WHITNEY REGULARITY OF THE IMAGE OF THE CHEVALLEY MAPPING

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ABSTRACT. A closed set F is Whitney 1-regular if for each compact $K \subset F$, the geodesic distance in K is equivalent to the Euclidean distance. Let P be the Chevalley map defined by an integrity basis of the algebra of polynomials invariant by a reflection group, this note gives the Whitney regularity of the image $P(\mathbb{R}^n)$. The proof uses ideas in [1], [3], [7] and [8] and needs a generalization to the Jacobian of the Chevalley mappings of a property of Van der Monde determinants.

1. INTRODUCTION

Definition 1.1. [12] A compact set $K \subset \mathbb{R}^n$, connected by rectifiable arcs, is Whitney 1-regular or has the Whitney property $\mathbf{P}_1$ if the geodesic distance in K is equivalent to the Euclidean distance.

In other words, K is 1-regular if there exists a constant $C_K > 0$, such that any couple of points x and x' in K may be joined by a rectifiable arc in K with length $\ell(x, x') \leq C_K|x - x'|$.

A closed set $\mathcal{F}$ is Whitney 1-regular when any compact set in $\mathcal{F}$ is Whitney 1-regular.

Let $W \subset O(n)$ be a finite reflection group. The algebra of W-invariant polynomials is generated by n algebraically independent W-invariant homogeneous polynomials [5] and the degrees of these polynomials are uniquely determined [6]. Let $p_1, \ldots, p_n$ be a set of such basic invariants and define the 'Chevalley' mapping

$$P : \mathbb{R}^n \ni x \mapsto (p_1(x), \ldots, p_n(x)) \in \mathbb{R}^n.$$

This note gives a proof of the Whitney property $\mathbf{P}_1$ for the image $P(\mathbb{R}^n)$ of the Chevalley mapping associated with a finite reflection group.

For $I_2(p)$, the 1-regularity of $P(\mathbb{R}^2)$ is clear since it is the salient part cut in the plane by a cusp. In particular the property holds for the group $G_2 = I_2(6)$. Since there is nothing to prove for $I_2(p)$, it will not be considered in what follows. The interested reader will find some details in appendix A.

For the symmetric group the property $\mathbf{P}_1$ of the image of the Newton mapping was conjectured in [3] with a proof in dimension less than or equal to 4. The general proof was given by Kostov [8] for the symmetric groups and for A_n groups .

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First we show that the results given for the symmetric group and announced for reflection groups by Givental in [7], actually hold for these groups.

Then we derive the property $\mathbf{P}_1$ for the image of the Chevalley map by using the methods in [8] or [1], and already in [3].

This property is not altered by diffeomorphism, so it does not depend on the choice of coordinates nor on the choice of the set of basic invariants since a change of basic invariants is an invertible polynomial map on the target.

The image sets of Chevalley maps associated with reflection groups are natural but not trivial examples of sets with this geometric property.

Finally, the property $\mathbf{P}_1$ shares with the convexity an interesting feature. Let $\mathcal{O}$ be an open set of $\mathbf{R}^n$ such that its closure $\mathcal{F} = \overline{\mathcal{O}}$ has property $\mathbf{P}_1$ and let $f \in C^m(\mathcal{O})$. If the field of Taylor polynomials of f has a continuous extension of degree m on $\mathcal{F}$, this extension is Whitney m-regular and by the Whitney extension theorem [11] there is one and actually there are infinitely many extensions of f on $\mathbb{R}^n$.

Theorem 1.2. [12] *Let $\mathcal{O}$ be an open set of $\mathbf{R}^n$ and assume that its closure $\mathcal{F} = \overline{\mathcal{O}}$ has property $\mathbf{P}_1$. If $f \in C^m(\mathcal{O})$is such that whenever $\mid k \mid= m$, the $\partial^{|k|} f/\partial x^k$ have a continuous extension to $\mathcal{F}$, then f has an extension $\tilde{f} \in C^m(\mathbb{R}^n)$.*

Thanks to the 1-regularity of $P(\mathbb{R}^n)$, the proof of the theorem of Chevalley in finite class of differentiability [4], would be easier. Let $f \in C^r(\mathbb{R}^n)^W$ be W invariant and $f = F \circ P$. The study of the continuity of the derivatives of F on the border of $P(\mathbb{R}^n)$, shows that the field of Taylor polynomials of F has a continuous extension of degree $[r/n]$ on $P(\mathbb{R}^n)$. By theorem 1.2 there is one, and actually there are infinitely many extensions of F, of class $C^{[r/n]}$ on $\mathbb{R}^n$. By lack of a stratified Whitney extension theorem, however, these extensions would loose the property that the class of differentiability on each stratum is governed by the isotropy subgroup of the stratum. The extension $\tilde{F}$ built in [4] on $\mathbb{R}^n$ by using an extension $\tilde{f}$ of f on $P^{-1}(\mathbb{R}^n) \in \mathbb{C}^n$ preserves this important property. Although one may multiply $\tilde{F}$ by any function of class C^r equal to 1 on a neighborhood of $P(\mathbb{R}^n)$, $\tilde{F}$ was, at leat in the neighborhood of $P(\mathbb{R}^n)$, the 'nicest extension'.

2. The Chevalley mapping, strata of its image and minors of its Jacobian

When W is reducible, it is a direct product of its irreducible components, say $W = W^0 \times W^1 \times \ldots \times W^s$ and we may write $\mathbb{R}^n$ as an orthogonal direct sum $\mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_s}$. The first component W^0 is the identity on the W-invariant subspace $\mathbb{R}^{n_0}$, and for $i = 1, \ldots, s$, W^i is a finite Coxeter group acting on $\mathbb{R}^{n_i}$.

For an irreducible W, unless explicitly stated, we will assume as we may that the degrees of the polynomials $p_1, \ldots, p_n$ are in increasing order: $2 = k_1 \leq \ldots \leq k_n = h$, where h is the Coxeter number of W. In the reducible case, using coordinates that fit with the orthogonal direct sum, if $w = w_0 w_1 \ldots w_s \in W$ with $w_i \in W^i$, $0 \leq i \leq s$, for all $x \in \mathbb{R}^n$ we have $w(x) = w(x^0, x^1, \ldots, x^s) = (x^0, w_1(x^1), \ldots, w_s(x^s))$. The direct product of the identity P^0 on $\mathbb{R}^{n_0}$ and Chevalley mappings P^i associated with W^i acting on $\mathbb{R}^{n_i}$, $1 \leq i \leq s$, is a Chevalley map associated with the action of W on $\mathbb{R}^n$. For each $i = 1 \ldots s$ the p^i_j are of degree k^i_j, for $j = 1$ to n_i. The k^i_1 are all equal to 2, and $k^i_{n_i}$ is the Coxeter number of W^i.

If for each i, $P^i(\mathbb{R}^{n_i})$ is 1-regular, so is $P(\mathbb{R}^n)$. As a consequence it is sufficient to prove the regularity when W is irreducible and from now on, we will assume W to be a finite Coxeter group.

Let $\mathcal{R}$ be the set of reflections different from identity in W. The number of these reflections is $\mathcal{R}^{\#} = d = \sum_{i=1}^{n}(k_i - 1)$. For each $\tau \in \mathcal{R}$, we consider a linear form λ_τ the kernel of which is the reflecting hyperplane $H_\tau = \{x \in \mathbf{R}^n | \tau(x) = x\}$.

The jacobian determinant of P is $|J_P| = c \prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through $\mathcal{R}$.

A Weyl Chamber C is a connected component of the regular set. The other connected components are obtained by the action of W and the regular set is $\bigcup_{w \in W} w(C)$. There is a stratification of $\mathbb{R}^n$ by the regular set, the reflecting hyperplanes H_τ and their intersections. The mapping P is neither injective nor surjective but it induces an analytic diffeomorphism of C onto the interior of $P(\mathbb{R}^n)$ and an homeomorphism that carries the stratification from the fundamental domain $\overline{C}$ onto $P(\mathbb{R}^n)$.

In the fundamental domain $\overline{C}$, the walls of the Weyl chamber C are contained in n hyperplanes $(H_\omega)_{\omega \in \Omega}$, where Ω is a subset of cardinal n in $\mathcal{R}$.

A stratum S of dimension k in $\overline{C}$ is the intersection of $(n-k)$ of the H_ω. The λ_τ that are linear combinations of the λ_ω vanishing on S will also vanish there, so that $p \geq n-k$ hyperplanes H_τ will intersect along S.

The points in S have the same isotropy subgroup W_S, generated by the reflections in the p hyperplanes H_τ containing S. In a neighborhood of S, since P is W_S invariant we can write $P = Q \circ V$, where Q is invertible and V is a Chevalley mapping for W_S.
We write $W_S = W^0 \times W^m \times \cdots \times W^\ell$ where W^0 is the identity on S and the W^ms are the irreducible components of W_S. We choose coordinates fitted to the orthogonal direct sum $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_\ell}$, and we have $V = \mathrm{Id}_k \times V_1 \times \cdots \times V_\ell$, where V_m is a Chevalley map associated with W^m.
The jacobian matrix J_V is block diagonal. Let $I_k, J_{V_1}, \ldots, J_{V_\ell}$ be the diagonal blocks, the determinants $|J_{V_1}|, \ldots, |J_{V_\ell}|$, all vanish on S. Moreover, the rank of the V_i on S is 0. The $k \times k$ minor $|I_k|$ in the upper left corner is 1 but all the bordering $(k+1) \times (k+1)$ minors vanish on S. The rank of V, which is the rank of the Chevalley mapping on S, is k the dimension of S.

When restricted to S, $P_{|S} = Q \circ V_{|S}$ is an analytic isomorphism on its image and onto each of its projections on the spaces $\mathbb{R}^\ell$, $k \leq \ell \leq n-1$.

3. The varieties $P_k^{-1}[P_k(x)]$.

We set $P_k = (p_1, p_2, \ldots, p_k) : \mathbb{R}^n \to \mathbb{R}^k$, and analogously $Q_k = (q_1, q_2, \ldots, q_k)$. We denote with Π_k the image $P_k(\mathbb{R}^n)$.

Let $m^k \in \Pi_k$, then

$$P_k^{-1}(m^k) = \{x \in \mathbb{R}^n ; p_1(x) = m_1, \ldots, p_k(x) = m_k\}$$

is a compact algebraic variety of co-dimension at most k.

Theorem 3.1. *For almost all m^k, the intersection of a fundamental domain $\overline{C}$ and $P_k^{-1}(m^k)$ is either empty or contractible.*

By Sard's lemma, for almost all m^k, the variety $P_k^{-1}(m^k)$ is a non singular manifold of dimension $n-k$, transversal to all strata in $\overline{C}$. If m^k is on the border of Π_k, the variety is singular. Even if m^k belongs to the interior of Π_k, it may be the image of both regular and singular points. In this case it is a singular image and $P_k^{-1}(m^k)$ is a singular manifold.

Lemma 3.2. *For almost all m_k, $P_k^{-1}(m^k)$ is a non singular manifold and on this manifold, p_{k+1} is a Morse function with no critical point on the strata of positive dimension in $P_k^{-1}(m^k) \cap \bar{C}$. Its critical points are the zero dimensional strata of this intersection, that is the non empty intersections of the manifold and the strata of dimension k in $\overline{C}$.*

See Lemma 1' - given in [7] without a complete proof.

Proof. On the non critical manifold $P_k^{-1}(m^k)$, there is a $k \times k$ minor of the matrix $\partial(p_1, \dots, p_k)/\partial(x_1, \cdots, x_n)$ that does not vanish and we may assume that it is the principal minor $\partial(p_1, \dots, p_k)/\partial(x_1, \dots, x_k)$ which is not 0.

Let $L = p_{k+1} - \mu_k(p_k - m_k) - \dots - \mu_1(p_1 - m_1)$ be the Lagrange function. The critical points of p_{k+1} on $P_k^{-1}(m^k)$ are the points where:

$$\frac{\partial p_{k+1}}{\partial x_i} - \mu_k \frac{\partial p_k}{\partial x_i} - \dots - \mu_1 \frac{\partial p_1}{\partial x_i} = 0, \quad \text{for } i = 1, \dots, n.$$

For this system to have solutions in the μ_is, $1 \le i \le k$, the $(k+1) \times (k+1)$ bordering minors $\partial(p_1, \dots, p_k, p_{k+1})/\partial(x_1, \dots, x_k, x_{k+s}),\ s = 1, \dots, n-k$ must all be zero.

Statement 3.3. *If a $k \times k$ minor*

$$\partial(p_1, \dots, p_k)/\partial(x_{i_1}, \dots, x_{i_k})$$

of the Jacobian of P does not vanish but the $(k+1) \times (k+1)$ minors

$$\partial(p_1, \dots, p_k, p_{k+1})/\partial(x_{i_1}, \dots, x_{i_k}, x_{i_j})$$

all vanish, then the rank of P is k.

If the group is crystallographic, we may assume without loss of generality that it is the principal minor

$$\partial(p_1, \dots, p_k)/\partial(x_1, \dots, x_k)$$

which is not 0 and then it is sufficient to show that if all the $(k+1) \times (k+1)$ minors

$$\partial(p_1, \dots, p_k, p_{k+1})/\partial(x_1, \dots, x_k, x_{k+s})$$

are 0, then so are all the $\partial(p_1, \dots, p_k, p_{k+s})/\partial(x_1, \dots, x_k, x_{k+1}),\ 1 \le s \le n-k$.

For the non-crystallographic groups H_3 and H_4 we should check that the statement is true when starting with any variables.

The statement, of interest by itself, is a consequence of the usual properties of the Van der Monde determinants for A_n and B_n. It also holds for D_n by the anti-symmetry of the minor determinants if we restrict ourselves as we may to a fundamental domain and when n is even, $n = 2k$, we take as k^{th} row in the jacobian determinant the row of derivatives of the x_i^n and as $k + 1^{st}$ row, the row of derivatives of $x_1 x_2 \dots x_n$. However in the general proof given below, for D_n it will be convenient to take $p_n = x_1 \dots x_n$ although it is not the invariant of maximal degree. For H_3, H_4 and F_4, the result has been checked by computations with Mathematica [1](see Appendix B).

Proof of the Statement.

[1]I am indebted to he University of Texas at Austin and Patrick Goetz from the Mathematics Department who provided me with the possibility of using Mathematica.

First observe that if all the $(k+1)\times(k+1)$ minors

$$\partial(p_1,\dots,p_{k+1})/\partial(x_1,\dots,x_k,x_{k+i}),\ i=1,\dots,n-k$$

are 0 then all the $(k+1)\times(k+1)$ minors $\partial(p_1,\dots,p_{k+1})/\partial(x_{j_1},\dots,x_{j_{k+1}})$ also vanish.

By Lagrange expansion along the $k+2^{nd}$ row, the

$$\partial(p_1,\dots,p_{k+2})/\partial(x_1,\dots,x_{k+1},x_{k+1+i}),\ i=1,\dots,n-k-1$$

vanish and so do all the $\partial(p_1,\dots,p_{k+2})/\partial(x_{j_1},\dots,x_{j_{k+2}})$.

By a finite induction, we get that the jacobian determinant itself is 0 and the rank is at most $n-1$. So we are on the closure of a stratum of dimension $n-1$, which is a fundamental domain for a group, say $W_{|H_i}$, acting on the hyperplane H_i of equation $\lambda_i(x)=0$ and generated by the reflections in the intersections of H_i and the other H_ω.

We consider a set of simple roots orthogonal to the H_ω. Let $y=(y_1,\dots,y_n)$ be the coordinates in this basis, we set $y=M(x)$ where M is the matrix of change of basis, the rows of which are the $(\lambda_s)_{s=1}^n$. We denote by $\overline{S_{n-1}^i}$ the closed stratum in H_i.

Let $U=(u_1,\dots,u_{n-1})$ be a system of basic invariants for $W_{|H_i}$. Since P restricted to H_i is also invariant by this group, on $\overline{S_{n-1}^i}$ we have: $P_{|\overline{S_{n-1}}}=R\circ U\circ M$, where R and M are invertible.

Denoting with $\hat{x}_i,\hat{y}_i$ the variables that do not show up, from the induction process we have:

$$\begin{gathered}\partial(p_1,\dots,p_{n-1})/\partial(x_1,\dots,\hat{x}_i,\dots,x_n)=\\ \partial(p_1,\dots,p_{n-1})/\partial(u_1,\dots,u_{n-1}).\partial(u_1,\dots,u_{n-1})/\partial(y_1,\dots,\hat{y}_i,\dots,y_n)\\ .\partial(y_1,\dots,\hat{y}_i,\dots,y_n)/\partial(x_1,\dots,\hat{x}_i,\dots,x_n)=0.\end{gathered}$$

On one hand $\partial(p_1,\dots,p_{n-1})/\partial(u_1,\dots,u_{n-1})\neq 0$ and on the other hand, for each Coxeter group there is an ordering of the simple roots, that is of the rows of the matrix M such that for $i=1,\dots,n$, $\partial(y_1,\dots,\hat{y}_i,\dots,y_n)/\partial(x_1,\dots,\hat{x}_i,\dots,x_n)\neq 0$, we then get:

$$\partial(u_1,\dots,u_{n-1})/\partial(y_1,\dots,\hat{y}_i,\dots,y_n)=0$$

.

As a consequence, for all $j=1,\dots,n$:

$$\begin{gathered}\partial(p_1,\dots,p_{n-1})/\partial(x_1,\dots\hat{x}_j,\dots,x_n)=\\ \partial(p_1,\dots,p_{n-1})/\partial(u_1,\dots,u_{n-1}).\partial(u_1,\dots,u_{n-1})/\partial(y_1,\dots,\hat{y}_i,\dots,y_{n-1})\\ .\partial(y_1,\dots,\hat{y}_i,\dots,y_{n-1})/\partial(x_1,\dots\hat{x}_j,\dots,x_n)=0,\end{gathered}$$

and

$$\begin{gathered}\partial(p_1,\dots,p_{n-2},p_n)/\partial(x_1,\dots\hat{x}_j,\dots,x_n)=\\ \partial(p_1,\dots,p_{n-2},p_n)/\partial(u_1,\dots,u_{n-1}).\partial(u_1,\dots,u_{n-1})/\partial(y_1,\dots,\hat{y}_i,\dots,y_{n-1})\\ .\partial(y_1,\dots,\hat{y}_i,\dots,y_{n-1})/\partial(x_1,\dots\hat{x}_j,\dots,x_n)=0.\end{gathered}$$

Hence the rank is at most $n-2$ on $\overline{S_{n-1}^i}$ and just the same on all the $\overline{S_{n-1}^j}$ in the walls H_j of $\overline{C}$, with $j=1,\dots,n$. So we are on the closure of a stratum of dimension $n-2$, say $\overline{S_{n-2}^m}$.

The process may be iterated until we reach a stratum of dimension k where the $k\times k$ determinant

$$\partial(p_1,\dots,p_k)/\partial(x_{i_1},\dots,x_{i_k})$$

is not 0 by assumption. On this stratum the rank cannot be more nor less than k, it is k and this achieves the proof of the statement. □

From Statement 3.3, the rank of P is k at the critical points of p_{k+1} on $P_k^{-1}(m^k)$, these points belong to strata S of dimension k in $\overline{C}$. They are points of 0-dimensional strata in the intersection $P_k^{-1}(m^k) \cap \overline{C}$.

If $P_k^{-1}(m^k)$ is not singular, strata of lower dimension do not intersect it since on such a stratum all $k \times k$ minors vanish. On strata of dimension $> k$, one of the $(k+1) \times (k+1)$ minors does not vanish and it must be one of the

$$\partial(p_1, \dots, p_k, p_{k+1})/\partial(x_1, \dots, x_k, x_{k+s}),\ s = 1, \dots, n-k$$

since if not, by statement 3.3, the rank would be $\leq k$. So p_{k+1} may not have a critical point on those strata.

Let us show that the critical points of p_{k+1} on $P_k^{-1}(m^k)$ are non degenerate. By an othogonal change of coordinates, we may assume that $\{x_1, \dots, x_k\}$ are the coordinates in S while $\{x_{k+1}, \dots, x_n\}$ are the coordinates in $S^\perp$.
At a critical point, for $1 \leq i \leq k, v_i = x_i$ and we have:

$$\frac{\partial p_{k+1}}{\partial x_i} - \mu_k \frac{\partial p_k}{\partial x_i} - \dots - \mu_1 \frac{\partial p_1}{\partial x_i} = \frac{\partial q_{k+1}}{\partial v_i} - \mu_k \frac{\partial q_k}{\partial v_i} - \dots - \mu_1 \frac{\partial q_1}{\partial v_i} = 0.$$

For $i \geq k$, on each $\mathbb{R}^{n_m}$, v^m is a Chevalley map for the irreducible component W_S^m of W_S, and we have:

$$\frac{\partial p_{k+1}}{\partial x_i} - \mu_k \frac{\partial p_k}{\partial x_i} - \dots - \mu_1 \frac{\partial p_1}{\partial x_i} = \sum_1^{n_m} \Big(\frac{\partial q_{k+1}}{\partial v_j} - \mu_k \frac{\partial q_k}{\partial v_j} - \dots - \mu_1 \frac{\partial q_1}{\partial v_j}\Big)\Big(\frac{\partial v_j}{\partial x_i}\Big) = 0,$$

but now since v_j is an homogeneous polynomial of degree at least 2, $\partial v_j/\partial x_i = 0$ on S.

The vectors $(\partial P/\partial x_1, \dots, \partial P/\partial x_k)$ span the k-dimensional plane tangent to $P(S)$. The $\partial P/\partial x_s, k+1 \leq s \leq n$ are linear combinations of them, hence the vanishing of the bordering minors.

The linearly independent vectors $\partial Q/\partial v_i$, $k+1 \leq i \leq n$, span the orthogonal complement of the tangent plane. Accordingly the minors of order $\geq k+1$ in the jacobian of Q do not vanish, and for $i \geq k+1$,

$$\frac{\partial q_{k+1}}{\partial v_i} - \mu_k \frac{\partial q_k}{\partial v_i} - \dots - \mu_1 \frac{\partial q_1}{\partial v_i} \neq 0.$$

When restricted to S, the second derivatives of L

$$\frac{\partial^2 p_{k+1}}{\partial x_i \partial x_j} - \mu_k \frac{\partial^2 p_k}{\partial x_i \partial x_j} - \dots - \mu_1 \frac{\partial^2 p_1}{\partial x_i \partial x_j}$$

gets no contribution for the variables x_i, x_j when $1 \leq i, j \leq k$. For each $\mathbb{R}^{n_m}$, we get:

$$\sum_{1 \leq r,t \leq n_m} \Big(\frac{\partial^2 q_{k+1}}{\partial v_r^m \partial v_t^m} - \mu_k \frac{\partial^2 q_k}{\partial v_r \partial v_t} - \dots - \mu_1 \frac{\partial^2 q_1}{\partial v_r \partial v_t}\Big)\Big(\frac{\partial v_r}{\partial x_i}\Big)\Big(\frac{\partial v_t}{\partial x_j}\Big) +$$
$$\Big(\frac{\partial q_{k+1}}{\partial v_r} - \mu_k \frac{\partial q_k}{\partial v_r} - \dots - \mu_1 \frac{\partial q_1}{\partial v_r}\Big)\Big(\frac{\partial^2 v_r}{\partial x_i \partial x_j}\Big).$$

As already mentioned the first derivatives $\partial v_r/\partial x_i$ all vanish and so do the mixed derivatives $\partial^2 v_r/\partial x_i \partial x_j$ either because v_r does not depend on x_i or x_j, or because it is an homogeneous polynomial of degree > 2 or a sum of squares of x_is. For the first two of these reasons, many terms in the pure derivatives also vanish. However for each $v_m : \mathbb{R}^{n_m} \to \mathbb{R}^{n_m}$, let v_m^1 be the first W_S^m-invariant which is the sum of the squares of the x_i in $\mathbb{R}^{n_m}$. We have $\partial^2 v_m^1/\partial x_i^2 = 2$, so that, with the above remark:

$$\frac{\partial^2 p_{k+1}}{\partial x_i^2} - \mu_k \frac{\partial^2 p_k}{\partial x_i^2} - \ldots - \mu_1 \frac{\partial^2 p_1}{\partial x_i^2} = 2\left(\frac{\partial q_{k+1}}{\partial v_m^1} - \mu_k \frac{\partial q_k}{\partial v_m^1} - \ldots - \mu_1 \frac{\partial q_1}{\partial v_m^1}\right) \neq 0.$$

For each W_S^i, the quadratic differential is definite with the sign of $\partial q_{k+1}/\partial v_m^1 - \mu_k \partial q_k/\partial v_m^1 - \ldots - \mu_1 \partial q_1/\partial v_m^1$ which may be different for different irreducible components. Overall the quadratic differential restricted to S, kernel of the first differential, is of the form $Q = \bigoplus_{i=1}^{l} Q_i$ where Q_i is a definite quadratic form on $\mathbb{R}^{n_i}$.

Accordingly $p_{k+1} = q_{k+1} \circ V$ is a Morse function on $P_k^{-1}(m^k)$. $\square$

By the equivariant Morse lemma [2], in the neighborhood of a critical point at the intersection of S and $P_k^{-1}(m^k)$, the restriction of p_{k+1} is W_S-locally equivalent to a W_S-invariant quadratic form which is the direct sum of definite quadratic forms in each $\mathbb{R}^{n_i}$. The pair (S, p_{k+1}) is locally equivalent to $(\mathbb{R}_+^u \oplus \mathbb{R}_+^v, Q_+ \oplus Q_-)$.

Lemma 3.4. [8] *The reconstruction of the topology of a level set of a function on the sum of quadrants $\mathbb{R}_+^a \oplus \mathbb{R}_+^b$ in the neighborhood of the critical point $0 \oplus 0$ with the quadratic differential $Q_+ \oplus Q_-$ is trivial if $a, b > 0$ and consists of the birth (death) of a simplex otherwise.*

Theorem 3.1 may now be proved by induction on k, see [7].

In particular, and this is what we need, for almost all $m^k \in \Pi_k$, $P_k^{-1}(m^k) \cap \overline{C}$ is connected.

Corollary 3.5. *Every variety $P_k^{-1}(m^k) \cap \overline{C}$, $k = 1, \ldots, n$, is connected or empty.*

The corollary may be derived from 3.1, exactly as in [8]. Basically if m^k is not in the regular image and $P_k^{-1}(m^k)$ is of dimension $n - s$, on the intersection with a stratum of dimension l the rank of P_k is s if $l \geq s$ and l if $l \leq s$. The intersection is of dimension $l - s$ in the first case and 0-dimensional in the second case. So, either they intersect transversally or the intersection is reduced to a finite number of points. When the intersection is transversal $P_k^{-1}(m^k)$ is connected since almost all the $P_k^{-1}(n^k)$ are, for n^k close enough to m^k. The second case is taken care of by an induction on k, using the following:

Lemma 3.6. [8] *Let K be a connected compact set in $\mathbb{R}^n$, and f be a function continuous on K. If all but a finite number of the level sets of f in K are connected, then they are all connected.*

We observe that if not empty$P_1^{-1}(m^1)$, intersection of $\overline{C}$ and a sphere, is connected and we assume that every non empty $P_k^{-1}(m^k) \cap \overline{C}$ is connected. If we study a $P_{k+1}^{-1}(m^{k+1})$ not covered by the previous cases, we apply 3.6 to the compact connected $P_k^{-1}(m^k) \cap \overline{C}$ containing it and p_{k+1} to reach the conclusion for the level set $P_{k+1}^{-1}(m^{k+1})$. $\square$

As a consequence, $P(P_k^{-1}(m^k))$ is connected or empty and we have:

Corollary 3.7. *The fibres of the projections: $\Pi_{k+1} \to \Pi_k$ are connected. They are points or intervals.*

4. Whitney 1-regularity of $P(\mathbb{R}^n)$

Let us consider the lift-up $(p_1, \dots, p_k) \mapsto p_{k+1}$ of the projection $P_k(S) \subset \Pi_k$ of a stratum of dimension k. The derivatives are obtained by solving the system:

$$\frac{\partial p_{k+1}}{\partial x_i} = \sum_{j=1}^{k} \frac{\partial p_{k+1}}{\partial p_j} \frac{\partial p_j}{\partial x_i}, \quad i = 1, \dots, k$$

by Cramer's method. Considering the subgroup of W generated by the reflections in the hyperplanes that do not contain S, for $j = 1, \dots, k$ the $\partial p_{k+1}/\partial p_j$, are quotients of two polynomials anti-invariant by this subgroup. So, they are rational fractions the numerator and the denominator of which are invariant homogeneous polynomials that do not vanish but at the origin. Since the degree of the numerator is greater than the degree of the denominator the rational fractions have continuous extensions on $\bar{S}$.

If we restrict ourselves to some compact subset K of $P(\mathbb{R}^n)$, determined for instance by $0 \leq p_1 \leq a$, the x_is and as a consequence the $\partial p_{k+1}/\partial p_j$, $j = 1, \dots, k$ are bounded on $\overline{S} \cap K$.

P is a homeomorphism of $\bar{S}$ onto its image $P(\bar{S})$, and so is P_k (on any compact it is continuous and one to one). Hence p_{k+1} which is continuous with respect to the variables $(x_1, \dots, x_k)$ on $\bar{S}$, is also continuous in the variables $(p_1, \dots, p_k)$ on $P_k(\bar{S})$ and moreover by the previous paragraph it is Lipschitz.

The border of Π_{k+1} is contained in the images $P_{k+1}(\bar{S})$ of closed strata of dimension k. These images are graphs of functions p_{k+1} on $P_k(\bar{S})$. By 3.7, the graph of one of the p_{k+1}, say $p_{k+1}^{\max}$, is above the others and another, say the graph of $p_{k+1}^{\min}$, is below the others. In Π_k, the images of the closure of strata of dimension k will intersect along the images of strata of lesser dimension. Above such points of intersection the mapping $p_{k+1}^{\max}$ (resp. $p_{k+1}^{\min}$) may and will change but globally $p_{k+1}^{\max}$ (resp. $p_{k+1}^{\min}$) will still be continuous and Lipschitz since the functions above the two strata are glued by their common value above the stratum of lesser dimension along which they intersect.

Now, it is easy to get for any two points A and B in a compact $K \subset P(\mathbb{R}^n)$ a continuous arc $AB \subset K$ of length $\ell(AB) \leq k_K|AB|$, following the method in [8] and already in [3].

We may also observe with [1] that the prism between graphs of Lipschitz functions over a Whitney 1-regular domain is Whitney 1-regular. Since Π_1 is Whitney 1-regular, by induction, assuming Π_k to be Whitney 1-regular, the prism Π_{k+1} over it and between the graphs of the Lipschitz functions $p_{k+1}^{\min}$ and $p_{k+1}^{\max}$ will also be Whitney 1-regular. Hence all the Π_k and in particular $\Pi_n = P(\mathbb{R}^n)$ are Whitney 1-regular and we can state:

Theorem 4.1. *The image of the Chevalley mapping is Whitney 1-regular.*

Appendix A. 1-regularity of $P(\mathbb{R}^2)$ for $I_2(k)$

The dihedral group $I_2(k)$ is generated by reflections in the two lines $x_2 = 0$ and $x_1 \sin\theta - x_2 \cos\theta = 0$ inclined at an angle $\theta = \pi/k$. A fundamental domain is the intersection of the half planes $x_2 \geq 0$ and $x_1 \sin\theta - x_2 \cos\theta \geq 0$.

The basic invariant polynomials are

$$p_1 = \sum_{j=1}^{k} (x_1 \cos 2j\theta + x_2 \sin 2j\theta)^2$$

$$p_2 = \sum_{j=1}^{k} (x_1 \cos 2j\theta + x_2 \sin 2j\theta)^k.$$

The image $P(\mathbb{R}^2)$ is homeomorphic to the fundamental domain, a parametric representation of its border (the singular image), is made of two branches, images of the reflection lines.

$$x_1 \geq 0,\ p_1 = x_1^2 \sum_{j=1}^{k} \cos^2 2j\theta,\ p_2 = x_1^k \sum_{j=1}^{k} \cos^k 2j\theta$$

$$x_1 \geq 0,\ p_1 = \frac{x_1^2}{\cos^2\theta} \sum \cos^2(2j-1)\theta,\ p_2 = \frac{x_1^k}{\cos^k\theta} \sum \cos^k(2j-1)\theta.$$

It is contained in $p_2^2 = A(\theta)p_1^k$ where $A(\theta)$ assumes two different values, one on each branch. The condition $p_1 \geq 0$ determines which connected component of the complement of the border is $P(\mathbb{R}^2)$.

Let A and B be two points in $P(\mathbf{R}^2)$. If the segment $[AB]$ is in $P(\mathbf{R}^2)$ there is nothing to prove. If not, let $[A_1B_1]$ be the intersection of $[AB]$ and the complement of $P(\mathbb{R}^2)$ (convexity shows that this intersection is a segment).

We replace the segment $[A_1B_1]$ by the arc $\widehat{A_1B_1}$ of the border. This is achieved by lifting up the projection of $[A_1B_1]$ on the p_1 axis by $p_1 \mapsto (p_1, \pm\sqrt{A(\theta)}p_2^{k/2})$ with the sign and the value of $A(\theta)$ determined by the extremities A_1 and B_1.

This lifting is Lipschitz since the derivative of p_2 is bounded by some C_K on the compact K (or rather its projection on the p_1-axis). Thus the length of $\widehat{A_1B_1}$ is estimated by $C_K d(A_1, B_1)$.

Finally the arc $[AA_1] \cup \widehat{A_1B_1} \cup [B_1B]$ is contained in $P(\mathbf{R}^2)$ and has a length which is less than or equal to $C_K d(A, B)$.

For any compact $K \subset P(\mathbb{R}^2)$ there exists a constant C_K such that any two points in K may be joined by an arc drawn in K of length $l \leq C_K d(A, B)$.

Appendix B. Statement 3.3 for D_n, F_4 and H_3

All the computations were achieved with Mathematica, using the invariants given in [9]. Below we give some of the main results and the methods we used.

B.1. D_n.

We consider the fundamental domain defined by the set of simple roots:

$$\alpha_n = e_{n-1} + e_n, \alpha_i = e_i - e_{i-1}, 1 \leq i \leq n-1.$$

In this fundamental domain, $x_1 \geq x_2 \geq ... \geq x_{n-1} \geq |x_n|$, so if $x_1x_2...x_k = 0$ and $k \leq n/2$, then $x_{k+1}...x_n$ is also 0. From this it is easy to check that the statement holds. For even $n = 2k$, the k^{th} row in the determinant should be the row of the x_i^{n-1} and the $(k+1)^{st}$ the row of derivatives of $x_1x_2 \dots x_n$.

B.2. F_4**.** We use the invariants:

$$p_1 = \sum_{1\le i<j\le 4} [(x_i + x_j)^2 + (x_i - x_j)^2]$$
$$p_2 = \sum_{1\le i<j\le 4} [(x_i + x_j)^6 + (x_i - x_j)^6]$$
$$p_3 = \sum_{1\le i<j\le 4} [(x_i + x_j)^8 + (x_i - x_j)^8]$$
$$p_4 = \sum_{1\le i<j\le 4} [(x_i + x_j)^{12} + (x_i - x_j)^{12}]$$

Rank 1.

We assume that one of the x_is, say x_1 for instance is not 0, but the $\partial(p_1,p_2)/\partial(x_i,x_j)$, $1 \le i < j \le 4$ are 0. As a matter of fact for this to happen it is sufficient that the $\partial(p_1,p_2)/\partial(x_1,x_j), 2 \le j \le 4$ be 0. Let us check that under these assumptions, $\partial(p_1,p_3)/\partial(x_1,x_2)$ and $\partial(p_1,p_4)/\partial(x_1,x_2)$ are both 0. Since F_4 is crystallographic, by symmetry this will be sufficient to prove the statement 3.3 for rank 1.

$$\partial(p_1,p_2)/\partial(x_1,x_2) = 288x_1x_2(x_1^2 - x_2^2)(x_1^2 + x_2^2 - 5x_3^2 - 5x_4^2)$$

We get $\partial(p_1,p_2)/\partial(x_1,x_3)$ and $\partial(p_1,p_2)/\partial(x_1,x_4)$ by replacing x_2 by x_3 and x_4 respectively.

$$\partial(p_1,p_3)/\partial(x_1,x_2) = 192x_1x_2(x_1^2 - x_2^2)A$$
$$A = 4x_1^4 + 18x_1^2x_2^2 + 4x_2^4 - 21x_1^2x_3^2 - 21x_2^2x_3^2 - 35x_3^4 - 21x_1^2x_4^2 - 21x_2^2x_4^2 - 35x_4^4$$

$$\partial(p_1,p_4)/\partial(x_1,x_2) = 288x_1x_2(x_1^2 - x_2^2)B$$
$$B = x_1^8 + 118x_1^6x_2^2 + 250x_1^4x_2^4 + 118x_1^2x_2^6 + 8x_2^8 - 55x_1^6x_3^2 - 55x_1^4x_2^2x_3^2 - 55x_2^6x_3^2 - 330x_1^4x_3^4$$
$$-330x_1^2x_2^2x_3^2 - 330x_2^4x_3^4 - 462x_1^2x_3^6 - 462x_2^2x_3^6 - 165x_3^8 - 55x_1^6x_4^2 - 55x_1^4x_2^2x_3^2 - 55x_1^2x_2^4x_4^2$$
$$-55x_2^6x_4^2 - 330x_1^4x_4^4 - 330x_1^2x_2^2x_4^4 - 330x_2^4x_4^4 - 462x_1^2x_4^6 - 462x_2^2x_4^6 - 165x_4^8$$

If $x_1x_2(x_1^2 - x_2^2) = 0$, both $\partial(p_1,p_3)/\partial(x_1,x_2)$ and $\partial(p_1,p_4)/\partial(x_1,x_2)$ vanish.

If not then $x_1^2 + x_2^2 - 5x_3^2 - 5x_4^2 = 0$ and for the other two minors to vanish there are several possibilities.

1) If $x_3 = 0$ and $x_1^2 = x_4^2$, then $x_1^2 + x_2^2 - 5x_3^2 - 5x_4^2 = x_2^2 - 4x_1^2 = 0$ and we can check that in this case both A and B vanish.

2) If $x_3 = 0$ and $x_1^2 + x_4^2 = 5x_2^2$, since $x_1^2 + x_2^2 = 5x_4^2$, we have $x_2^2 = x_4^2$ and then $x_1^2 = 4x_2^2$. Again we can check that A and B are 0.

Of course if we interchange the parts played by x_3 and x_4, we get the same results.

3) If $x_1^2 = x_3^2 = x_4^2$, $x_2^2 = 9x_1^2$, and $A = B = 0$.

4) Finally if

$$x_1^2 + x_2^2 - 5x_3^2 - 5x_4^2 = x_1^2 + x_3^2 - 5x_2^2 - 5x_4^2 = x_1^2 + x_4^2 - 5x_2^2 - 5x_3^2 = 0,$$

$x_2^2 = x_3^2 = x_4^2$, so $x_1^2 = 9x_2^2$ and again A and B are 0.

The other possibilities (*e.g.* $x_3 = x_4 = 0$) would bring that all the x_is are 0 and do not fit with the rank 1.

Rank 2.

Assuming the minors:

$\partial(p_1,p_2,p_3)/\partial(x_1,x_2,x_3) = -69120x_1x_2x_3(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2)A$

$A = x_1^4+x_2^4+x_3^4-2x_1^2x_2^2-2x_1^2x_3^2-2x_2^2x_3^2-6x_1^2x_4^2-6x_2^2x_4^2-6x_3^2x_4^2+21x_4^4,$

$\partial(p_1,p_2,p_3)/\partial(x_1,x_2,x_4) = -69120x_1x_2x_4(x_1^2-x_2^2)(x_1^2-x_4^2)(x_2^2-x_4^2)B$

$B = x_1^4+x_2^4+x_4^4-2x_1^2x_2^2-2x_1^2x_4^2-2x_2^2x_4^2-6x_1^2x_3^2-6x_2^2x_3^2-6x_3^2x_4^2+21x_3^4,$

$\partial(p_1,p_2,p_3)/\partial(x_2,x_3,x_4) = -69120x_2x_3x_4(x_2^2-x_3^2)(x_2^2-x_4^2)(x_3^2-x_4^2)C$

$C = 21x_1^4+x_2^4+x_3^4+x_4^4-6x_1^2(x_2^2+x_3^2+x_4^2)-2x_2^2x_3^2-2x_2^2x_4^2-2x_3^2x_4^2$

to vanish, we show that so does $\partial(p_1,p_2,p_4)/\partial(x_1,x_2,x_3)$.

Actually

$\partial(p_1,p_2,p_4)/\partial(x_1,x_2,x_3) = -34560x_1x_2x_3(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2)D$

$$\begin{aligned}D = {}& 25x_1^8+25x_2^8+25x_3^8+9x_1^6x_2^2+9x_2^6x_3^2+9x_1^2x_3^6+9x_2^2x_3^6+9x_1^2x_2^6+9x_1^6x_3^2-68x_1^4x_2^4\\ &-68x_1^4x_3^4-68x_2^4x_3^4-227x_1^4x_2^2x_3^2-227x_1^2x_2^4x_3^2-227x_1^2x_2^2x_3^4-118x_1^4x_2^4\\ &-536x_1^4x_2^2x_4^2-536x_1^2x_2^4x_4^2-118x_2^6x_4^2-536x_1^4x_3^2x_4^2-822x_1^2x_2^2x_3^2x_4^2-536x_2^4x_3^2x_4^2\\ &-536x_1^2x_3^4x_4^2-536x_2^2x_3^4x_4^2-118x_3^6x_2^4+385x_1^4x_4^4+319x_1^2x_2^2x_4^4+385x_2^2x_4^4+319x_1^2x_3^2x_4^4\\ &+319x_2^2x_3^2x_4^4+385x_3^4x_4^4+792x_1^2x_4^6+792x_2^2x_4^6+792x_3^2x_4^6+660x_4^8\end{aligned}$$

If $x_1x_2x_3(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2) = 0$, we are done.
If not then $A = 0$ and there are several possibilities for the second minor to be 0.

1)If $x_4 = 0$, or $x_1^2 = x_4^2$, or $x_2^2 = x_4^2$, under any of these conditions A and D factor, A divides D and $A = 0$ implies $D = 0$.

2)If $B = 0$, since $A-B = 4(x_3^2-x_4^2)(x_1^2+x_2^2-5x_3^2-5x_4^2) = 0$, either $x_3^2-x_4^2 = 0$ or $x_1^2+x_2^2-5x_3^2-5x_4^2 = 0$.
In the first case A divides D.
In the second case we consider the third minor which is also 0.
Since $A-C = -4(x_1^2-x_4^2)(5x_1^2+5x_4^2-x_2^2-x_3^2)$, in any case $\partial(p_1,p_2,p_4)/\partial(x_1,x_2,x_3)$ vanishes, either because $x_1^2-x_3^2 = 0$, or because A divides D when $x_1^2 = x_4^2$ or $x_3^2 = x_4^2$.

B.3. H_3**.** The invariants from [9] are:

$p_1(x) = (gx_1+x_2)^2+(gx_1-x_2)^2+(gx_2+x_3)^2+(gx_2-x_3)^2+(gx_3+x_1)^2+(gx_3-x_1)^2$

$p_2(x) = (gx_1+x_2)^6+(gx_1-x_2)^6+(gx_2+x_3)^6+(gx_2-x_3)^6+(gx_3+x_1)^6+(gx_3-x_1)^6$

$p_3(x) = (gx_1+x_2)^{10}+(gx_1-x_2)^{10}+(gx_2+x_3)^{10}+(gx_2-x_3)^{10}+(gx_3+x_1)^{10}+(gx_3-x_1)^{10}$

where g is the golden ratio $(1+\sqrt{5}/2$. We may assume that we are in the fundamental domain defined by $x_2 \geq 0, x_3 \geq 0, g^{-1}x_1 - x_2 - gx_3 \geq 0$.

We just have to study the case when the rank is one. Assume that one of x_1, x_2, x_3, say x_1 for instance is not 0, and the 2×2 determinants $\partial(p_1,p_2)/\partial(x_i,x_j)$, $1 \leq i < j \leq 3$ all vanish. As a matter of fact if both $\partial(p_1,p_2)/\partial(x_1,x_2)$ and $\partial(p_1,p_2)/\partial(x_1,x_3)$ vanish, then so does the third one, $\partial(p_1,p_2)/\partial(x_2,x_3)$. For the computations however, it may be convenient to consider the three of them.

Let us show that under these assumptions the determinants $\partial(p_1,p_3)/\partial(x_i,x_j)$, $1 \leq i < j \leq 3$, vanish. This will bring that the $\partial(p_2,p_3)/\partial(x_i,x_j), 1 \leq i < j \leq 3$ also vanish and that the rank of the jacobian determinant is 1.

$$\partial(p_1,p_2)/\partial(x_1,x_2) = 12(5+\sqrt{5})x_1x_2A$$

$$A = (15+7\sqrt{5})x_1^4+(5+3\sqrt{5})x_2^4+10(7+3\sqrt{5})x_2^2x_3^2-10(2+\sqrt{5})x_3^4-10x_1^2(2(2+\sqrt{5})x_2^2+(3+\sqrt{5})x_3^2$$

$$\partial(p_1,p_2)/\partial(x_1,x_3) = -12(5+\sqrt{5})x_1x_3B$$

$B = (15+7\sqrt{5})x_1^4-10(2+\sqrt{5})x_2^4-10(3+\sqrt{5})x_2^2x_3^2+(15+7\sqrt{5})x_3^4+10x_1^2((7+3\sqrt{5})x_2^2-2(2+\sqrt{5})x_3^2$

We assume that $\partial(p_1,p_2)/\partial(x_1,x_2) = \partial(p_1,p_2)/\partial(x_1,x_3) = 0$ and we check if $\partial(p_1,p_3)/\partial(x_1,x_2) = 0$. Since H_3 is not crystallographic, we also checked $\partial(p_1,p_3)/\partial(x_1,x_3)$, but the computations are very similar.

If $x_1x_2 = 0$, the result is clear since x_1x_2 is a factor in $\partial(p_1,p_3)/\partial(x_1,x_2)$.

If $x_1x_2 \neq 0$, $A = 0$ and there are 2 possibilities for the other minors to vanish:

1) $x_3 = 0$, then

$$\partial(p_1,p_2)/\partial(x_1,x_2)(x_1,x_2,0) = ax_1x_2(x_1+gx_2)(x_1-gx_2)(x_1+g^{-2}x_2)(x_1-g^{-2}x_2).$$

If any of the last 4 factors vanishes, we are on a 1-dimensional stratum where the rank of the jacobian is 1.

2) The last case, $A = B = 0$, is not possible in the fundamental domain since each cone $A = 0$ and $B = 0$ intersects the fundamental domain along a different edge of the pyramid. This entails that $x_3 = 0$ or $x_2 = 0$ and that the rank is 1.

Since H_3 is not crystallographic, we also checked the cases $x_2 \neq 0$ and $x3 \neq 0$, and the computations were quite similar.

For H_4, the computations and the results are analogous but the computations are heavier in particular when studying the rank 2.

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