Generalized notions of symmetry of ODE's and reduction procedures

Giampaolo Cicogna*

Dipartimento di Fisica "E.Fermi" dell'Università di Pisa
and Istituto Nazionale di Fisica Nucleare, Sez. di Pisa

Largo B. Pontecorvo 3, Ed. B-C, I-56127, Pisa, Italy

Abstract
This paper describes the notion of σ-symmetry, which extends the one of λ-symmetry, and its application to reduction procedures of systems of ordinary differential equations and of dynamical systems as well. We also consider orbital symmetries, which give rise to a different form of reduction of dynamical systems. Finally, we discuss how dynamical systems can be transformed into higher-order ordinary differential equations, and how these symmetry properties of the dynamical systems can be transferred into reduction properties of the corresponding ordinary differential equations. Many examples illustrate the various situations.

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Introduction

It is well known that if an ordinary differential equation (ODE) of order $q > 1$ admits a Lie point-symmetry, then the order of the equation can be lowered by *one* (*two* in some cases, e.g. when the equation comes from a variational problem), see e.g. [1, 2, 3, 4, 5].

It is also known that the same is true even if the equation admits a λ-symmetry, a notion which has been introduced by C. Muriel and J. Romero in

*Email: cicogna@df.unipi.it

2001 [6, 7] and which has received a number of applications and extensions (see e.g. [8] with references therein, and [9] for a more recent contribution).

We have recently further extended this result [10, 11, 12]. Let us fix our notations. We will always denote by t the independent variable, in order to unify the notations, as a large part of this paper will be concerned with dynamical systems, where time t is typically the independent variable. The ODE will be denoted by

$$\mathcal{E} \,=\, \mathcal{E}\big(t, u^{(k)}(t)\big) \,=\, 0 \qquad \big(u^{(k)}(t) = \mathrm{d}^k u/\mathrm{d}t^k \ ,\ k = 0, \ldots, q\big)$$

and the generators of Lie point-symmetries will be written in the form

$$X \,=\, \varphi(t,u)\frac{\partial}{\partial u} + \tau(t,u)\frac{\partial}{\partial t} \ .$$

According to a by now standard abuse of language, we will denote by X both the symmetry and its Lie generator.

We will consider, instead of a *single* vector field X, a set $\mathcal{X}$ of $s > 1$ vector fields X_α in involution

$$[X_\alpha, X_\beta] \,=\, \nu_{\alpha\beta\gamma} X_\gamma \qquad (\alpha, \beta, \gamma = 1, \ldots, s) \tag{1}$$

together with a system of ODE's $\mathcal{E}_a = 0$, $a = 1, \ldots, n$. This leads to the introduction of the notion of "combined" *joint-λ-symmetries*, or *σ-symmetries* for short. The precise definition and its application to the reduction of systems of ODE's will be given in the next Section. Using the same idea, we will show (Sect. 2) that also dynamical systems (DS), i.e. systems of first-order ODE's, can be suitably reduced when they admit a σ-symmetry. In Sect. 3, we include the case of *orbital* symmetries, which give rise to a different form of reduction of DS. Finally, in Sect. 4, we discuss how DS can be transformed into a higher-order ODE, and how these symmetry properties of the DS can be transferred into reduction properties of the corresponding ODE. Several new examples will illustrate the various situations. All the objects (functions, vector fields) considered in this paper are assumed to be smooth enough.

The presence of σ-symmetries admits interesting geometrical interpretations and algebraic aspects: for a full discussion of these arguments and several other details we refer to [10, 11, 12] and references therein.

This is a full paper presented within ICNAAM 2012; a very short and preliminary sketch of part of these results can be found in the Enlarged Abstracts of the Conference Proceedings [13].

1 Basic definitions and reduction of ODE's

First of all, we need the two following definitions.

Definitions

i) *Given $n > 1$ variables $u \equiv \{u^a(t)\}$, $(a = 1, \ldots, n)$, and $s > 1$ vector fields $\mathcal{X} \equiv \{X_\alpha\}$, $(\alpha = 1, \ldots, s)$, a σ-prolongation is a deformed prolongation rule* *which involves a given $s \times s$ matrix $\sigma = \sigma(t, u, \dot{u})$: the first σ-prolongation $Y_\alpha^{[1]}$ of $X_\alpha = \varphi_\alpha \cdot \nabla_u + \tau_\alpha \partial/\partial t$ is defined by*

$$Y_\alpha^{[1]} := X_\alpha^{[1],\sigma} \; = \; X_\alpha^{[1]} + \sigma_{\alpha\beta}(\varphi_\beta^a - \dot{u}^a \tau_\beta) \frac{\partial}{\partial \dot{u}^a}$$

where $X_\alpha^{[1]}$ is the first standard prolongation. Higher order prolongations $Y_\alpha^{[k]}$ can be easily obtained by recursion.

ii) *A system of n ODE's $\mathcal{E} \equiv \{\mathcal{E}_a\big(t, u^{(k)}(t)\big)\} = 0$ for the n variables $u(t)$, of order $q > 1$, is σ-symmetric under the set $\mathcal{X}$ if*

$$Y_\alpha^{[q]} \mathcal{E}|_{\mathcal{E}=0} \; = \; 0$$

i.e. if $\mathcal{E}$ is invariant under the σ-prolongations $Y_\alpha^{[q]}$ of all the X_α.

It can be remarked that the case $s = 1$ would correspond to λ-symmetries.

Based on the above definitions, we can state the following result.

Theorem 1. *Let a system of n ODE's $\mathcal{E} = 0$ of order $q > 1$ be σ-symmetric under a set $\mathcal{X}$ of vector fields X_α $(\alpha = 1, \ldots, s > 1)$ in involution with constant rank r $(r \leq s; r \leq n)$; if the involution relations are preserved in their q-th σ-prolongations $Y_\alpha^{[q]}$, then – under standard regularity and nondegeneracy conditions – the order of r ODE's can be lowered by one. This is obtained in terms of some r new variables η_α which are invariant under the 1^{st} σ-prolongations $Y_\alpha^{[1]}$.*

Sketch of the proof. The main ingredient of the proof is the following *completely algebraic* result, which holds for general vector fields $X_\alpha = \varphi_\alpha \cdot \nabla_u + \tau_\alpha \partial/\partial t$

$$[D_t, Y_\alpha^{[k+1]}] \; = \; -\sigma_{\alpha\beta} Y_\beta^{[k]} + (D_t \tau_\alpha + \sigma_{\alpha\beta}\tau_\beta) D_t \tag{2}$$

where D_t is the total derivative, and its consequence

$$Y_\alpha^{[k+1]} \, \frac{D_t \zeta_1^{[k]}}{D_t \zeta_2^{[k]}} \; = \; 0 \tag{3}$$

where $\zeta_i^{[k]}$ is any k-order differential invariant under $Y_\alpha^{[k]}$. Assume for simplicity (but the general result holds in general) that the X_α are *vertical* vector fields, i.e. that $\tau_\alpha = 0$: then, the time t is a common invariant under all the X_α. Assume also, for the moment, that $n = r$. Then, no other variable is admitted with this property. Considering the first σ-prolonged vector fields $Y_\alpha^{[1]}$, there exist, according to Frobenius theorem, exactly n common differential invariants of order 1 under $Y_\alpha^{[1]}$. Let us denote these by η_α $(\alpha = 1, \ldots, r = n)$. Using (3) with $k = 1$, choosing as ζ_1 any of these η_α and $\zeta_2 = t$, we deduce that $D_t \eta_\alpha = \dot{\eta}_\alpha$ are second-order differential quantities which are common invariants under the second σ-prolongation $Y_\alpha^{[2]}$, and so on. This is called *invariance by differentiation property.* The σ-invariance of the system $\mathcal{E} = 0$, then implies that all the equations of this system must contain, apart from t, only the common invariant variables with their derivatives. Choosing η_α as new variables, the equations of our system thus become equations of order $q-1$. If instead $n > r$, then, still thanks to Frobenius theorem, there are, in addition to t, other $(n-r)$ variables w_j $(j = 1, \ldots, n-r)$ of order zero which are common invariants under X_α. Therefore, thanks to (3), also $\dot{w}_j$ are $(n-r)$ common invariants under the first σ-prolongation $Y_\alpha^{[1]}$, in addition to other r invariants η_α, and so on. In other words, starting from the invariants w_j and η_α, one obtains all higher-order differential invariants. As before, our system must be written in terms of these invariant quantities; then the system of ODE's can be split into a subsystem of r equations of order $q-1$ in the variables t and η_α, and another system of $n-r$ equations of order q. •

Example 1. Consider the system of ODE's (in the examples we will usually write as $u_1, u_2, \ldots$ instead of u^a to avoid confusion, and $\dot{u}_1 = \mathrm{d}u_1/\mathrm{d}t$, etc.)

$$\begin{cases} \ddot{u}_1 = t\ddot{u}_2 + t\dot{u}_2 + 2\dot{u}_2 + u_2 + h_1(t,u) \\ \ddot{u}_2 = \dot{u}_1 - \dot{u}_2 + h_2(t,u) \\ \ddot{u}_3 = u_2 + t\dot{u}_2 + h_3(t,u) \end{cases} \tag{4}$$

where h_a are arbitrary functions of t and of the quantities $u_1 - u_2 - u_3, u_1 - u_2 - \dot{u}_1 + tu_2, u_1 - u_2 - \dot{u}_2$. For generic h_1, h_2, h_3 there is no standard Lie symmetry for this system, but it is σ-symmetric under the vector fields (then $n = 3, r = 2$)

$$X_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} \quad , \quad X_2 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}$$

with

$$\sigma = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} .$$

The first σ-prolongations are

$$Y_1^{[1]} = X_1 + t\frac{\partial}{\partial \dot{u}_1} + t\frac{\partial}{\partial \dot{u}_3} \quad , \quad Y_2^{[1]} = X_2 + \frac{\partial}{\partial \dot{u}_1} + \frac{\partial}{\partial \dot{u}_2} .$$

In the new σ-symmetry adapted variables $w = u_1 - u_2 - u_3$, $\eta_1 = u_1 - u_2 - \dot{u}_1 + tu_2$, $\eta_2 = u_1 - u_2 - \dot{u}_2$ the above equations become, in agreement with Theorem 1,

$$\ddot{\eta}_1 = -\dot{\eta}_1 + \dot{\eta}_2 + h_1(\eta_1, \eta_2, w) \ , \ \dot{\eta}_2 = -h_2(\eta_1, \eta_2, w) \ , \ \ddot{w} = \dot{\eta}_2 + \dot{\eta}_1 - h_3(\eta_1, \eta_2, w)$$

△

It can be observed that if one of the equations of the system of ODE's is of order 1 and this is lowered according to Theorem 1, then one is left with an *algebraic* equation for the variables t and η_α. This happens for instance if in Example above one of the equations is replaced by

$$\dot{u}_1 = u_1 - u_2 + tu_2 + h_0(t,u)$$

which is reduced to

$$\eta_1 + h_0(\eta_1, \eta_2, w) \ = \ 0 \ .$$

Notice that this algebraic equation is actually a first-order differential equation for the initial variables u^a (the presence of an "auxiliary" first-order differential equation is indeed standard in λ-type symmetries).

This remark introduces the special and specially interesting case of dynamical systems, which will be considered in detail in the next sections.

2 Reduction of Dynamical Systems

Dynamical systems are systems of first-order time-evolution differential equations of the form

$$\dot{u}^a \ = \ f^a(t,u) \qquad a = 1, \ldots, n$$

It is not too restrictive to consider *autonomous* DS, and vertical vector fields with φ_α *independent of time*, i.e.

$$\dot{u} = f(u) \qquad X_\alpha \ = \ \varphi^a_\alpha \frac{\partial}{\partial u^a} \equiv \varphi_\alpha \cdot \nabla_u \tag{5}$$

Given a DS, the σ-determining equations, i.e. the equations giving the conditions for the DS to be invariant under the first σ-prolongations $Y^{[1]}_\alpha$ of X_α, when restricted to the solution manifold of the DS, take the particularly simple form

$$[X_\alpha, F] \ = \ \sigma_{\alpha\beta} X_\beta \qquad (\alpha, \beta = 1, \ldots, s) \tag{6}$$

having introduced the "dynamical" vector field

$$F \ = \ f \cdot \nabla_u \ .$$

In particular, the restriction to the solution manifold of the DS $\dot{u} = f(u)$, implies that σ may be chosen as a function of t, u only, indeed $\sigma\big(t, u, f(t,u)\big) = \overline{\sigma}(t,u)$. From (6), one may directly recover for this case the invariance by differentiation property: indeed, if w_j satisfies $X_\alpha w_j = 0$, then

$$X_\alpha(D_t w_j) = X_\alpha(f \cdot \nabla_u) w_j = X_\alpha F \, w_j = (F X_\alpha + \sigma_{\alpha\beta} X_\beta) w_j = 0$$

i.e. $D_t w_j$ is also invariant under all the X_α.

As well known, given a set $\mathcal{X}$ of vector fields in involution, it is not granted in general that their prolongations are still in involution (see [10, 12] for a discussion and some examples on this point). However, in the case of DS, we have the following useful result (in the following, we will simply write Y_α instead of $Y^{[1]}_\alpha$):

Lemma. *Let a DS satisfy (6) with a set of vector fields X_α in involution. Then, restricting to the solution manifold of the DS, the first σ-prolonged vector fields Y_α satisfy the same involution property.*

`Proof`. We first have

$$Y_\alpha \;=\; X_\alpha + \left(D_t \varphi^a_\alpha + \sigma_{\alpha\beta}\varphi^a_\beta\right)\frac{\partial}{\partial \dot u^a} = X_\alpha + X_\alpha\, f \nabla_{\dot u}$$

thanks to (6). Then

$$[Y_\alpha, Y_\beta] \;=\; [X_\alpha, X_\beta] + \nu_{\alpha\beta\gamma} X_\gamma f \nabla_{\dot u} = \nu_{\alpha\beta\gamma} Y_\gamma$$

using the involution properties of X_α. •

Then we have:

Theorem 2. *In the above simplifying assumptions (5), let a DS be σ-symmetric under a set $\mathcal{X}$ of vector fields X_α ($\alpha = 1, \ldots, s > 1$) in involution, with rank $r < n$; then the DS can be locally reduced to a DS involving $n - r$ variables w_j:*

$$\dot w_j \;=\; W_j(w)$$

plus a system of r "reconstruction equations " depending on the solutions of the reduced system.

As for the above case of general ODE's, the proof is based on the introduction of n symmetry-adapted variables: precisely of $(n - r)$ variables w_j which are the entries of the reduced DS and are common invariants under X_α, and of r first-order differential Y_α-invariants η_α.

It can be noticed that this reduction to a $n-r$-dimensional DS holds exactly as in the case of standard (exact) symmetries. See also [14] for the case of DS admitting λ-symmetries.

Example 2. This is a very trivial example, given to provide a clear illustration of the procedure. The DS

$$\begin{cases} \dot u_1 \;=\; h_1(u_1, u_2, u_3) + g_1(u_1 - u_3) \\ \dot u_2 \;=\; h_2(u_1, u_2, u_3) + g_2(u_1 - u_3) \\ \dot u_3 \;=\; h_1(u_1, u_2, u_3) + g_3(u_1 - u_3) \end{cases}$$

where h_a, g_a are arbitrary functions of the indicated arguments, admits the two vector fields

$$X_1 \;=\; \partial/\partial u_1 + \partial/\partial u_3 \quad , \quad X_2 \;=\; \partial/\partial u_2$$

as σ-symmetry, as can be easily verified, with

$$\sigma \;=\; \begin{pmatrix} \partial h_1/\partial u_1 + \partial h_1/\partial u_3 & \partial h_2/\partial u_1 + \partial h_2/\partial u_3 \\ \partial h_1/\partial u_2 & \partial h_2/\partial u_2 \end{pmatrix} .$$

In terms of the symmetry-adapted variables $w = u_1 - u_3$, $\eta_1 = \dot u_1 - h_1$, $\eta_2 = \dot u_2 - h_2$, the DS becomes

$$\dot w = g_1(w) - g_3(w) \ , \ \eta_1 = g_1(w) \ , \ \eta_2 = g_2(w)$$

where the first equation is the reduced system and the other two the reconstruction equations. △

In the following, it will be convenient to rewrite equation (6) in the more transparent form, with evident notations,

$$[\varphi_\alpha, f] \;=\; \sigma_{\alpha\beta}\varphi_\beta \qquad (\alpha,\beta = 1,\dots,s) \tag{7}$$

An important property of σ-symmetric DS is given by the following proposition, which can be easily verified, using (6) (or (7)):

Proposition 1. *Let $\dot u = f$ be a DS admitting a set $\mathcal{X}$ of vector fields $X_\alpha = \varphi_\alpha \cdot \nabla_u$ in involution as a* standard *symmetry. Then, for any choice of s functions $\mu_\alpha(u)$, the new DS*

$$\dot u \;=\; f^* := f + \sum_{\alpha=1}^{s} \mu_\alpha \varphi_\alpha$$

admits the set $\mathcal{X}$ as σ-symmetry, where σ is given by

$$\sigma_{\alpha\beta} \;=\; X_\alpha(\mu_\beta) + \mu_\gamma \nu_{\alpha\gamma\beta} \ .$$

For a partial converse of this result, see [11]. The above proposition is clearly useful for constructing explicit examples of σ-symmetric DS (it is known that, given a DS, it may be very difficult to determine its σ-symmetries, because the σ-determining equations are in general differential functional equations: see [10, 12] for a discussion on this aspect).

Example 3. As a special case of the above Proposition, consider $f = Au$ for some matrix A; then obviously $\varphi_\alpha = B_\alpha u$, with B_α matrices such that $[A, B_\alpha] = 0$, provide standard symmetry vector fields X_α for $\dot u = Au$. These matrices will satisfy $[B_\alpha, B_\beta] = c_{\alpha\beta\gamma} B_\gamma$ and hence $[X_\alpha, X_\beta] = -c_{\alpha\beta\gamma} X_\gamma$ and the vector fields X_α provide a σ-symmetry for the DS (we can take $B_0 = A$)

$$\dot u \;=\; A\,u \;+\; \sum_{\alpha=0}^{s} \mu_\alpha(u)\, B_\alpha u \ .$$

for any functions μ_α. As a concrete example, consider the DS

$$\dot u_1 \;=\; u_1 - u_2 \quad , \quad \dot u_2 \;=\; -u_1 + u_2 \quad , \quad \dot u_3 \;=\; a u_3$$

where a is any constant: it admits the two standard symmetries

$$X_1 \;=\; u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3} \ , \ X_2 \;=\; u_2 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}$$

Using Prop.1 with $\mu_1 = u_1$, $\mu_2 = u_3$, we obtain the new DS

$$\begin{cases} \dot u_1 \;=\; u_1 - u_2 + u_1^2 + u_1 u_2 \\ \dot u_2 \;=\; -u_1 + u_2 + u_1 u_2 + u_1 u_3 \\ \dot u_3 \;=\; a u_3 + u_1 u_3 \end{cases}$$

which admits the above vector fields as σ-symmetry. Accordingly (here $n - r = 1$), we get in terms of the common invariant variable $w = (u_1^2 - u_2^2)/u_3^2$ the reduced equation $\dot w = 2(1-a)w$. $\triangle$

3 Orbital symmetries

If in the equation (6) (or (7)) the indices α, β run from 0 to s, and some $\sigma_{\alpha 0} := \theta_\alpha \neq 0$, i.e. if

$$[\varphi_\alpha, f] = \theta_\alpha f + \sigma_{\alpha\beta}\varphi_\beta \qquad (\alpha, \beta = 1, \ldots, s) \tag{8}$$

the case of *orbital* σ-symmetries is included.

Let us recall that in the case of *proper* orbital symmetries (i.e. when $\sigma_{\alpha\beta} = 0$, some $\theta_\alpha \neq 0$) we have:

Proposition 2. *If a n-dimensional DS $\dot{u} = f(u)$ admits an involutive set of $s \geq 1$ vector fields $\mathcal{X} \equiv \{X_\alpha\}$ as a (proper) orbital symmetry, then: i) X_α map solution orbits into solution orbits; ii) there is a scalar nonzero function $\rho(u)$ such that the DS*

$$\dot{u} = \rho\, f(u)$$

is standardly symmetric under X_α; iii) the DS $\dot{u} = \rho\, f(u)$ is orbitally equivalent *to $\dot{u} = f(u)$, i.e. the two DS have the same solutions orbits and the same constants of motion. The initial DS can then "orbitally reduced", i.e. there are $n - r$ variables w_j (where $r < n$ is the rank of $\mathcal{X}$), invariant under X_α, and a nonzero scalar function $\omega(u)$ such that*

$$\dot{w}_j = \omega(u)\, W_j(w) \tag{9}$$

i.e. we get a reduction "up to a common scalar factor".

In the general case of orbital-σ-symmetries (8) we have essentially the same result:

Theorem 3. *In the simplifying assumptions as above, if a DS admits an involutive set $\mathcal{X}$ of orbital σ-symmetries, then the DS can be orbitally reduced as in Proposition 2.*

As is clear from (9), if we have at least two variables w_j, say w_1, w_2, we can obtain from (9) a reduced equation of the form

$$\frac{\mathrm{d}w_1}{dw_2} = \Psi(w_1, w_2)\ .$$

Example 4. Let us use in this case for simplicity the notations $u \equiv (x, y, z)$ and $r^2 = x^2 + y^2$, $\theta = \arctan(y/x)$. Consider the DS

$$\begin{cases} \dot{x} = h_1(x,y,z)x + h_2(x,y,z)y \\ \dot{y} = h_1(x,y,z)y - h_2(x,y,z)x \\ \dot{z} = h_3(x,y,z)z \end{cases}$$

and the rotation vector field $X = y\partial/\partial x - x\partial/\partial y = \partial/\partial\theta$; then $w_1 = r$, $w_2 = z$. We distinguish the following cases:

a) all the h_i are functions of $r^2 = x^2 + y^2$ and z only, then X is a standard symmetry, and a complete reduction is obtained:

$$\dot{r} = h_1(r,z)r \quad , \quad \dot{z} = h_3(r,z)z \quad , \quad \dot{\theta} = -h_2(r,z)$$

b) only h_1 and h_3 are functions of r and z, then X is a λ-symmetry, and $\dot{r}$ and $\dot{z}$ are as in a), but $\dot{\theta} = -h_2(r, z, \theta)$

c) h_2/h_1 and h_3/h_1 are functions of r and z, then X is a orbital symmetry and we have reduction up to a common factor

$$\dot{r} = h_1(r, z, \theta) r \quad , \quad \dot{z} = h_1(r, z, \theta)\chi_a(r, z) z \quad , \quad \dot{\theta} = -h_1(r, z, \theta)\chi_b(r, z)$$

giving

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \Psi_1(r, z) \quad , \quad \frac{\mathrm{d}\theta}{dr} = \Psi_2(r, z)$$

d) only h_3/h_1 is function of r and z, then X is a orbital σ-symmetry, and $\dot{r}$ and $\dot{z}$ are as in c), but $\dot{\theta} = -h_2(r, z, \theta)$.

Another related result, concerning the presence of constants of motions of the DS having the property of being simultaneously invariant under the symmetry is the following:

Corollary. *In the above hypotheses, if a DS admits a rank r involutive set of σ-symmetries, or orbital σ-symmetries, there are $n - r - 1$ constants of motion, independent of time, of the DS, which are also invariant under all the σ-symmetries X_α.*

This is obtained (using again Frobenius theorem) looking for common invariants of the extended $(s+1)$-dimensional set $\widehat{\mathcal{X}} := \{F, X_\alpha\}$ (with $F = f \cdot \nabla_u$ as before), or $\widehat{\varphi} := \{f, \varphi_\alpha\}$. An extension to non-autonomous DS and time dependent constants of motion can be easily obtained.

4 From DS to higher-order ODE's

Any ODE $\mathcal{E}\big(u(t)\big) = 0$ of order $n > 1$ can be transformed into a DS, as well known. Writing $u^{(n)} = p(t, u, \dot{u}, \ldots)$, if the ODE does not contain explicitly the independent variable t, then one can put as usual

$$u = u_1 \ , \ \dot{u}_1 = u_2 \ , \ldots , \ \dot{u}_n = p(u, \dot{u}, \ldots)$$

if instead p depends on t, one simply includes the new variable $u_0 = t$ and the equation $\dot{u}_0 = 1$. The converse is "in principle" (locally, and apart from degenerate cases) also true (see [15, 11]), but the transformation of a DS into an ODE requires the inversion of some implicit expressions.

Let us show the procedure in the case of a DS with 3 dependent variables u^a. If the DS is autonomous, $\dot{u}^a = f^a(u)$, then one puts

$$u_1 := y_1 := y \ , \ \dot{u}_1 = f_1(u) := y_2 = \dot{y},$$

$$\dot{y}_2 = D_t f_1(u) = f \cdot \nabla_u \, f_1(u) := \Phi(u) := y_3 = \ddot{y}$$

then one has to express u_2 and u_3 in terms of y_1, y_2, y_3 using the two above definitions, and finally one gets

$$\dot{y}_3 = \dddot{y} = D_t \Phi\big(u(y)\big) := p(y)$$

which produces the ODE

$$\dddot{y} = p(y, \dot{y}, \ddot{y}) \ .$$

If the DS is non-autonomous, then it can be "autonomized" introducing as usual $u_0 = t$, and the above procedure can be adapted accordingly.

The procedure of transforming a DS into a higher-order ODE opens interesting possibilities of reducing the ODE. If indeed the DS admits some symmetry (including σ-symmetries and orbital symmetries), then we have shown that the DS can be reduced in terms of suitable symmetry-adapted variables. This reduction is immediately transferred, up to the change of variables described above, to the higher-order ODE. Observing that not all symmetries of the DS become automatically Lie point-symmetries of the ODE, we get a sort of "reduction of the ODE's without symmetries". There are several possibilities in this direction, as we will show in the following examples.

To illustrate the procedure, we give first an example of DS admitting a standard symmetry; we construct the corresponding higher order ODE and show how the symmetry property of the DS can be used to obtain a reduced equation for the ODE. In this example, the reduced equation can be easily solved and this procedure provides thus an alternative way to get the full solution of the ODE. Examples 6 and 7 deal with ODE's deduced from DS admitting resp. a λ-symmetry and a σ-symmetry.

Example 5. The DS

$$\dot{u}_1 = u_1 + u_1^2 u_2 \quad , \quad \dot{u}_2 = u_2 + u_1 u_2^2$$

admits the standard symmetry $X = u_1 \partial/\partial u_1 - u_2 \partial/\partial u_2$. An invariant variable under X is $w = u_1 u_2$, which satisfies the reduced equation $\dot{w} = 2w + 2w^2$. The ODE obtained through the positions $u_1 = y$, $\dot{u}_1 = y_2 = \dot{y}$ etc. is

$$\ddot{y} \ = \ -2\dot{y} + 3\frac{\dot{y}^2}{y} \ .$$

Integrating the reduced equation for w and passing to the new variable y we obtain the reduced equation for the ODE

$$\frac{\dot{y}}{y} \ = \ \frac{c\ \exp(2t)}{1 - c\ \exp(2t)} - 1$$

and from this the full solution of the ODE

$$y \ = \ \frac{c' \exp(t)}{\sqrt{c\ \exp(2t) - 1}}$$

where c, c' are constants. $\triangle$

Example 6. This is a simple example with a λ-symmetry. We start from the DS

$$\dot{u}_1 \ = \ u_2 \quad , \quad \dot{u}_2 \ = \ 2u_2^2/u_1$$

having a (standard) dilation symmetry $X = u_1 \partial/\partial u_1 + u_2 \partial/\partial u_2$. Using Prop. 1 with $\mu = u_1$, the new DS

$$\dot{u}_1 \ = \ u_2 + u_1^2 \quad , \quad \dot{u}_2 \ = \ 2u_2^2/u_1 + u_1 u_2 \tag{10}$$

admits the above X as a λ-symmetry. With $u_1 = y$, $u_2 = \dot{y} - y^2$, according to the above described procedure, we get the ODE

$$\ddot{y} \ = \ 2\frac{\dot{y}^2}{y} - y\dot{y} + y^3 \ .$$

The DS (10) can be reduced by the X-invariant variable $w = u_2/u_1$, indeed $\dot{w} = w^2$; the same reduction holds for the new variable $\widetilde{w} = (\dot{y}/y) - y$, as easily checked. The reduced equation for w can be immediately solved producing the (time-dependent) first integral for the ODE $\kappa = t + y/(\dot{y} - y^2) = \mathrm{const}$. This equation for $y(t)$ can be further integrated giving the general solution of the ODE

$$y(t) \ = \ \Big((c-t)(c' + \log(c-t)\Big)^{-1}$$

where c, c' are constants. As above, the solution of the ODE could be obtained (although not too easily) also by standard methods, but this example can be useful to further illustrate this symmetry-based procedure. $\triangle$

Example 7. This is an example where an ODE is constructed starting from a DS admitting a σ-symmetry. The very simple DS

$$\dot{u}_1 = 1 \quad , \quad \dot{u}_2 = u_3 \quad , \quad \dot{u}_3 = u_2$$

admits the two standard symmetries

$$X_1 \ = \ \frac{\partial}{\partial u_1} \quad , \quad X_2 \ = \ u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3} \ .$$

Using Prop.1 with $\mu_1 = u_3$, $\mu_2 = 1/u_1$, we obtain the new DS

$$\dot{u}_1 = 1 + u_3 \quad , \quad \dot{u}_2 = u_3 + u_2/u_1 \quad , \quad \dot{u}_3 = u_2 + u_3/u_1 \tag{11}$$

which then admits the two vector fields X_1, X_2 as σ-symmetry. A common invariant under these vector fields is $w = u_2/u_3$ which satisfies the equation $\dot{w} = 1 - w^2$. The ODE which is deduced from the above DS (11) is

$$\dddot{y} = \ \dot{y} - 1 + 2\frac{\ddot{y}}{y} + \frac{(\dot{y}-1)^2}{y^2} \ .$$

After integration of the equation for w, passing to the new variable y, one obtains the reduced equation for $y(t)$

$$\frac{y\ddot{y} - \dot{y} + 1}{y(\dot{y}-1)} = \frac{\exp(2t) - c}{\exp(2t) + c}$$

where c is a constant. $\triangle$

The two next and final examples deal with the case of σ-orbital symmetries. According to Prop.2, we can construct orbitally symmetric DS starting from a σ-symmetric (or standardly symmetric as well) by multiplication by an arbitrary function $\rho(u)$. A good choice for this function may be, e.g., $\rho = 1/f_1(u)$ with usual notations, in such a way that the new DS becomes, renaming for convenience the variables $u_1, \ldots, u_n$ as $v_0, \ldots, v_{n-1}$

$$\dot{v}_0 = 1 \quad , \quad \dot{v}_1 = f_2(v)/f_1(v) \quad , \ldots , \quad \dot{v}_{n-1} = f_n(v)/f_1(v) \tag{12}$$

i.e. in the form of an "autonomized" DS where $v_0 = t$ and then $v_1 = y$, $\dot{v}_1 = f_2(v)/f_1(v) = \dot{y}$, $\dot{v}_2 = D_t\big(f_2(v)/f_1(v)\big) = \ddot{y}$, etc. The ODE deduced from this DS will then be of order $(n-1)$. We shall adopt this choice for the function $\rho(u)$ in both the following examples.

As said above, in the case of orbital symmetries, we need *at least two* invariants w_j under the vector fields X_α in order to have a reduced equation. This may be reached either with two invariants under a single vector field (hence in the case of a single standard symmetry, or also a λ-symmetry as considered in Example 8 below), or with two common invariants under two vector fields (standard, or also σ-symmetry as in Example 9).

Example 8. The DS

$$\dot{u}_1 \;=\; u_1 u_2 \quad , \quad \dot{u}_2 \;=\; u_1/u_3 \quad , \quad \dot{u}_3 = u_3$$

admits the standard symmetry $X = u_1\partial/\partial u_1 + u_3\partial/\partial u_3$. Using Prop.1 with $\mu = u_2$ and then Prop.2 with $\rho = 1/(u_1 u_2)$ we get the DS, using the notations introduced in (12),

$$\dot{v}_0 \;=\; 1 \quad , \quad \dot{v}_1 \;=\; \frac{1}{2v_1v_2} \quad , \quad \dot{v}_2 \;=\; v_2\frac{1+v_1}{2v_0v_1}$$

which admits the above vector field X as an orbital σ- (actually, a λ) -symmetry. The ODE which can be deduced from this DS is

$$\ddot{y} \;=\; -\frac{\dot{y}(2t\dot{y}+y+1)}{2ty} \;.$$

There are two independent invariants under the above vector field X, namely $w_1 = u_2 = v_1$, $w_2 = u_1/u_3 = v_0/v_2$; they satisfy the equations

$$\dot{w}_1 = \frac{1}{v_0}\frac{w_2}{2w_1} \quad , \quad \dot{w}_2 = \frac{1}{v_0}w_2\frac{w_1-1}{2w_1}$$

and from these we obtain the reduced equation $\mathrm{d}w_2/\mathrm{d}w_1 = w_1 - 1$ which can be easily integrated giving in the new variable y the reduced equation for $y(t)$ (and a first integral for the ODE)

$$2ty\dot{y} - \frac{1}{2}y^2 + y \;=\; \mathrm{const} \;.$$

Example 9. The simple DS

$$\dot{u}_1 = 0 \quad , \quad \dot{u}_2 = u_3 \quad , \quad \dot{u}_3 = u_4 \quad , \quad \dot{u}_4 = u_2$$

admits the two standard symmetries

$$X_1 \, = \, \frac{\partial}{\partial u_1} \quad , \quad X_2 \, = \, u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3} + u_4 \frac{\partial}{\partial u_4} \ .$$

Thanks to Prop.1 and 2, with $\mu_1 = u_2$, $\mu_2 = u_1$ and $\rho = 1/u_2$ we obtain the DS, with the notations as in (12)

$$\dot{v}_0 = 1 \quad , \quad \dot{v}_1 = v_0 + v_2/v_1 \quad , \quad \dot{v}_2 = (v_3 + v_0 v_2)/v_1 \quad , \quad \dot{v}_3 = 1 + v_0 v_3/v_1 \ .$$

Two independent common invariants under X_1, X_2 are $w_1 = u_2/u_3 = v_1/v_2$, $w_2 = u_4/u_3 = v_3/v_2$. The corresponding ODE is

$$\dddot{y} = \frac{1}{y^2} \Big(1 - (\dot{y} - t)^3 - ty + 3ty\ddot{y} - 4y\dot{y}\ddot{y} \Big)$$

which admits the reduced equation

$$\frac{\mathrm{d}w_1}{\mathrm{d}w_2} \, = \, \frac{1 - w_1 w_2}{w_1 - w_2^2} \ .$$

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