

On parametric Gevrey asymptotics for singularly perturbed partial differential equations with delays

Alberto Lastra,* Stéphane Malek†

July 17, 2013

Abstract

We study a family of singularly perturbed q -difference-differential equations in the complex domain. We provide sectorial holomorphic solutions in the perturbation parameter ϵ . Moreover, we achieve the existence of a common formal power series in ϵ which represents each actual solution, and establish q -Gevrey estimates involved in this representation. The proof of the main result rests on a new version of the so-called Malgrange-Sibuya Theorem regarding q -Gevrey asymptotics. A particular Dirichlet like series is studied on the way.

Key words: q -difference-differential equations, singular perturbations, formal power series, Borel-Laplace transform, Borel summability, q -Gevrey asymptotic expansions. 2010 MSC: 35C10, 35C20.

1 Introduction

We study a family of q -difference-differential equations of the form

$$(1) \quad \epsilon \partial_t \partial_z^S X(\epsilon, t, z) + a \partial_z^S X(\epsilon, t, z) = \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} b_{\underline{\kappa}}(\epsilon, z) (\partial_t^{\kappa_0} \partial_z^{\kappa_1} X)(\epsilon, q^{m_{\underline{\kappa},1}} t, q^{m_{\underline{\kappa},2}} z),$$

under appropriate initial conditions

$$(2) \quad (\partial_z^j X)(\epsilon, t, 0) = \phi_j(\epsilon, t), \quad 0 \leq j \leq S-1.$$

Here, S is an integer with $S \geq 1$, and $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. \mathcal{N} stands for a finite subset of \mathbb{N}^2 , where $\mathbb{N} := \{0, 1, 2, \dots\}$ is the set of nonnegative integers. For every $(\kappa_0, \kappa_1) \in \mathcal{N}$, $b_{\underline{\kappa}}(\epsilon, z)$ turns out to be a polynomial in the variable z with holomorphic and bounded coefficients in a neighborhood of the origin in the parameter, and $m_{\underline{\kappa},1}, m_{\underline{\kappa},2} \in \mathbb{N}$.

From now on, q stands for a fixed real number with $0 < q < 1$.

We construct actual holomorphic solutions $X(\epsilon, t, z)$ for the previous Cauchy problem in $\mathcal{E} \times \mathcal{T} \times \mathbb{C}$, where \mathcal{E} is a bounded open sector in the complex plane with vertex at the origin, and \mathcal{T} is an unbounded well-chosen open set. The procedure is based on the use of the map $t \mapsto t/\epsilon$

*The author is partially supported by the project MTM2012-31439 of Ministerio de Ciencia e Innovacion (Spain), and Caja de Burgos Obra Social

†The author is partially supported by the french ANR-10-JCJC 0105 project and the PHC Polonium 2013 project No.28217SG

which was firstly considered by M. Canalis-Durand, J. Mozo-Fernandez and R. Schäfke in [10] to transform a singularly perturbed equation into an auxiliary regularly perturbed equation, easier to handle. This celebrated technique has also been used in the study of singularly perturbed partial differential equations (see [29] and [33] for example), q -difference-differential equations (like in [32] or [28]), and more recently to the study of difference-differential equations (see [34]).

Indeed, the present work is motivated by a previous work [34], where the second author studies a singularly perturbed difference-differential equation with small delay. This work can be seen as a continuation of that one. The dynamics appearing in that previous work involve a small shift in variable t with respect to ϵ , meaning that they are of the form $(\epsilon, t, z) \mapsto (\epsilon, t + \kappa_2\epsilon, z)$, whereas the actual work deals with a shrinking behaviour in both t and z variables.

In [34], a Gevrey 1+ phenomenon, with estimates associated to the sequence $\left(\left(\frac{n}{\log n}\right)^n\right)_{n \geq 0}$, is observed for the series solution of the problem. This sequence naturally appears when working with difference equations (see [9], [8] for example). Now, a q -Gevrey like behaviour, related to the sequence of estimates $(q^{-n^2})_{n \geq 0}$, appears. This behaviour comes up in the context of q -difference equations (see [17], [36]). One can observe that 1+ sequence is asymptotically upper bounded by Gevrey sequence $(n!)_{n \geq 0}$, and this one is upper bounded by q -Gevrey sequence $(q^{-n^2})_{n \geq 0}$.

The main aim of this work is to construct actual holomorphic solutions $X(\epsilon, t, z)$ of (1)+(2) and obtain sufficient conditions for the existence and unicity of a formal power series in the parameter ϵ , $\hat{X}(\epsilon, t, z) = \sum_{\beta \geq 0} \hat{X}_\beta(t, z) \frac{\epsilon^\beta}{\beta!}$, owing its coefficients in an adequate functional space, and such that X is represented by \hat{X} in a sense to precise (see Theorem 5). This representation is measured in terms of q -Gevrey bounds due to the appearance of q -difference operators on the right-hand side in (1).

The Cauchy problem (1)+(2) we consider in this paper comes also within the framework of the asymptotic analysis of linear differential and partial differential equations with multiplicative delays.

In the context of differential equations most of the statements in the literature are dedicated to linear problems of the form

$$(3) \quad x'(t) = F(t, x(\lambda_1 t), \dots, x(\lambda_n t), x'(\lambda_1 t), \dots, x'(\lambda_n t))$$

where F are vector valued polynomial functions in t and linear in its other arguments, where $0 < \lambda_j < 1$, for $1 \leq j \leq n$ are real numbers, and concern the study of asymptotic behaviour of some of their solutions $x(t)$ as t tends to infinity for given initial data $x(0)$. When F is real or matrix valued and with constant coefficients, we quote [11], [21], [24], [25]. For polynomial F in t , we notice [14], [20]. For studies in a complex variable t , we refer to [19], [39]. For more general delay functional equations, we indicate [13].

In the framework of linear partial differential equations, we mention a series of papers devoted to general results on the existence and unicity of holomorphic solutions to generalized Cauchy-Kowalevski type problems with shrinkings of the form

$$\partial_t^m u(t, x) = f(t, x, u(t, x), (\partial_x^l u(t, x), \partial_x^p u(\alpha(t)t, x), \partial_x^q u(t, \beta(t, x)x))_{(l,p,q) \in I})$$

for some integer $m \geq 1$, a finite set I , and where f is analytic or of Gevrey type function and such that the functions $\alpha(t)$ and $\beta(t, x)$ satisfy the shrinking constraints $|\alpha(t)| < 1$ and $|\beta(t, x)| < 1$ for given initial data $(\partial_t^j u)(0, x)$, $0 \leq j \leq m - 1$ that belong to some functional space. We refer to [3], [26], [27]. For partial differential problems with contractions dealing with less regular solution spaces like Sobolev spaces, we quote [37], for instance.

Let us briefly reproduce the strategy followed. We consider a finite family of sectors with vertex at the origin $(\mathcal{E}_i)_{0 \leq i \leq \nu}$ which provides a good covering at 0 in the variable ϵ (see Definition 4). Let $i \in \{0, 1, \dots, \nu - 1\}$. One can consider an auxiliary Cauchy problem

$$\partial_z^S W(\epsilon, \tau, z) = \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \frac{b_{\underline{\kappa}}(\epsilon, z)}{(a - \tau)q^{m_{\underline{\kappa}, 1}(\kappa_0 + 1)}} \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} (\partial_z^{\kappa_1} W)(\epsilon, q^{-m_{\underline{\kappa}, 1}\tau}, q^{m_{\underline{\kappa}, 2}z}),$$

with initial conditions $(\partial_z^j W)(\epsilon, \tau, 0) = W_j(\epsilon, \tau)$, $0 \leq j \leq S - 1$. We assume W_j is a holomorphic function in $(D(0, r_0) \setminus \{0\}) \times D(0, \hat{R}_0)$ for some $r_0, \hat{R}_0 > 0$, for every $0 \leq j \leq S - 1$, which is upper bounded in terms of q -Gevrey bounds (see (28)). Moreover, we assume each W_j can be extended to $\mathcal{E}_i \times \mathcal{S}$, where \mathcal{S} is a sector with vertex at the origin, and verifying q -Gevrey bounds in $\mathcal{E}_i \times S_0$, with $S_0 := \{z \in \mathcal{S} : |z| \geq R_0\}$ (see (14)). Under these hypotheses, one can construct a formal solution to the auxiliary Cauchy problem, $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$, where $W_\beta(\epsilon, \tau)$ turns out to be a holomorphic function in $(D(0, r_0) \setminus \{0\}) \times (D_\beta \setminus \{0\})$. Here, D_β is a disc centered at the origin with radius decreasing to 0 whenever β tends to infinity, and reproducing q -Gevrey bounds given by the initial conditions (see (29)). Moreover, each $W_\beta(\epsilon, \tau)$ can be extended to $\mathcal{E}_i \times S_\beta$ under q -Gevrey bounds (see (15)), where $S_\beta := \{z \in \mathcal{S} : |z| > R_\beta\}$, with $(R_\beta)_{\beta \geq 0}$ being a sequence of positive numbers that decrease to 0. We assume $S_\beta \cap D_\beta \neq \emptyset$ for every $\beta \geq 0$. The decrease rate of both R_β and the radius of D_β has to be chosen adequately, in accordance to the elements of a q -Gevrey sequence such as $(q^{\alpha\beta^2})_{\beta \geq 0}$ for some $\alpha > 0$.

The main difficulty in this work is the occurrence of propagation of singularities in the coefficients of the auxiliary problem which leads to a small divisor phenomenon. The singular points form a sequence of complex numbers tending to 0. As a result, one can only obtain a formal solution for the auxiliary problem. In [30], a small divisor phenomenon comes from the Fuchsian operator studied in the main Cauchy problem. There, $q \in \mathbb{C}$ is chosen to have $|q| > 1$, whilst in the present work $q \in \mathbb{R}$ with $0 < q < 1$. A suchlike phenomenon also appears in [29], where the asymptotics in the parameter suffers the effect of a small divisor, and it is solved studying a Dirichlet like series.

General Dirichlet series of the form

$$\sum_{n \geq 0} a_n e^{-\lambda_n z}$$

have been thoroughly studied in the case when $(\lambda_n)_{n \geq 0}$ is an increasing sequence of real numbers to ∞ (see [22], [38], [2]) or a sequence of complex numbers with $|\lambda_n| \rightarrow \infty$ (see [31]). This theory has also been developed when working with almost periodic functions, introduced by H. Bohr (see [6], [5], [15]), which are the uniform limits in \mathbb{R} of exponential polynomials $\sum_{k=1}^n a_k e^{is_k x}$, where the values s_k belong to the so-called spectrum $\Lambda \subseteq \mathbb{R}$. However, we are more interested in the behaviour of the sum when x tends to ∞ in the positive imaginary axis. Our technique rests on Euler-Mac-Laurin formula, Watson's Lemma and the equivalence between null q -Gevrey asymptotics and the fact of being q -exponentially small.

In [29], we solve the problem by means of a Dirichlet series with a spectrum being of the form $(\frac{1}{(k+1)^\alpha})_{k \geq 0}$. Now, the spectrum which helps us to achieve our purpose is of geometric nature (see Lemma 10).

The growth properties of W_β for $\beta \geq 0$ allow us to apply a Laplace like transform on each of them with respect to the variable τ in order to provide a holomorphic solution $X_i(\epsilon, \tau, z)$ of the main problem, defined in $\mathcal{E}_i \times \mathcal{T} \times \mathbb{C}$, for some appropriate unbounded open set \mathcal{T} . In addition to this, one has null q -Gevrey asymptotic bounds for the difference of X_i and X_{i+1} when the

domain of the variable z is restricted to a bounded set, meaning that for every $\rho > 0$, there exist $L_1, L_2 > 0$ such that

$$\sup_{\substack{t \in \tau \\ z \in D(0, \rho)}} |X_{i+1}(\epsilon, t, z) - X_i(\epsilon, t, z)| \leq L_1 e^{-L_2 \frac{1}{(-\log(q))^2} \log^2 |\epsilon|},$$

for every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$.

Finally, a novel version regarding q -Gevrey asymptotics of Malgrange-Sibuya Theorem (Theorem 4) leads us to the main result in the present work (Theorem 5), where we guarantee the existence of a formal power series in ϵ ,

$$\hat{X}(\epsilon, t, z) = \sum_{\beta \geq 0} \frac{X_\beta(t, z)}{\beta!} \epsilon^\beta \in \mathbb{H}_{\mathcal{T}, \rho}[[\epsilon]],$$

with coefficients in the Banach space of bounded holomorphic functions defined in $\mathcal{T} \times D(0, \rho)$, which is common for every $0 \leq i \leq \nu - 1$, and such that X_i admits \hat{X} as its q -Gevrey asymptotic expansion of some positive type in the variable ϵ (see (60)).

It is worth pointing out that a q -Gevrey version of Malgrange-Sibuya Theorem was already obtained in [28], when dealing with $q \in \mathbb{C}$, $|q| > 1$. There, the type in the asymptotic expansion involved suffers some increase. This is so due to the need of extension results in ultradifferentiable classes of functions (see [7], [12]) to be applied along the proof. Here, the geometry of the problem changes so that we are able to maintain the type q -Gevrey. The proof rests on the classical Malgrange-Sibuya Theorem (see [23]).

The paper is organized as follows.

In Section 2 and Section 3, we introduce Banach spaces of formal power series in order to solve auxiliary Cauchy problems with the help of fixed point results involving complete metric spaces. In Section 2, this result is achieved when dealing with formal power series with holomorphic coefficients in a product of a finite sector with vertex at the origin times an infinite sector, while in Section 3 the result is obtained when dealing with a product of two punctured discs at 0.

In Section 4, we first recall the definition and main properties of a Laplace like transform, and q -Gevrey asymptotic expansions (Subsection 4.1). Next, we construct analytic solutions for the main problem and determine flat q -Gevrey bounds for the difference of two solutions when the intersection of the domains in the perturbation parameter is not empty (Subsection 4.2). In the proof, a Dirichlet type series is studied. The section is concluded proving the existence of a formal power series in the perturbation parameter which represents every solution in some sense which is specified (Subsection 4.3).

2 A Cauchy problem in weighted Banach spaces of Taylor power series

$M, A_1, C, \delta_1 > 0$ are fixed positive real numbers throughout the present work. Let $q \in \mathbb{R}$ with $0 < q < 1$ and $(R_\beta)_{\beta \geq 0}$ be a sequence of positive real numbers.

We consider an open and bounded sector \mathcal{E} with vertex at the origin and we fix an open and unbounded sector \mathcal{S} with vertex at the origin having positive distance to a fixed complex number $a \in \mathbb{C}^*$, it is to say, there exists $M_1 > 0$ such that $|\tau - a| > M_1$ for every $\tau \in \mathcal{S}$. We write S_β for the subset of \mathcal{S} defined by

$$S_\beta := \{z \in \mathcal{S} : |z| > R_\beta\}.$$

The incoming definition of Banach spaces of functions and formal power series turns out to be an adaptation of the corresponding one in [28]. Here, the symmetry of these norms at 0 and the point of infinity in the τ variable has to be removed, so that a Laplace like transform of the elements in these Banach spaces makes sense.

Definition 1 Let $\epsilon \in \mathcal{E}$ and $\beta \in \mathbb{N}$. $E_{\beta, \epsilon, S_\beta}$ denotes the vector space of functions $v \in \mathcal{O}(S_\beta)$ such that

$$\|v(\tau)\|_{\beta, \epsilon, S_\beta} := \sup_{\tau \in S_\beta} \left\{ \frac{|v(\tau)|}{e^{M \log^2 \left(\frac{|\tau|}{|\epsilon|} + \delta_1 \right)}} \left| \frac{\tau}{\epsilon} \right|^{-C\beta} \right\} q^{-A_1 \beta^2}$$

is finite.

Let $\delta > 0$. $H(\epsilon, \delta, \mathcal{S})$ denotes the complex vector space of all formal power series $v(\tau, z) = \sum_{\beta \geq 0} v_\beta(\tau) \frac{z^\beta}{\beta!}$ with $v_\beta \in \mathcal{O}(S_\beta)$ for every $\beta \geq 0$ and such that

$$\|v(\tau, z)\|_{(\epsilon, \delta, \mathcal{S})} := \sum_{\beta \geq 0} \|v_\beta(\tau)\|_{\beta, \epsilon, S_\beta} \frac{\delta^\beta}{\beta!} < \infty.$$

It is straightforward to check that the pair $(H(\epsilon, \delta, \mathcal{S}), \|\cdot\|_{(\epsilon, \delta, \mathcal{S})})$ is a Banach space.

For our purposes, the elements in the sequence $(R_\beta)_{\beta \geq 0}$ are chosen to be related to the ones in a q -Gevrey sequence. This choice would provide that S_β tends to \mathcal{S} when $\beta \rightarrow \infty$.

Let $(\mathbb{E}_\beta)_{\beta \geq 0}$ be a family of complex functional Banach spaces. For every $v(\tau, z) = \sum_{\beta \geq 0} v_\beta(\tau) \frac{z^\beta}{\beta!} \in (\cup_{\beta \geq 0} \mathbb{E}_\beta)[[z]]$, we consider the formal integration operator ∂_z^{-1} defined on $(\cup_{\beta \geq 0} \mathbb{E}_\beta)[[z]]$ by

$$\partial_z^{-1}(v(\tau, z)) := \sum_{\beta \geq 1} v_{\beta-1}(\tau) \frac{z^\beta}{\beta!}.$$

Lemma 1 Let $s, \ell_0, \ell_1, m_1, m_2 \in \mathbb{N}$, $\delta > 0$ and $\epsilon \in \mathcal{E}$. We assume that

$$(4) \quad C(\ell_1 + s) - \ell_0 - 2m_1 M(-\log(q)) \geq 0.$$

In addition to this, we consider the elements in $(R_\beta)_{\beta \geq 0}$ are such that

$$(5) \quad R_\beta \geq q^{m_1} R_{\beta - \ell_1 - s},$$

for every $\beta \geq \ell_1 + s$. Moreover, we assume there exist constants $d_1, d_2 > 0$ such that

$$(6) \quad R_\beta \geq d_1 q^{d_2 \beta},$$

for every $\beta \geq 0$. In addition to this, we assume

$$(7) \quad m_2 - 2A_1(\ell_1 + s) - m_1 C + d_2 [\ell_0 - 2m_1 M \log(q) - C(\ell_1 + s)] > 0.$$

Under the previous assumptions, there exists a positive constant C_{11} , which does not depend on ϵ nor δ , such that

$$\left\| z^s \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \frac{1}{q^{m_1(\ell_0+1)}} (\partial_z^{-\ell_1} v)(\tau q^{-m_1}, z q^{m_2}) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq C_{11} \delta^{\ell_1 + s} \|v(\tau, z)\|_{(\epsilon, \delta, \mathcal{S})},$$

for every $v \in H(\epsilon, \delta, \mathcal{S})$.

Proof Let $v(\tau, z) = \sum_{\beta \geq 0} v_\beta(\tau) \frac{z^\beta}{\beta!} \in H(\epsilon, \delta, \mathcal{S})$. We have that

$$(8) \quad \left\| z^s \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \frac{1}{q^{m_1(\ell_0+1)}} (\partial_z^{-\ell_1} v)(\tau q^{-m_1}, z q^{m_2}) \right\|_{(\epsilon, \delta, \mathcal{S})} \\ = \left\| \sum_{\beta \geq \ell_1 + s} v_{\beta - \ell_1 - s}(\tau q^{-m_1}) q^{m_2(\beta - s) - m_1(\ell_0 + 1)} \frac{\beta!}{(\beta - s)!} \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \frac{z^\beta}{\beta!} \right\|_{(\epsilon, \delta, \mathcal{S})}.$$

From (5), one derives that for every $\tau \in S_\beta$, $v_{\beta - \ell_1 - s}(\tau q^{-m_1})$ is well defined and the function $\tau \mapsto v_{\beta - \ell_1 - s}(q^{-m_1} \tau)$ is holomorphic in S_β for every $\beta \geq \ell_1 + s$. The expression in (8) equals

$$(9) \quad \sum_{\beta \geq \ell_1 + s} \left\| v_{\beta - \ell_1 - s}(\tau q^{-m_1}) q^{m_2(\beta - s) - m_1(\ell_0 + 1)} \frac{\beta!}{(\beta - s)!} \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \right\|_{\beta, \epsilon, S_\beta} \frac{\delta^\beta}{\beta!}.$$

Let $\beta \geq \ell_1 + s$. From the definition of the norm $\|\cdot\|_{\beta, \epsilon, S_\beta}$, we get

$$(10) \quad \left\| v_{\beta - \ell_1 - s}(\tau q^{-m_1}) q^{m_2(\beta - s) - m_1(\ell_0 + 1)} \frac{\beta!}{(\beta - s)!} \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \right\|_{\beta, \epsilon, S_\beta} \\ = \sup_{\tau \in S_\beta} \left\{ |v_{\beta - \ell_1 - s}(\tau q^{-m_1})| \left(\frac{|\tau| q^{-m_1}}{|\epsilon|} \right)^{-C(\beta - \ell_1 - s)} e^{-M \log^2 \left(\frac{|\tau|}{|\epsilon|} q^{-m_1 + \delta_1} \right)} \right. \\ \left. \times \left| \frac{\tau}{\epsilon} \right|^{\ell_0} \left(\frac{|\tau| q^{-m_1}}{|\epsilon|} \right)^{C(\beta - \ell_1 - s)} e^{M \log^2 \left(\frac{|\tau| q^{-m_1}}{|\epsilon|} + \delta_1 \right)} \left| \frac{\tau}{\epsilon} \right|^{-C\beta} e^{-M \log^2 \left(\frac{|\tau|}{|\epsilon|} + \delta_1 \right)} \right\} \\ \times \frac{\beta!}{(\beta - s)!} q^{m_2(\beta - s) - m_1(\ell_0 + 1)} q^{-A_1 \beta^2} q^{A_1(\beta - \ell_1 - s)^2} q^{-A_1(\beta - \ell_1 - s)^2}.$$

It is immediate to check that

$$(11) \quad e^{-M \log^2 \left(\frac{|\tau|}{|\epsilon|} + \delta_1 \right) + M \log^2 \left(\frac{|\tau| q^{-m_1}}{|\epsilon|} + \delta_1 \right)} \leq C_{01} e^{-M \log^2 \left(\frac{|\tau|}{|\epsilon|} \right) + M \log^2 \left(\frac{|\tau| q^{-m_1}}{|\epsilon|} \right)} \leq C_{02} \left(\frac{|\tau|}{|\epsilon|} \right)^{-2m_1 M \log(q)},$$

for some positive constants C_{01} and C_{02} only depending on q, m_1, M . Moreover,

$$(|\tau| q^{-m_1})^{C(\beta - \ell_1 - s)} = C_{03} q^{-m_1 C \beta} |\tau|^{C(\beta - \ell_1 - s)},$$

for some constant $C_{03} > 0$ depending on q, m_1, ℓ_1, s . This last equality and (4) yield

$$\left| \frac{\tau}{\epsilon} \right|^{\ell_0} \left(\frac{|\tau| q^{-m_1}}{|\epsilon|} \right)^{C(\beta - \ell_1 - s)} \left| \frac{\tau}{\epsilon} \right|^{-C\beta} \left(\frac{|\tau|}{|\epsilon|} \right)^{-2m_1 M \log(q)} \\ = C_{04} \left(\frac{|\epsilon|}{|\tau|} \right)^{-\ell_0 + C(\ell_1 + s) + 2m_1 M \log(q)} \\ \leq C_{05} |\tau|^{\ell_0 - 2m_1 M \log(q) - C(\ell_1 + s)} q^{-m_1 C \beta},$$

for some positive constants C_{04} and C_{05} depending on $q, m, \ell_0, \ell_1, s, C$ and \mathcal{E} . From the hypothesis (6) on R_β , the last expression is upper bounded by

$$C_{05} q^{(-m_1 C + d_2(\ell_0 - 2m_1 M \log(q) - C(\ell_1 + s)))\beta},$$

for some positive constant C_{05} only depending on $q, m_1, \ell_0, \ell_1, s, \mathcal{E}, C$ and d_1 . Now, from (7) one gets that $\beta!/(\beta-s)!q^{p_1(\beta)}$ is upper bounded by a constant $C_{06} > 0$ which does not depend on β , where

$$p_1(\beta) = m_2(\beta-s) - m_1(\ell_0+1) - A_1\beta^2 + A_1(\beta-\ell_1-s)^2 - m_1C\beta + d_2[\ell_0 - 2m_1M \log(q) - C(\ell_1+s)].$$

Taking into account all these computations, one achieves that (10) can be upper bounded by

$$C_{05} \sup_{\tau \in S_\beta} \left\{ |v_{\beta-\ell_1-s}(\tau q^{-m_1})| \left(\frac{|\tau|q^{-m_1}}{|\epsilon|} \right)^{-C(\beta-\ell_1-s)} e^{-M \log^2\left(\frac{|\tau|q^{-m_1}}{|\epsilon|} + \delta_1\right)} \right\} q^{-A_1(\beta-\ell_1-s)}.$$

The lemma follows bearing in mind (5) and the definition of the norms in $E_{\beta-(\ell_1+s), \epsilon, S_{\beta-(\ell_1+s)}}$ and of $H(\epsilon, \delta, \mathcal{S})$. \square

Remark: The hypotheses made in (5), (6) and (7) are verified if one departs from $R_\beta = d_1q^{d_2\beta}$ for some small enough positive d_2 , and any $d_1 > 0$, provided (4) is satisfied and $m_2 - 2A_1(\ell_1+s) - m_1C > 0$.

Lemma 2 *Let $F(\epsilon, \tau)$ be a holomorphic and bounded function defined on $\mathcal{E} \times \mathcal{S}$. Then, there exists a constant $C_{12} = C_{12}(F, \mathcal{E}, \mathcal{S}) > 0$ such that*

$$\|F(\epsilon, \tau)v_\epsilon(\tau, z)\|_{(\epsilon, \delta, \mathcal{S})} \leq C_{12} \|v_\epsilon(\tau, z)\|_{(\epsilon, \delta, \mathcal{S})}$$

for every $\epsilon \in \mathcal{E}$, every $\delta > 0$ and all $v_\epsilon \in H(\epsilon, \delta, \mathcal{S})$.

Proof Direct calculations on the definition of the norms in the space $H(\epsilon, \delta, \mathcal{S})$ allow us to conclude when taking $C_{12} := \max\{|F(\epsilon, \tau)| : \epsilon \in \mathcal{E}, \tau \in \mathcal{S}\}$. \square

Let $S \geq 1$, and \mathcal{N} be a finite subset of \mathbb{N}^2 . We also fix $a \in \mathbb{C} \setminus \mathbb{R}_+$, where \mathbb{R}_+ stands for the set $\{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0, \operatorname{Im}(z) = 0\}$.

For every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$, let $m_{\underline{\kappa},1}, m_{\underline{\kappa},2}$ be nonnegative integers and $b_{\underline{\kappa}}(\epsilon, z) \in \mathcal{O}(D(0, r_0))[z]$, where $r_0 > 0$ is such that $\bar{\mathcal{E}} \subseteq D(0, r_0)$. We write $b_{\underline{\kappa}}(\epsilon, z) = \sum_{s \in I_{\underline{\kappa}}} b_{\underline{\kappa},s}(\epsilon)z^s$, where $I_{\underline{\kappa}}$ is a finite subset of \mathbb{N} for every $\underline{\kappa} \in \mathcal{N}$. We assume that $1 \leq \kappa_1 < S$ for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$.

We consider the functional equation

$$(12) \quad \partial_z^S W(\epsilon, \tau, z) = \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \frac{b_{\underline{\kappa}}(\epsilon, z)}{(a-\tau)q^{m_{\underline{\kappa},1}(\kappa_0+1)}} \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} (\partial_z^{\kappa_1} W)(\epsilon, q^{-m_{\underline{\kappa},1}\tau}, q^{m_{\underline{\kappa},2}z})$$

with initial conditions

$$(13) \quad (\partial_z^j W)(\epsilon, \tau, 0) = W_j(\epsilon, \tau) \quad , 0 \leq j \leq S-1,$$

where the function $(\epsilon, \tau) \mapsto W_j(\epsilon, \tau)$ is an element in $\mathcal{O}(\mathcal{E} \times \mathcal{S})$ for every $0 \leq j \leq S-1$.

We make the following

Assumption (A) For every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$ and every $s \in I_{\underline{\kappa}}$, we assume

$$C(S - \kappa_1 + s) - \kappa_0 - 2m_{\underline{\kappa},1}M(-\log(q)) \geq 0,$$

$$[C(S - \kappa_1 + s) - \kappa_0 - 2m_{\underline{\kappa},1}M(-\log(q))]d_2 < m_{\underline{\kappa},2} - 2A_1(S - \kappa_1 + s) - m_{\underline{\kappa},1}C.$$

Assumption (B) $R_\beta \geq q^{m_{\underline{\kappa},1}}R_{\beta-\kappa_1-s}$, and there exist $d_1, d_2 > 0$ with $R_\beta \geq d_1q^{d_2\beta}$, for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$ and every $s \in I_{\underline{\kappa}}$.

Theorem 1 *Let Assumption (A) and Assumption (B) be fulfilled. We assume that the initial conditions in (13) verify there exist $\Delta > 0$ and $0 < \tilde{M} < M$ such that for every $0 \leq j \leq S-1$*

$$(14) \quad |W_j(\epsilon, \tau)| \leq \Delta e^{\tilde{M} \log^2\left(\frac{|\tau|}{|\epsilon|} + \delta_1\right)} |\epsilon|^{K_0},$$

for every $\tau \in S_0$, $\epsilon \in \mathcal{E}$, where $K_0 = \max\{\kappa_0 : (\kappa_0, \kappa_1) \in \mathcal{N}\}$. Then, there exists $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$, formal solution of (12)+(13), where $W_\beta \in \mathcal{O}(\mathcal{E} \times S_\beta)$.

Then, there exist positive constants C_{13} , and C_{14} (only depending on $q, d_1, d_2, C, S, \delta_1, A_1$), and $\delta > 0$ such that

$$(15) \quad |W_\beta(\epsilon, \tau)| \leq C_{13} \beta! \left(\frac{C_{14}}{\delta}\right)^\beta e^{M \log^2\left(\frac{|\tau|}{|\epsilon|} + \delta_1\right)} \left|\frac{\tau}{\epsilon}\right|^{C\beta} q^{A_1 \beta^2},$$

for every $\beta \geq 0$, all $\epsilon \in \mathcal{E}$ and every $\tau \in S_\beta$.

Proof Let $\epsilon \in \mathcal{E}$. We put $\mathbb{E} := \{\mathcal{O}(S_\beta) : \beta \geq 0\}$ and define the map \mathcal{A}_ϵ from $\mathbb{E}[[z]]$ into itself by

$$(16) \quad \mathcal{A}_\epsilon(\tilde{W}(\tau, z)) := \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \frac{b_{\underline{\kappa}}(\epsilon, z)}{(a - \tau) q^{m_{\underline{\kappa}, 1}(\kappa_0 + 1)}} \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} \left[(\partial_z^{\kappa_1 - S} \tilde{W})(q^{-m_{\underline{\kappa}, 1}} \tau, q^{m_{\underline{\kappa}, 2}} z) \right. \\ \left. + \partial_z^{\kappa_1} w_\epsilon(q^{-m_{\underline{\kappa}, 1}} \tau, q^{m_{\underline{\kappa}, 2}} z) \right],$$

where $w_\epsilon(\tau, z) = \sum_{j=0}^{S-1} W_j(\epsilon, \tau) \frac{z^j}{j!}$. For an appropriate choice of $\delta, \Delta > 0$, the map \mathcal{A}_ϵ turns out to be a Lipschitz shrinking map.

Lemma 3 *There exist $R, \delta, \Delta > 0$ (not depending on ϵ) such that:*

1. $\left\| \mathcal{A}_\epsilon(\tilde{W}(\tau, z)) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq R$ for every $\tilde{W}(\tau, z) \in B(0, R)$. $B(0, R)$ denotes the closed ball centered at 0 with radius R in $H(\epsilon, \delta, \mathcal{S})$.

2.

$$\left\| \mathcal{A}_\epsilon(\tilde{W}_1(\tau, z)) - \mathcal{A}_\epsilon(\tilde{W}_2(\tau, z)) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq \frac{1}{2} \left\| \tilde{W}_1(\tau, z) - \tilde{W}_2(\tau, z) \right\|_{(\epsilon, \delta, \mathcal{S})}$$

for every $\tilde{W}_1, \tilde{W}_2 \in B(0, R)$.

Proof Let $R > 0$ and $\delta > 0$. In order to prove the first enunciate, we take $\tilde{W}(\tau, z) \in B(0, R) \subseteq H(\epsilon, \delta, \mathcal{S})$. From Lemma 1 and Lemma 2 we deduce that

$$(17) \quad \left\| \mathcal{A}_\epsilon(\tilde{W}(\tau, z)) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \sum_{s \in I_{\underline{\kappa}}} \frac{M_{\underline{\kappa}, s}}{M_1} \left[C_{01} \delta^{S - \kappa_1 + s} \left\| \tilde{W}(\tau, z) \right\|_{(\epsilon, \delta, \mathcal{S})} \right. \\ \left. + \left\| \frac{z^s}{q^{m_{\underline{\kappa}, 1}(\kappa_0 + 1)}} \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} \partial^{\kappa_1} w_\epsilon(q^{-m_{\underline{\kappa}, 1}} \tau, q^{m_{\underline{\kappa}, 2}} z) \right\|_{(\epsilon, \delta, \mathcal{S})} \right],$$

with $M_{\underline{\kappa}, s} = \sup_{\epsilon \in \mathcal{E}} |b_{\underline{\kappa}, s}(\epsilon)| < \infty$ for every $\underline{\kappa} \in \mathcal{N}$ and $s \in I_{\underline{\kappa}}$.

Let us fix $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$ and $s \in I_{\underline{\kappa}}$. Taking into account the definition of $H(\epsilon, \delta, \mathcal{S})$, we derive

$$\left\| \frac{z^s}{q^{m_{\underline{\kappa}, 1}(\kappa_0 + 1)}} \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} \partial^{\kappa_1} w_\epsilon(q^{-m_{\underline{\kappa}, 1}} \tau, q^{m_{\underline{\kappa}, 2}} z) \right\|_{(\epsilon, \delta, \mathcal{S})}$$

$$\begin{aligned}
&= \sum_{j=s}^{S-1-\kappa_1-s} \left\| W_{j+\kappa_1-s}(\epsilon, q^{-m_{\kappa_1,1}\tau}) \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} \right\|_{j,\epsilon,S_j} q^{m_{\kappa_1,2}(j-s)-m_{\kappa_1,1}(\kappa_0+1)} \frac{j!}{(j-s)!} \frac{\delta^j}{j!} \\
(18) \quad &\leq C_{14} \sum_{j=s}^{S-1-\kappa_1-s} \sup_{\tau \in S_j} |W_{j+\kappa_1-s}(\epsilon, q^{-m_{\kappa_1,1}\tau})| \left| \frac{\tau}{\epsilon} \right|^{\kappa_0} \left| \frac{\tau}{\epsilon} \right|^{-Cj} e^{-M \log^2\left(\frac{|\tau|}{|\epsilon|} + \delta_1\right)} \delta^j,
\end{aligned}$$

for some $C_{14} > 0$ which only depends on the parameters defining equation (12). The terms of the form $|\epsilon|^{Cj}$ in the previous expression can be upper bounded by an adequate constant. Taking into account (14), usual estimates in (18) derive

$$\left\| \mathcal{A}_\epsilon(\tilde{W}(\tau, z)) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \sum_{s \in I_{\underline{\kappa}}} \frac{M_{\underline{\kappa}, s}}{M_1} \left[C_{01} \delta^{S-\kappa_1+s} \left\| \tilde{W}(\tau, z) \right\|_{(\epsilon, \delta, \mathcal{S})} + C_{15} \right],$$

for some C_{15} depending on the parameters defining the equation, and such that tends to 0 whenever both Δ and δ tend to 0. An appropriate choice for these constants allow us to conclude the first part of the proof.

The second part of the lemma follows similar arguments as before. Let $\tilde{W}_1, \tilde{W}_2 \in B(0, R) \subseteq H(\epsilon, \delta, \mathcal{S})$. One has

$$\left\| \mathcal{A}_\epsilon(\tilde{W}_1) - \mathcal{A}_\epsilon(\tilde{W}_2) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \sum_{s \in I_{\underline{\kappa}}} \frac{M_{\underline{\kappa}, s}}{M_1} C_{01} \delta^{S-\kappa_1+s} \left\| \tilde{W}_1 - \tilde{W}_2 \right\|_{(\epsilon, \delta, \mathcal{S})}.$$

The result is achieved with an adequate choice of $\delta > 0$. \square

Let R, Δ and δ be as in the previous lemma. Bearing in mind Lemma 3 one can apply the shrinking map theorem on complete metric spaces to guarantee the existence of a fixed point for \mathcal{A}_ϵ in $B(0, R) \subseteq H(\epsilon, \delta, \mathcal{S})$, say \tilde{W}_ϵ , which verifies $\left\| \tilde{W}_\epsilon(\tau, z) \right\|_{(\epsilon, \delta, \mathcal{S})} \leq R$, and $\mathcal{A}_\epsilon(\tilde{W}_\epsilon(\tau, z)) = \tilde{W}_\epsilon(\tau, z)$. Let us define

$$(19) \quad W_\epsilon(\tau, z) = \partial_z^{-S} \tilde{W}_\epsilon(\tau, z) + w_\epsilon(\tau, z).$$

We put $\tilde{W}(\epsilon, \tau, z) := \tilde{W}_\epsilon(\tau, z)$, and $W(\epsilon, \tau, z) := \partial_z^{-S} \tilde{W}(\epsilon, \tau, z) + w_\epsilon(\tau, z)$. Then, $W(\epsilon, \tau, z)$ can be written as a formal power series in z ,

$$W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!},$$

where $W_{\beta+S}(\epsilon, \tau) = \tilde{W}_{\beta, \epsilon}(\tau)$ for every $\beta \geq 0$.

From the construction of $W(\epsilon, \tau, z)$, we have $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$ is a formal solution of (12)+(13). Moreover, from the domain of holomorphy of the initial conditions in (13) and the recursion formula satisfied by the coefficients in $W(\epsilon, \tau, z)$:

$$(20) \quad \frac{W_{h+S}(\epsilon, \tau)}{h!} = \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \sum_{h_1+h_2=h, h_1 \in I_{\underline{\kappa}}} b_{\underline{\kappa}, h_1}(\epsilon) \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} \frac{q^{m_{\kappa_1,2}h_2}}{(a-\tau)h_2!q^{m_{\kappa_1,1}(\kappa_0+1)}} W_{h_2+\kappa_1}(\epsilon, q^{-m_{\kappa_1,1}\tau}),$$

we can conclude the function $(\epsilon, \tau) \mapsto W_\beta \in \mathcal{O}(\mathcal{E} \times \mathcal{S})$ for every $\beta \geq 0$.

Finally, the estimates in (15) are obtained for every $\beta \geq 0$ from the fact that $\tilde{W}_\epsilon \in B(0, R) \subseteq H(\epsilon, \delta, \mathcal{S})$. The definition of the elements in $H(\epsilon, \delta, \mathcal{S})$ lead us to

$$\left\| \tilde{W}_{\beta, \epsilon} \right\|_{\beta, \epsilon, S_\beta} \leq R \beta! \left(\frac{1}{\delta} \right)^\beta,$$

so that

$$|W_\beta(\epsilon, \tau)| = |\tilde{W}_{\beta-S, \epsilon}(\tau)| \leq R(\beta - S)! \left(\frac{1}{\delta}\right)^{\beta-S} e^{M \log^2\left(\frac{|\tau|}{|\epsilon|} + \delta_1\right)} \left|\frac{\tau}{\epsilon}\right|^{C(\beta-S)} q^{A_1(\beta-S)^2},$$

for every $\beta \geq S$. In addition to this, Assumption (B) and usual estimates allow us to refine the previous estimates leading to

$$|W_\beta(\epsilon, \tau)| \leq C_{13}\beta! \left(\frac{C_{14}}{\delta}\right)^\beta e^{M \log^2\left(\frac{|\tau|}{|\epsilon|} + \delta_1\right)} \left|\frac{\tau}{\epsilon}\right|^{C\beta} q^{A_1\beta^2},$$

for some constants $C_{13} > 0$ and $C_{14} > 0$ which only depend on $q, d_1, d_2, C, S, \delta_1$ and A_1 . This is valid for every $\epsilon \in \mathcal{E}$ and $\tau \in S_\beta$. The hypothesis (14) in the enunciate allows us to affirm that (15) is also valid for $0 \leq \beta \leq S - 1$. \square

Remark: One derives holomorphy of W_β in the variable τ in the whole sector \mathcal{S} , and not only in S_β for every $\beta \geq S$ whilst the estimates are only given for $\tau \in S_\beta$. It is also worth saying that $R > 0$ can be arbitrarily chosen whenever $s > 0$ for every $s \in I_{\underline{\kappa}}, \underline{\kappa} \in \mathcal{N}$.

3 Second Cauchy problem in a weighted Banach space of Taylor series

We provide the solution of a Cauchy problem with analogous equation as the one studied in the previous section, written as a formal power series in z with coefficients in an appropriate Banach space of functions in the variable τ and the perturbation parameter ϵ . In Section 2, the domain of holomorphy of the coefficients remains invariant from the domain of holomorphy of the initial conditions. This happens so because the dilation operator $\tau \mapsto q^{-1}\tau$ sends points in any infinite sector in the complex plane with vertex at the origin into itself. Now, the domain of holomorphy of the coefficients for the formal solution of the Cauchy problem under study depends on the index considered. More precisely, if the initial conditions present a singularity at some point $a \in \mathbb{C}$ in the variable τ , the coefficients of the formal solution of the Cauchy problem have singularities in τ that tend to 0, providing a small divisor phenomenon.

For every $\rho > 0$, \dot{D}_ρ stands for the set $D(0, \rho) \setminus \{0\}$. We preserve the value of the positive constants M, A_1, C and δ_1 from the previous section. Let $r_0 > 0$ with $\bar{\mathcal{E}} \subseteq D(0, r_0)$ and $(\hat{R}_\beta)_{\beta \geq 0}$ be a sequence of positive real numbers.

Definition 2 Let $\beta \in \mathbb{N}$. For $r_0 > 0$ and $\epsilon \in D(0, r_0) \setminus \{0\}$, $E_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}}^2$ stands for the vector space of functions $v \in \mathcal{O}(\dot{D}_{\hat{R}_\beta})$ such that

$$|v(\tau)|_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}} := \sup_{\tau \in \dot{D}_{\hat{R}_\beta}} \left\{ |v(\tau)| \frac{|\epsilon|^{C\beta}}{e^{M \log^2(|\tau| + \delta_1)}} \right\} q^{-A_1\beta^2}$$

is finite. Let $\delta > 0$. We write $H_2(\epsilon, \delta)$ for the vector space of all formal power $v(\tau, z) = \sum_{\beta \geq 0} v_\beta(\tau) z^\beta / \beta!$ such that $v_\beta \in E_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}}^2$ with

$$|v(\tau, z)|_{(\epsilon, \delta)} := \sum_{\beta \geq 0} |v_\beta(\tau)|_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}} \frac{\delta^\beta}{\beta!} < \infty.$$

The pair $(H_2(\epsilon, \delta), |\cdot|_{(\epsilon, \delta)})$ is a Banach space.

Lemma 4 Let $s, \ell_0, \ell_1, m_1, m_2 \in \mathbb{N}$, $\delta > 0$ and $\epsilon \in D(0, r_0) \setminus \{0\}$. We assume that

$$(21) \quad C(\ell_1 + s) - \ell_0 \geq 0, \quad m_2 - 2A_1(\ell_1 + s) > 0.$$

Moreover, we assume that the elements of the sequence $(\hat{R}_\beta)_{\beta \geq 0}$ are such that

$$(22) \quad \hat{R}_\beta \leq q^{m_1} \hat{R}_{\beta - \ell_1 - s},$$

for every $\beta \geq \ell_1 + s$.

Under the previous assumptions, there exists a positive constant C_{21} which depends on $C, q, m_1, m_2, s, \ell_0, \ell_1, M, A_1, \delta_1, r_0$ (not depending on ϵ nor δ), such that

$$\left| z^s \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \frac{1}{q^{m_1(\ell_0+1)}} (\partial_z^{-\ell_1} v)(\tau q^{-m_1}, zq^{m_2}) \right|_{(\epsilon, \delta)} \leq C_{21} \delta^{\ell_1+s} |v(\tau, z)|_{(\epsilon, \delta)},$$

for every $v \in H_2(\epsilon, \delta)$.

Proof Let $v(\tau, z) = \sum_{\beta \geq 0} v_\beta(\tau) \frac{z^\beta}{\beta!}$ be an element of $H_2(\epsilon, \delta)$. We have

$$(23) \quad \left| z^s \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \frac{1}{q^{m_1(\ell_0+1)}} (\partial_z^{-\ell_1} v)(\tau q^{-m_1}, zq^{m_2}) \right|_{(\epsilon, \delta)} \\ = \left| \sum_{\beta \geq \ell_1 + s} v_{\beta - \ell_1 - s}(\tau q^{-m_1}) q^{m_2(\beta - s) - m_1(\ell_0 + 1)} \frac{\beta!}{(\beta - s)!} \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \frac{z^\beta}{\beta!} \right|_{(\epsilon, \delta)}.$$

From (22), one derives that for every $\tau \in \dot{D}_{\hat{R}_{\beta - \ell_1 - s}}$, $v_{\beta - \ell_1 - s}(\tau q^{-m_1})$ is well defined. In addition to this, the function $\tau \mapsto v_{\beta - \ell_1 - s}(\tau q^{-m_1})$ is holomorphic in $\dot{D}_{\hat{R}_\beta}$ for every $\beta \geq \ell_1 + s$. The expression in (23) equals

$$(24) \quad \sum_{\beta \geq \ell_1 + s} \left| v_{\beta - \ell_1 - s}(\tau q^{-m_1}) q^{m_2(\beta - s) - m_1(\ell_0 + 1)} \frac{\beta!}{(\beta - s)!} \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \right|_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}} \frac{\delta^\beta}{\beta!}.$$

Let $\beta \geq \ell_1 + s$. From the definition of the norm $|\cdot|_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}}$, we get

$$(25) \quad \left| v_{\beta - \ell_1 - s}(\tau q^{-m_1}) q^{m_2(\beta - s) - m_1(\ell_0 + 1)} \frac{\beta!}{(\beta - s)!} \left(-\frac{\tau}{\epsilon} \right)^{\ell_0} \right|_{\beta, \epsilon, \dot{D}_{\hat{R}_\beta}} \\ = \sup_{\tau \in \dot{D}_{\hat{R}_\beta}} \left\{ |v_{\beta - \ell_1 - s}(\tau q^{-m_1})| \frac{|\epsilon|^{C(\beta - \ell_1 - s)}}{e^{M \log^2 \left(\frac{|\tau|}{q^{m_1}} + \delta_1 \right)}} \right. \\ \left. \times \left| \frac{\tau}{\epsilon} \right|^{\ell_0} |\epsilon|^{C(\ell_1 + s)} e^{M \left(\log^2 \left(\frac{|\tau|}{q^{m_1}} + \delta_1 \right) - \log^2(|\tau| + \delta_1) \right)} \right\} q^{p_1(\beta)} \frac{\beta!}{(\beta - s)!} q^{-A_1(\beta - \ell_1 - s)^2},$$

with $p_1(\beta) = m_2(\beta - s) - m_1(\ell_0 + 1) - A_1\beta^2 + A_1(\beta - \ell_1 - s)^2$.

The result follows provided that one is able to estimate the expression

$$q^{p_1(\beta)} \frac{\beta!}{(\beta - s)!} |\tau|^{\ell_0} |\epsilon|^{C(\ell_1 + s) - \ell_0} e^{M \left(\log^2 \left(\frac{|\tau|}{q^{m_1}} + \delta_1 \right) - \log^2(|\tau| + \delta_1) \right)}.$$

From the first of the hypotheses made in (21), $|\epsilon|^{C(\ell_1+s)-\ell_0}$ is upper bounded by a constant. Also, taking into account (22), there exists $\hat{R} > 0$ such that $|\tau| \leq \hat{R}$ for every $\tau \in \cup_{\beta \geq 0} \dot{D}_{\hat{R}\beta}$, so that

$$|\tau|^{\ell_0} \exp\left(M(\log^2(|\tau|q^{-m_1} + \delta_1) - \log^2(|\tau| + \delta_1))\right) \leq D_{21}(q, m_1, \delta_1, M, \ell_0),$$

for some positive constant D_{21} . The result immediately follows from (21) that guarantees that $\beta!/(\beta-s)!q^{p_1(\beta)}$ is bounded from above. \square

Let $\hat{R} > 0$ be as in the proof of the previous lemma, i.e. $\hat{R} \geq \hat{R}_\beta$ for every $\beta \geq 0$.

Lemma 5 *Let $F(\epsilon, \tau)$ be a holomorphic and bounded function defined on $D(0, r_0) \times D(0, \hat{R})$. Then, there exists a constant $C_{22} = C_{22}(F) > 0$ such that*

$$|F(\epsilon, \tau)v_\epsilon(\tau, z)|_{(\epsilon, \delta)} \leq C_{22} |v_\epsilon(\tau, z)|_{(\epsilon, \delta)}$$

for every $\epsilon \in D(0, r_0) \setminus \{0\}$, every $\delta > 0$ and all $v_\epsilon \in H_2(\epsilon, \delta)$.

Proof Direct calculations on the definition of the norms in the space $H_2(\epsilon, \delta)$ allow us to conclude when taking $C_{22} := \max\{|F(\epsilon, \tau)| : \epsilon \in D(0, r_0), \tau \in D(0, \hat{R})\}$. \square

Let $S \geq 1$, and \mathcal{N} be a finite subset of \mathbb{N}^2 . We also fix $a \in \mathbb{C} \setminus \mathbb{R}_+$ such that $|a| \geq \hat{R}$, with \hat{R} as before.

Let $m_{\underline{\kappa}, 1}$, $m_{\underline{\kappa}, 2}$ and $b_{\underline{\kappa}}$ be as in Section 2, for every $\underline{\kappa} = (\kappa_0, \kappa_1)$.

We consider the functional equation

$$(26) \quad \partial_z^S W(\epsilon, \tau, z) = \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} \frac{b_{\underline{\kappa}}(\epsilon, z)}{(a - \tau)q^{m_{\underline{\kappa}, 1}(\kappa_0+1)}} \left(-\frac{\tau}{\epsilon}\right)^{\kappa_0} (\partial_z^{\kappa_1} W)(\epsilon, q^{-m_{\underline{\kappa}, 1}}\tau, q^{m_{\underline{\kappa}, 2}}z)$$

with initial conditions

$$(27) \quad (\partial_z^j W)(\epsilon, \tau, 0) = W_j(\epsilon, \tau) \quad , 0 \leq j \leq S-1,$$

where the function $(\epsilon, \tau) \mapsto W_j(\epsilon, \tau)$ is an element in $\mathcal{O}((D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_0})$ for every $0 \leq j \leq S-1$.

We make the following

Assumption (A') For every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$ and every $s \in I_{\underline{\kappa}}$, we assume

$$C(S - \kappa_1 + s) - \kappa_0 \geq 0, \quad m_{\underline{\kappa}, 2} - 2A_1(S - \kappa_1 + s) > 0.$$

Remark: Observe that Assumption (A) implies Assumption (A').

Assumption (B') We assume $\hat{R}_\beta \leq q^{m_{\underline{\kappa}, 1}}\hat{R}_{\beta-\kappa_1-s}$ for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$, every $s \in I_{\underline{\kappa}}$ and every $\beta \geq \kappa_1 + s$.

We first state a result which provides a concrete value for the elements in $(\hat{R}_\beta)_{\beta \geq 0}$ under Assumption (B'). The choice is made in two respects: first, to clarify how the singularities suffer propagation in the formal solution of (26)+(27), with respect to the variable τ ; and second, to provide acceptable domains of holomorphy for such coefficients when regarding this phenomenon of propagation of singularities. Any other appropriate choice for the elements in $(\hat{R}_\beta)_{\beta \geq 0}$ regarding these issues would also be fairish for our purpose.

Lemma 6 Let $\hat{d}_1, \hat{d}_2 > 0$ and $\epsilon \in D(0, r_0) \setminus \{0\}$.

We put $\hat{R}_\beta := \hat{R}_0$ for $\beta = 0, 1, \dots, S-1$, and $\hat{R}_\beta = \hat{d}_1 q^{\hat{d}_2 \beta}$ for every $\beta \geq S$. Let us assume that (26)+(27) has a formal solution in z , $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$. Then, there exists \hat{d}_{20} such that for every $\hat{d}_2 \geq \hat{d}_{20}$, the function $\tau \mapsto W_\beta(\epsilon, \tau)$ belongs to $\mathcal{O}(\dot{D}_{\hat{R}_\beta})$ for every $\beta \geq S$ and all $\epsilon \in D(0, r_0) \setminus \{0\}$.

Proof Let $W(\epsilon, \tau, z)$ be a formal power series in z of the form $\sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$. One can plug the formal power series into equation (26) to obtain the recursion formula in (20) for the coefficients $(W_\beta)_{\beta \geq S}$. From this recurrence, one derives the domain of holomorphy for W_{h+S} in the variable τ depends on the domain of holomorphy on τ of $W_{h_2+\kappa_1}$ and also on $q^{-m_{\underline{\kappa},1}}$ for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$, every $0 \leq h_2 \leq h$ such that $h - h_2 \in I_{\underline{\kappa}}$.

The initial conditions W_0, \dots, W_{S-1} are holomorphic functions in $\dot{D}_{\hat{R}_0}$.

Lemma 7 For every $N \geq 1$ the coefficients $W_{NS-(N-1)\kappa_{10}}, \dots, W_{(N+1)S-N\kappa_{10}}$ turn out to be holomorphic functions in $\dot{D}_{q^{N\bar{m}_{\underline{\kappa},1}} \hat{R}_0}$, for $\kappa_{10} := \max\{\kappa_1 : (\kappa_0, \kappa_1) \in \mathcal{N}\}$ and $\bar{m}_{\underline{\kappa},1} := \max\{m_{\underline{\kappa},1} : \underline{\kappa} \in \mathcal{N}\}$.

Proof

We prove it by recurrence on N and regarding the recursion formula (20).

Let $N = 1$. One has $h_2 + \kappa_1 \leq S - 1$ for any h_2, κ_1 as in (20) if and only if $h_2 \leq S - 1 - \kappa_1$ for every $(\kappa_0, \kappa_1) \in \mathcal{N}$, it is to say, if and only if $h_2 \leq S - 1 - \kappa_{10}$. In this case, W_{h+S} only depends on the initial conditions $(W_j)_{0 \leq j \leq S-1}$. Moreover,

$$h + S \in \{S, S + 1, \dots, 2S - \kappa_{10} - 1\},$$

and the dilation on the variable τ allow us to obtain that $W_S, \dots, W_{2S-\kappa_{10}-1}$ are holomorphic functions in $\dot{D}_{q^{\bar{m}_{\underline{\kappa},1}} \hat{R}_0}$.

The proof can be followed recursively for every $N \geq 2$ by considering analogous blocks of indices as before. \square

Regarding Lemma 7, the proof of Lemma 6 is concluded if one can check that for every $N \geq 1$, $\hat{R}_\beta \leq \hat{R}_0 q^{N\bar{m}_{\underline{\kappa},1}}$ whenever

$$\beta \in \{NS - (N-1)\kappa_{10}, \dots, (N+1)S - N\kappa_{10} - 1\} = \{N(S - \kappa_{10}) + \kappa_{10}, \dots, N(S - \kappa_{10}) + S - 1\}.$$

Let $N \geq 1$ and $\beta = N(S - \kappa_{10}) + L$, with $\kappa_{10} \leq L \leq S - 1$. Let $\hat{d}_1 \leq \hat{R}_0$. We have $\hat{R}_\beta = \hat{d}_1 q^{\hat{d}_2 [N(S - \kappa_{10}) + L]} \leq \hat{R}_0 q^{N\bar{m}_{\underline{\kappa},1}}$ if and only if $N\bar{m}_{\underline{\kappa},1} \leq \hat{d}_2 [N(S - \kappa_{10}) + L]$. The result follows for any $\hat{d}_2 \geq \frac{\bar{m}_{\underline{\kappa},1}}{S - \kappa_{10}}$. \square

Lemma 8 Let \hat{R}_β be defined as in Lemma 6. Then, $(\hat{R}_\beta)_{\beta \geq 0}$ satisfies Assumption (B').

Proof From the definition of \hat{R}_β , the lemma follows when taking $\tilde{d}_2 \geq \frac{m_{\underline{\kappa},1}}{\kappa_1 + s}$ for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$, and every $s \in I_{\underline{\kappa}}$. \square

Assumption (B''): We assume $\hat{R}_\beta = \hat{R}_0$ for $0 \leq \beta \leq S - 1$ and $\hat{R}_\beta := \hat{d}_1 q^{\hat{d}_2 \beta}$ for any $\hat{d}_2 \geq \hat{d}_{20}$, with $\hat{d}_{20} > 0$ as in Lemma 6.

As it has been pointed out before, the Assumption (B') is substituted in the present work by Assumption (B'') with the cost of losing some generality, but giving concrete values for \hat{R}_β , for every $\beta \geq 0$. The incoming theorem is valid when considering any other choice of the elements in $(\hat{R}_\beta)_{\beta \geq 0}$ satisfying Assumption (B').

Theorem 2 *Let Assumption (A') and Assumption (B'') be fulfilled. We also make the next assumption on the initial conditions (27): there exist $\Delta > 0$ and $0 < \tilde{M} < M$ such that*

$$(28) \quad |W_j(\epsilon, \tau)| \leq \Delta e^{\tilde{M} \log^2(|\tau| + \delta_1)} |\epsilon|^{K_0},$$

for every $\tau \in D(0, \hat{R}_0)$, $\epsilon \in D(0, r_0) \setminus \{0\}$ and $0 \leq j \leq S - 1$, where $K_0 = \max\{\kappa_0 : (\kappa_0, \kappa_1) \in \mathcal{N}\}$. Then, there exists a formal power series $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$, with $W_\beta(\epsilon, \tau) \in \mathcal{O}((D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_\beta})$, which provides a formal solution of (26)+(27). Moreover, there exist positive constants C_{23} and C_{24} (only depending on $r_0, \hat{R}_0, q, C, S, A_1, \delta_1, M$), and $0 < \delta < 1$ such that

$$(29) \quad |W_\beta(\epsilon, \tau)| \leq C_{23} \beta! \left(\frac{C_{24}}{\delta} \right)^\beta |\epsilon|^{-C\beta} e^{M \log^2(|\tau| + \delta_1)} q^{A_1 \beta^2},$$

for every $\epsilon \in D(0, r_0) \setminus \{0\}$, $\tau \in \dot{D}_{\hat{R}_\beta}$, and for every $\beta \in \mathbb{N}$.

Proof The proof follows analogous steps as the one of Theorem 1, so we do not enter into details not to repeat arguments.

Let $\epsilon \in D(0, r_0) \setminus \{0\}$ and $0 < \delta < 1$. The set \mathbb{E} is taken to be $\{\mathcal{O}(\dot{D}_{\hat{R}_\beta}) : \beta \geq 0\}$. We consider the map \mathcal{A}_ϵ from $\mathbb{E}[[z]]$ into itself defined in the same way as in (16).

From Lemma 6 and Assumption (B''), the unique formal solution of (26)+(27), determined by the recursion formula (20), $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$, is such that $W_\beta(\epsilon, \tau) \in \mathcal{O}(\dot{D}(0, \hat{R}_\beta))$ for every $\beta \geq 0$.

Regarding the initial conditions of the Cauchy problem, one can reduce \hat{d}_1 , if necessary, so that $\hat{R}_{j+s} q^{-m_{\underline{\kappa}, 1}} \leq \hat{R}_0$ and so the map $\tau \mapsto W_{j+\kappa_1}(\epsilon, q^{-m_{\underline{\kappa}, 1}} \tau)$ is well defined, for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$, every $s \in I_{\underline{\kappa}}$ and $j = 0, 1, \dots, S - 1 - \kappa_1$. Moreover, from (28), the expression

$$\left| \frac{z^s}{q^{m_{\underline{\kappa}, 1}(\kappa_0 + 1)}} \left(-\frac{\tau}{\epsilon} \right)^{\kappa_0} \partial_z^{\kappa_1} w_\epsilon(q^{-m_{\underline{\kappa}, 1}} \tau, q^{m_{\underline{\kappa}, 2}} z) \right|_{(\epsilon, \delta)}$$

can be estimated in an analogous manner as in the corresponding step of the proof of Theorem 1, for every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$ and all $s \in I_{\underline{\kappa}}$. \square

4 Analytic solutions in a parameter of singularly perturbed Cauchy problem

4.1 Laplace transform and q -Gevrey asymptotic expansion

In this subsection, we recall some identities for the Laplace transform, and state some definitions and first results on q -Gevrey asymptotic expansions. The next lemma can be found in [34].

Lemma 9 *Let $m \in \mathbb{N}$, and $w_1(\tau)$ be a holomorphic function in an unbounded sector U such that there exist $C, K > 0$ with*

$$(30) \quad |w(\tau)| \leq C \exp(K|\tau|),$$

for every $\tau \in U$. Let \mathcal{D} be an unbounded sector with vertex at 0 which verifies that

$$d + \arg(t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right), \quad \cos(d + \arg(t)) \geq \delta_2,$$

for some $d \in \mathbb{R}$ and $\delta_2 > 0$. Then,

$$t \mapsto \int_{L_d} w(\tau) e^{-t\tau} d\tau$$

is a holomorphic and bounded function defined for $t \in \mathcal{D} \cap \{|t| > K/\delta_2\}$. Moreover, the following identities hold:

$$(31) \quad \int_{L_d} \tau^m e^{-t\tau} d\tau = \frac{m!}{t^{m+1}}, \quad \partial_t \left(\int_{L_d} w(\tau) e^{-t\tau} d\tau \right) = \int_{L_d} (-\tau) w(\tau) e^{-t\tau} d\tau,$$

where $L_d = \mathbb{R}_+ e^{id} \subseteq U \cup \{0\}$, for all $t \in \mathcal{D} \cap \{|t| > K/\delta_2\}$.

In the sequel, we work with functions which satisfy more restrictive bounds than the ones in (30). Indeed, we deal with bounds of the form $C \exp(K \log^2 |\tau|)$, for some $C, K > 0$. This alters the asymptotic behaviour of the Laplace transform and causes the appearance of q -Gevrey asymptotic expansions, associated to estimates related to the sequence $(q^{-n^2})_{n \geq 0}$.

For any open sector $S = \{z \in \mathbb{C} : a < \arg(z) < b, |z| < \rho\}$ in the complex plane with vertex at 0 with ρ finite or infinite, and $0 \leq a < b \leq 2\pi$, we say the finite sector \tilde{S} with vertex at the origin is a proper subsector of S , and we denote it $\tilde{S} \prec S$, if $\tilde{S} = \{\tau \in \mathbb{C}^* : a_1 < \arg(\tau) < b_1, |z| < \tilde{\rho}\}$ for some $0 \leq a < a_1 < b_1 < b \leq 2\pi$, and some $\tilde{\rho} \in \mathbb{R}$, $0 < \tilde{\rho} < \rho$.

\mathbb{H} stands for a complex Banach space.

We preserve the Definition of q -Gevrey asymptotic expansion established in [28], in order to be coherent with the definitions in that work.

Definition 3 Let S be a sector in \mathbb{C}^* with vertex at the origin, and $A > 0$. We say a holomorphic function $f : S \rightarrow \mathbb{H}$ admits the formal power series $\hat{f} = \sum_{n \geq 0} f_n \epsilon^n \in \mathbb{H}[[\epsilon]]$ as its q -Gevrey asymptotic expansion of type A in S if for every $\tilde{S} \prec S$ there exist $C_1, H > 0$ such that

$$(32) \quad \left\| f(\epsilon) - \sum_{n=0}^N f_n \epsilon^n \right\|_{\mathbb{H}} \leq C_1 H^N q^{-A \frac{N^2}{2}} \frac{|\epsilon|^{N+1}}{(N+1)!}, \quad N \geq 0,$$

for every $\epsilon \in \tilde{S}$.

The next proposition, detailed in [28] in the more general geometry of q -spirals, characterises null q -Gevrey asymptotic expansion.

Proposition 1 Let $A > 0$ and $f : S \rightarrow \mathbb{H}$ a holomorphic function in a sector S with vertex at the origin. Then,

- i) If f admits the power series with null coefficients, which is denoted by $\hat{0}$, as its q -Gevrey asymptotic expansion of type A , then for every $\tilde{S} \prec S$ there exists $C_1 > 0$ with

$$\|f(\epsilon)\|_{\mathbb{H}} \leq C_1 e^{-\frac{1}{\tilde{a}} \frac{1}{2(-\log(q))} \log^2 |\epsilon|},$$

for every $\epsilon \in \tilde{S}$ and every $\tilde{a} > A$.

- ii) If for every $\tilde{S} \prec S$ there exists $C_1 > 0$ with

$$\|f(\epsilon)\|_{\mathbb{H}} \leq C_1 e^{-\frac{1}{\tilde{a}} \frac{1}{2(-\log(q))} \log^2 |\epsilon|},$$

for every $\epsilon \in \tilde{S}$ then f admits $\hat{0}$ as its q -Gevrey asymptotic expansion of type \tilde{a} in S , for every $\tilde{a} > A$.

4.2 Analytic solutions in a parameter of singularly perturbed Cauchy problem

We recall the definition of a good covering.

Definition 4 Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a finite family of open sectors with vertex at the origin and finite radius ϵ_0 . We assume that $\mathcal{E}_i \cap \mathcal{E}_{i+1} \neq \emptyset$ for $0 \leq i \leq \nu-1$ (we put $\mathcal{E}_\nu := \mathcal{E}_0$), and also that $D(0, \nu_0) \setminus \{0\} \subseteq \cup_{i=0}^{\nu-1} \mathcal{E}_i$ for some $\nu_0 > 0$. Then, the family $\{\mathcal{E}_i\}_{1 \leq i \leq \nu-1}$ is known as a good covering in \mathbb{C}^* .

Definition 5 Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a good covering in \mathbb{C}^* . We consider a family $\{\{\tilde{S}_i\}_{0 \leq i \leq \nu-1}, \mathcal{T}\}$ such that:

1. There exist $d_i \in [0, 2\pi)$, $0 < \theta_i < \frac{\pi}{2}$ such that

$$\tilde{S}_i = \tilde{S}_i(d_i, \theta_i) := \left\{ t \in \mathbb{C}^* : |\arg(t) - d_i| < \frac{\theta_i}{2} \right\},$$

for every $0 \leq i \leq \nu-1$.

2. \mathcal{T} is an unbounded subset of an open sector with vertex at the origin. We assume $|t| \geq r\tau$ for every $t \in \mathcal{T}$.
3. For every $0 \leq i \leq \nu-1$ and $\tau \in \tilde{S}_i$, there exists $\delta_3 > 0$ such that $|a - \tau| > \delta_3$.
4. For every $0 \leq i \leq \nu-1$, $t \in \mathcal{T}$ and $\epsilon \in \mathcal{E}_i$, one has $t/\epsilon \in \tilde{S}_i$.

Under the previous settings, we say the family $\{\{\tilde{S}_i\}_{0 \leq i \leq \nu-1}, \mathcal{T}\}$ is associated to the good covering $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$.

Let us consider a good covering in \mathbb{C}^* , $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$.

Let $S \geq 1$ and $a \in \mathbb{C} \setminus \mathbb{R}_+$. We consider a finite subset of \mathbb{N}^2 , \mathcal{N} . For every $\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}$, let $m_{\underline{\kappa},1}, m_{\underline{\kappa},2} \in \mathbb{N}$, and $b_{\underline{\kappa}}(\epsilon, z)$ a holomorphic and bounded function on $D(0, r_0) \times \mathbb{C}$, for some $r_0 > 0$. For each $0 \leq i \leq \nu-1$, we consider the main Cauchy problem in the present work:

$$(33) \quad \epsilon \partial_t \partial_z^S X_i(\epsilon, t, z) + a \partial_z^S X_i(\epsilon, t, z) = \sum_{\underline{\kappa}=(\kappa_0, \kappa_1) \in \mathcal{N}} b_{\underline{\kappa}}(\epsilon, z) (\partial_t^{\kappa_0} \partial_z^{\kappa_1} X_i)(\epsilon, q^{m_{\underline{\kappa},1}} t, q^{m_{\underline{\kappa},2}} z),$$

with initial conditions

$$(34) \quad (\partial_z^j X_i)(\epsilon, t, 0) = \phi_{i,j}(\epsilon, t) \quad 0 \leq j \leq S-1,$$

where the functions $\phi_{i,j}(\epsilon, t)$ are constructed as follows. Let $\{\{\tilde{S}_i\}_{0 \leq i \leq \nu-1}, \mathcal{T}\}$ be a family of open sets associated to the good covering $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$.

From now on, we assume the values of $(R_\beta)_{\beta \geq 0}$ and $(\hat{R}_\beta)_{\beta \geq 0}$ are those in the preceding sections. If necessary, one can adjust the values of d_1, d_2, \hat{d}_1 and \hat{d}_2 so that $R_\beta < \hat{R}_\beta$ for every $\beta \geq 0$ so that $\dot{D}_{\hat{R}_\beta} \cap \tilde{S}_{i,\beta} \neq \emptyset$ for every $\beta \geq 0$ and every $0 \leq i \leq \nu-1$. Here, we have put

$$\tilde{S}_{i,\beta} := \left\{ \tau \in \mathbb{C}^* : \tau \in \tilde{S}_i, |\tau| > R_\beta \right\}, \quad \beta \geq 0, 0 \leq i \leq \nu-1.$$

For every $0 \leq j \leq \nu - 1$, we assume that $(\epsilon, \tau) \mapsto W_j(\epsilon, \tau)$ is a bounded and holomorphic function on $(D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_j}$ verifying

$$|W_j(\epsilon, \tau)| \leq \Delta e^{\tilde{M} \log^2(\frac{|\tau|}{|\epsilon|} + \delta_1)} |\epsilon|^{K_0},$$

for every $(\epsilon, \tau) \in (D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_j}$. Here $\tilde{M}, K_0, \Delta, \delta_1$ are the constants provided in Theorem 1. Assume moreover that $W_j(\epsilon, \tau)$ can be extended to an analytic function $(\epsilon, \tau) \mapsto W_{\mathcal{E}_i, S_j, j}(\epsilon, \tau)$ defined on $\mathcal{E}_i \times \tilde{S}_{i, j}$ and

$$|W_{\mathcal{E}_i, \tilde{S}_{i, j}, j}(\epsilon, \tau)| \leq \Delta e^{\tilde{M} \log^2(|\tau| + \delta_1)} |\epsilon|^{K_0},$$

for every $(\epsilon, \tau) \in \mathcal{E}_i \times \tilde{S}_{i, j}$.

Take γ_i such that $\mathbb{R}_+ e^{\gamma_i \sqrt{-1}} \subseteq D_{\hat{R}_j} \cup \tilde{S}_{i, j}$. We put

$$\phi_{i, j}(t, \epsilon) := \int_{L_{\gamma_i}} W_{\mathcal{E}_i, S_j, j}(\epsilon, \tau) e^{-\frac{t\tau}{\epsilon}} d\tau,$$

for every $(\epsilon, t) \in \mathcal{E}_i \times \mathcal{T}$. One can check that $\phi_{i, j}$ is well defined and holomorphic in $\mathcal{T} \times \mathcal{E}_i$. Indeed, there exists $\delta_2 > 0$ such that $\cos(\gamma_j - \arg(t/\epsilon)) > \delta_2$ for every $(t, \epsilon) \in \mathcal{E}_i \times \mathcal{T}$. Moreover, from the growth properties of $W_{\mathcal{E}_i, \tilde{S}_{i, j}, j}(\epsilon, \tau)$, one deduces

$$\left| \int_{L_{\gamma_i}} W_{\mathcal{E}_i, S_j, j}(\epsilon, \tau) e^{-\frac{t\tau}{\epsilon}} d\tau \right| \leq \int_0^\infty |W_{\mathcal{E}_i, S_j, j}(\epsilon, se^{\sqrt{-1}\gamma_i})| e^{-\frac{|t|\delta_2 s}{|\epsilon|}} ds \leq \Delta |\epsilon|^{K_0} \int_0^\infty e^{\tilde{M} \log^2(\frac{s}{|\epsilon|} + \delta_1)} e^{-\frac{s\delta_2 r \mathcal{T}}{|\epsilon|}} ds,$$

which is convergent for every $(\epsilon, t) \in \mathcal{E}_i \times \mathcal{T}$.

Theorem 3 *Let Assumptions (A), (B) and (B'') be fulfilled. For every $0 \leq i \leq \nu - 1$, we consider the problem (33)+(34) with initial conditions constructed as above. Then, the problem (33)+(34) admits a solution $X_i(\epsilon, t, z)$ which is holomorphic and bounded in $\mathcal{E}_i \times \mathcal{T} \times \mathbb{C}$.*

Moreover, for every $0 \leq i \leq \nu - 1$ and for every $\Delta > 1$ there exists $E_1 > 0$ (not depending on ϵ), such that

$$(35) \quad \sup_{\substack{t \in \mathcal{T} \\ z \in D(0, \rho)}} |X_{i+1}(\epsilon, t, z) - X_i(\epsilon, t, z)| \leq E_1 e^{-\frac{A_1}{2d_2^2 \Delta} \frac{1}{(-\log(q))} \log^2 |\epsilon|},$$

for every $\epsilon \in \mathcal{E}_i \cap \tilde{\mathcal{E}}_{i+1}$ (where, by convention, $X_\nu := X_0$).

Proof Let $0 \leq i \leq \nu - 1$ and $\epsilon \in \mathcal{E}_i$. We consider the Cauchy problem (12) with initial conditions given by

$$(36) \quad (\partial_z^j W)(\epsilon, \tau, 0) = W_{\mathcal{E}_i, \tilde{S}_{i, j}, j}(\epsilon, \tau), \quad 0 \leq j \leq S - 1.$$

Theorem 1 shows that the problem (14)+(36) has a formal solution $W(\epsilon, \tau, z) = \sum_{\beta \geq 0} W_\beta(\epsilon, \tau) \frac{z^\beta}{\beta!}$, with $W_\beta \in \mathcal{O}(\mathcal{E}_i \times \tilde{S}_{i, \beta})$ for every $\beta \geq 0$. Moreover, for every $\beta \geq 0$ one has

$$(37) \quad |W_\beta(\epsilon, \tau)| \leq C_{13} \beta! \left(\frac{C_{14}}{\delta} \right)^\beta e^{M \log^2(\frac{|\tau|}{|\epsilon|} + \delta_1)} \left| \frac{\tau}{\epsilon} \right|^{C\beta} q^{A_1 \beta^2},$$

for every $(\epsilon, \tau) \in \mathcal{E}_i \times \tilde{S}_{i,\beta}$, where C_{13}, C_{14} and δ are positive constants provided in the proof of Theorem 1. In a parallel direction, one can consider the same Cauchy problem with initial conditions given by

$$(38) \quad (\partial_z^j W)(\epsilon, \tau, 0) = W_j(\epsilon, \tau), \quad 0 \leq j \leq S-1,$$

where $W_j \in \mathcal{O}((D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_j})$ are as above.

From Theorem 2, one concludes that the formal power series $W(\epsilon, \tau, z)$ is such that W_β can be extended to a holomorphic function defined in $(D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_\beta}$, for every $\beta \geq 0$. We preserve notation for these extensions. Moreover, for every $\beta \geq 0$ one has

$$(39) \quad |W_\beta(\epsilon, \tau)| \leq C_{23}\beta! \left(\frac{C_{24}}{\delta}\right)^\beta |\epsilon|^{-C\beta} e^{M \log^2(|\tau| + \delta_1)} q^{A_1\beta^2},$$

for every $(\epsilon, \tau) \in (D(0, r_0) \setminus \{0\}) \times \dot{D}_{\hat{R}_\beta}$, and some positive constants C_{23} and C_{24} determined in the proof of Theorem 2.

We put $X_i(\epsilon, \tau, z) = \sum_{\beta \geq 0} X_{i,\beta}(t, \epsilon) \frac{z^\beta}{\beta!}$, where

$$X_{i,\beta}(\epsilon, t) := \int_{L_{\gamma_i}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau.$$

We first check that X_i is, at least formally, a solution of (33)+(34). From (31), one can check by inserting the formal power series X_i in (33), that it turns out to be a formal solution in the variable z of (33)+(34) if and only if $W(\epsilon, \tau, z)$ is a formal solution of (12)+(13) and (26)+(27).

Bearing in mind that W_β verifies (38) and (39), one derives $X_{i,\beta}$ is well defined in $\mathcal{E}_i \times \mathcal{T}$, for every $\beta \geq 0$. We now state a proof for the fact that $(\epsilon, \tau, z) \mapsto X_i(\epsilon, \tau, z)$ is indeed a holomorphic solution of (33)+(34) in $\mathcal{E}_i \times \mathcal{T} \times \mathbb{C}$. Let $\epsilon \in \mathcal{E}_i$, $t \in \mathcal{T}$, and $\beta \in \mathbb{N}$. One has

$$(40) \quad \left| \int_{L_{\gamma_i}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| \leq \left| \int_{L_{\gamma_{i,1}}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| + \left| \int_{L_{\gamma_{i,2}}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| + \left| \int_{L_{\gamma_{i,3}}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right|,$$

where $L_{\gamma_{i,1}} := L_{\gamma_i} \cap D_{\hat{R}_\beta}$, $L_{\gamma_{i,2}} := L_{\gamma_i} \cap \tilde{S}_{i,\beta}$ and $L_{\gamma_{i,3}} := L_{\gamma_i} \cap \dot{D}_{\hat{R}_\beta} \cap \tilde{S}_{i,\beta}$. We only give details on the first and second integrals appearing on the right-hand side of the previous inequality. The first integral on the right-hand side of (40) can be upper bounded by means of (39), and the choice of direction γ_i .

$$(41) \quad \begin{aligned} \left| \int_{L_{\gamma_{i,1}}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| &\leq \int_0^{\hat{R}_\beta} |W_\beta(se^{\sqrt{-1}\gamma_i}, \epsilon)| e^{-\frac{s|t|}{|\epsilon|} \cos(\gamma_i - \arg(t/\epsilon))} ds \\ &\leq C_{23}\beta! \left(\frac{C_{24}}{\delta}\right)^\beta q^{A_1\beta^2} \int_0^{\hat{R}_\beta} |\epsilon|^{-C\beta} e^{M \log^2(\frac{s}{|\epsilon|} + \delta_1)} e^{-\frac{s\delta_2 r \mathcal{T}}{|\epsilon|}} ds. \end{aligned}$$

One has

$$|\epsilon|^{-C\beta} e^{M \log^2(\frac{s}{|\epsilon|} + \delta_1)} e^{-\frac{s\delta_2 r \mathcal{T}}{|\epsilon|}} \leq \tilde{C}_{23} |\epsilon|^{-C\beta} e^{-\frac{s\delta_2 r \mathcal{T}}{2|\epsilon|}},$$

for some $\tilde{C}_{23} > 0$. Now, the function $x \mapsto x^{-C\beta} e^{-\frac{s\delta_2 r \mathcal{T}}{|\epsilon|}}$ attains its maximum at $x = \frac{C\beta 2}{s\delta_2 r \mathcal{T}}$. One can reduce r_0 , if necessary, to conclude that this function is increasing for $x \in [0, \epsilon_0]$. The expression in (41) is upper bounded by

$$C_{23}\beta! \left(\frac{C_{24}}{\delta r_0^C}\right)^\beta q^{A_1\beta^2} \int_0^\infty e^{-\frac{s\delta_2 r \mathcal{T}}{2r_0}} ds.$$

This yields

$$(42) \quad \left| \int_{L_{\gamma_i,1}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| \leq C_{31}\beta! C_{32}^\beta q^{A_1\beta^2},$$

for some constants $C_{31}, C_{32} > 0$ only depending on $\delta, r_0, R_0, q, S, A_1, C, \delta_1, M, r_{\mathcal{T}}, \delta_2$. We now consider the second integral appearing on the right-hand side of (40). From (37) and similar estimates as before we get

$$(43) \quad \left| \int_{L_{\gamma_i,2}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| \leq C_{13}\beta! \left(\frac{C_{14}}{\delta} \right)^\beta q^{A_1\beta^2} \int_{R_\beta}^\infty e^{M \log^2\left(\frac{s}{|\epsilon|} + \delta_1\right)} \left(\frac{s}{|\epsilon|} \right)^{C\beta} e^{-\frac{s\delta_2 r_{\mathcal{T}}}{|\epsilon|}} ds.$$

The function $x \mapsto g_1(x) = e^{M \log^2(x + \delta_1)} x^{C\beta}$, $x \geq 0$ is such that $g_1(x) \leq g_2(x)$ for all $x \geq 0$, where $g_2(x) = \tilde{C}_{13} e^{M \log^2(x)} x^{C\beta}$, for some positive constant \tilde{C}_{13} , not depending on β . g_2 attains its maximum value at $x_0 = \exp(-\frac{C\beta}{2M})$ so that $g(x) \leq g(x_0) = \exp(-\frac{C^2\beta^2}{4M})$, for every $x > 0$. This implies

$$e^{M \log^2\left(\frac{s}{|\epsilon|} + \delta_1\right)} \left(\frac{s}{|\epsilon|} \right)^{C\beta} e^{-\frac{s\delta_2 r_{\mathcal{T}}}{|\epsilon|}} \leq e^{-\frac{C^2}{4M}\beta^2} e^{-\frac{s\delta_2 r_{\mathcal{T}}}{|\epsilon|}}.$$

From (43) we derive

$$(44) \quad \left| \int_{L_{\gamma_i,2}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| \leq C_{13} \int_0^\infty e^{-\frac{s\delta_2 r_{\mathcal{T}}}{2r_0}} ds \beta! q^{A_1\beta^2} e^{-\frac{C^2}{4M}\beta^2} e^{-\frac{R_\beta\delta_2 r_{\mathcal{T}}}{2r_0}} = \tilde{C}_{14}\beta! q^{\left(A_1 - \frac{C^2}{4M \log(q)}\right)\beta^2},$$

for some $\tilde{C}_{14} > 0$.

From (42) and (44), we lead to the existence of positive constants C_{41}, C_{42} , not depending on β , such that

$$\left| \sum_{\beta \geq 0} X_{i,\beta}(t, \epsilon) \frac{z^\beta}{\beta!} \right| \leq C_{41} \sum_{\beta \geq 0} C_{42}^\beta q^{A_1\beta^2} |z|^\beta,$$

for every $z \in \mathbb{C}$. This allows us to conclude the first part of the proof.

Let $0 \leq i \leq \nu - 1$ and $\rho > 0$. For every $(\epsilon, t, z) \in (\mathcal{E}_i \cap \mathcal{E}_{i+1}) \times \mathcal{T} \times D(0, \rho)$ we have

$$|X_{i+1}(\epsilon, t, z) - X_i(\epsilon, t, z)| \leq \sum_{\beta \geq 0} |X_{i+1,\beta}(\epsilon, t) - X_{i,\beta}(\epsilon, t)| \frac{\rho^\beta}{\beta!}.$$

We can write

$$\begin{aligned} X_{i+1,\beta}(\epsilon, t) - X_{i,\beta}(\epsilon, t) &= \int_{L_{\gamma_{i+1},2}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau - \int_{L_{\gamma_i,2}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \\ &\quad + \int_{L_{\gamma_{i+1},4} - L_{\gamma_i,4}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau, \end{aligned}$$

where $L_{\gamma_{i+1},4} - L_{\gamma_i,4}$ stands for the path consisting of two parts: the first one going from $R_\beta e^{\sqrt{-1}\gamma_{i+1}}$ to 0 along the segment $[0, R_\beta e^{\sqrt{-1}\gamma_{i+1}}]$ and the path going from 0 to $R_\beta e^{\sqrt{-1}\gamma_i}$ following direction γ_i .

This integral has already been estimated in (44), for the first part of the proof, so we omit the details. We also omit the details on the integral concerning the path $L_{\gamma_{i+1},2}$ which is analogous.

In order to estimate the integral along the path $L_{\gamma_{i+1},4} - L_{\gamma_i,4}$, one can observe that the function involved in the integrand does not depend on the index i considered, for this function is well defined for $(\epsilon, \tau) \in (D(0, r_0) \setminus \{0\}) \times \dot{D}_{R_\beta}$. One can apply Cauchy Theorem to derive

$$\int_L W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau = 0,$$

where $L = L_{\gamma_{i+1},4} - L_{\gamma_i,4} - L_1$ is the closed path with $s \in [\gamma_i, \gamma_{i+1}] \rightarrow L_1(s) = R_\beta e^{\sqrt{-1}s}$. Moreover, $\left| \int_{L_{\gamma_{i+1},4} - L_{\gamma_i,4}} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right|$ equals

$$\begin{aligned} \left| \int_{L_1} W_\beta(\tau, \epsilon) e^{-\frac{t\tau}{\epsilon}} d\tau \right| &\leq C_{23}\beta! \left(\frac{C_{24}}{\delta} \right)^\beta |\epsilon|^{-C\beta} e^{M \log^2(R_\beta + \delta_1)} q^{A_1\beta^2} R_\beta \int_{\gamma_i}^{\gamma_{i+1}} e^{-\frac{R_\beta|t| \cos(\theta - \arg(t/\epsilon))}{|\epsilon|}} d\theta \\ &\leq \hat{C}_{23}\beta! \left(\frac{C_{24}}{\delta} \right)^\beta |\epsilon|^{-C\beta} q^{A_1\beta^2} R_\beta \int_{\gamma_i}^{\gamma_{i+1}} e^{-\frac{R_\beta r \mathcal{T} \delta_2}{2|\epsilon|}} ds \\ (45) \quad &\leq \check{C}_{23}\beta! \left(\frac{C_{24}}{\delta} \right)^\beta q^{A_1\beta^2} |\epsilon|^{-C\beta} e^{-\frac{R_\beta r \mathcal{T} \delta_2}{2|\epsilon|}} e^{-\frac{R_\beta r \mathcal{T} \delta_2}{2|\epsilon|}}. \end{aligned}$$

for some $\hat{C}_{23}, \check{C}_{23} > 0$. It only rests to take into account that the function $x \in (0, r_0) \mapsto x^{-C\beta} e^{-\frac{R_\beta r \mathcal{T} \delta_2}{2x}}$ is monotone increasing in $(0, r_0)$, so that $|\epsilon|^{-C\beta} e^{-\frac{R_\beta r \mathcal{T} \delta_2}{2|\epsilon|}}$ can be included in the constants \check{C}_{23} and C_{24} .

From (44) and (45) one gets the existence of positive constants C_6, C_7 such that

$$|X_{i+1,\beta}(\epsilon, t) - X_{i,\beta}(\epsilon, t)| \leq C_6\beta! C_7^\beta q^{A_1\beta^2} e^{-\frac{d_1\delta_2 r \mathcal{T}}{2} \frac{q^{d_2\beta}}{|\epsilon|}},$$

for every $(\epsilon, t) \in (\mathcal{E}_i \cap \mathcal{E}_{i+1}) \times \mathcal{T}$. Taking this last estimate into the expression of $X_{i+1} - X_i$ one can conclude that

$$|X_{i+1}(\epsilon, t, z) - X_i(\epsilon, t, z)| \leq C_6 \sum_{\beta \geq 0} (C_7\rho)^\beta q^{A_1\beta^2} e^{-\frac{d_1\delta_2 r \mathcal{T}}{2} \frac{q^{d_2\beta}}{|\epsilon|}},$$

for every $(\epsilon, t, z) \in (\mathcal{E}_i \cap \mathcal{E}_{i+1}) \times \mathcal{T} \times D(0, \rho)$. The proof of the second statement in the theorem leans on the incoming lemma whose proof is left until the end of the current section. It provides information on the estimates for a Dirichlet type series. A similar argument concerning a Dirichlet series of different nature can be found in [29], Lemma 9, when dealing with Gevrey asymptotic expansions.

Lemma 10 *Let A_1, D_1, D_2, d_2 be positive constants, with $D_2 > 1$. Then, for every $\Delta > 1$ there exist $E_1 > 0$ and $\delta > 0$ such that*

$$(46) \quad \sum_{\beta \geq 0} D_1^\beta q^{A_1\beta^2} e^{-D_2 \frac{q^{d_2\beta}}{\epsilon}} \leq E_1 e^{-\frac{A_1}{d_2^2 \Delta} \frac{1}{(-\log(q))^2} \log^2 \epsilon},$$

for every $\epsilon \in (0, \delta]$.

□

The proof of Lemma 10 heavily rests on the q -Gevrey version of some preliminary results which are classical in Gevrey case (see [29] and the references therein). Their proofs do not differ from the classical ones, so we omit them.

Lemma 11 Let $b > 0$ and $f : [0, b] \rightarrow \mathbb{C}$ a continuous function having the formal expansion $\sum_{n \geq 0} a_n t^n \in \mathbb{C}[[t]]$ as its q -asymptotic expansion of type $A_1 > 0$ at 0, meaning there exist $C, \tilde{H} > 0$ such that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| \leq CH^N q^{-\frac{A_1 N^2}{2}} \frac{|t|^N}{N!},$$

for every $N \geq 1$ and $t \in [0, \delta]$, for some $0 < \delta < b$.

Then, the function

$$I(x) = \int_0^b f(s) e^{-\frac{s}{x}} ds$$

admits the formal power series $\sum_{n \geq 0} a_n n! x^{n+1} \in \mathbb{C}[[\epsilon]]$ as its q -Gevrey asymptotic expansion of type A_1 at 0. It is to say, there exist $\tilde{C}, \tilde{H} > 0$ such that

$$\left| I(x) - \sum_{n=0}^{N-1} a_n n! x^{n+1} \right| \leq \tilde{C} \tilde{H}^{N+1} q^{-\frac{A_1 (N+1)^2}{2}} \frac{|x|^{N+1}}{(N+1)!},$$

for every $N \geq 0$ and $x \in [0, \delta']$ for some $0 < \delta' < b$.

One can adapt the proof of Proposition 4 in [28] in our framework.

Lemma 12 Let $A_1, \delta > 0$ and $\psi : [0, \delta] \rightarrow \mathbb{C}$ be a continuous function. Then,

1. If there exist $C, H > 0$ such that $|\psi(x)| \leq CH^n q^{-\frac{A_1 n^2}{2}} \frac{|x|^n}{n!}$, for every $n \in \mathbb{N}$, $n \geq 0$ and $x \in [0, \delta]$, then for every $\tilde{A}_1 > A_1$ there exists $\tilde{C} > 0$ such that

$$|\psi(x)| \leq \tilde{C} e^{-\frac{1}{\tilde{A}_1} \frac{1}{(-\log(q))^2} \log^2 |x|},$$

for every $x \in (0, \delta]$.

2. If there exists $C > 0$ such that $|\psi(x)| \leq C e^{-\frac{1}{\tilde{A}_1} \frac{1}{(-\log(q))^2} \log^2 |x|}$, for every $n \in \mathbb{N}$, and $x \in [0, \delta]$, then for every $\tilde{A}_1 > A_1$ there exists $\tilde{C}, \tilde{H} > 0$ such that

$$|\psi(x)| \leq \tilde{C} \tilde{H}^n q^{-\frac{\tilde{A}_1 n^2}{2}} \frac{|x|^n}{n!},$$

for every $n \in \mathbb{N}$ and for every $x \in (0, \delta]$.

proof of Lemma 10:

Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a C^1 function. For every $n \in \mathbb{N}$, one can apply Euler-Mac-Laurin formula

$$\sum_{\kappa=0}^n f(\kappa) = \frac{1}{2}(f(0) + f(n)) + \int_0^n f(t) dt + \int_0^n B_1(t - [t]) f'(t) dt,$$

where $B_1(s) = s - \frac{1}{2}$ is the Bernoulli polynomial and $[\cdot]$ stands for the floor function, to $f(s) = D_1^s q^{A_1 s^2} e^{-D_2 \frac{q^{d_2 s}}{\epsilon}}$. One leads to

$$(47) \quad \sum_{\kappa=0}^n D_1^\kappa q^{A_1 \kappa^2} e^{-D_2 \frac{q^{d_2 \kappa}}{\epsilon}} = \frac{1}{2} \left(e^{-\frac{D_2}{\epsilon}} + D_1^n q^{A_1 n^2} e^{-D_2 \frac{q^{d_2 n}}{\epsilon}} \right) + \int_0^n D_1^t q^{A_1 t^2} e^{-D_2 \frac{q^{d_2 t}}{\epsilon}} dt \\ + \int_0^n B_1(t - [t]) D_1^t q^{A_1 t^2} e^{-D_2 \frac{q^{d_2 t}}{\epsilon}} (\log(D_1) + \log(q) A_1 2t - D_2 \frac{\log(q) d_2}{\epsilon}) dt.$$

Taking the limit when n tends to infinity in the previous expression we arrive at an equality for a convergent series:

$$\begin{aligned} \sum_{\kappa=0}^{\infty} D_1^\kappa q^{A_1 \kappa^2} e^{-D_2 \frac{q^{d_2 \kappa}}{\epsilon}} &= \frac{1}{2} e^{-\frac{D_2}{\epsilon}} + \int_0^\infty D_1^t q^{A_1 t^2} e^{-D_2 \frac{q^{d_2 t}}{\epsilon}} dt \\ &+ \int_0^\infty B_1(t - [t]) D_1^t q^{A_1 t^2} e^{-D_2 \frac{q^{d_2 t}}{\epsilon}} (\log(D_1) + \log(q) A_1 2t - D_2 \frac{\log(q) d_2}{\epsilon}) dt. \end{aligned}$$

Let $I_1 := \int_0^\infty f(t) dt$, and $I_2 := \int_0^\infty B_1(t - [t]) \left(\log(D_1) + \log(q) A_1 2t - D_2 \frac{\log(q) d_2}{\epsilon} q^{\frac{D_2 t}{\epsilon}} \right) f(t) dt$.

From the fact that $B_1(t - [t]) \leq 1/2$ for every $t \geq 0$ and the change of variable $D_2 q^{d_2 t} = u$, one gets

$$I_1 = \int_0^{D_2} f_1(u) e^{-\frac{u}{\epsilon}} du, \quad \text{where } f_1(u) := D_1^{\frac{\log(u/D_2)}{\log(q) d_2}} q^{\frac{A_1 \log^2(u/D_2)}{\log^2(q) d_2^2}} \frac{1}{d_2(-\log(q))u}$$

and

$$I_2 \leq \frac{1}{2} \int_0^{D_2} (f_2(u) + f_3(u) + f_4(u)) e^{-\frac{u}{\epsilon}} du,$$

with $f_2(u) := \log(D_1) f_1(u)$, $f_3(u) := \frac{2A_1}{d_2} \log(u/D_2) f_1(u)$, $f_4 := \frac{(-\log(q) d_2)}{\epsilon} e^{\frac{D_2 \log(u/D_2)}{d_2 \epsilon}} f_1(u)$, for every $u \in (0, D_2]$. Bearing in mind that $f_3(u) < 0$ for $u \in (0, D_2]$, and from usual estimates we derive $I_2 \leq C_1 (1 + \frac{1}{\epsilon}) \int_0^{D_2} f_1(u) e^{-\frac{u}{\epsilon}} du$ for some $C_1 > 0$. The proof is complete if one can estimate $e^{-D_2/\epsilon}$, I_1 and $1/\epsilon I_1$ appropriately. The first expression is clearly upper bounded according to (46).

From usual estimates we arrive at

$$I_1 \leq C_3 \int_0^{D_2} e^{-\frac{A_1}{(-\log(q)) d_2^2} \log^2(u/D_2)} e^{-\frac{u}{\epsilon}} du = C_3 \int_0^{D_3} \tilde{f}(u) e^{-\frac{u}{\epsilon}},$$

for some $C_3 > 0$. From Lemma 12, the function \tilde{g} defined by $u \in [0, 1] \mapsto \tilde{f}(D_2 u)$ (extended by continuity to $u = 0$) is such that for every $\tilde{\Delta} > 1$ there exist $\tilde{C}, \tilde{H} > 0$ with

$$|\tilde{g}(u)| \leq \tilde{C} \tilde{H}^n q^{-\frac{d_2^2 \tilde{\Delta}^2}{A_1} n^2} \frac{|u|^n}{n!},$$

for every $u \in [0, D_2]$ and for every $n \geq 0$. From Lemma 11, the functions $I_1(\epsilon)$ and $\frac{1}{\epsilon} I_1(\epsilon)$ admit the series with null coefficients as q -asymptotic expansion of type $(d_2^2 \tilde{\Delta})/A_1$. Again, from Lemma 12, one can conclude that for every $\Delta > \tilde{\Delta}$ there exists $C_4 > 0$ such that both I_1 and $\frac{1}{\epsilon} I_1$ are upper bounded by $C_4 e^{-\frac{A_1}{d_2^2 \Delta} \frac{1}{(-\log(q))^2} \log^2(\epsilon)}$, for every $\epsilon \in (0, \epsilon_1]$, for some $\epsilon_1 > 0$. \square

4.3 Existence of formal series solutions in the complex parameter

In this last subsection we obtain a q -Gevrey version of a Malgrange-Sibuya type Theorem. A result in this direction has already been obtained by the authors in [28] when dealing with $q \in \mathbb{C}$, $|q| > 1$. In that work, the geometry of the problem differs from the one in the present work. Indeed, the result is settled in terms of discrete q -spirals tending to the origin, and with $q \in \mathbb{C}$.

Given $q \in \mathbb{C}$ with $0 < |q| < 1$ and a nonempty open subset $U \subset \mathbb{C}^*$, the discrete q -spiral associated to U and q consists of the products of an element in U and q^m , for some $m \in \mathbb{N}$. For

our purpose, q is a real number and U is chosen in such a way that the discrete q -spiral turns out to be a sector with vertex at the origin.

The proof of the q -Gevrey version of Malgrange-Sibuya Theorem in [28] is based on the use of extension results on ultradifferential spaces of weighted functions which preserve the information of q -Gevrey bounds but causes that the q -Gevrey type involved in the q -Gevrey asymptotic suffers an increase. Here, one can follow similar steps as for the classical proof Malgrange-Sibuya theorem based on Cauchy-Heine transform, so that the q -Gevrey type is preserved. In [34], an analogous demonstration for the Gevrey version of the result can be found. We have decided to include the whole proof of the result in order to facilitate comprehension and clarity of the result.

Theorem 4 (*q-MS*)

Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a Banach space over \mathbb{C} and $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a good covering in \mathbb{C}^* . For all $0 \leq i \leq \nu-1$, let G_i be a holomorphic function from \mathcal{E}_i into the Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and let the cocycle $\Delta_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon)$ be a holomorphic function from the sector $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$ into \mathbb{E} (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$ and $G_\nu = G_0$). We make the following assumptions.

- 1) The functions $G_i(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_i$ tends to the origin in \mathbb{C} , for all $0 \leq i \leq \nu-1$.
- 2) Δ_i has a q -exponential decreasing of some type $L > 0$, for every $0 \leq i \leq \nu-1$, meaning there exists $C_i > 0$ such that

$$(48) \quad \|\Delta_i(\epsilon)\|_{\mathbb{E}} \leq C_i e^{-\frac{1}{L} \frac{1}{(-\log(q))^2} \log^2 |\epsilon|},$$

for every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, and $0 \leq i \leq \nu-1$.

Then, there exists a formal power series $\hat{G}(\epsilon) \in \mathbb{E}[[\epsilon]]$ such that $G_i(\epsilon)$ admits $\hat{G}(\epsilon)$ as its q -Gevrey asymptotic expansion of type L on \mathcal{E}_i , for every $0 \leq i \leq \nu-1$.

Proof

We first state an auxiliary result.

Lemma 13 For all $0 \leq i \leq \nu-1$, there exist bounded holomorphic functions $\Psi_i : \mathcal{E}_i \rightarrow \mathbb{C}$ such that

$$(49) \quad \Delta_i(\epsilon) = \Psi_{i+1}(\epsilon) - \Psi_i(\epsilon)$$

for all $\epsilon \in Z_i$, where by convention $\Psi_\nu(\epsilon) = \Psi_0(\epsilon)$. Moreover, there exist $\varphi_m \in \mathbb{E}$, $m \geq 0$, such that for each $0 \leq l \leq \nu-1$, any $\hat{L} > L$ and every $\mathcal{W} \prec \mathcal{E}_l$, there exist $\hat{K}_l, \hat{M}_l > 0$ with

$$(50) \quad \|\Psi_l(\epsilon) - \sum_{m=0}^{M-1} \varphi_m \epsilon^m\|_{\mathbb{E}} \leq \hat{K}_l (\hat{M}_l)^M q^{-\hat{L} \frac{(M-1)^2}{2}} \frac{|\epsilon|^M}{M!}$$

for all $\epsilon \in \mathcal{W}$, all $M \geq 2$.

Proof

We follow analogous arguments as in Lemma XI-2-6 from [23] with appropriate modifications in the asymptotic expansions of the functions constructed with the help of the Cauchy-Heine transform.

For all $0 \leq l \leq \nu - 1$, we choose a segment

$$C_l = \{te^{\sqrt{-1}\theta_l}, t \in [0, r]\} \subset \mathcal{E}_l \cap \mathcal{E}_{l+1}.$$

These ν segments divide the open punctured disc $D(0, r) \setminus \{0\}$ into ν open sectors $\tilde{\mathcal{E}}_0, \dots, \tilde{\mathcal{E}}_{\nu-1}$ where

$$\tilde{\mathcal{E}}_l = \{\epsilon \in \mathbb{C}^* / \theta_{l-1} < \arg(\epsilon) < \theta_l, |\epsilon| < r\}, \quad 0 \leq l \leq \nu - 1,$$

where by convention $\theta_{-1} = \theta_{\nu-1}$. Let

$$\Psi_l(\epsilon) = \frac{-1}{2\pi\sqrt{-1}} \sum_{h=0}^{\nu-1} \int_{C_h} \frac{\Delta_h(\xi)}{\xi - \epsilon} d\xi$$

for all $\epsilon \in \tilde{\mathcal{E}}_l$, for $0 \leq l \leq \nu - 1$, be defined as a sum of Cauchy-Heine transforms of the functions $\Delta_h(\epsilon)$. By deformation of the paths C_{l-1} and C_l without moving their endpoints and letting the other paths C_h , $h \neq l-1, l$ untouched (with the convention that $C_{-1} = C_{\nu-1}$), one can continue analytically the function Ψ_l onto \mathcal{E}_l . Therefore, Ψ_l defines a holomorphic function on \mathcal{E}_l , for all $0 \leq l \leq \nu - 1$.

Now, take $\epsilon \in \mathcal{E}_l \cap \mathcal{E}_{l+1}$. In order to compute $\Psi_{l+1}(\epsilon) - \Psi_l(\epsilon)$, we write

$$(51) \quad \Psi_l(\epsilon) = \frac{-1}{2\pi\sqrt{-1}} \int_{\tilde{C}_l} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi + \frac{-1}{2\pi\sqrt{-1}} \sum_{h=0, h \neq l}^{\nu-1} \int_{C_h} \frac{\Delta_h(\xi)}{\xi - \epsilon} d\xi,$$

$$\Psi_{l+1}(\epsilon) = \frac{-1}{2\pi\sqrt{-1}} \int_{\tilde{C}_l} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi + \frac{-1}{2\pi\sqrt{-1}} \sum_{h=0, h \neq l}^{\nu-1} \int_{C_h} \frac{\Delta_h(\xi)}{\xi - \epsilon} d\xi$$

where the paths \hat{C}_l and \check{C}_l are obtained by deforming the same path C_l without moving its endpoints in such a way that:

(a) $\hat{C}_l \subset \mathcal{E}_l \cap \mathcal{E}_{l+1}$ and $\check{C}_l \subset \mathcal{E}_l \cap \mathcal{E}_{l+1}$,

(b) $\Gamma_{l,l+1} := -\check{C}_l + \hat{C}_l$ is a simple closed curve with positive orientation whose interior contains ϵ .

Therefore, due to the residue formula, we can write

$$(52) \quad \Psi_{l+1}(\epsilon) - \Psi_l(\epsilon) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{l,l+1}} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi = \Delta_l(\epsilon)$$

for all $\epsilon \in \mathcal{E}_l \cap \mathcal{E}_{l+1}$, for all $0 \leq l \leq \nu - 1$ (with the convention that $\Psi_\nu = \Psi_0$).

In a second step, we derive asymptotic properties of Ψ_l . We fix an $0 \leq l \leq \nu - 1$ and a proper closed sector \mathcal{W} contained in \mathcal{E}_l . Let \tilde{C}_l (resp. \tilde{C}_{l-1}) be a path obtained by deforming C_l (resp. C_{l-1}) without moving the endpoints in order that \mathcal{W} is contained in the interior of the simple closed curve $\tilde{C}_{l-1} + \gamma_l - \tilde{C}_l$ (which is itself contained in \mathcal{E}_l), where γ_l is a circular arc joining the two points $re^{\sqrt{-1}\theta_{l-1}}$ and $re^{\sqrt{-1}\theta_l}$. We get the representation

$$(53) \quad \Psi_l(\epsilon) = \frac{-1}{2\pi\sqrt{-1}} \int_{\tilde{C}_l} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi + \frac{-1}{2\pi\sqrt{-1}} \int_{\tilde{C}_{l-1}} \frac{\Delta_{l-1}(\xi)}{\xi - \epsilon} d\xi \\ + \frac{-1}{2\pi\sqrt{-1}} \sum_{h=0, h \neq l, l-1}^{\nu-1} \int_{C_h} \frac{\Delta_h(\xi)}{\xi - \epsilon} d\xi$$

for all $\epsilon \in \mathcal{W}$. One assumes that the path \tilde{C}_l is given as the union of a segment $L_l = \{te^{\sqrt{-1}w_l} : t \in [0, r_1]\}$ where $r_1 < r$ and $w_l > \theta_l$ and a curve $\Gamma_l = \{\mu_l(\tau) : \tau \in [0, 1]\}$ such that $\mu_l(0) = r_1 e^{\sqrt{-1}w_l}$, $\mu_l(1) = r e^{\sqrt{-1}\theta_l}$ and $r_1 \leq |\mu_l(\tau)| < r$ for all $\tau \in [0, 1]$. We also assume that there exists a positive number $\sigma < 1$ with $|\epsilon| \leq \sigma r_1$ for all $\epsilon \in \mathcal{W}$. By construction of the path Γ_l , we get that the function $\epsilon \mapsto \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_l} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi$ defines an analytic function on the open disc $D(0, r_1)$.

It remains to give estimates for the integral $\frac{1}{2\pi\sqrt{-1}} \int_{L_l} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi$. Let $M \geq 0$ be an integer. From the usual geometric series expansion, one can write

$$(54) \quad \frac{1}{2\pi\sqrt{-1}} \int_{L_l} \frac{\Delta_l(\xi)}{\xi - \epsilon} d\xi = \sum_{m=0}^M \alpha_{l,m} \epsilon^m + \epsilon^{M+1} E_{l,M+1}(\epsilon)$$

where

$$(55) \quad \alpha_{l,m} = \frac{1}{2\pi\sqrt{-1}} \int_{L_l} \frac{\Delta_l(\xi)}{\xi^{m+1}} d\xi, \quad E_{l,M+1}(\epsilon) = \frac{1}{2\pi\sqrt{-1}} \int_{L_l} \frac{\Delta_l(\xi)}{\xi^{M+1}(\xi - \epsilon)} d\xi$$

for all $\epsilon \in \mathcal{W}$.

Gathering (48) and (55), we get

$$(56) \quad \|\alpha_{l,m}\|_{\mathbb{E}} \leq \frac{K_l}{2\pi} \int_0^{r_1} \frac{e^{-\frac{1}{L} \frac{1}{(-\log(q))^2} \log^2 \tau}}{\tau^{m+1}} d\tau$$

The changes of variable $\log(\tau) = s$ first, and $s = \sqrt{2L(-\log(q))}t$ afterwards, transform the right-hand side of (56) into

$$\begin{aligned} \frac{K_l}{2\pi} \int_{-\infty}^{\log(r_1)} \frac{e^{-\frac{1}{L} \frac{1}{(-\log(q))^2} s^2}}{e^{sm}} ds &= \frac{K_l \sqrt{2L(-\log(q))}}{2\pi} \int_{-\infty}^{\frac{\log(r_1)}{\sqrt{2L(-\log(q))}}} \exp(-t^2 - m\sqrt{2L(-\log(q))}t) dt \\ &\leq \frac{K_l \sqrt{2L(-\log(q))}}{2\pi} \int_{-\infty}^{\infty} \exp(-t^2 - m\sqrt{2L(-\log(q))}t) dt. \end{aligned}$$

The application of

$$e^{\frac{a^2}{4}} \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2 - ax} dx,$$

for every $a \in \mathbb{R}$, which can be found in [1] (Chapter 10, p. 498), leads us to

$$(57) \quad \|\alpha_{l,m}\|_{\mathbb{E}} \leq \frac{\sqrt{2L(-\log(q))} K_l}{2\sqrt{\pi}} q^{-L \frac{m^2}{2}}.$$

Moreover, as above, one can choose a positive number $\eta > 0$ (depending on \mathcal{W}) such that $|\xi - \epsilon| \geq |\xi| \sin(\eta)$ for all $\xi \in L_l$ and all $\epsilon \in \mathcal{W}$. Again by (48) and (55), and following analogous calculations as before we obtain

$$(58) \quad \|E_{l,M+1}(\epsilon)\|_{\mathbb{E}} \leq \frac{K_l}{2\pi \sin(\eta)} \int_0^{r_1} \frac{e^{-\frac{1}{L} \frac{1}{2(-\log(q))} \log^2 \tau}}{\tau^{M+2}} d\tau \leq \frac{\sqrt{2L(-\log(q))} K_l}{2 \sin(\nu) \sqrt{\pi}} q^{-L \frac{(M+1)^2}{2}}$$

for all $\epsilon \in \mathcal{W}$. Using comparable arguments, one can give analogous estimates when estimating the other integrals

$$\frac{-1}{2\pi\sqrt{-1}} \int_{\tilde{C}_{l-1}} \frac{\Delta_{l-1}(\xi)}{\xi - \epsilon} d\xi, \quad \frac{-1}{2\pi\sqrt{-1}} \int_{C_h} \frac{\Delta_h(\xi)}{\xi - \epsilon} d\xi$$

for all $h \neq l, l-1$.

As a consequence, for any $0 \leq l \leq \nu-1$, there exist $\varphi_{l,m} \in \mathbb{E}$, for all $m \geq 0$ and a constant $\hat{K}_l > 0$ such that

$$(59) \quad \|\Psi_l(\epsilon) - \sum_{m=0}^{M-1} \varphi_{l,m} \epsilon^m\|_{\mathbb{E}} \leq \hat{K}_l q^{-L \frac{M^2}{2}} |\epsilon|^M$$

for all $M \geq 2$, all $\epsilon \in \mathcal{W}$.

Taking into account Proposition 1, we deduce that for every $\hat{\mathcal{E}}_{i,i+1} \prec \mathcal{E}_l \cap \mathcal{E}_{l+1}$ and for every $\hat{L} > L$, the function $\Psi_{l+1}(\epsilon) - \Psi_l(\epsilon)$ has the formal series $\hat{0}$ as q -Gevrey asymptotic expansion of type \hat{L} in $\hat{\mathcal{E}}_{i,i+1}$. From the unicity of the asymptotic expansions on sectors, we deduce that all the formal series $\sum_{m \geq 0} \varphi_{l,m} \epsilon^m$, $0 \leq l \leq \nu-1$, are equal to some formal series denoted $\hat{G}(\epsilon) = \sum_{m \geq 0} \varphi_m \epsilon^m \in \mathbb{E}[[\epsilon]]$. □

We consider now the bounded holomorphic functions

$$a_i(\epsilon) = G_i(\epsilon) - \Psi_i(\epsilon)$$

for all $0 \leq i \leq \nu-1$, all $\epsilon \in \mathcal{E}_i$. By definition, for any $i \in \{0, \dots, \nu-1\}$, we have that

$$a_{i+1}(\epsilon) - a_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon) - \Delta_i(\epsilon) = 0$$

for all $\epsilon \in Z_i$. Therefore, each $a_i(\epsilon)$ is the restriction on \mathcal{E}_i of a holomorphic function $a(\epsilon)$ on $D(0, r) \setminus \{0\}$. Since $a(\epsilon)$ is moreover bounded on $D(0, r) \setminus \{0\}$, the origin turns out to be a removable singularity for $a(\epsilon)$ which, as a consequence, defines a convergent power series on $D(0, r)$.

Finally, one can write

$$G_i(\epsilon) = a(\epsilon) + \Psi_i(\epsilon)$$

for all $\epsilon \in \mathcal{E}_i$, all $0 \leq i \leq \nu-1$. Moreover, $a(\epsilon)$ is a convergent power series, and for every $\hat{L} > L$, $\Psi_i(\epsilon)$ has the series $\hat{G}(\epsilon) = \sum_{m \geq 0} \varphi_m \epsilon^m$ as q -Gevrey asymptotic expansion of type \hat{L} on \mathcal{E}_i , for all $0 \leq i \leq \nu-1$. □

We are under conditions to enunciate the main result in the present work.

Theorem 5 *Let $\rho > 0$.*

Under the same hypotheses as in Theorem 3, we denote $\mathbb{H}_{\mathcal{T}, \rho}$ the Banach space of holomorphic and bounded functions in $\mathcal{T} \times D(0, \rho)$ with the supremum norm. Then, there exists a formal power series

$$\hat{X}(\epsilon, t, z) = \sum_{k \geq 0} \frac{X_k(t, z)}{k!} \epsilon^k \in \mathbb{H}_{\mathcal{T}, \rho}[[\epsilon]],$$

formal solution of

$$\epsilon \partial_t \partial_z^S \hat{X}(\epsilon, t, z) + a \partial_z^S \hat{X}(\epsilon, t, z) = \sum_{\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}} b_{\underline{\kappa}}(\epsilon, z) (\partial_t^{\kappa_0} \partial_z^{\kappa_1} \hat{X})(\epsilon, q^{m_{\underline{\kappa}, 1}} t, q^{m_{\underline{\kappa}, 2}} z).$$

Moreover, for every $0 \leq i \leq \nu-1$ and every $L_2 > \frac{d_z^2}{A_1}$, the function $X_i(\epsilon, t, z)$ constructed in Theorem 3 admits $\hat{X}(\epsilon, t, z)$ as its q -Gevrey asymptotic expansion of type L_2 in \mathcal{E}_i , meaning

that for every $0 \leq i \leq \nu - 1$ and $\tilde{\mathcal{E}}_i \prec \mathcal{E}_i$, there exist $L_0, L_1 > 0$ such that

$$(60) \quad \sup_{\substack{t \in \mathcal{T} \\ z \in D(0, \rho)}} \left| X(\epsilon, t, z) - \sum_{k=0}^N \frac{X_k(t, z)}{k!} \epsilon^k \right| \leq L_0 L_1^N q^{-L_2 \frac{N^2}{2}} \frac{|\epsilon|^{N+1}}{(N+1)!},$$

for every $N \geq 0$ and all $\epsilon \in \tilde{\mathcal{E}}_i$.

Proof Let us consider the family $(X_i(\epsilon, t, z))_{0 \leq i \leq \nu-1}$ constructed in Theorem 3. For every $0 \leq i \leq \nu - 1$, we define the function $\epsilon \in \mathcal{E}_i \mapsto G_i(\epsilon) := X_i(\epsilon, t, z)$, which belongs to the space $\mathbb{E}_{\mathcal{T}, \rho}$. From (35), we derive the cocycle $\Delta_i := G_{i+1}(\epsilon) - G_i(\epsilon)$ verifies (48) in Theorem 4, with $1/L := \frac{A_1}{d_2^2 \Delta}$ for some fixed $0 < \Delta < 1$. Theorem 4 guarantees the existence of a formal power series $\hat{X}(\epsilon) \in \mathbb{H}_{\mathcal{T}, \rho}[[\epsilon]]$, such that for every $L_2 > L$, $G_i(\epsilon)$ admits $\hat{G}(\epsilon)$ as its q -Gevrey asymptotic expansion of type L_2 on \mathcal{E}_i . This is valid for every $0 \leq i \leq \nu - 1$. This concludes the second part of the result.

It only rests to verify that \hat{X} is a formal solution of (33)+(34).

If we write $\hat{X}(\epsilon, t, z) := \sum_{k \geq 0} \frac{X_k(t, z)}{k!} \epsilon^k$, we have

$$(61) \quad \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon \in \tilde{\mathcal{E}}_i}} \sup_{(t, z) \in \mathcal{T} \times D(0, \rho)} \left| \partial_\epsilon^\ell X_i(t, z, \epsilon) - X_\ell(t, z) \right| = 0,$$

for every $0 \leq i \leq \nu - 1$ and all $\ell \geq 0$.

Let $0 \leq i \leq \nu - 1$. By construction, X_i satisfies (33)+(34). We differentiate in the equality (33) $\ell \geq 1$ times with respect to ϵ . By Leibniz's rule, we deduce that $\partial_\epsilon^\ell X_i(\epsilon, t, z)$ satisfies

$$\begin{aligned} & \epsilon \partial_\epsilon^\ell \partial_t \partial_z^S X_i(\epsilon, t, z) + \ell \partial_\epsilon^{\ell-1} \partial_t \partial_z^S X_i(\epsilon, t, z) + a \partial_z^S X_i(\epsilon, t, z) \\ &= \sum_{\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}} \sum_{\ell_0 + \ell_1 = \ell} \frac{\ell!}{\ell_0! \ell_1!} \partial_\epsilon^{\ell_1} b_{\underline{\kappa}}(\epsilon, z) (\partial_\epsilon^{\ell_1} \partial_t^{\kappa_0} \partial_z^{\kappa_1} X_i)(\epsilon, q^{m_{\underline{\kappa}, 1}} t, q^{m_{\underline{\kappa}, 2}} z), \end{aligned}$$

for every $(\epsilon, t, z) \in \mathcal{E}_i \times \mathcal{T} \times D(0, \rho)$. Let $\epsilon \rightarrow 0$ in the previous expression. From (61) we obtain

$$(62) \quad \begin{aligned} & \partial_t \partial_z^S \left(\frac{X_{\ell-1}(t, z)}{(\ell-1)!} \right) + a \partial_z^S \left(\frac{X_\ell(t, z)}{\ell!} \right) \\ &= \sum_{\underline{\kappa} = (\kappa_0, \kappa_1) \in \mathcal{N}} \sum_{\ell_0 + \ell_1 = \ell} \left(\frac{\partial_\epsilon^{\ell_0} b_{\underline{\kappa}}(0, z)}{\ell_0!} \right) \left(\frac{\partial_t^{\kappa_0} \partial_z^{\kappa_1} X_{\ell_1}(q^{m_{\underline{\kappa}, 1}} t, q^{m_{\underline{\kappa}, 2}} z)}{\ell_1!} \right). \end{aligned}$$

$b_{\underline{\kappa}}(z, \epsilon)$ is holomorphic with respect to ϵ for every $\underline{\kappa} \in \mathcal{N}$. This entails $b_{\underline{\kappa}}(\epsilon, z) = \sum_{h \geq 0} \frac{\partial_\epsilon^h b_{\underline{\kappa}}(0, z)}{h!} \epsilon^h$, for every (ϵ, z) in a neighborhood of the origin in \mathbb{C}^2 . From this, and (62), we deduce $\hat{X}(\epsilon, t, z) = \sum_{k \geq 0} \frac{X_k(t, z)}{k!} \epsilon^k \in \mathbb{H}_{\mathcal{T}, \rho}[[\epsilon]]$ is a formal solution of (33)+(34). \square

References

- [1] G. E. Andrews, R. Askey, R. Roy, Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.
- [2] T. Apostol, Modular functions and Dirichlet series in number theory, Springer, first edition 1976, second edition, 1990.

- [3] A. Augustynowicz, H. Leszczyński, W. Walter, *Cauchy-Kovalevskaia theory for equations with deviating variables*. Aequationes Math. 58 (1999), no. 1-2, 143–156.
- [4] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New-York, 2000.
- [5] A. S. Besicovitch, *Almost periodic functions*, Dover, New York, 1954.
- [6] H. Bohr, *Almost periodic functions*, Chelsea, New York, 1947.
- [7] J. Bonet, R.W. Braun, R. Meise, B. A. Taylor *Whitney's extension theorem for nonquasi-analytic classes of ultradifferentiable functions*, Stud. Math. 99 (2) (1991), 155–184.
- [8] B. Braaksma, B. Faber, *Multisummability for some classes of difference equations*. Ann. Inst. Fourier (Grenoble) 46 (1996), no. 1, 183–217.
- [9] B. Braaksma, B. Faber, G. Immink, *Summation of formal solutions of a class of linear difference equations*. Pacific J. Math. 195 (2000), no. 1, 35–65.
- [10] M. Canalis-Durand, J. Mozo-Fernandez, R. Schäfke, *Monomial summability and doubly singular differential equations*, J. Differential Equations 233 (2007), no. 2, 485–511.
- [11] J. Carr, J. Dyson, *The matrix functional differential equation $y'(x) = Ay(\lambda x) + By(x)$* . Proc. Roy. Soc. Edinburgh Sect. A 75 (1975/76), no. 1, 5–22.
- [12] J. Chaumat, A.-M. Chollet, *Surjectivité de l'application restriction à un compact dans des classes de fonctions ultradifférentiables*, Math. Ann. 298 (1994), 7-40.
- [13] J. Čermák, *The Abel equation in the asymptotic theory of functional differential equations*. Aequationes Math. 64 (2002), no. 1-2, 89–103.
- [14] J. Čermák, *On the differential equation with power coefficients and proportional delays*. Tatra Mt. Math. Publ. 38 (2007), 57–69.
- [15] C. Corduneanu, *Almost periodic functions*, Wiley Interscience, New York, 1961.
- [16] O. Costin, *Asymptotics and Borel summability*, Chapman Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.
- [17] L. Di Vizio, J.-P. Ramis, J.Sauloy, C. Zhang, *Équations aux q -différences*. Gaz. Math. No. 96 (2003), 20–49.
- [18] G. Derfel, *Functional-differential equations with compressed arguments and polynomial coefficients: asymptotics of the solutions*. J. Math. Anal. Appl. 193 (1995), no. 2, 671–679.
- [19] G. Derfel, A. Iserles, *The pantograph equation in the complex plane*. J. Math. Anal. Appl. 213 (1997), no. 1, 117–132.
- [20] G. Derfel, F. Vogl, *On the asymptotics of solutions of a class of linear functional-differential equations*. European J. Appl. Math. 7 (1996), no. 5, 511–518.
- [21] L. Fox, D. Mayers, J. Ockendon, A. Tayler, *On a functional differential equation*. J. Inst. Math. Appl. 8 (1971), 271–307.

- [22] G. H. Hardy, M. Riesz, *The general theory of Dirichlet series*, Cambridge University Press, 1915.
- [23] P. Hsieh, Y. Sibuya, *Basic theory of ordinary differential equations*. Universitext. Springer-Verlag, New York, 1999.
- [24] A. Iserles, *On the generalized pantograph functional-differential equation*. European J. Appl. Math. 4 (1993), no. 1, 1–38.
- [25] T. Kato, J.B. McLeod, *The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$* . Bull. Amer. Math. Soc. 77 1971 891–937.
- [26] M. Kawagishi, *A generalized Cauchy-Kovalevskaja-Nagumo theorem with shrinkings*. Sci. Math. Jpn. 54 (2001), no. 1, 39–50.
- [27] M. Kawagishi, T. Yamanaka, *The heat equation and the shrinking*. Electron. J. Differential Equations 2003, No. 97, 14 pp.
- [28] A. Lastra, S. Malek, *On q -Gevrey asymptotics for singularly perturbed q -difference-differential problems with an irregular singularity*. Abstr. Appl. Anal., Art. ID 860716, (2012) 35 pp.
- [29] A. Lastra, S. Malek, J. Sanz, *Gevrey solutions of threefold singular nonlinear partial differential equations*, to appear in J. Differential Equations, 2013.
- [30] A. Lastra, S. Malek, J. Sanz, *On q -asymptotics for q -difference-differential equations with Fuchsian and irregular singularities*, J. Differential Equations, vol. 252, Issue 10 (2012), 5185-5216.
- [31] A. F. Leont'ev, *Entire functions. Series of exponentials* (in Russian), Nauka, Moscow, 1983.
- [32] S. Malek, *Singularly perturbed q -difference-differential equations with irregular singularity*, J. Dynam. Control. Syst. 17 (2011), no. 2.
- [33] S. Malek, *On the summability of formal solutions for doubly nonlinear partial differential equations*, J. Dyn. Control Syst. 18 (2012), no. 1, 45-82.
- [34] S. Malek, *Singularly perturbed small step size difference-differential nonlinear PDEs*, to appear in the Journal of Difference Equations, 2013.
- [35] B. Malgrange, *Travaux d'Écalle et de Martinet-Ramis sur les systèmes dynamiques*, Séminaire N.Bourbaki, 1981–1982, exp n 582, p. 59–73.
- [36] J-P. Ramis, *About the growth of entire functions solutions of linear algebraic q -difference equations* Annales de la faculté des sciences de Toulouse Sér. 6, 1 no. 1 (1992), 53–94.
- [37] L. Rossovskii, *On the boundary value problem for the elliptic functional-differential equation with contractions*. Funct. Differ. Equ. 8 (2001), no. 3-4, 395–406.
- [38] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, second edition, 1939.
- [39] C. Zhang, *Analytic continuation of solutions of the pantograph equation by means of θ modular formula*, preprint ArXiv (2012).