Exponentially small lower bounds for the splitting of separatrices to whiskered tori with frequencies of constant type *

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We study the splitting of invariant manifolds of whiskered tori with two frequencies in nearly-integrable Hamiltonian systems, such that the hyperbolic part is given by a pendulum. We consider a 2-dimensional torus with a fast frequency vector $\omega/\sqrt{\varepsilon}$, with $\omega = (1, \Omega)$ where Ω is an irrational number of constant type, i.e. a number whose continued fraction has bounded entries. Applying the Poincaré–Melnikov method, we find exponentially small lower bounds for the maximal splitting distance between the stable and unstable invariant manifolds associated to the invariant torus, and we show that these bounds depend strongly on the arithmetic properties of the frequencies.

Keywords: splitting of separatrices, Melnikov integrals, numbers of constant type.

1. Introduction

The aim of this paper is to introduce a methodology for measuring the exponentially small splitting of separatrices in a perturbed Hamiltonian system, associated to a 2-dimensional whiskered torus (invariant hyperbolic torus), with *fast frequencies*:

$$\omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}, \quad \omega = (1, \Omega), \quad \varepsilon > 0. \tag{1}$$

This phenomenon requires a careful study due to the *singular* character of the problem with respect to the perturbation parameter ε and, on the other hand, to the presence of *small divisors* associated to the frequencies of the torus. For this reason, the first results on asymptotic estimates for the splitting are recent, and have been obtained assuming concrete frequency ratios Ω, such as the golden mean, and other quadratic irrational numbers. In this paper, we show that a partial generalization of such techniques allows us to obtain *lower bounds* for the splitting, for *all* sufficiently small values of ε, assuming that the frequency ratio is an irrational *number of constant type* (also called a badly approximable number), i.e. a number whose continued fraction has bounded entries. In this way, we establish the existence of splitting for a much wider (uncountable) class of frequency ratios.

As the unperturbed system, we consider an integrable Hamiltonian H_0 with 3 degrees of freedom having 2-dimensional whiskered tori with coincident stable and unstable whiskers (invariant manifolds). In

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general, for a perturbed Hamiltonian

$$H = H_0 + \mu H_1 \tag{2}$$

where μ is small, the stable and unstable whiskers of a given torus do not coincide anymore, giving rise to the phenomenon called *splitting of separatrices*, discovered by Poincaré [Poincaré, 1890]. In order to give a measure for the splitting, one often describes it by a periodic vector function $\mathcal{M}(\theta)$, $\theta \in \mathbb{T}^2$, usually called *splitting function*, giving the distance between the invariant manifolds in the complementary directions, on a transverse section. The most popular tool to measure the splitting is the *Poincaré–Melnikov method*, introduced in [Poincaré, 1890] and rediscovered later [Melnikov, 1963; Arnold, 1964]. This method provides a first order approximation

$$\mathcal{M}(\theta) = \mu M(\theta) + \mathcal{O}(\mu^2), \tag{3}$$

where $M(\theta)$ is called the *Melnikov function* and is defined by an integral. In fact, it was established [Eliasson, 1994; Delshams & Gutiérrez, 2000] that both vector functions are the gradients of scalar functions: the *splitting potential* and the *Melnikov potential*, denoted $\mathcal{L}(\theta)$ and $L(\theta)$ respectively. This result implies the existence of *homoclinic orbits* (i.e. intersections between the stable and unstable whiskers) in the perturbed system.

We consider in (2) a whiskered torus with fast frequencies as in (1), assuming a relation between the parameters ε and μ of the form $\mu = \varepsilon^p$ for some $p > 0$, having in this way a singular problem. The interest for such a setting lies in its relation to the normal form of a nearly-integrable Hamiltonian, with ε as the perturbation parameter, in the vicinity of a simple resonance [Niederman, 2000; Delshams & Gutiérrez, 2001].

In such a singular problem, one can show that the splitting is *exponentially small* with respect to ε. The first results on exponentially small splitting concerned the case of one and a half degrees of freedom, i.e. for 1 frequency, providing upper bounds [Neishtadt, 1984]. The problem of establishing *asymptotic estimates*, or at least *lower bounds*, for the exponentially small splitting, is more difficult, due to the fact that the Melnikov function is exponentially small in ε and the error of the method could overcome the main term in (3). The first result justifying the Poincaré–Melnikov method and, hence, providing an asymptotic estimate for the exponentially small splitting was obtained in [Lazutkin, 2003] for the Chirikov standard map. Later, this was extended to the case of a Hamiltonian with one and a half degrees of freedom [Delshams & Seara, 1992, 1997; Gelfreich, 1997] or an area-preserving map [Delshams & Ramírez-Ros, 1998]. In the quoted papers specific perturbations H_1 were assumed, but a more general (meromorphic) perturbation was recently considered in [Guardia & Seara, 2012]. It is worth remarking that, in some cases, the Poincaré–Melnikov method does not predict correctly the size of the splitting, as shown for instance in [Baldomá *et al.*, 2012].

For 2 or more frequencies, it turns out that *small divisors* appear in the splitting function and, as first noticed in [Lochak, 1992], the arithmetic properties of the frequency vector ω play an important rôle. This was established in [Simó, 1994], and rigorously proved in [Delshams *et al.*, 1997] for the quasi-periodically forced pendulum. A different technique was used in [Lochak *et al.*, 2003] (see also [Rudnev & Wiggins, 2000]), namely the parametrization of the whiskers as solutions of Hamilton–Jacobi equation, to obtain exponential small estimates of the splitting, and the existence of transverse homoclinic orbits for some intervals of the perturbation parameter ε. Moreover, it was shown in [Delshams & Gutiérrez, 2004] the continuation of the exponentially small estimates and the transversality of the splitting, for all sufficiently small values of ε, under a certain condition on the phases of the perturbation. Otherwise, homoclinic bifurcations can occur, studied in [Simó & Valls, 2001] for the Arnold's example. The quoted papers considered the case of 2 frequencies, and assuming in most cases that the frequency ratio is the famous *golden mean*: $\Omega = (\sqrt{5} - 1)/2$. A generalization to some other 2-dimensional quadratic frequencies as well to 3-dimensional cubic frequencies was studied in [Delshams & Gutiérrez, 2003; Delshams *et al.*, 2013], and the case of 2-dimensional frequencies of constant type had also been considered in [Rudnev & Wiggins, 1998]. For a more complete background and references concerning exponentially small splitting, see for instance [Lochak *et al.*, 2003; Delshams *et al.*, 2004].

In the present paper we pay attention to non-quadratic frequency vectors (1) of constant type. It turns out that the methodology developed in [Delshams & Gutiérrez, 2003; Delshams *et al.*, 2013] can be partially applied and that, using arithmetic properties of numbers of constant type, we obtain exponentially small *lower bounds* for the *maximal splitting distance* (and, consequently, show the existence of splitting).

It is well-known that the property of being a number of constant type is equivalent to satisfying a *Diophantine condition* with the minimal exponent (see, for instance, [Lang, 1995]): there exists $\nu > 0$ such that

$$|q\Omega - p| \geq \frac{\nu}{q}, \qquad \forall p, q \in \mathbb{Z},\ q \geq 1. \tag{4}$$

This condition can also be expressed in terms of the vector $k = (-p, q)$: for some $\gamma > 0$,

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{|k|}, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}. \tag{5}$$

One of the goals of this paper is to show, for the above frequencies, that we can detect the integer vectors $k \in \mathbb{Z}^2 \setminus \{0\}$ that fit better the inequality in (5). We show that the "least" small divisors (relatively to the size of $|k|$) are related to *principal convergents* p_n/q_n of the continued fraction of Ω, and we call such vectors $k = v(n) = (-p_n, q_n)$ the *resonant convergents* of ω. This allows us to detect the dominant harmonic in the splitting function $\mathcal{M}(\theta)$, for each small enough value of the perturbation parameter ε.

In the main result of this paper (see Theorem 1), we establish exponentially small lower bounds for the *maximal splitting distance*, valid in the case of frequencies of constant type, giving in this way a partial generalization of the asymptotic estimates obtained in [Delshams *et al.*, 2013] for quadratic frequencies. We point out that the set of irrational numbers of constant type is uncountable, in contrast to quadratic irrational numbers that form a countable set. We also stress that, for some purposes, it is not necessary to establish the transversality of the splitting, and can be enough to provide lower bounds of the maximal splitting distance. Indeed, such lower bounds imply the existence of splitting between the invariant manifolds, which provides a strong indication of the non-integrability of the system near the given torus, and opens the door to the application of topological methods [Gidea & Llave, 2006] for the study of Arnold diffusion in such systems.

1.1. *Setup and main result*

In order to formulate our main result, let us describe the Hamiltonian considered, which is analogous to the one considered in [Delshams *et al.*, 2004] and other related works. In symplectic coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$, we consider a perturbed Hamiltonian (2), with H_0, H_1 of the form

$$H_0(x, y, I) = \langle \omega_\varepsilon, I\rangle + \frac{1}{2}\langle \Lambda I, I\rangle + \frac{y^2}{2} + \cos x - 1, \tag{6}$$

$$H_1(x, \varphi) = h(x)\, f(\varphi), \tag{7}$$

$$h(x) = \cos x, \qquad f(\varphi) = \sum_{\substack{k\in\mathbb{Z}^2\\ k_2\geq 0}} \mathrm{e}^{-\rho|k|} \cos(\langle k, \varphi\rangle - \sigma_k), \tag{8}$$

where the restriction in the sum is introduced in order to avoid repetitions. This Hamiltonian is a generalization of the Arnold example (introduced in [Arnold, 1964] to illustrate the transition chain mechanism in Arnold diffusion). It provides a particular model for the behavior of a nearly-integrable Hamiltonian system (with ε as the perturbation parameter) in the vicinity of a *simple resonance*, after carrying out one step of resonant normal form, and a rescaling that gives rise to the fast frequencies (1) (see [Niederman, 2000; Delshams & Gutiérrez, 2001] for details). The parameters ε and μ should not be regarded as independent, but linked by a relation of the type $\mu = \varepsilon^p$.

Notice that the unperturbed system H_0 consists of the pendulum given by $P(x, y) = y^2/2 + \cos x - 1$ and 2 rotors with fast frequencies: $\dot\varphi = \omega_\varepsilon + \Lambda I$, $\dot I = 0$. The pendulum has a hyperbolic equilibrium at the origin, and the (upper) separatrix can be parametrized by $(x_0(s), y_0(s)) = (4\arctan \mathrm{e}^s, 2/\cosh s)$, $s \in \mathbb{R}$. The rotors system (φ, I) has the solutions $\varphi = \varphi_0 + (\omega_\varepsilon + \Lambda I_0)\, t$, $I = I_0$. Consequently, H_0 has a 2-parameter family of 2-dimensional whiskered invariant tori, with coincident stable and unstable whiskers. Among the family of whiskered tori, we will focus our attention on the torus located at $I = 0$, whose frequency vector is ω_ε as in (1), in our case a frequency vector of constant type. We also assume the condition of *isoenergetic nondegeneracy*

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \tag{9}$$

When adding the perturbation μH_1, the *hyperbolic KAM theorem* can be applied (see for instance [Niederman, 2000]) thanks to the Diophantine condition (5) and the isoenergetic nondegeneracy (9). For μ small enough, the whiskered torus persists with some shift and deformation, as well as its local whiskers.

In general, for $\mu \neq 0$ the (global) whiskers do not coincide anymore, and one can introduce a *splitting function* giving the distance between the whiskers in the directions of the action coordinates $I \in \mathbb{R}^2$: denoting $\mathcal{J}^\pm(\theta)$ parameterizations of a transverse section of both whiskers, one can define $\mathcal{M}(\theta) := \mathcal{J}^-(\theta) - \mathcal{J}^+(\theta)$, $\theta \in \mathbb{T}^2$ (see [Delshams & Gutiérrez, 2000, §5.2], and also [Eliasson, 1994]). This function turns out to be the gradient of the (scalar) *splitting potential*: $\mathcal{M}(\theta) = \nabla \mathcal{L}(\theta)$. Notice that the nondegenerate critical points of $\mathcal{L}$ correspond to simple zeros of $\mathcal{M}$ and give rise to transverse homoclinic orbits to the whiskered torus.

Applying the Poincaré–Melnikov method, the first order approximation (3) is given by the (vector) *Melnikov function* $M(\theta)$, which is the gradient of the *Melnikov potential*: $M(\theta) = \nabla L(\theta)$. The latter one can be defined by integrating the perturbation H_1 along a trajectory of the unperturbed homoclinic manifold, starting at the point of the section $s = 0$ with a given phase θ:

$$L(\theta) = -\int_{-\infty}^{\infty} [h(x_0(t)) - h(0)] f(\theta + \omega_\varepsilon t)\, dt. \tag{10}$$

In order to emphasize the rôle played by the arithmetic properties of the frequencies, we have chosen for the perturbation the special form given in (7–8). This form was considered in [Lochak *et al.*, 2003; Delshams & Gutiérrez, 2004], and allows us to deal with the Melnikov function and obtain asymptotic estimates for the splitting. Notice that the constant $\rho > 0$ in the Fourier expansion of $f(\varphi)$ in (8) gives the complex width of analyticity of this function. The phases σ_k can be chosen arbitrarily for our purpose in this paper, since we are focused in lower bounds for the splitting (instead, some restriction on the phases has to be imposed in order to the study the transversality of the splitting, as in [Delshams & Gutiérrez, 2004]).

Now we can formulate our main result, providing a lower estimate for the *maximal splitting distance* in the case of frequencies of constant type. We use the notation $|f| \succeq |g|$ if we can bound $|f| \geq a|g|$ with some positive constant a not depending on ε, μ.

Theorem 1 [main result]. *For the Hamiltonian system introduced in (2), (6–8) with 3 degrees of freedom, satisfying the isoenergetic condition (9), assume that ε is small enough and $\mu = \varepsilon^p$ with $p > 3$. If Ω in (1) is a number of constant type, then the following lower bound holds:*

$$\max_{\theta \in \mathbb{T}^2} |\mathcal{M}(\theta)| \succeq \frac{\mu}{\varepsilon^{1/2}} \exp\left\{-\frac{C}{\varepsilon^{1/4}}\right\}, \tag{11}$$

where $C = C(\Omega, \rho)$ is a positive constant, defined in (32).

In the proof of this result, we put emphasis on the constructive part of the proofs, using the arithmetic properties of the frequencies in order to provide a methodology which can be applied to the case of numbers of constant type, stressing the similarities and differences with the quadratic frequencies considered in [Delshams *et al.*, 2013]. We show that, for a given ε small enough, the dominant harmonic of the Melnikov function $M(\theta)$ can be related to a resonant convergent, and, consequently, we obtain an estimate for the maximal value of this function. In a further step, the first order approximation can be validated showing that the dominant harmonics of the splitting function $\mathcal{M}(\theta)$ correspond to the dominant harmonics of the Melnikov function, as done in [Delshams & Gutiérrez, 2004].

The paper is organized as follows. In Section 2 we study the arithmetic properties of irrational numbers of constant type, and in Section 3 we find, for frequencies of constant type, the dominant harmonic of the splitting potential, whose size allows us to provide a lower bound for the maximal splitting distance, as established in Theorem 1.

2. Arithmetic properties of numbers of constant type

2.1. *Continued fractions and principal convergents*

Let $0 < \Omega < 1$ be an irrational number. It is well-known that it has an infinite continued fraction

$$\Omega = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}\,, \qquad a_n \in \mathbb{Z}^+,\ n \geq 1 \quad (\text{and } a_0 = 0). \tag{12}$$

Its entries, which are integers $a_n \geq 1$, are called the *partial quotients* of the continued fraction. It is also well-known that the rational numbers $\dfrac{p_n}{q_n} = [a_1, \ldots, a_n]$, $n \geq 1$, called the *(principal) convergents* of Ω, provide successive best rational approximations to Ω. Thus, if we consider the vectors $w(n) := (q_n, p_n)$, we obtain approximations to the direction of the vector $\omega = (1, \Omega)$ (see, for instance, [Schmidt, 1980] and [Lang, 1995] as general references on continued fractions).

Now, in order to obtain approximations to the orthogonal line $\langle \omega \rangle^\perp$, i.e. to the *quasi-resonances* of ω, we introduce the following sequence of vectors, that we call the *"resonant convergents"*:

$$v(n) := (-p_n, q_n). \tag{13}$$

We see from the standard recurrences

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2}\,, &\quad q_0 &= 1, &\quad q_{-1} &= 0, \\ p_n &= a_n p_{n-1} + p_{n-2}\,, &\quad p_0 &= a_0 = 0\,, &\quad p_{-1} &= 1, \end{aligned}$$

that the vectors $w(n)$ and $v(n)$ are also given by recurrence relations:

$$w(n) = a_n w(n-1) + w(n-2), \qquad w(0) = (1,0),\ \ w(-1) = (0,1), \tag{14}$$

$$v(n) = a_n v(n-1) + v(n-2), \qquad v(0) = (0,1),\ \ v(-1) = (-1,0). \tag{15}$$

The following result provides alternative expressions for $w(n)$ and $v(n)$ in terms of products of unimodular matrices. We point out that similar products for $w(n)$ appear in [Liardet & Stambul, 1998; Stambul, 2000].

Proposition 1. *Let $0 < \Omega < 1$ be irrational. We define $A_m = \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix}$, $m \geq 1$, where a_m are the partial quotients of the continued fraction (12). Then, for $n \geq 1$ one has:*

(a) $w(n) = A_1 \cdots A_n w(0), \quad w(0) = (1,0),$

(b) $v(n) = (-1)^n A_1^{-1} \cdots A_n^{-1} v(0), \quad v(0) = (0,1).$

Proof. It is enough to prove the relations

$$A_1 \cdots A_n = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}, \qquad A_1^{-1} \cdots A_n^{-1} = (-1)^n \begin{pmatrix} p_{n-1} & -p_n \\ -q_{n-1} & q_n \end{pmatrix}, \tag{16}$$

since the right multiplication of each equation with $w(0)$ and $v(0)$, respectively, implies the assertions (a) and (b). To prove the equalities (16), we use induction and use the recurrence relations (14–15). For $n = 1$, the equalities are easily verified. Assuming that the equalities are true for n we prove them for $n+1$:

$$A_1 \cdots A_n A_{n+1} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_{n+1} & q_n \\ p_{n+1} & p_n \end{pmatrix},$$

$$A_1^{-1} \cdots A_n^{-1} A_{n+1}^{-1} = (-1)^n \begin{pmatrix} p_{n-1} & -p_n \\ -q_{n-1} & q_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a_{n+1} \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} p_n & -p_{n+1} \\ -q_n & q_{n+1} \end{pmatrix}.$$

$\square$

2.2. *Numbers of constant type*

In this paper we are interested in the study of arithmetic properties of the *numbers of constant type*, i.e. irrational numbers whose continued fraction has bounded partial quotients. Such numbers are also called *badly approximable*. We refer to [Schmidt, 1980], [Lang, 1995] and to the survey [Shallit, 1992] as general references.

Some explicit examples of non-quadratic numbers of constant type appear in [Shallit, 1979, 1992]. For instance, for the number

$$\Omega = 2\sum_{k=1}^{\infty} 2^{-2^k} = [1,1,1,2,1,1,1,1,1,1,1,2,1,1,1,\ldots] \simeq 0.632843018043786, \tag{17}$$

all partial quotients are 1 or 2, and can be obtained using a recurrence relation, given in [Shallit, 1979, Th. 11].

The property of being a number of constant type is equivalent to the Diophantine condition with the minimal exponent (5). In view of this, we define as in [Delshams & Gutiérrez, 2003] the following *"numerators"*:

$$\gamma_k := |\langle k,\omega\rangle| \cdot |k| \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\} \tag{18}$$

(for integer vectors, we use the norm $|\cdot| = |\cdot|_1$, i.e. the sum of absolute values of the components of the vector). Our aim is to study the integer vectors k which give the smallest values γ_k. We denote

$$\gamma^* := \liminf_{|k|\to\infty} \gamma_k > 0. \tag{19}$$

We can also define analogous numerators from the expression (4) of the Diophantine condition, as well as their associated asymptotic value (see also [Cassels, 1957; Schmidt, 1980]):

$$\nu_q := q\,\|q\Omega\| \quad \forall q \geq 1, \qquad \nu^* := \liminf_{q\to\infty} \nu_q > 0, \tag{20}$$

where we denote $\|a\| := |a - \mathrm{rint}(a)| = \min_{p\in\mathbb{Z}} |a-p|$, i.e. the distance to the closest integer. It is easy to check that $\nu^* = \gamma^*/(1+\Omega)$. Indeed, writing $k = (-p,q)$ in (5), we have that, for a fixed $q \geq 1$, the small divisor $|\langle k,\omega\rangle| = |q\Omega - p|$ is minimized for $p = \mathrm{rint}(q\Omega)$, and we have $|k| = p+q \approx (1+\Omega)q$, as $q \to \infty$.

Quadratic numbers. We give a brief summary of the results of [Delshams & Gutiérrez, 2003], concerning quadratic irrational numbers (i.e. real roots of quadratic polynomials with integer coefficients). Notice that all quadratic numbers belong to the class of numbers of constant type. Indeed, this is a consequence of the well-known fact that for a quadratic number Ω, the continued fraction (12) is eventually periodic, with some period $m \geq 1$, that is, there exists $l \geq 1$ such $a_{n+m} = a_n$ for $n \geq l$.

A technique for studying resonances for quadratic frequencies was developed in [Delshams & Gutiérrez, 2003], where the periodicity of the continued fraction of quadratic numbers was used to construct a unimodular matrix T having the vector $\omega = (1,\Omega)$ as an eigenvector with eigenvalue $\lambda > 1$. Clearly, the matrix T provides approximations to the direction of ω. Then, the associated quasi-resonances are given by the matrix $U := (T^{-1})^\top$. In fact, the study can be restricted to the case of purely periodic numbers $\Omega = [\overline{a_1,\ldots,a_m}]$. Otherwise for a quadratic number with non-purely periodic continued fraction $\hat\Omega = [b_1,\ldots,b_l,\overline{a_1,\ldots,a_m}] = [b_1,\ldots,b_l,\Omega]$ a linear change given by a unimodular matrix can be done between $\omega = (1,\Omega)$ and $\hat\omega = (1,\hat\Omega)$. If $\Omega \in (0,1)$ is a quadratic irrational number with a purely m-periodic continued fraction, then $T = A_1 \cdots A_m$ and $U = A_1^{-1} \cdots A_m^{-1}$, where the matrices A_n have been introduced in Proposition 1. All the integer vectors $k \in \mathbb{Z}^2 \setminus \{0\}$ with $|\langle k,\omega\rangle| < 1/2$ can be subdivided into *resonant sequences*:

$$s(j,n) := U^n k^0(j), \qquad n = 0,1,2,\ldots \tag{21}$$

where the initial vector $k^0(j) = (-\mathrm{rint}(j\Omega), j)$, $j \in \mathbb{Z}^+$, satisfies

$$\frac{1}{2\lambda} < |\langle k^0(j),\omega\rangle| < \frac{1}{2}. \tag{22}$$

For each $j \in \mathbb{Z}^+$ satisfying (22), it was proved in [Delshams & Gutiérrez, 2003, Th. 2] (see also [Delshams *et al.*, 2013]) that, asymptotically, the resonant sequence $s(j,n)$ exhibits an geometric growth as $n \to \infty$, with ratio λ, and that the sequence of the numerators $\gamma_{s(j,n)}$ has a limit γ_j^*. Since the lower bounds for γ_j^*, also provided in [Delshams & Gutiérrez, 2003], are increasing in j, we can select the minimal of them corresponding to some j_0 and, thus, get the value for γ^* defined in (19). The integer vectors of the corresponding sequence $s(j_0,n)$ are called *the primary resonances*, and the *secondary resonances* are the integer vectors belonging to any of the remaining sequences $s(j,n)$, $j \neq j_0$.

It is also easy to deduce that $Uv(n) = (-1)^m v(n+m)$, where $v(n)$ is defined in (13). This implies that for quadratic frequencies the sequence of resonant convergents $v(n)$ is divided into m resonant sequences (21) (the sign is not relevant).

Non-quadratic numbers of constant type. If Ω is a number of constant type, non-quadratic, there is no periodicity in its continued fraction and, hence, we cannot construct a matrix like U, or a sequence analogous to the primary resonances. Alternatively, we can use in this case the sequence of resonant convergents $v(n)$, defined in (13), to obtain some similar results.

In the next result we provide an upper and a lower bound for ν^* in (20), and, hence, for γ^* in (19), and we show that it can be obtained by restricting q to the denominators q_n of the convergents.

Lemma 1. *For the numbers $0 < \Omega < 1$ of constant type, we have*

$$\frac{1}{3} \leq \nu^* = \liminf_{n\to\infty} \nu_{q_n} \leq \frac{1}{\sqrt{5}}, \qquad \limsup_{n\to\infty} \nu_{q_n} \leq 1.$$

Proof. We use several results in [Schmidt, 1980, §I.5] (namely, Theorems I.5A, I.5B, I.5C and Lemma I.3E), concerning the properties of the convergents of any irrational number. On one hand, for a given q and $p = \mathrm{rint}(q\Omega)$, if $|q\Omega - p| < 1/2q$ and p/q is a reduced fraction, i.e. $\nu_q < 1/2$, then q is a convergent q_n, and an infinite number of convergents satisfies such inequality (otherwise, if it is not a reduced fraction then we can write $p/q = p_n/q_n$ for some convergent, and $\nu_{q_n} < \nu_q$). This implies that the limit in (20) can be restricted to the convergents q_n. In fact, an infinite number of convergents satisfy the sharper inequality $\|q_n\Omega\| < 1/(\sqrt{5}\,q_n)$, which gives the upper bound $\nu^* \leq 1/\sqrt{5}$. On the other hand, since all convergents satisfy the inequality $\|q_n\Omega\| < 1/q_n$, we get the upper bound $\limsup_{n\to\infty} \nu_{q_n} \leq 1$. Finally, a classical Markoff's theorem [Markoff, 1879] implies that $\nu^* \geq 1/3$. □

Now we define

$$E = E(\Omega) := \left(\frac{1}{\nu^*} \limsup_{n\to\infty} \nu_{q_n} \right)^{1/2} = \left(\frac{\limsup_{n\to\infty} \nu_{q_n}}{\liminf_{n\to\infty} \nu_{q_n}} \right)^{1/2}, \tag{23}$$

which is a finite number (the bounds of Lemma 1 imply that $1 \leq E \leq \sqrt{3}$), and the bounds $\nu^* \leq \nu_{q_n} \leq E^2\nu^*$ hold asymptotically, as $n \to \infty$. (We use an expression like "$a_n \leq b_n$ as $n \to \infty$" if $\limsup_{n\to\infty}(a_n/b_n) \leq 1$.)

Consequently, the smallest numerators γ_k can be found among the resonant convergents $k = v(n)$ introduced in (13). Moreover, we have the following asymptotic bounds, as $n \to \infty$:

$$\gamma^* \leq \gamma_{v(n)} \leq E^2\gamma^*. \tag{24}$$

The following result provides a *geometric lower and upper bound* for the convergents q_n, generalizing in some sense the geometric asymptotic estimate of resonant sequences of a quadratic number given in [Delshams & Gutiérrez, 2003, Th. 2].

Proposition 2. *Let $\Omega = [a_1, a_2, \ldots]$ be a number of constant type, and define $M = M(\Omega) := 1 + \max_n a_n$. For any $n \geq 2$, one has:*

$$1 + \frac{1}{M} \leq \frac{q_n}{q_{n-1}} \leq M.$$

Proof. This is a simple consequence of a well-known general formula: $\dfrac{q_n}{q_{n-1}} = a_n + [a_{n-1}, \dots, a_1]$ (see [Schmidt, 1980, Lemma §I.3F]), which implies the inequalities $a_n + \dfrac{1}{a_{n-1}+1} = a_n + [a_{n-1}, 1] \le \dfrac{q_n}{q_{n-1}} \le a_n + [a_{n-1}, \infty] \le a_n + 1$. □

3. Lower bounds for the maximal splitting distance

This section is devoted to the proof of Theorem 1, giving a lower bound for the maximal splitting distance between the invariant manifolds of the whiskered torus, when the frequency ratio is a number of constant type. We start with a brief description of our approach. As said in the introduction, a measure for the splitting distance on a transverse section is given by the splitting function, which is the gradient of the splitting potential: $\mathcal{M}(\theta) = \nabla\mathcal{L}(\theta)$, $\theta \in \mathbb{T}^2$. In order to obtain a lower bound for the splitting function, we consider the Fourier expansion of the splitting potential:

$$\mathcal{L}(\theta) = \sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\} \\ k_2\ge 0}} \mathcal{L}_k \cos(\langle k, \theta\rangle - \tau_k).$$

For the splitting function, its vector Fourier coefficients $\mathcal{M}_k$ are related to the scalar coefficients of the splitting potential: $|\mathcal{M}_k| = |k|\,\mathcal{L}_k$.

The Poincaré–Melnikov method provides the first order approximation (3) in terms of the Melnikov function, which is the gradient of the splitting potential: $M(\theta) = \nabla L(\theta)$. The Fourier coefficients L_k of the Melnikov potential can be computed explicitly, which allows us to detect the dominant harmonic, corresponding to some $k = S(\varepsilon)$ for any sufficiently small ε. Such dominance, which holds for the Poincaré–Melnikov approximation and is found in a constructive way, can also be validated for the whole splitting function when the error term in (3) is added, in our singular case $\mu = \varepsilon^p$, analogously to [Delshams & Gutiérrez, 2004] (where the case of the golden number was considered).

In this way, we obtain an exponentially small asymptotic estimate for the coefficient dominant harmonic, as well as an exponentially small upper bound for the sum of all other harmonics, showing that the dominant harmonic is large enough to ensure that the maximal splitting distance can be approximated by the size of this coefficient:

$$\max_{\theta\in\mathbb{T}^2} |\mathcal{M}(\theta)| \approx |\mathcal{M}_S| \quad \text{as } \varepsilon \to 0, \qquad S = S(\varepsilon). \tag{25}$$

(We use the expression "$f(\varepsilon) \approx g(\varepsilon)$ as $\varepsilon \to 0$" if $\lim\limits_{\varepsilon\to 0}(f(\varepsilon)/g(\varepsilon)) = 1$.) The asymptotic estimate for $|\mathcal{M}_S|$ is given in Proposition 3(a), in terms of an oscillating function $h_1(\varepsilon)$. If Ω is a quadratic irrational number, the function $h_1(\varepsilon)$ is periodic with respect to $\ln\varepsilon$ (see [Delshams & Gutiérrez, 2003; Delshams *et al.*, 2013]). Instead, in the case of a number of constant type Ω under consideration, the function $h_1(\varepsilon)$ has a more complicated behavior but can be bounded from below and above, which gives rise to exponentially small upper bounds (as in [Simó, 1994], [Delshams *et al.*, 2004] and other works) and to exponentially small lower bounds as in the statement of Theorem 1.

3.1. *Dominant harmonics of the splitting potential*

We put our functions f and h defined in (8) into the integral (10) and, calculating it by residues, we get the Fourier expansion of the Melnikov potential:

$$L(\theta) = \sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\} \\ k_2\ge 0}} L_k \cos(\langle k, \theta\rangle - \sigma_k), \qquad L_k = \frac{2\pi|\langle k, \omega_\varepsilon\rangle|\,\mathrm{e}^{-\rho|k|}}{\sinh|\frac{\pi}{2}\langle k, \omega_\varepsilon\rangle|}\,.$$

Using (1) and (18), we can present the coefficients in the form

$$L_k = \alpha_k\,\mathrm{e}^{-\beta_k}, \qquad \alpha_k \approx \frac{4\pi\gamma_k}{|k|\sqrt{\varepsilon}}\,, \quad \beta_k = \rho|k| + \frac{\pi\gamma_k}{2|k|\sqrt{\varepsilon}}\,, \tag{26}$$

where an exponentially small term has been neglected in the denominator of α_k. For any given ε, the harmonics with largest coefficients $L_k(\varepsilon)$ correspond essentially to the smallest exponents $\beta_k(\varepsilon)$. Thus, we have to study the dependence on ε of such exponents.

With this aim, we introduce for any X, Y the function

$$G(\varepsilon; X, Y) := \frac{Y^{1/2}}{2}\left[\left(\frac{\varepsilon}{X}\right)^{1/4} + \left(\frac{X}{\varepsilon}\right)^{1/4}\right],$$

having its minimum at $\varepsilon = X$, with the minimum value $G(X; X, Y) = Y^{1/2}$. Then, the exponents $\beta_k(\varepsilon)$ in (26) can be presented in the form

$$\beta_k(\varepsilon) = \frac{C_0}{\varepsilon^{1/4}}\, g_k(\varepsilon), \qquad g_k(\varepsilon) := G(\varepsilon; \varepsilon_k, \tilde{\gamma}_k), \qquad C_0 := (2\pi\rho\gamma^*)^{1/2}, \tag{27}$$

where

$$\varepsilon_k := D_0 \frac{\tilde{\gamma}_k^2}{|k|^4}, \qquad \tilde{\gamma}_k := \frac{\gamma_k}{\gamma^*}, \qquad D_0 := \left(\frac{\pi\gamma^*}{2\rho}\right)^2. \tag{28}$$

Notice that, to define the *"normalized numerators"* $\tilde{\gamma}_k$, we have taken into account the limit numerator γ^* introduced in (19). Consequently, for all k we have $\beta_k(\varepsilon) \geq \dfrac{C_0\tilde{\gamma}_k^{1/2}}{\varepsilon^{1/4}}$, which provides an asymptotic estimate for the maximum value of the coefficient $L_k(\varepsilon)$ of each harmonic. We point out that similar estimates of the size of a given harmonic from the arithmetic properties of frequencies, for the case of constant type, were already obtained in [Rudnev & Wiggins, 1998].

We define, for any given ε, the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ as the first and the second minima of the values $g_k(\varepsilon)$, $k \in \mathbb{Z}^2 \setminus \{0\}$, and we denote $S = S(\varepsilon)$ the integer vector that gives the first minimum:

$$h_1(\varepsilon) := \min_k g_k(\varepsilon) = g_S(\varepsilon), \qquad h_2(\varepsilon) := \min_{k \neq S} g_k(\varepsilon). \tag{29}$$

The function $h_1(\varepsilon)$ is continuous, and provides an estimate of the size of the most dominant coefficient of the Melnikov potential. The function $h_2(\varepsilon)$ is also continuous, and we have $h_2 > h_1$ excepting at some isolated values of ε where h_1 and h_2 coincide because of a change in the vector $S(\varepsilon)$ giving the dominant harmonic. Notice also that $S(\varepsilon)$ remains constant in each interval between two of such consecutive values of ε.

In the particular case of quadratic frequencies, the function $h_1(\varepsilon)$ is periodic in $\ln\varepsilon$ (see [Delshams & Gutiérrez, 2003; Delshams *et al.*, 2013]), due to a periodicity in the graphs of the functions $g_k(\varepsilon)$, which can be deduced from the asymptotic geometric growth of the resonant sequences (21). Unfortunately, in the more general case of non-quadratic frequencies of constant type, the graphs of $g_k(\varepsilon)$ do no exhibit a periodicity in $\ln\varepsilon$. In general, it is hard to provide an analytic description of the function $h_1(\varepsilon)$, but we can obtain an upper bound for it, as $\varepsilon \to 0$ (see Section 3.2), which gives rise to a lower bound for the dominant harmonic and, in view of (25), also for the maximal splitting distance. As an illustration, the function $h_1(\varepsilon)$ is represented in Figure 1 for the concrete case of the frequency ratio Ω introduced in (17).

Now we give, from the function $h_1(\varepsilon)$, an estimate for the dominant harmonic in the Fourier expansion of the splitting function, as well as for the sum of all other harmonics. We use the notation $f \sim g$ if $c_1|g| \leq |f| \leq c_2|g|$ with some positive constants c_1, c_2 not depending on ε, μ.

Proposition 3. *For ε small enough and $\mu = \varepsilon^p$ with $p > 3$, one has:*

(a) $|\mathcal{M}_S| \sim \mu\,|S|\,L_S \sim \dfrac{\mu}{\varepsilon^{1/2}} \exp\left\{-\dfrac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}$;

(b) $\displaystyle\sum_{k \neq S} |\mathcal{M}_k| \sim \frac{\mu}{\varepsilon^{1/2}} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}$.

Proof. The proof is similar to the analogous result in [Delshams *et al.*, 2013, Prop. 4]. By the Poincaré–Melnikov method (3), the coefficients of the splitting function can be approximated by the expression

$$|\mathcal{M}_k| = |k|\,\mathcal{L}_k \sim \mu\,|k|\,L_k = \mu\,|k|\,\alpha_k\,\mathrm{e}^{-\beta_k},$$

where we have neglected the error term of (3) in this first approximation, and we have used the expression (26) of the coefficients of the Melnikov potential. As mentioned previously, the main behavior of the coefficients $L_k(\varepsilon)$ is given by the exponents $\beta_k(\varepsilon)$, which have been written in (27) in terms of the functions $g_k(\varepsilon)$. In particular, the coefficient L_S, associated to the dominant harmonic $S = S(\varepsilon)$, can be expressed in terms of the function $h_1(\varepsilon)$ introduced in (29). Thus, the exponential factor in (a) is directly given by $\mathrm{e}^{-\beta_S}$. On the other hand, the polynomial factor comes from the estimate $|S|\,\alpha_S \sim 1/\sqrt{\varepsilon}$, which comes directly from the approximation of α_S given by (26), using also the estimate $\gamma_S \sim 1$ (this is a consequence of Lemma 2, see the first remark in the next section). In this way, we have obtained an asymptotic estimate for the size $|M_S|$ of the dominant coefficient of the Melnikov function. To complete the proof of part (a), one has to see that the same estimate is valid for $\mathcal{M}_S$, i.e. when the error term in the Poincaré–Melnikov approximation (3) is not neglected, in our singular case $\mu = \varepsilon^p$. We omit the details for this step, since it can be worked out as it was done in [Delshams & Gutiérrez, 2004, Lemma 5] for the case of the golden number ($\Omega = (\sqrt{5}-1)/2$), using upper bounds for the error term provided in [Delshams *et al.*, 2004, Th. 10].

The proof of part (b) is carried out in similar terms. For the second dominant harmonic, we get an exponentially small estimate with the function $h_2(\varepsilon)$, defined in (29). This estimate is also valid if one considers the whole sum in (b), since for any given ε the terms of this sum can be bounded by geometric series and, hence, it can be estimated by its dominant term (see [Delshams & Gutiérrez, 2004, Lemma 4] for more details). □

3.2. *Upper and lower bounds for $h_1(\varepsilon)$*

To conclude the proof of Theorem 1, we provide bounds for the function $h_1(\varepsilon)$. In particular, an upper bound for this function gives rise to a lower bound for the splitting function, in view of the approximation (25)

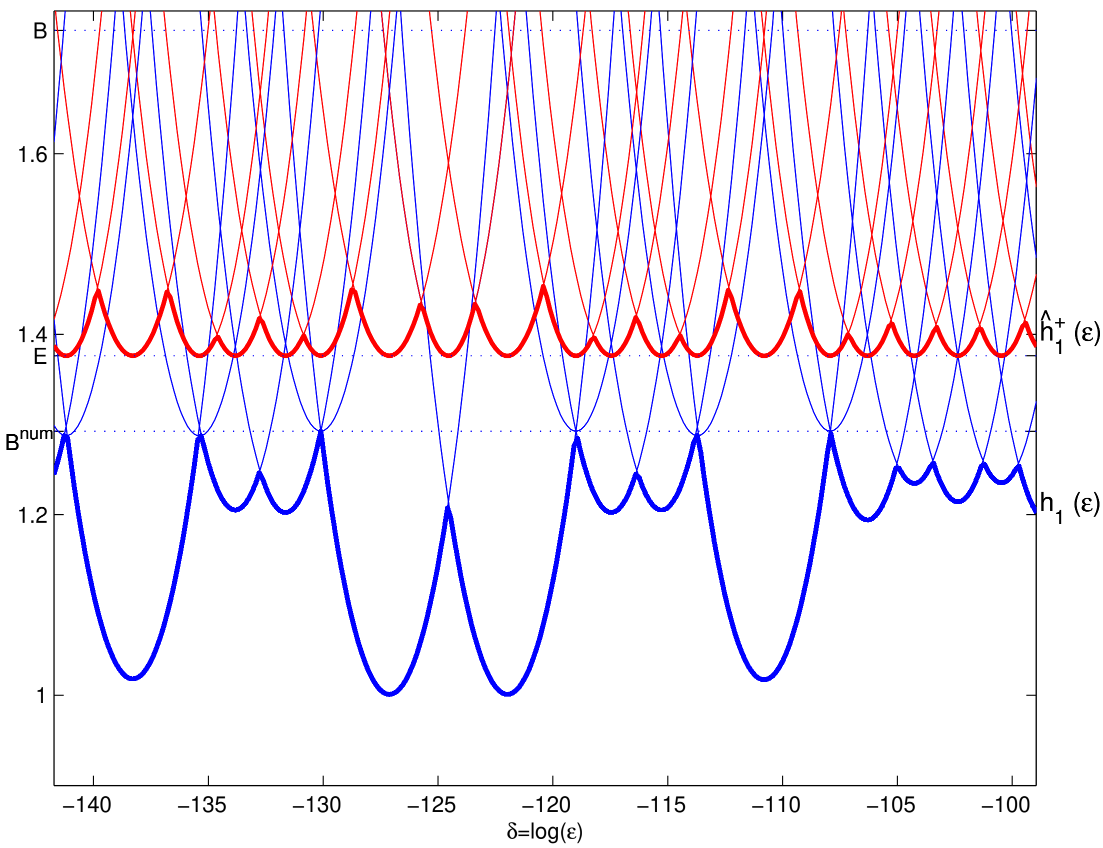

Fig. 1. *Graphs of the functions $\widehat{g}_n(\varepsilon)$, $h_1(\varepsilon)$ (in thin and thick blue) and $\widehat{g}_n^+(\varepsilon)$, $\widehat{h}_1^+(\varepsilon)$ (in thin and thick red) using a logarithmic scale for ε, for the frequency ratio Ω given in (17). (Constants: $M = 3$, $E \approx 1.3761$, $B \approx 1.7366$, $B^{\mathrm{num}} \approx 1.2925$.)*

and Proposition 3(a).

We are going to show that, to provide an upper bound for $h_1(\varepsilon)$, we can restrict our study of the minimum in (29) to the harmonics associated to resonant convergents $v(n)$, defined in (13), which play an analogous rôle as the primary resonances for quadratic frequencies considered in [Delshams & Gutiérrez, 2003]. We are going to use the results on arithmetic properties of frequencies of constant type (see Section 2). Thus, we define

$$\widehat{h}_1(\varepsilon) := \min_n \widehat{g}_n(\varepsilon). \qquad \widehat{g}_n(\varepsilon) := g_{v(n)}(\varepsilon) = G(\varepsilon; \varepsilon_{v(n)}, \tilde{\gamma}_{v(n)}),$$

We know from (24) that the minimum values of the functions $\widehat{g}_n(\varepsilon)$ satisfy the bounds: $1 \le \tilde{\gamma}_{v(n)}^{1/2} \le E$ (asymptotically, as $n \to \infty$), where the constant $E = E(\Omega)$ has been defined in (23). With this in mind, we also define the functions

$$\widehat{h}_1^+(\varepsilon) := \min_n \widehat{g}_n^+(\varepsilon), \qquad \widehat{g}_n^+(\varepsilon) := G(\varepsilon; \varepsilon_{v(n)}, E^2), \tag{30}$$

which are represented in Figure 1 for the concrete case of the frequency ratio Ω inroduced in (17). It is clear that $h_1(\varepsilon) \le \widehat{h}_1(\varepsilon) \le \widehat{h}_1^+(\varepsilon)$ for any ε. Since all the functions $\widehat{g}_n^+(\varepsilon)$ have the same minimum value E, it is much simpler to study the behavior of function $\widehat{h}_1^+(\varepsilon)$ and, in particular, to provide an upper bound for it. As shown in the next lemma, this upper bound is given by the constant

$$B = B(\Omega) := \frac{E}{2}\left[(EM)^{1/2} + (EM)^{-1/2}\right], \tag{31}$$

where E and M have been defined in (23) and in Proposition 2, respectively.

Lemma 2. *The function $h_1(\varepsilon)$ defined in (29) satisfies the asymptotic bounds:* $1 \le h_1(\varepsilon) \le B$, *as* $\varepsilon \to 0$.

Proof. The lower bound is very simple: the function $h_1(\varepsilon)$ is defined from the functions $g_k(\varepsilon)$ in (27), whose minimum values are $\tilde{\gamma}_k^{1/2}$. Recall that the normalized numerators were introduced in (28), and satisfy the asymptotic bound $\tilde{\gamma}_k \ge 1$, as $|k| \to \infty$.

Now we are going to obtain an upper bound for the function $\widehat{h}_1^+(\varepsilon)$ defined in (30), and hence for $h_1(\varepsilon)$. According to (27) and (28), each function $\widehat{g}_n^+(\varepsilon)$ takes its minimum value at $\varepsilon_{v(n)} = D_0 \tilde{\gamma}_{v(n)}^2 / |v(n)|^4$. Using the asymptotic bounds (24), as well as the geometric growth of the convergents described in Proposition 2, we see that $\varepsilon_{v(n-1)}/\varepsilon_{v(n)} \le (EM)^4$, as $n \to \infty$. In other words, using a logarithmic scale for ε (as in Figure 1), the distance between the minimum points $\varepsilon_{v(n-1)}$ and $\varepsilon_{v(n)}$ of two consecutive functions, $\widehat{g}_{n-1}^+$ and $\widehat{g}_n^+$, is $\le 4\ln(EM)$. Their value at the "middle point" (which provides an approximate intersecting point) is the constant B defined in (31), which provides an upper bound, as $\varepsilon \to 0$, for $\widehat{h}_1^+(\varepsilon)$ and also for $h_1(\varepsilon)$. □

Remarks.

(1) In the example considered in Figure 1, for all values of ε the dominant harmonic $S = S(\varepsilon)$ is given by a resonant convergent $v(N)$, $N = N(\varepsilon)$, and we have $h_1(\varepsilon) = \widehat{h}_1(\varepsilon) \le \widehat{h}_1^+(\varepsilon)$. In other examples, it may happen that for some small intervals of ε the dominant harmonic S is a non-convergent vector, having $h_1(\varepsilon) < \widehat{h}_1(\varepsilon)$ in such intervals. In any case, the minimum value $\tilde{\gamma}_S^{1/2}$ of the function $g_S(\varepsilon)$ associated to the dominant harmonic $S = S(\varepsilon)$ is always contained in the interval $[1, B(\Omega)]$, hence we can write $\gamma_S = \tilde{\gamma}_S \cdot \gamma^* \sim 1$, an estimate that was used in the proof of Proposition 3.

(2) The upper bound $h_1(\varepsilon) \le B$ bound is not sharp, since it considers the worst possible case in the bounds of (24) and Proposition 2. A much sharper upper bound $h_1(\varepsilon) \le B^{\mathrm{num}}$ can be obtained numerically (see Figure 1).

Now we can complete the proof of Theorem 1. Indeed, applying the upper bound $h_1(\varepsilon) \le B$ to the exponent in Proposition 3(a) and using (25), we get the lower bound (11), and the constant in the exponent is given by

$$C(\Omega, \rho) = C_0(\Omega, \rho) \cdot B(\Omega), \tag{32}$$

where C_0 and B have been defined in (28) and (31) respectively.

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