kappa_1^2-\alpha_1)-(\alpha_2+k_2)^2-(\kappa_1^2-\alpha_1)^2} \leq CA^lB^\beta$$

and

$$\frac{1}{|q|^{P(l,\beta)-P(l-\alpha_1+k_1,\beta-\alpha_2-k_2)+m_{1,k}\beta-m_{0,k}l}} = (|q|^{-\alpha_2-k_2-2(-\alpha_1+k_1)+m_{0,k}})^l \times (|q|^{-m_{1,k}-\alpha_1+k_1+2(\alpha_2+k_2)})^\beta|q|^{-(\alpha_2+k_2)(k_1-\alpha_1)-(\alpha_2+k_2)^2-(k_1-\alpha_1)^2} \leq CA^lB^\beta$$

for all $l \geq \alpha_1$, all $\beta \geq \alpha_2 + k_2$. So that we get a constant $C_{1,1} > 0$ (depending on the constants $|q|$,$c$,$c_0$,$\alpha$,$k$,$m_{0,k}$,$m_{1,k}$ and $Q_j(X)$, $1 \leq j \leq \kappa_1^1$) and a constant $C_{1,2} > 0$ (depending on $\alpha$,$k$,$m_{0,k}$,$m_{1,k}$) such that

$$\mathcal{M}_{l,\beta} \leq C_{1,1}T^{\alpha_1-\kappa_1^2}X^{\alpha_2+k_2} \quad , \quad \mathcal{M}_{l,\beta}^1 \leq C_{1,2}T^{\alpha_1-k_1}X^{\alpha_2+k_2} \tag{42}$$

for all $l \geq \alpha_1$, all $\beta \geq \alpha_2 + k_2$. Finally, from (41) and (42), we deduce (17). □

### 3.4 A functional equation in $\mathbb{SE}(T, X, |q|, c, c_0)$

**Proposition 5** *We consider the following functional equation*

$$\phi(t, z, \underline{u}) = \sum_{k=(k_1,k_2)\in\mathcal{L}}\sum_{\alpha\in J_k} a_{\alpha,k}\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}(\phi(t, z, \underline{u})) + b(t, z, \underline{u}) \tag{43}$$

*where $q \in \mathbb{C}$ with $|q| > 1$, $\mathcal{L}$ is a finite subset of $\mathbb{N}^2$, $J_k$ are subsets of $\mathbb{N}^3$ and $m_{0,k}, m_{1,k} \geq 1$ are positive integers. We assume that there exist two real numbers $c, c_0 > 0$ such that*

$$\alpha_0 \leq c_0 \quad , \quad dk_1 \leq c(\alpha_2 + k_2) \quad , \tag{44}$$
$$1 \leq m_{0,k} < \alpha_2 + k_2 - 2\alpha_1 \quad , \quad -\alpha_1 + k_1 + 2(\alpha_2 + k_2) + \frac{ck_1\log(3/2)}{\log(|q|)} < m_{1,k},$$

*for all* $k = (k_1, k_2) \in \mathcal{L}$*, all* $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in J_k$*. We assume that* $a_{\alpha,k} \in \mathbb{C}$ *for all* $k \in \mathcal{L}$*, all* $\alpha \in J_k$ *and that* $b(t, z, \underline{u}) \in \mathbb{SE}(T_0, X_0, |q|, c, c_0)$*, for some* $T_0, X_0 > 0$*.*

*Then, for* $X$ *sufficiently small such that* $0 < X \leq X_0$ *(depending on* $T_0$*,*$|q|$*,*$c$*,*$c_0$*,*$k \in \mathcal{L}$*,*$\alpha \in J_k$*,*$a_{\alpha,k}$*,*$m_{0,k}$*,*$m_{1,k}$*,*$Q_j(X)$*,* $1 \leq j \leq k_1$*) the equation (43) has a unique solution* $\phi$ *is the Banach space* $\mathbb{SE}(T_0, X, |q|, c, c_0)$*.*

**Proof** We consider the map $\mathcal{N}$ from $\mathbb{C}[[t, z, \underline{u}]]$ into itself defined by

$$\mathcal{N}(\phi) = \sum_{k=(k_1,k_2)\in\mathcal{L}} \sum_{\alpha\in J_k} a_{\alpha,k}\mathcal{H}_{q,\alpha,k,m_{0,k},m_{1,k}}(\phi)$$

for all $\phi \in \mathbb{C}[[t, z, \underline{u}]]$. Under the assumption (44), we deduce from the proposition 4, that $\mathcal{N}$ is a linear map from $\mathbb{SE}(T_0, X, |q|, c, c_0)$ into itself satisfying the estimates

$$(45) \qquad ||\mathcal{N}(\phi)||_{(T_0,X)} \leq (C_1 \sum_{k=(k_1,k_2)\in\mathcal{L}} \sum_{\alpha=(\alpha_0,\alpha_1,\alpha_2)\in J_k} |a_{\alpha,k}|(1 + \frac{1}{T_0})^{k_1} T_0^{\alpha_1} X^{\alpha_2+k_2})||\phi||_{(T_0,X)}$$

for all $0 < X \leq X_0$, for all $\phi \in \mathbb{SE}(T_0, X, |q|, c, c_0)$. From the assumptions (44), we have that $\alpha_2 + k_2 \geq 1$, for all $k \in \mathcal{L}$, all $\alpha \in J_k$. So that if we choose $X$ sufficiently small such that $0 < X \leq X_0$ (depending on $C_1$,$T_0$, $a_{\alpha,k}$, for all $k \in \mathcal{L}$, all $\alpha \in J_k$), we get that

$$(46) \qquad ||\mathcal{N}(\phi)||_{(T_0,X)} \leq \frac{1}{2}||\phi||_{(T_0,X)}$$

for all $\phi \in \mathbb{SE}(T_0, X, |q|, c, c_0)$. If Id denotes the identity map on $\mathbb{SE}(T_0, X, |q|, c, c_0)$ defined by $\mathrm{Id}(\phi) = \phi$, we get that the map $\mathrm{Id} - \mathcal{N}$ is invertible from $\mathbb{SE}(T_0, X, |q|, c, c_0)$ into itself. By construction, we have that $b \in \mathbb{SE}(T_0, X, |q|, c, c_0)$. So that finally $\phi = (\mathrm{Id} - \mathcal{N})^{-1}b$ is the unique solution of (43) in $\mathbb{SE}(T_0, X, |q|, c, c_0)$. □

# 4 Linear $q$-difference-differential equations

In this section, we state the main result of this paper.

**Theorem 1** *Let* $u(t)$ *be a solution of a nonlinear differential equation (8) satisfying the assumptions* **1.** *and* **2.** *from the section 3.1. We consider the following linear Cauchy problem*

$$(47) \qquad \partial_z^S v(t, z) = \sum_{h=(h_1,h_2)\in\mathcal{S}} a_h(u(t), t, z)(\partial_t^{h_1}\partial_z^{h_2} v)(q^{m_{0,h}}t, zq^{-m_{1,h}})$$

*for initial conditions*

$$(48) \qquad (\partial_z^j v)(t, 0) = \omega_j(u(t), t) \quad , \quad 0 \leq j \leq S - 1,$$

*where* $S \geq 1$ *is an integer,* $\mathcal{S}$ *is a finite subset of* $\mathbb{N}^2$*,*

$$a_h(u, t, z) = \sum_{\alpha=(\alpha_0,\alpha_1,\alpha_2)\in J_h} a_{\alpha,h} u^{\alpha_0} t^{\alpha_1} z^{\alpha_2} \in \mathbb{C}[u, t, z],$$

*$\omega_j(u,t) \in \mathbb{C}[u,t]$, for $0 \leq j \leq S-1$, and $m_{0,h}, m_{1,h} \geq 1$ are integers, for all $h \in \mathcal{S}$. We assume that there exists a constant $c > 0$ such that*

$$(49) \quad (1+M)^c < |q|^{1/2} \quad , \quad dh_1 \leq c(\alpha_2 + S - h_2) \quad ,$$
$$S > h_2 \quad , \quad m_{0,h} \leq \alpha_2 \quad , \quad 1 \leq m_{0,h} < \alpha_2 + S - h_2 - 2\alpha_1,$$
$$-\alpha_1 + h_1 + 2(\alpha_2 + S - h_2) + \frac{ch_1 \log(3/2)}{\log(|q|)} < m_{1,h},$$

*for all $h = (h_1,h_2) \in \mathcal{S}$, all $\alpha = (\alpha_0,\alpha_1,\alpha_2) \in J_h$ (with the constant $M$ defined in the assumption* **2.***). If one writes*

$$\omega_j(u,t) = \sum_{(l,\beta)\in\Omega_j} \omega_{j,l,\beta} u^l t^\beta,$$

*where $\Omega_j$ are finite subsets of $\mathbb{N}^2$, we consider the polynomials*

$$|\omega|_j(u,t) = \sum_{(l,\beta)\in\Omega_j} |\omega_{j,l,\beta}| u^l t^\beta,$$

*for all $0 \leq j \leq S-1$.*

*Then, there exists a holomorphic function $v(t,z)$ on $q^{-\mathbb{N}} S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$ which solves the problem (47), (48). Moreover, the function $v(t,z)$ satisfies the following estimates :*

**i)** *There exist a constant $M' > 0$ (depending on $u(t)$ and $q$), a constant $c_0 > 0$ (depending on $a_h(u,t,z)$, $\omega_j(u,t)$ for $h \in \mathcal{S}$, $0 \leq j \leq S-1$) and $C_v > 0$ (depending on $v$) such that, for all $X_0 > 0$, all $T_0 > 0$, there exists $0 < X \leq X_0$ such that*

$$(50) \quad |v(t,z)| \leq \sum_{j=0}^{S-1} |\omega|_j(M'|t-t_0|^{-m\mu}, |t_0|+r_0)\frac{{X_0}^j}{j!}$$
$$+ C_v X_0^S \exp(\frac{1}{2\log(|q|)}(\log(2(|t_0|+r_0)/T_0))^2)$$
$$\times (1+M'|t-t_0|^{-m\mu})^{c_0} \exp(H_{1,c}(M)(\log(H_{2,c_0}(M)(1+M'|t-t_0|^{-m\mu})^c \frac{X_0}{X}))^2)$$

*for all $t \in S_{d_0,r_0,\delta_0}(t_0)$, for all $|z| < X_0$, where the functions $H_{1,c}(r)$ and $H_{2,c_0}(r)$ are defined in the proposition 1.*

**ii)** *There exist a constant $M' > 0$ (depending on $u(t)$ and $q$), a constant $c_0 > 0$ (depending on $a_h(u,t,z)$, $\omega_j(u,t)$ for $h \in \mathcal{S}$, $0 \leq j \leq S-1$) and $C_v > 0$ (depending on $v$) such that, for all $X_0 > 0$, all $T_0 > 0$, there exists $0 < X \leq X_0$ such that*

$$(51) \quad |v(t,z)| \leq \sum_{j=0}^{S-1} |\omega|_j(M, |t_0|+r_0)\frac{{X_0}^j}{j!} + C_v X_0^S \exp(\frac{1}{2\log(|q|)}(\log(2(|t_0|+r_0)/T_0))^2)$$
$$\times (1+M'|q^k t-t_0|^{-m\mu})^{c_0} \exp(H_{1,c}(M)(\log(H_{2,c_0}(M)(1+M'|q^k t-t_0|^{-m\mu})^c \frac{X_0}{X}))^2)$$

*for all $t \in q^{-k} S_{d_0,r_0,\delta_0}(t_0)$, for all $k \geq 1$, for all $|z| < X_0$.*

**Proof** Let us consider the function

$$I(t,z) = \sum_{j=0}^{S-1} \omega_j(u(t),t)\frac{z^j}{j!},$$

for all $(t,z) \in q^{-\mathbb{N}} S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$. Using Lemma 2 we get that

$$(\partial_t^{h_1}\partial_z^{h_2} I)(q^{m_{0,h}}t, z/q^{m_{1,h}}) = \sum_{j=0}^{S-1-h_2} \tilde{\omega}_j(u(q^{m_{0,h}}t),t)\frac{z^j}{j!}$$

for all $h=(h_1,h_2) \in \mathcal{S}$, where $\tilde{\omega}_j(u,t) \in \mathbb{C}[u,t]$, for $0 \leq j \leq S-1-h_2$. Now, we choose $c_0 > 0$ large enough such that $c_0 \geq \alpha_0$, for all $\alpha = (\alpha_0,\alpha_1,\alpha_2) \in J_h$, all $h \in \mathcal{S}$ and such that

$$b_h(t,z,u_0,u_{m_{0,h}}) = a_h(u_0,t,z)(\sum_{j=0}^{S-1-h_2} \tilde{\omega}_j(u_{m_{0,h}},t)\frac{z^j}{j!}) \in \mathbb{SE}(T_0,X_0,|q|,c,c_0)$$

for all $T_0, X_0 > 0$, for all $h \in \mathcal{S}$. In the following, we put

$$b(t,z,\underline{u}) := \sum_{h=(h_1,h_2)\in\mathcal{S}} b_h(t,z,u_0,u_{m_{0,h}}).$$

We consider now the functional equation

$$\phi(t,z,\underline{u}) = \sum_{h=(h_1,h_2)\in\mathcal{S}} \sum_{\alpha \in J_h} a_{\alpha,h}\mathcal{H}_{q,\alpha,(h_1,S-h_2),m_{0,h},m_{1,h}}(\phi(t,z,\underline{u})) + b(t,z,\underline{u}). \tag{52}$$

From the assumptions (49), we get that the assumptions (44) are fulfilled for the equation (52). From the proposition 5, we deduce that for $0 < X \leq X_0$ small enough, the equation (52) has a unique solution $\phi$ is the Banach space $\mathbb{SE}(T_0,X,|q|,c,c_0)$. Now, we consider the function $w(t,z) = \phi(t,z,(u(q^j t))_{j\geq 0})$, which is, from Proposition 2, holomorphic on $q^{-\mathbb{N}} S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$. From the proposition 3, we have that the function $w$ solves the equation

$$w(t,z) = \sum_{h=(h_1,h_2)\in\mathcal{S}} a_h(u(t),t,z)(\partial_t^{h_1}\partial_z^{h_2-S}w)(q^{m_{0,h}}t, zq^{-m_{1,h}}) + b(t,z,(u(q^j t))_{j\geq 0}) \tag{53}$$

for all $(t,z) \in q^{-\mathbb{N}} S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$. We consider the function $v(t,z) = I(t,z) + \partial_z^{-S}w(t,z)$ which is holomorphic for all $(t,z) \in q^{-\mathbb{N}} S_{d_0,r_0,\delta_0}(t_0) \times \mathbb{C}$. Using (53), one checks that $v(t,z)$ solves the problem (47), (48). Moreover, from the proposition 2, $w(t,z)$ satisfies estimates of the form (10). We deduce that the function $\partial_z^{-S}w(t,z)$ satisfies the following estimates : there exist two constants $M' > 0$ (depending on $u(t)$ and $q$) and $C_\phi$ (depending on $\phi$) such that

$$\begin{aligned} |\partial_z^{-S}w(t,z)| \leq X_0^S C_\phi \exp(\frac{1}{2\log(|q|)}(\log(2(|t_0|+r_0)/T_0))^2) \\ \times (1+M'|q^k t - t_0|^{-m\mu})^{c_0} \exp(H_{1,c}(M)(\log(H_{2,c_0}(M)(1+M'|q^k t - t_0|^{-m\mu})^c \frac{X_0}{X}))^2) \end{aligned} \tag{54}$$

for all $X_0 > 0$, for all $t \in q^{-k} S_{d_0,r_0,\delta_0}(t_0)$, for all $k \geq 0$, for all $|z| < X_0$, where the functions $H_{1,c}(r)$ and $H_{2,c_0}(r)$ are defined in the proposition 1. On the other hand, we have that

$$|I(t,z)| \leq \sum_{j=0}^{S-1} |\omega|_j(|u(t)|, |t_0|+r_0)\frac{X_0^j}{j!} \tag{55}$$

for all $X_0 > 0$, for all $t \in q^{-k} S_{d_0,r_0,\delta_0}(t_0)$, for all $k \geq 0$, for all $|z| < X_0$. Finally, from (54) and (55), we get the estimates (50), (51). □

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