Time dependent Schroedinger equation for black holes: No information loss

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Abstract

Introducing a black hole's *effective temperature*, we obtained an Honorable Mention in the 2012 Gravity Research Foundation Essay Competition interpreting black hole's quasi-normal modes naturally in terms of quantum levels. This permitted us to show that quantities which are fundamental to realize the underlying quantum gravity theory, like Bekenstein-Hawking entropy, its sub-leading corrections and the number of microstates, are function of the black hole's excited state, i.e. of the black hole's quantum level, which is symbolized by the quantum "overtone" number n that denotes the countable sequence of quasi-normal modes. Here, we improve the analysis by finding a fundamental equation that directly connects the probability of emission of an Hawking quantum with the two emission levels which are involved in the transition. That equation permits us to interpret the correspondence between Hawking radiation and black hole quasi-normal modes in terms of a *time dependent Schroedinger system.* In such a system, the quasi-normal modes energies, which are also the total energies emitted by the black hole in correspondence of the various quantum levels, represent the eigenvalues of the unperturbed Hamiltonian of the system and the Hawking quanta represent the energy transitions among the eigenvalues, which correspond to a perturbed Hamiltonian $\propto \delta(t)$. In this way, we explicitly write down a *time dependent Schroedinger equation* for the system composed by Hawking radiation and black hole quasi-normal modes. The states of the correspondent *Schroedinger wave-function* can be written in terms of a *unitary* evolution matrix instead of a density matrix. Thus, they result to be *pure* states instead of mixed ones. Hence, we conclude with the non-trivial consequence that information comes out in black hole's evaporation. This issue is also a confirmation of the assumption by 't Hooft that Schroedinger equations can be used universally for all dynamics in the universe, further endorsing the conclusion that black hole evaporation must be information preserving.

The introduction of a black hole's *effective temperature* [1, 2], which takes into account both of the non-strictly thermal [3, 4] and non-strictly continuous [1] characters of Hawking radiation [5], recently permitted us to re-analyse black hole's quasi-normal modes and to interpret them naturally in terms of quantum levels for emissions of particles [1, 2]. Here, we further improve the analysis. A fundamental equation that directly connects the probability of emission of an Hawking quantum with the two emission levels which are involved in the transition is found. That equation enables to interpret the correspondence between black hole quasi-normal modes and Hawking radiation in terms of a time dependent *Schroedinger system.* The quasi-normal modes energies, which are also the total energies emitted by the black hole in correspondence of the various quantum levels, represent the eigenvalues of the unperturbed Hamiltonian of the system and the Hawking quanta represent the energy transitions among the eigenvalues corresponding to a perturbed Hamiltonian $\propto \delta(t)$. Then, we can write down a *time dependent Schroedinger equation* for the system composed by Hawking radiation and black hole's quasi-normal modes. The states of the correspondent *Schroedinger wave-function* are written in terms of a *unitary* evolution matrix instead of a density matrix. Thus, they result to be *pure* states instead of mixed ones. The conclusion implies the non-trivial consequence that there is not information loss in black hole evaporation, in agreement with the use of Feynman sum over histories [6], with the results of string theory [7], and with the recent result based on the Kerr-Schild formalism in Kerr-Newman black holes [8]. Hence, the final conclusion is that black hole evaporation must be information preserving, confirming the assumption by 't Hooft that Schroedinger equations can be used universally for all dynamics in the universe [24].
Working with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units), the strictly thermal approximation gives the probability of emission [1]-[5]

$$\Gamma \sim \exp(-\frac{\omega}{T_H}), \tag{1}$$

where ω is the energy-frequency of the emitted particle and $T_H \equiv \frac{1}{8\pi M}$ the Hawking temperature. A more precise computation in the tunnelling framework, which takes into account the energy conservation, i.e. the contraction of the black hole which enables a back reaction due to the varying geometry, gives a remarkable correction [3, 4]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1-\frac{\omega}{2M})], \tag{2}$$

where the additional term $\frac{\omega}{2M}$ is present [3, 4]. By introducing the *effective temperature* [1, 2]

$$T_E(\omega) \equiv \frac{2M}{2M-\omega}T_H = \frac{1}{4\pi(2M-\omega)}, \tag{3}$$

we can re-write eq. (3) in a Boltzmann-like form similar to eq. (1)

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}), \tag{4}$$

where $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor,* with $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$. In other words, the effective temperature replaces the Hawking temperature in the equation of the probability of emission [1, 2]. The physical interpretation is the following. In various fields of Science, one takes into account the deviation from the thermal spectrum of an emitting body by introducing an effective temperature which represents the temperature of a black body that would emit the same total amount of radiation. We introduced the concept of effective temperature also in the black hole's physics [1, 2]. The effective temperature depends on the energy-frequency of the emitted radiation and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the radiation spectrum of a black hole from the strictly thermal feature [1, 2]. The introduction of the effective temperature enables the introduction of others *effective quantities.* Following [1, 2], let us consider the initial mass of the black hole *before* the emission, M, and the final mass of the hole *after* the emission, $M-\omega$. We can introduce the *effective mass* and the *effective horizon*

$$M_E \equiv M - \frac{\omega}{2}, \; r_E \equiv 2M_E \tag{5}$$

of the black hole *during* its contraction, i.e. *during* the emission of the particle [1, 2]. Such effective quantities are average quantities [1, 2]. In fact, r_E is the average of the initial and final horizons while M_E is the average of the initial and final masses [1, 2]. The effective temperature T_E is the inverse of the average value of the inverses of the initial and final Hawking temperatures (*before* the emission we have $T_{H\text{ initial}} = \frac{1}{8\pi M}$, *after* the emission we have $T_{H\text{ final}} = \frac{1}{8\pi(M-\omega)}$ respectively) [1, 2]. It is important to recall that the tunnelling is a *discrete* instead of *continuous* process [1]. In fact, we have two different *countable* black hole's physical states, the black hole's state before the emission of the particle and the black hole's state after the emission of the particle [1]. Thus, the emission of the particle can be interpreted like a *quantum transition* of frequency ω between the two discrete states [1]. In the language of the tunnelling approach, a trajectory in imaginary or complex time joins two separated classical turning points [1, 3]. The consequence is that the radiation spectrum is also discrete [1]. It is better to clarify this important issue in details. If we consider a well fixed Hawking temperature, the statistical probability distribution (2) is a continuous function. On the other hand, the Hawking temperature in (2) varies in time with a character which is *discrete.* The reason is that the forbidden region which the emitting particle traverses has a *finite* size [3]. That is exactly the reason because the effective temperature (3) has been introduced. Considering a strictly thermal approximation, the turning points have zero separation and it is not clear what joining trajectory has to be considered as there is not barrier [3]. The problem is solved when one argues that the forbidden finite region from $r_{initial} = 2M$ to $r_{final} = 2(M-\omega)$ that the tunnelling particle traverses works like barrier [3]. In other words, the intriguing explanation is that it is the particle itself which generates a secret tunnel through the black hole horizon [3]. The physical consequences are that the spectrum is not strictly thermal [3, 4] and the Hawking temperature *has a* *discrete character in time*. Hence, also the statistical probability distribution (2) is *discrete in time*. We stress that the emitted energies are not only discrete, but also *countable*, as it has been shown in [9, 10]. Interesting proposals on the non strictly continuous character of Hawking radiation can be found in the literature [11, 12]. It is straightforward that discrete energy spectra rather than continuous ones are in general associated to quantum systems of finite size [11]. In the tunnelling framework, the dynamics responsible of the discrete character of the spectrum refer not only to the finite region enclosed by the horizon of the black hole like in [11], but in particular to the finite size of the forbidden region that is traversed by the tunnelling particle [3]. The discrete character of the spectrum is also very important for the physical interpretation of black hole's quasi-normal modes [1, 2]. The remarkable idea that black hole's quasi-normal modes can release information about the area quantization arises from the works [13, 14]. The original results in [13, 14] found various objections [15, 16], which have been well addressed in [15], where the initial proposal in [13, 14] has been refined. Black hole's quasi-normal frequencies are labelled as ω_{nl}, where l is the angular momentum quantum number ($l \geq 2$ for gravitational perturbations) and n ($n = 1, 2, ...$) is the quantum "overtone" number which denotes a countable sequence of quasi-normal modes for each l [1, 2, 13, 14, 15, 16]. The quasi-normal modes of the Schwarzschild black hole become independent of l for large n, and, in strictly thermal approximation, their countable sequence is [1, 2, 13, 14, 15, 16]

$$\begin{array}{c} \omega_n = \ln 3 \times T_H + 2\pi i (n + \frac{1}{2}) \times T_H + \mathcal{O}(n^{-\frac{1}{2}}) = \\ \\ = \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}). \end{array} \tag{6}$$

The quasi-normal frequencies (6) are interpreted like superposition of the damped oscillations [1, 2, 15]

$$\exp(-i\omega_I t)[a \sin \omega_R t + b \cos \omega_R t] \tag{7}$$

which have a spectrum of complex frequencies $\omega = \omega_R + i\omega_I$. The equation governing a damped harmonic oscillator $\mu(t)$ is [1, 2, 15]

$$\ddot{\mu} + K\dot{\mu} + \omega_0^2 \mu = F(t), \tag{8}$$

where ω_0 is the proper frequency of the harmonic oscillator, $F(t)$ an external force per unit mass and K the damping constant. If one assumes that the external force per unit mass scales like a Dirac delta function, i.e.

$$F(t) \propto \delta(t), \tag{9}$$

$\mu(t)$ results to be a superposition of a term oscillating as $\exp(i\omega t)$ and of a term oscillating as $\exp(-i\omega t)$, see [15] for details. Then, through the identifications [1, 2, 15]

$$\frac{K}{2} = \omega_I, \qquad \sqrt{\omega_0^2 - \frac{K}{4}^2} = \omega_R, \tag{10}$$

which give

$$\omega_0 = \sqrt{\omega_R^2 + \omega_I^2}, \tag{11}$$

a damped harmonic oscillator reproduces the behavior (7). The identification $\omega_0 = \omega_R$ is correct only in the approximation $\frac{K}{2} \ll \omega_0$, i.e. only for very long-lived modes [1, 2, 15]. For highly excited modes, which represent a lot of black hole's quasi-normal modes, the opposite limit is the correct one. By using this observation, some aspects of quantum physics of black holes that were discussed in previous literature assuming that the relevant frequencies were $(\omega_R)_n$ rather than $(\omega_0)_n$ can be re-examined in a correct way [1, 2, 15].

We recall that ideas on the continuous character of Hawking radiation did not agree with attempts to interpret the frequency of the discrete quasi-normal modes (6), preventing to associate quasi-normal modes to Hawking radiation [17] as the discrete behavior of the energy spectrum (6) results incompatible with the spectrum of Hawking radiation whose energies are of the same order but continuous [17]. On the other hand, the non-strictly thermal behavior which also implies, as we have shown above, the non-strictly continuous character of Hawking radiation, removes the above difficulty [1]. Thus, the discrete behavior of Hawking radiation permits to interpret the quasi-normal frequencies (6) also like energies of physical Hawking quanta [1]. Quasi-normal modes represent the reaction of a black hole to small, discrete perturbations in terms of damped oscillations [1, 2, 13, 14, 15]. The capture of a particle which causes an increase in the horizon area is a type of discrete perturbation [13, 14, 15]. Thus, it is very natural to assume that the emission of a particle which causes a decrease in the horizon area is also a perturbation which generates a reaction in terms of countable quasi-normal modes as it is a discrete instead of continuous process [1]. Based on such a natural correspondence between Hawking radiation and black hole's quasi-normal modes, one can consider quasi-normal modes in terms of quantum levels for emitted energies too [1, 2]. This important point is in agreement with the idea that black holes can be considered in terms of highly excited states in the underlying quantum gravity theory [1, 2, 15]. When one enables the correspondence between black hole's quasi-normal frequencies and the emission and/or the absorption of particles, the validity of Bohr Correspondence Principle [18] is implicitly assumed. That Principle states that "transition frequencies at large quantum numbers should equal classical oscillation frequencies" [13, 14]. We applied the concept of effective temperature to the analysis of physical interpretation of the spectrum of black hole's quasi-normal modes in [1, 2]. Another key point of the analysis was that eq. (6) is an approximation as it implies the assumption that the black hole's radiation spectrum is strictly thermal. The deviation from the thermal spectrum in eq. (2) is taken into due account when one replaces the Hawking temperature T_H with the effective temperature T_E in eq. (6) [1, 2]. In that way, the correct expression for the quasi-normal modes of the Schwarzschild black hole becomes [1, 2]

$$\begin{array}{c}
\omega_n = \ln 3 \times T_E(|\omega_n|) + 2\pi i(n+\frac{1}{2}) \times T_E(|\omega_n|) + \mathcal{O}(n^{-\frac{1}{2}}) = \\
= \frac{\ln 3}{4\pi(2M-|\omega_n|)} + \frac{2\pi i}{4\pi(2M-|\omega_n|)}(n+\frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}).
\end{array} \quad (12)$$

The correct derivation of eq. (12) can be found in[1, 2] (precise details of that derivation are in the Appendix of [25]. Here we recall its physical interpretation. The imaginary part of (6) is explained as it follows. As on the given background the quasi-normal modes determine the position of poles of a Green's function, the Euclidean black hole solution converges to a thermal circle at infinity with the inverse temperature $\beta_H = \frac{1}{T_H}$ [19]. Hence, the spacing of the poles in eq. (6) coincides with the spacing $2\pi i T_H$ expected for a thermal Green's function [19]. Indeed, as the spectrum is not strictly thermal, one naturally assumes that the Euclidean black hole solution converges to a *non-thermal* circle at infinity [1, 2]. Thus, the replacement [1, 2]

$$\beta_H = \frac{1}{T_H} \rightarrow \beta_E(\omega) = \frac{1}{T_E(\omega)}, \quad (13)$$

takes into due account the non-strictly thermal feature of the radiation spectrum of a black hole. Then, the spacing of the poles in eq. (12) coincides with the spacing [1, 2]

$$2\pi i T_E(\omega) = 2\pi i T_H(\frac{2M}{2M-\omega}), \quad (14)$$

expected for a *non-thermal* Green's function [1, 2]. In fact a dependence on the frequency is now present [1, 2].

The physical solution for the absolute values of the frequencies (12) has been found in [1, 2]

$$\begin{array}{c}
(\omega_0)_n \equiv E_n = |\omega_n| = M - \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n+\frac{1}{2})^2}} \\
\simeq M - \sqrt{M^2 - \frac{1}{2}(n+\frac{1}{2})}.
\end{array} \quad (15)$$

$(\omega_0)_n$ is interpreted like the total energy emitted by the black hole at that time [1, 2]. Now, we show that the energy emitted in an arbitrary transition $n \rightarrow m$, with $m > n$, is proportional to the effective temperature associated to the transition. Considering an emission from the ground state to a state with large n the mass of the black hole changes from M to

$$M_n \equiv M - E_n. \quad (16)$$

On the other hand, in the transition from the state with n to the state with $m > n$ the mass of the black hole changes again from M_n to

$$M_m \equiv M - E_n + \Delta E_{n\rightarrow m} = M - E_m, \quad (17)$$

as it is $\Delta E_{n\rightarrow m} \equiv E_m - E_n$ [1, 2]. Considering eq. (15), eqs. (16) and (17) read

$$M_n = \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n+\frac{1}{2})^2}} \tag{18}$$

and

$$M_m = \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(m+\frac{1}{2})^2}}. \tag{19}$$

Now, we set

$$\Delta E_{n\to m} \equiv E_m - E_n =$$

$$= \sqrt{M^2 - \tfrac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n+\tfrac{1}{2})^2}} - \sqrt{M^2 - \tfrac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(m+\tfrac{1}{2})^2}} =$$

$$= M_n - M_m = K \left[T_E\right]_{n\to m}, \tag{20}$$

where $\left[T_E\right]_{n\to m}$ is the effective temperature associated to the transition $n \to m$, and M_n and M_m are given by eq. (17). Let us see if there are values of the constant K for which eq. (20) is satisfied. We recall that

$$\left[T_E\right]_{n\to m} = \frac{K}{4\pi\left(M_n + M_m\right)}, \tag{21}$$

as the effective temperature is the inverse of the average value of the inverses of the initial and final Hawking temperatures. Thus, eq. (20) can be rewritten as

$$= M_n^2 - M_m^2 = \frac{K}{4\pi}. \tag{22}$$

By using eqs. (18) and (19), for large m and n eq. (22) becomes

$$\frac{1}{2}\left(m-n\right) = \frac{K}{4\pi}, \tag{23}$$

which implies that eq. (20) is satisfied for $K = 2\pi\left(m-n\right)$. Hence, we can rewrite (20) as

$$\Delta E_{n\to m} = E_m - E_n = 2\pi\left(m-n\right)\left[T_E(\omega)\right]_{n\to m}. \tag{24}$$

By using eq. (4), the probability of emission between the two levels n and m can be written in the intriguing form

$$\Gamma_{n\to m} \sim \exp -\left\{\frac{\Delta E_{n\to m}}{\left[T_E(\omega)\right]_{n\to m}}\right\} = \exp\left[-2\pi\left(m-n\right)\right]. \tag{25}$$

Thus, we have find the fundamental result that the probability of emission between two arbitrary levels characterized by the two "overtone" quantum numbers n and m scales like $\exp\left[-2\pi\left(m-n\right)\right]$. In particular, for $m = n+1$ the probability of emission has its maximum value $\sim \exp(-2\pi)$, i.e. the probability is maximum for two adjacent levels, as one can intuitively expect. Notice that, if one fixes n, the probabilities (25) can be normalized to the unity in the following way

$$\sum_{m=n}^{m_{max}} \Gamma_{n\to m} = \sum_{m=n}^{m_{max}} \alpha \exp\left[-2\pi\,(m-n)\right] = 1, \tag{26}$$

where α is the prefactor of eq. (25), m_{max} is the maximum value for the "overtone" number m and $m = n$ corresponds to the probability that the black hole does not emit. Let us compute m_{max}. Following [1], as $(\omega_0)_m = E_m$ is the total emitted energy for a black hole excited at a level m, one needs

$$M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(m+\frac{1}{2})^2} \geq 0 \tag{27}$$

in eq. (15). In fact, a black hole does not emit more energy than its total mass. This implies that the countable sequence of quasi-normal modes cannot be infinity [1]. Eq. (27) can be solved giving a maximum value for the "overtone" number m [1]

$$m \leq m_{max} = 2\pi^2 \left(\sqrt{16M^4 - (\frac{\ln 3}{\pi})^2} - 1\right). \tag{28}$$

The maximum value (28) corresponds to $(\omega_0)_{m_{max}} = E_{m_{max}} = M$. On the other hand, by using the Generalized Uncertainty Principle, the approach in [20] proposed that the total evaporation of a black hole can be prevented exactly like the Uncertainty Principle prevents the hydrogen atom from total collapse. In such an analysis, the collapse is prevented by dynamic instead of by symmetry as *Planck scale* is approached [20]. In that way, eq. (27) has to be slightly modified, becoming (notice that the Planck mass is equal to 1 in Planck units) [25]

$$M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(m+\frac{1}{2})^2} \geq 1. \tag{29}$$

The solution of eq. (29) gives a different value of the maximum value for for the "overtone" number m [25]

$$m \leq m_{max} = 2\pi^2 \left(\sqrt{16(M^2-1)^2 - (\frac{\ln 3}{\pi})^2} - 1\right). \tag{30}$$

Putting $k = m - n$ and $\exp\left[-2\pi\right] = X$ eq. (26) becomes

$$\sum_{k=0}^{k_{max}} \Gamma_{0\to k} = \alpha \sum_{k=0}^{k_{max}} X^k = 1. \tag{31}$$

The sum in eq. (31) is well known. It is the *kth* partial sum of the geometric series and can be solved as [21]

$$\sum_{k=0}^{k_{max}} X^k = \alpha \frac{1 - X^{(k_{max}+1)}}{1 - X}. \tag{32}$$

Thus, one gets

$$\alpha \frac{1 - X^{(k_{max}+1)}}{1 - X} = 1, \tag{33}$$

which permits to solve for α

$$\alpha \equiv \alpha_n = \frac{1 - X}{1 - X^{(k_{max}+1)}} = \frac{1 - \exp - [2\pi]}{1 - \exp\left[-2\pi\left(m_{max} - n + 1\right)\right]}. \tag{34}$$

Hence, we find that the constant of proportionality α depends on the black hole's quantum level n. Notice that for $m_{max} \gg n$ one finds that such a dependence can be neglected

$$\alpha \simeq 1 - \exp - [2\pi] \simeq 1 - 1.87 * 10^{-3} \sim 1. \tag{35}$$

This is not surprising as for $\Delta E_{n\to m} \ll M_n$, i.e. when the emitted energy is much minor than mass of the black hole and the condition $m_{max} \gg m$, n is guaranteed, the thermal approximation is excellent as the back reaction due to the energy conservation can be neglected. The dependence of the constant of proportionality α on the black hole's quantum level n becomes very important when m, $n \sim m_{max}$, i.e. near the final stages of the black hole's evaporation. In that case, in which we label the constant of proportionality α_n, the thermal approximation breaks down as the condition $\Delta E_{n\to m} \ll M_n$ is no more guaranteed and one needs to use the correct formula (34). An intermediate case can be considered too, i.e. when it is $m_{max} \sim m \gg n$. In that case, eq. (35) can be used even if the condition $\Delta E_{n\to m} \ll M_n$ is not guaranteed and the thermal approximation breaks down. On the other hand, eq. (25) shows that transitions in which $m \gg n$ are highly improbable.

Inserting the result (34) in eq. (25) we fix the probability of emission between the two levels n and m as

$$\begin{array}{c}
\Gamma_{n\to m} = \alpha_n \exp - \left\{\frac{\Delta E_{n\to m}}{\left[T_E(\omega)\right]_{n\to m}}\right\} = \alpha_n \exp\left[-2\pi\left(m-n\right)\right] = \\
= \left\{\frac{1-\exp-[2\pi]}{1-\exp[-2\pi(m_{max}-n+1)]}\right\} \exp\left[-2\pi\left(m-n\right)\right].
\end{array} \tag{36}$$

We recall that the quasi-normal frequencies (12) are the eigenvalues of the equation [1, 2, 17, 25]

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \omega^2\right)\phi, \tag{37}$$

where

$$V(x) \equiv V\left[x(r)\right] = \left(1 - \frac{2M_E}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M_E}{r^3}\right) \tag{38}$$

is the (time independent) *effective Regge-Wheeler potential* [1, 2] and the relation between the Regge-Wheeler "tortoise" coordinate x and the radial coordinate r is [1, 2]

$$\begin{array}{c} x = r + 2M_E \ln\left(\frac{r}{2M_E} - 1\right) \\ \frac{\partial}{\partial x} = \left(1 - \frac{2M_E}{r}\right)\frac{\partial}{\partial r}. \end{array} \tag{39}$$

Eq. (37) is interpreted like a *Schroedinger equation* [1, 2, 17]. In fact, from a mathematical point of view, the quasi-normal frequencies are discrete quasi-normal states, analogous to the quasi-stationary states of quantum mechanics which frequency is allowed to be complex [17]. From the quantum mechanical point of view, one can physically interpret Hawking radiation like energies of quantum jumps among the unperturbed levels (15) [1, 2]. Hence, we have a perturbation of the type in eq. (9) which acts on the quasi-normal modes [1, 2]. Such a perturbation can be described by an operator [22]

$$U(t) = \begin{array}{c} W(t) \quad for\ 0 \leq t \leq \tau \\ \\ 0 \quad for\ t < 0\ and\ t > \tau. \end{array} \tag{40}$$

We need $W(t)$ to be conform to eq. (9). Thus, in an appropriate orthonormal basis [22], the matrix elements of $W(t)$ can be written as

$$W_{ij}(t) \equiv A_{ij}\delta(t), \tag{41}$$

where the A_{ij} are real. Thus, the complete (time dependent) Hamiltonian is described by the operator [22]

$$H(x,t) \equiv V(x) + U(t), \tag{42}$$

which permits to write the correspondent *time dependent Schroedinger equation* for the system [22]

$$i\frac{d|\psi(x,t)>}{dt} = [V(x) + U(t)]\,|\psi(x,t)> = H(x,t)|\psi(x,t)> . \tag{43}$$

The state which satisfies eq. (43) is [22]

$$|\psi(x,t)> = \sum_m a_m(t)\exp\left(-i\omega_m t\right)|\varphi_m(x)>, \tag{44}$$

where the $\varphi_m(x)$ are the eigenfunctions of eq. (37) and the ω_m are the correspondent eigenvalues. In order to solve the complete quantum mechanical problem described by the operator (42) one needs to know the probability amplitudes $a_m(t)$ due to the application of the perturbation descripted by the time dependent operator (40) [22], which represents the perturbation associated to the emission of an Hawking quantum. For $t < 0$, i.e. before the perturbation operator (40) starts to work, the system is in a stationary state $|\varphi_n t(x)>$, at the quantum level n, with energy $E_n = |\omega_n|$ given by eq. (15). Therefore, in eq. (44) only the term

$$|\psi_n(x,t)>= \exp(-i\omega_n t)\,|\varphi_n(x)>, \tag{45}$$

is not null for $t<0$. This implies $a_m(t)=\delta_{mn}$ for $t<0$. When the perturbation operator (40) stops to work, i.e. after that a quantum has been emitted, for $t>\tau$ the probability amplitudes $a_m(t)$ return to be time independent, having the value $a_{n\to m}(\tau)$ [22]. In other words, for $t>\tau$ the system is described by the *Schroedinger wave-function*

$$|\psi_{final}(x,t)>= \sum_{m=n}^{m_{max}} a_{n\to m}(\tau)\exp(-i\omega_m t)\,|\varphi_m(x)>\,. \tag{46}$$

Thus, the probability to find the system in an eigenstate having energy $E_m = |\omega_m|$ is given by [22]

$$\Gamma_{n\to m}(\tau) = |a_{n\to m}(\tau)|^2. \tag{47}$$

In order to solve the problem, one proceeds following [22]. By using a standard analysis, one can obtain the following differential equation from eq. (46) [22]

$$i\frac{d}{dt}a_{n\to m}(t) = \sum_{l=m}^{m_{max}} W_{ml}a_{n\to l}(t)\exp\left[i\left(\Delta E_{l\to m}\right)t\right]. \tag{48}$$

To first order in $U(t)$, by using the Dayson series, one gets the solution [22]

$$a_{n\to m} = -i\int_0^t \left\{W_{mn}(t')\exp\left[i\left(\Delta E_{n\to m}\right)t'\right]\right\}dt'. \tag{49}$$

By inserting (41) in (49) we obtain

$$a_{n\to m} = -iA_{mn}\int_0^t \left\{\delta(t')\exp\left[i\left(\Delta E_{n\to m}\right)t'\right]\right\}dt' = -\frac{i}{2}A_{mn} \tag{50}$$

Combining this equation with eqs. (36) and (47) one gets

$$\begin{aligned} &\alpha_n\exp\left[-2\pi\left(m-n\right)\right] = \tfrac{1}{4}A_{mn}^2 \\ &A_{mn} = 2\sqrt{\alpha_n}\exp\left[-\pi\left(m-n\right)\right] \\ &a_{n\to m} = -i\sqrt{\alpha_n}\exp\left[-\pi\left(m-n\right)\right]. \end{aligned} \tag{51}$$

We stress that, as $\sqrt{\alpha_n}\sim 1$, for $m=n+1$, i.e. when A_{mn} is maximum for an emission, one gets $A_{mn}\sim 10^{-2}$. This implies that the error in our solution is $\sim 10^{-4}$, i.e. our approximation is very good. Clearly, for $m>n+1$ the approximation is better. Thus, one can write down the final form of the Schroedinger wave-function of the system

$$|\psi_{final}(x,t)>=\sum_{m=n}^{m_{max}}-i\sqrt{\alpha_n}\exp\left[-\pi\left(m-n\right)\left(-i\omega_m t\right)\right]|\varphi_m(x)>. \qquad (52)$$

Notice that the *Schroedinger wave-function* of the system (52) represents a *pure final state instead of a mixed final state.* Thus, the states are written in terms of a *unitary* evolution matrix instead of a density matrix and this implies the fundamental conclusion that information is not loss in black hole evaporation, in agreement with the results of string theory [7], with the use of Feynman sum over histories [6], and with the using of the Kerr-Schild formalism in Kerr-Newman black holes [8]. Notice that we found this fundamental result, which is in full agreement with quantum mechanics, by using a pure semi-classical analysis, i.e. without extending the assumptions of our theoretical approach like in the cited cases of string theory [7] and of the use of Feynman sum over histories [6]. The result is also in agreement with the assumption by 't Hooft that Schroedinger equations can be used universally for all dynamics in the universe [24].

In summary, in this letter we found a fundamental equation that directly connects the probability of emission of an Hawking quantum with the two emission levels which are involved in the transition. Such an equation permitted us to interpret the correspondence between Hawking radiation and black hole quasi-normal modes in terms of a time dependent Schroedinger system. In that system, the energies of the quasi-normal modes, which are also the total energies emitted by the black hole in correspondence of the various quantum levels, represent the eigenvalues of the unperturbed Hamiltonian of the system and the Hawking quanta represent the energy transitions among the eigenvalues, which correspond to a perturbed Hamiltonian $\propto \delta(t)$. In this way, we explicitly wrote down a time dependent Schroedinger equation for the system composed by Hawking radiation and black hole quasi-normal modes. The states of the correspondent Schroedinger wave-function are written in terms of a *unitary* evolution matrix instead of a density matrix, and, in turn, result to be pure states instead of mixed ones. In other words, the black hole obeys a time dependent Schroedinger equation which allows pure states to evolve into pure states. This implies the non-trivial consequence that information comes out in black hole's evaporation. As a consequence, the underlying quantum gravity theory should be also unitary. We stress that the result of this paper are in full agreement with the results of string theory [7], with the use of Feynman sum over histories [6], and with the using of the Kerr-Schild formalism in Kerr-Newman black holes [8]. On the other hand, an important difference is that here we found this fundamental result, which is in full agreement with quantum mechanics, by using a pure semi-classical analysis, i.e. without extending the assumptions of our theoretical approach like in the cited cases of string theory [7] and of the use of Feynman sum over histories [6].

We also emphasize that the results of this lettr are correct only for $n \gg 1$, i.e. only for highly excited black holes. This is the reason because we assumed an emission from the ground state to a state with large n in our discussion.

On the other hand, a state with large n is always reached at late times, maybe not through a sole emission from the ground state, but, indeed, through various subsequent emissions of Hawking quanta.

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References

[1] C. Corda, Int. Journ. Mod. Phys. D 21, 1242023 (2012, Honorable Mention in the Gravity Research Foundation Competition).

[2] C. Corda, JHEP 1108, 101 (2011).

[3] M. K. Parikh, Gen. Rel. Grav. 36, 2419 (2004, First Award in the Gravity Research Foundation Competition).

[4] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).

[5] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[6] S. W. Hawking, Phys. Rev. D72, 084013 (2005).

[7] S. D. Mathur, Int. J. Mod. Phys. D, 21, 1241002 (2012, Second Award in the Gravity Research Foundation Competition).

[8] A. Burinskii, Gen. Rel. Grav. 41, 2281-2286 (2009).

[9] M. Parikh, (hep-th/0402166).

[10] M. A. Arzano, J. M. Medved, and E. C. Vagenas, JHEP 0509, 037 (2005).

[11] J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. B 360, 7 (1995).

[12] Youngsub Yoon, arXiv:1210.8355 (2012).

[13] S. Hod, Phys. Rev. Lett. 81 4293 (1998).

[14] S. Hod, Gen. Rel. Grav. 31, 1639 (1999, Fifth Award in the Gravity Research Foundation Competition).

[15] M. Maggiore, Phys. Rev. Lett. 100, 141301 (2008).

[16] R. A. Konoplya and A. Zhidenko, Rev. Mod. Phys. 83, 793-836 (2011).

[17] L. Motl, Adv. Theor. Math. Phys. 6, 1135 (2003).

[18] N. Bohr, Zeitschrift für Physik 2 (5), 423–478 (1920).

[19] L. Motl and A. Neitzke, Adv. Theor. Math. Phys. 7, 307 (2003).

[20] R. J. Adler, P. Chen and D. I. Santiago, Gen. Rel. Grav. 3, 2101-2108 (2001, Third Award at Gravity Research Foundation).

[21] J. Stewart, *Calculus*, 5th ed., Brooks Cole. ISBN 978-0-534-39339-7 (2002).

[22] J. J. Sakurai, *Modern Quantum Mechanic*s, Pearson Education, 2006.

[23] H. P. Nollert, Phys. Rev. D 47, 5253-5258 (1993).

[24] G. 't Hooft, Int. J. Mod. Phys. A 11, 4623-4688 (1996).

[25] C. Corda, arXiv:1210.7747.