Non-strictly black body spectrum from the tunnelling mechanism

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Abstract

A modern and largely used approach to obtain Hawking radiation is the tunnelling mechanism. However, in various papers in the literature, the analysis concerned almost only to obtain the Hawking temperature through a comparison of the probability of emission of an outgoing particle with the Boltzmann factor.

In a interesting and well written paper, Banerjee and Majhi improved the approach, by explicitly finding a black body spectrum associated with black holes. On the other hand, this result, which has been obtained by using a reformulation of the tunnelling mechanism, is in contrast which the remarkable result by Parikh and Wilczek, that, indeed, found a probability of emission which is compatible with a non-strictly thermal spectrum.

By using our recent introduction of an effective state for a black hole, here we solve such a contradiction, through a slight modification of the analysis by Banerjee and Majhi. The final result will be a non-strictly black body spectrum from the tunnelling mechanism.

1 Introduction

In recent years, the tunnelling mechanism has been an elegant and largely used approach to obtain Hawking radiation [1], see for example [2]-[6] and refs. within. A problem on such an approach was that, in the cited and in other papers in the literature, the analysis has been finalized almost only to obtain the Hawking temperature through a comparison of the probability of emission of an outgoing particle with the Boltzmann factor. The problem was apparently solved in the interesting work [7], where, through a reformulation of the tunnelling mechanism, a black body spectrum associated with black holes has been found. In any case, this result is in contrast which the remarkable result in [2, 3], that, indeed, found a probability of emission which is compatible with a non-strictly thermal spectrum.

By introducing an *effective state*, we recently interpreted black hole's quasi-normal modes in terms of quantum levels by finding a natural connection between Hawking radiation and quasi-normal modes [8, 9]. Here we show that the *effective quantities* permit also to solve the above cited contradiction, through a slight modification of the analysis in [7]. The final result will be a non-strictly black body spectrum from the tunnelling mechanism.

For the sake of simplicity, in this paper we refer to the Schwarzschild black hole and we work with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units).

2 A review of the strictly thermal analysis

We emphasize that in this Section we closely follow [7]. Let us consider a Schwarzschild black hole. The Schwarzschild line element is (but see [11] for clarifying historical notes to this notion)

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2). \qquad (1)$$

The event horizon is defined by $r_H = 2M$ [7, 10]. As we want to discuss Hawking radiation like tunnelling, the radial trajectory is relevant [2, 3, 7]. Hence, we consider only the $(r\text{—}t)$ sector of the line element (1) [7]. Let us consider a Klein-Gordon massless field φ which obeys to [7]

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = 0, \qquad (2)$$

which, considering only the $(r\text{—}t)$ sector, becomes in the Schwarzschild space-time (1) [7]

$$-\frac{1}{1 - \frac{2M}{r})}\partial_t^2\varphi + \frac{1}{4M}\partial_r\varphi + (1 - \frac{2M}{r})\partial_r^2\varphi = 0, \qquad (3)$$

where $\frac{1}{4M}$ is the black hole's surface gravity.

The standard WKB ansatz gives [7]

$$\varphi(r,t) = \exp\left[-iS(r,t)\right]. \qquad (4)$$

$S(r,t)$ can be expanded as [7]

$$S(r,t) = S_0(r,t) + \sum_{m=1}^{m=\infty} S_m(r,t). \qquad (5)$$

Inserting eq. (5) in eq. (3) and taking the semi-classical limit (the Planck constant $\to 0$), one gets [7]

$$\partial_t S_0(r,t) = \pm(1 - \frac{2M}{r})\partial_R S_0(r,t), \qquad (6)$$

which is the well know semi-classical Hamilton-Jacobi equation [4, 7, 12]. Eq. (6) can be also obtained starting from Dirac [7, 13] or Maxwell equations [7, 14]. Eq. (6) depends also on imposing the chirality (holomorphic) condition on φ with the WKB ansatz in eq. (4), with the $+\,-$ solutions standing for the left (right) modes [15].

Considering the timelike Killing vector of the stationary Schwarzschild line element (1), one chooses an ansatz for $S_0(r,t)$ as [7]

$$S_0(r,t) = \omega t + \tilde{S}_0(r). \tag{7}$$

ω in eq. (7) is the conserved quantity which corresponds to the timelike Killing vector [7]. Inserting eq. (7) in eq. (6) one gets a solution for $\tilde{S}_0(r)$, that re-inserted in eq. (7) gives [7]

$$S_0(r,t) = \omega(t \pm r_*), \tag{8}$$

where $r_* \equiv \int \frac{dr}{1-\frac{2M}{r}}$. Now, one introduces the sets of null tortoise coordinates defined as [7]

$$\begin{aligned} u &\equiv t - r_* \\ v &\equiv t + r_*. \end{aligned} \tag{9}$$

These coordinates are defined inside and outside the event horizon [7]. If one expresses eq. (8) in terms of the tortoise coordinates (9) and then substitutes in eq. (4) , the right and left modes for both sectors can be written down as [7]

$$\begin{aligned} \left[\varphi^{(R)}\right]_{in} &= \left[\exp\left(-i\omega u\right)\right]_{in}; \qquad \left[\varphi^{(L)}\right]_{in} = \left[\exp\left(-i\omega v\right)\right]_{in} \\ \left[\varphi^{(R)}\right]_{out} &= \left[\exp\left(-i\omega u\right)\right]_{out} \qquad \left[\varphi^{(L)}\right]_{out} = \left[\exp\left(-i\omega v\right)\right]_{out}. \end{aligned} \tag{10}$$

In the tunnelling framework, after the production of a virtual pair of particles, one member of such a pair tunnels through the horizon in a quantum mechanical way [7]. The nature of the coordinates changes while the particle crosses the horizon [2, 3, 7]. One takes into account this by using with Kruskal coordinates, which are a type of coordinates viable on both sides of the horizon [7]. The Kruskal time coordinate T and the Krsuskal space coordinate X are defined, inside and outside the horizon, as [7, 10, 16]

$$\begin{aligned} T_{in} &\equiv \left[\exp\left(\tfrac{r_{*in}}{4M}\right)\right]\cosh\left(\tfrac{t_{in}}{4M}\right); \qquad X_{in} \equiv \left[\exp\left(\tfrac{r_{*in}}{4M}\right)\right]\sinh\left(\tfrac{t_{in}}{4M}\right) \\ T_{out} &\equiv \left[\exp\left(\tfrac{r_{*out}}{4M}\right)\right]\sinh\left(\tfrac{t_{out}}{4M}\right); \qquad X_{out} \equiv \left[\exp\left(\tfrac{r_{*out}}{4M}\right)\right]\cosh\left(\tfrac{t_{out}}{4M}\right). \end{aligned} \tag{11}$$

The connection between the two sets of coordinates is given by [7, 10, 16]

$$\begin{aligned} t_{in} &\to t_{out} - 2\pi i M \\ r_{*in} &\to r_{*out} + 2\pi i M. \end{aligned} \tag{12}$$

Eq. (12) implies $T_{in} \to T_{out}$ and $T_{in} \to T_{out}$ [7]. The relations which connect the null coordinates defined inside and outside the horizon can be obtained using the definition (9) [7]

$$\begin{aligned} u_{in} = t_{in} - r_{*_{in}} \to u_{out} - 2\pi i M \\ v_{in} = t_{in} + r_{*_{in}} \to v_{out}. \end{aligned} \tag{13}$$

The two eqs. (13) imply that the inside and outside modes are connected by [7]

$$\begin{aligned} \left[\varphi^{(R)}\right]_{in} = \exp\left(-4\pi M\omega\right)\left[\varphi^{(R)}\right]_{out} \\ \left[\varphi^{(L)}\right]_{in} = \left[\varphi^{(L)}\right]_{out}. \end{aligned} \tag{14}$$

In order to find the spectrum, one starts to consider n non-interacting virtual pairs created inside the black hole [7]. The modes defined in the first set of eq. (10) represent each pair [7]. In that way, when observed from outside, the physical state of the system can be written down by using the transformations (14) [7]

$$|\Psi>= N\sum_n |n_{in}^{(L)}> \otimes |n_{in}^{(R)}> \to N\sum_n \exp\left(-4\pi n M\omega\right)|n_{out}^{(L)}> \otimes |n_{out}^{(R)}> \tag{15}$$

N in eq. (15) is a normalization constant, while represents $|n_{out}^{(L)}>$ the number n of left going modes [7]. By applying the normalization condition $<\Psi|\Psi>=1$ one gets [7]

$$N = \left(\sum_n \exp\left(-8\pi n M\omega\right)\right)^{-\frac{1}{2}}. \tag{16}$$

We recall that $n = 0, 1, 2, 3, ...$ for bosons and $n = 0, 1$ for fermions respectively [7]. Hence, the two values of the normalization constant are [7]

$$\begin{aligned} N_{boson} = \left(1 - \exp\left(-8\pi n M\omega\right)\right)^{\frac{1}{2}} \\ N_{fermion} = \left(1 + \exp\left(-8\pi n M\omega\right)\right)^{-\frac{1}{2}}. \end{aligned} \tag{17}$$

Thus, one writes down the (normalized) physical states of the system for bosons and fermions as [7]

$$\begin{aligned} |\Psi>_{boson} = \left(1 - \exp\left(-8\pi n M\omega\right)\right)^{\frac{1}{2}} \sum_n \exp\left(-4\pi n M\omega\right)|n_{out}^{(L)}> \otimes |n_{out}^{(R)}> \\ |\Psi>_{fermion} = \left(1 + \exp\left(-8\pi n M\omega\right)\right)^{-\frac{1}{2}} \sum_n \exp\left(-4\pi n M\omega\right)|n_{out}^{(L)}> \otimes |n_{out}^{(R)}> \end{aligned} \tag{18}$$

Hereafter we focus the analysis only on bosons. In fact, for fermions the analysis is identical [7]. The density matrix operator of the system is [7]

$$\hat{\rho}_{boson} \equiv \Psi >_{boson} < \Psi|_{boson}$$

$$= (1 - \exp(-8\pi n M\omega)) \sum_{n,m} \exp[-4\pi(n+m)M\omega] |n_{out}^{(L)} > \otimes |n_{out}^{(R)} >< m_{out}^{(R)}| \otimes < m_{out}^{(L)}|. \quad (19)$$

If one traces out the ingoing modes, the density matrix for the outgoing (right) modes reads [7]

$$\hat{\rho}_{boson}^{(R)} = (1 - \exp(-8\pi n M\omega)) \sum_n \exp(-8\pi n M\omega) |n_{out}^{(R)} >< n_{out}^{(R)}|. \quad (20)$$

This implies that the average number of particles detected at infinity is [7]

$$< n >_{boson} = tr\left[\hat{\rho}_{boson}^{(R)}\right] = \frac{1}{\exp(-8\pi n M\omega) - 1}, \quad (21)$$

where the trace has been taken over all the eigenstates and the final result has been obtained through a bit of algebra, see [7] for details. The result of eq. (21) is the well known Bose-Einstein distribution. A similar analysis works also for fermions [7], and one easily gets the well known Fermi-Dirac distribution

$$< n >_{fermion} = \frac{1}{\exp(-8\pi n M\omega) + 1}, \quad (22)$$

Both the distributions correspond to a black body spectrum with the Hawking temperature [1, 7]

$$T_H \equiv \frac{1}{8\pi M}. \quad (23)$$

3 Non-thermal correction

The result in [7], that we reviewed in previous Section, is remarkable. In fact, we have see that, through a reformulation of the tunnelling mechanism, one can found a black body spectrum associated with black holes which is in perfect agreement with the famous original result by Hawking [1]. On the other hand, it is in contrast with another remarkable result [2, 3]. In fact, the probability of emission connected with the two distributions (21) and (22) is given by [1, 2, 3]

$$\Gamma \sim \exp(-\frac{\omega}{T_H}). \quad (24)$$

But in [2, 3] a remarkable correction, through an exact calculation of the action for a tunnelling spherically symmetric particle, has been found, yielding

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1 - \frac{\omega}{2M})]. \quad (25)$$

This important result, which is clearly in contrast with the result in [7], that we reviewed in previous Section, enables a correction, the additional term $\frac{\omega}{2M}$

[2, 3]. The important difference is that the authors of [7] did not taken into due account the conservation of the energy, which generates a dynamical instead of static geometry of the black hole [2, 3]. In other words, the energy conservation forces the black hole to contract during the process of radiation [2, 3]. Therefore, the horizon recedes from its original radius, and, at the end of the emission, the radius becomes smaller [2, 3]. The consequence is that black holes do not strictly emit like black bodies [2, 3].

It is important to recall that the tunnelling is a *discrete* instead of *continuous* process [8]. In fact, two different *countable* black hole's physical states must be considered, the physical state before the emission of the particle and the physical state after the emission of the particle [8]. Thus, the emission of the particle can be interpreted like a *quantum transition* of frequency ω between the two discrete states [8]. In the language of the tunnelling mechanism, a trajectory in imaginary or complex time joins two separated classical turning points [2, 3]. Another important consequence is that the radiation spectrum is also discrete [8]. Let us clarify this important issue in a better way. At a well fixed Hawking temperature and the statistical probability distribution (25) are continuous functions. On the other hand, the Hawking temperature in (25) varies in time with a character which is *discrete*. In fact, the forbidden region traversed by the emitting particle has a *finite* size [3]. Considering a strictly thermal approximation, the turning points have zero separation. Therefore, it is not clear what joining trajectory has to be considered because there is not barrier [3]. The problem is solved if we argue that the forbidden finite region from $r_{initial} = 2M$ to $r_{final} = 2(M-\omega)$ that the tunnelling particle traverses works like barrier [3]. Thus, the intriguing explanation is that it is the particle itself which generates a tunnel through the horizon [3].

A good way to take into due account the dynamical geometry of the black hole during the emission of the particle is to introduce the black hole's *effective state*. By introducing the *effective temperature* [8, 9]

$$T_E(\omega) \equiv \frac{2M}{2M-\omega} T_H = \frac{1}{4\pi(2M-\omega)}, \tag{26}$$

one re-writes eq. (26) in a Boltzmann-like form similar to the original probability found by Hawking

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}), \tag{27}$$

where $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor,* with $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$ [8, 9]. Hence, the effective temperature replaces the Hawking temperature in the equation of the probability of emission [8, 9]. Let us discuss the physical interpretation. In various fields of science, we can takes into account the deviation from the thermal spectrum of an emitting body by introducing an effective temperature. It represents the temperature of a black body that would emit the same total amount of radiation. We introduced the concept of effective temperature in the black hole's physics in [8, 9]. $T_E(\omega)$ depends on the energy-frequency of the emitted radiation and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the radiation spectrum of a black hole from the strictly thermal feature [8, 9].

The introduction of $T_E(\omega)$ permits the introduction of others *effective quantities*. In fact, let us consider the initial mass of the black hole *before* the emission, M, and the final mass of the hole *after* the emission, $M-\omega$ respectively [8, 9]. The *effective mass* and the *effective horizon* of the black hole *during* its contraction, i.e. *during* the emission of the particle, are defined as [8, 9]

$$M_E \equiv M - \frac{\omega}{2}, \ r_E \equiv 2M_E. \tag{28}$$

The above effective quantities are average quantities [8, 9]. r_E is the average of the initial and final horizons and M_E is the average of the initial and final masses [8, 9]. Therefore, T_E is the inverse of the average value of the inverses of the initial and final Hawking temperatures (*before* the emission $T_{H \text{ initial}} = \frac{1}{8\pi M}$, *after* the emission $T_{H \text{ final}} = \frac{1}{8\pi(M-\omega)}$ respectively) [8, 9]. Thus, the Hawking temperature *has a discrete character in time*.

We stress that the introduction of the effective temperature does not degrade the importance of the Hawking temperature. Indeed, as the Hawking temperature changes with a discrete behavior in time, it is not clear which value of such a temperature has to be associated to the emission of the particle. Has one to consider the value of the Hawking temperature *before* the *emission* or the value of the Hawking temperature after the emission? The answer is that one must consider an *intermediate* value, the effective temperature, which is the inverse of the average value of the inverses of the initial and final Hawking temperatures. In a certain sense, it represents the value of the Hawking temperature *during* the emission. $T_E(\omega)$ takes into account the non-strictly thermal character of the radiation spectrum and the non-strictly continuous character of subsequent emissions of Hawking quanta.

Therefore, one can define two further effective quantities. The *effective Schwarzschild line element* is given by

$$ds^2 \equiv -(1 - \frac{2M_E}{r})dt^2 + \frac{dr^2}{1 - \frac{2M_E}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{29}$$

and, consequently, the *effective surface gravity* is defined as $\frac{1}{4M_E}$. Thus, the effective line element (29) takes into account the *dynamical* geometry of the black hole during the emission of the particle. Now, one can replace eq. (3) for the $(r\text{—}t)$ sector with the *effective equation*

$$-\frac{1}{1 - \frac{2M_E}{r})}\partial_t^2\varphi + \frac{1}{4M_E}\partial_r\varphi + (1 - \frac{2M_E}{r})\partial_r^2\varphi = 0. \tag{30}$$

In analogous way, putting $r_{*E} \equiv \int \frac{dr}{1 - \frac{2M_E}{r}}$ the two eqs. (9) are replaced by the *effective tortoise coordinates*

$$\begin{aligned} u &\equiv t - r_{*E} \\ v &\equiv t + r_{*E}. \end{aligned} \tag{31}$$

Clearly, if one follows step by step the analysis in Section 2, at the end obtains the correct physical states for boson and fermions as

$$|\Psi>_{boson}= (1-\exp{(-8\pi n M_E \omega)})^{\frac{1}{2}} \sum_n \exp{(-4\pi n M_E \omega)} \, |n_{out}^{(L)}> \otimes |n_{out}^{(R)}>$$

$$|\Psi>_{fermion}= (1+\exp{(-8\pi n M_E \omega)})^{-\frac{1}{2}} \sum_n \exp{(-4\pi n M_E \omega)} \, |n_{out}^{(L)}> \otimes |n_{out}^{(R)}> \tag{32}$$

and the correct distributions as

$$\begin{array}{c} <n>_{boson}= \frac{1}{\exp(-8\pi n M_E\omega)-1} = \frac{1}{\exp[-4\pi n(M-\omega)\omega]-1} \\ \\ <n>_{fermion}= \frac{1}{\exp(-8\pi n M_E\omega)+1} = \frac{1}{\exp[-4\pi n(M-\omega)\omega]+1}, \end{array} \tag{33}$$

which represent the distributions associated to the probability of emission (25).

We recall that, this deviation from strict thermality is consistent with unitarity [3, 8, 9] and has profound implications for the black hole information puzzle because arguments that information is lost during black hole's evaporation rely in part on the assumption of strict thermal behavior of the spectrum [3, 8, 9, 17]. In other words, the process of black hole's evaporation should be unitary, information should be preserved and the underlying quantum gravity theory should be unitary too.

4 Conclusion remarks

In the remarkable paper [7] the tunnelling approach on Hawking radiation has been improved by explicitly finding a black body spectrum associated with black holes. But a problem is this result, which has been obtained by using a reformulation of the tunnelling mechanism, is in contrast which the other remarkable result in [2, 3], that, indeed, found a probability of emission which is compatible with a non-strictly thermal spectrum.

By using our recent introduction of an effective state for a black hole [8, 9] in this paper we solved such a contradiction, through a slight modification of the analysis in [7]. The final result consists in a non-strictly black body spectrum from tunnelling mechanism.

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