COADJOINT DYNAMICAL SYSTEMS OF SOLVABLE LIE GROUPS

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Abstract. In this paper we approach some basic topological properties of dual spaces of solvable Lie groups using suitable dynamical systems related to the coadjoint action. One of our main results is that the coadjoint dynamical system of any exponential solvable Lie group is a piecewise pullback of group bundles.

2010 MSC: 22E27; 22A22; 46L55

Keywords: C^*-algebra; exponential solvable Lie group; dynamical system; groupoid

1. Introduction

If G is any Lie group whose exponential map $\exp_G\colon \mathfrak{g} \to G$ is bijective, then it is well known that G is a solvable Lie group, but the classification of these Lie groups is an open problem. It is known however that G is isomorphic to a closed subgroup of the invertible upper-triangular real matrices of suitable size, and there exists a canonical homeomorphism from the space of coadjoint orbits $\mathfrak{g}^*/G$ onto the dual space $\widehat{G}$, which consists of the equivalence classes of unitary irreducible representations of G. The dynamical system $(G, \mathfrak{g}^*, \mathrm{Ad}_G^*)$ defined by the coadjoint action $\mathrm{Ad}_G^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ holds a key role in this picture by means of its orbit space $\mathfrak{g}^*/G$, but the topological properties of the orbits also encode certain properties of their corresponding representations of G. Thus, a coadjoint orbit is closed if and only if its corresponding unitary irreducible representation of G is CCR, that is, the corresponding image of $C^*(G)$ is equal to the C^*-algebra of compact operators on the representation space; see also [BB15] for the significance of open coadjoint orbits.

Several central problems in representation theory of the exponential solvable Lie group G are related to further topological properties of the orbit space $\mathfrak{g}^*/G$ and their C^*-algebraic interpretations. More specifically, it is known that $\mathfrak{g}^*/G$ is in general a locally quasi-compact space which does not have the Hausdorff property, and whose points (more exactly, its one-point subsets) are locally closed. Therefore it is interesting to study larger locally closed subsets of $\mathfrak{g}^*/G$ which are locally closed and whose relative topology has the Hausdorff property. Examples of such subsets are the spectra of continuous-trace subquotients of $C^*(G)$, and we have shown in [BB15] that they play a crucial role in computing the real rank of $C^*(G)$. Along these lines, for any connected, simply connected nilpotent Lie group G, we

Date: 28 November 2015.

This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS–UEFISCDI, project number PN-II-RU-TE-2014-4-0370.

have shown in [BBL14] that $C^*(G)$ is a solvable C^*-algebra, that is, it has a finite series $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = C^*(G)$ consisting of closed two-sided ideals such that all the subquotients $\mathcal{J}_k/\mathcal{J}_{k-1}$ for $k = 1, \dots, n$ are continuous-trace C^*-algebras which are (strongly) Morita equivalent to commutative C^*-algebras.

One of the main themes of the present paper is to contribute to the above discussion, by exploring some properties of locally closed Hausdorff subsets $\Gamma \subseteq \mathfrak{g}^*/G$ which are related to dynamical systems, in the sense of transformation groups. Specifically, if $q\colon \mathfrak{g}^* \to \mathfrak{g}^*/G$ denotes the quotient map, then $\Xi := q^{-1}(\Gamma) \subseteq \mathfrak{g}^*$ is a G-invariant locally compact Hausdorff space. One can consider the corresponding dynamical system $(G, \Xi, \mathrm{Ad}^*_G|_{G\times\Xi})$, whose orbit space has the Hausdorff property, unlike the orbit space of $(G, \mathfrak{g}^*, \mathrm{Ad}^*_G)$. We will develop some abstract tools that allow us to study the C^*-algebra of that dynamical system and its relation to topological properties of the coadjoint orbits contained in Γ. For that purpose we will establish the main results on the natural level of generality, which is the theory of C^*-algebras of second countable, locally compact groupoids with left Haar systems.

For clarity, before proceeding with description of the structure of the present paper, it is worthwhile to briefly discuss here, on a purely algebraic level, the method of our investigation. This will also explain the crucial role of bundles of coadjoint isotropy groups in representation theory of exponential solvable Lie groups.

All groupoids are pullbacks of group bundles. Let $\mathcal{G}$ be any groupoid, viewed as a discrete topological space, with its domain/range maps $d, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ and the quotient map $q\colon \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}/\mathcal{G}$ onto its set of orbits. Assume that we have fixed a map $\gamma\colon \mathcal{G}^{(0)}/\mathcal{G} \to \mathcal{G}^{(0)}$ with $q \circ \gamma = \mathrm{id}$, that is, γ is a cross-section of q. Denote $\Xi := \gamma(\mathcal{G}^{(0)}/\mathcal{G}) \subseteq \mathcal{G}^{(0)}$, and fix another map $\sigma\colon \mathcal{G}^{(0)} \to \mathcal{G}$ with $d \circ \sigma = \mathrm{id}$. The set Ξ intersects every $\mathcal{G}$-orbit at exactly one point.

Then define the bundle of isotropy groups $\Gamma := \bigsqcup\limits_{x\in\Xi} \mathcal{G}(x)$, and let $\Pi\colon \Gamma \to \Xi$ be the canonical projection with $\Pi^{-1}(x) = \mathcal{G}(x)$ for all $x \in \Xi$. Define $\theta := \gamma \circ q\colon \mathcal{G}^{(0)} \to \Xi$ and the *pullback of* Π *by* θ

$$\theta^{\downarrow\downarrow}(\Pi) := \{(x, h, y) \in \mathcal{G}^{(0)} \times \Gamma \times \mathcal{G}^{(0)} \mid \theta(x) = \Pi(h) = \theta(y)\}.$$

The projections on the first and third coordinates, regarded as domain/range maps, define a groupoid $\theta^{\downarrow\downarrow}(\Pi) \rightrightarrows \mathcal{G}^{(0)}$ and it is straightforward to check that the map

$$\Phi\colon \mathcal{G} \to \theta^{\downarrow\downarrow}(\Pi), \quad \Phi(g) := (r(g), \sigma(r(g))g\sigma(d(g))^{-1}, d(g)) \tag{1.1}$$

is a groupoid isomorphism, with its inverse

$$\Phi^{-1}\colon \theta^{\downarrow\downarrow}(\Pi) \to \mathcal{G}, \quad \Phi^{-1}(x, h, y) = \sigma(y)^{-1}h\sigma(x) \tag{1.2}$$

(see [MRW87] and [Bu03] for the special case of transitive groupoids). In this way every groupoid is (noncanonically) isomorphic to the pullback of the group bundle obtained as the restriction of its isotropy subgroupoid to any cross-section of its space of orbits. In particular, when viewed as discrete groupoids, *all transformation groups are pullbacks of group bundles.*

Structure of this paper. Loosely speaking, one of our main results (Theorem 5.2) is a version of the above italicized statement for the dynamical system (transformation group) defined by the coadjoint action of any exponential solvable Lie group G. It is not difficult to see from (1.1)–(1.2) that the key points in the proof of that result are the conditions which ensure that the above cross-sections γ and σ can be constructed satisfying appropriate topological conditions. These topological conditions should be strong enough for obtaining information on the C^*-algebra of the coadjoint dynamical system of G, in the sense that the corresponding C^*-algebra turns out to be piecewise Morita equivalent to C^*-algebras of group bundles as above. Along the way we also need to establish some facts on locally compact groupoids, which may hold an independent interest, and which we were unable to locate in the literature. But from the present perspective, the significance of these facts is that they belong to a framework in which transformation groups are obtained from group bundles by a procedure that preserves the Morita-equivalence class of their C^*-algebras. Therefore the bulk of this paper (Sections 2–4) is devoted to establishing the topological framework in which the pullback operation works suitably. Then Section 5 contains our main result along with some specific examples of solvable Lie groups.

2. Preliminaries

2.1. **Basic notation.** *Throughout the paper, unless otherwise mentioned, $\mathcal{G}$ will be a* **second countable locally compact Hausdorff** *groupoid with its space of objects $\mathcal{G}^{(0)}$, space of morphisms $\mathcal{G}$, and its domain/range maps $d, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$. We usually summarize this setting by the symbol $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$.* For short we will say that $\mathcal{G}$ is a **locally compact groupoid**.

Hence one has a category whose objects constitute the set $\mathcal{G}^{(0)}$, all morphisms are invertible and constitute the set $\mathcal{G}$, and the set of composable pairs of morphisms is defined by $\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} \mid d(g) = r(h)\}$; the sets $\mathcal{G}^{(0)}$ and $\mathcal{G}$ are endowed with second countable locally compact topologies for which the structural maps (domain, range, inversion of morphisms $\iota\colon \mathcal{G} \to \mathcal{G}$, and composition of morphisms $\mathcal{G}^{(2)} \to \mathcal{G}$) are continuous. Moreover, we assume that the canonical inclusion map $\mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$, which maps every object to its identity morphism, is a homeomorphism onto its image, and the domain map d is open (hence so is the range map $r = \iota \circ d$).

For any point $x \in \mathcal{G}^{(0)}$, its isotropy group is $\mathcal{G}(x) := \{g \in \mathcal{G} \mid d(g) = r(g) = x\}$ and its $\mathcal{G}$-orbit is $\mathcal{G}.x := \{r(g) \mid g \in \mathcal{G},\ d(g) = x\}$. The set of all $\mathcal{G}$-orbits is denoted by $\mathcal{G}^{(0)}/\mathcal{G}$. If there exists only one orbit then $\mathcal{G}$ is called a *transitive groupoid.* If $\mathcal{G}(x) = \{\mathbf{1}\}$ for all $x \in \mathcal{G}^{(0)}$, then $\mathcal{G}$ is called a *principal groupoid.* A principal transitive groupoid is called a *pair groupoid.*

A subset $A \subseteq \mathcal{G}^{(0)}$ is said to be $\mathcal{G}$*-invariant* if for every $x \in A$ one has $\mathcal{G}.x \subseteq A$. If moreover $\mathcal{G}(x) = \{\mathbf{1}\}$ for all $x \in A$, then A is called a *principal* (invariant) set.

Now assume that in addition $\mathcal{G}$ is endowed with a left Haar system, that is, a family $\lambda = \{\lambda^x\}_{x \in \mathcal{G}^{(0)}}$ where λ^x is a Borel measure supported on $r^{-1}(x) \subseteq \mathcal{G}$ for every $x \in \mathcal{G}^{(0)}$, satisfying the continuity condition that $\mathcal{G}^{(0)} \ni x \mapsto \lambda(\varphi) := \int \varphi \mathrm{d}\lambda^x \in \mathbb{C}$ is continuous and the invariance condition $\int \varphi(gh)\mathrm{d}\lambda^{d(g)}(h) = \int \varphi(h)\mathrm{d}\lambda^{r(g)}(h)$ for all $g \in \mathcal{G}$ and $\varphi \in \mathcal{C}_c(\mathcal{G})$.

Then one can define a convolution on the space $\mathcal{C}_c(\mathcal{G})$ by the formula

$$(\varphi_1 * \varphi_2)(x) := \int\limits_{r^{-1}(r(g))} \varphi_1(gh^{-1})\varphi_2(h)\mathrm{d}\lambda^{r(g)} \text{ for } g \in \mathcal{G} \text{ and } \varphi_1, \varphi_2 \in \mathcal{C}_c(\mathcal{G}).$$

This makes $\mathcal{C}_c(\mathcal{G})$ into an associative $*$-algebra with the involution defined by $\varphi^*(g) := \overline{\varphi(g^{-1})}$ for all $g \in \mathcal{G}$ and $\varphi \in \mathcal{C}_c(\mathcal{G})$. There also exists a natural algebra norm on $\mathcal{C}_c(\mathcal{G})$ defined by

$$\|f\|_I := \max\Bigl\{\sup_{x\in\mathcal{G}^{(0)}} \int |\varphi|\mathrm{d}\lambda^x, \sup_{x\in\mathcal{G}^{(0)}} \int |\varphi^*|\mathrm{d}\lambda^x\Bigr\}.$$

Then $C^*(\mathcal{G})$ is defined as the completion of $\mathcal{C}_c(\mathcal{G})$ with respect to the norm

$$\|\varphi\| := \sup_{\pi} \|\pi(\varphi)\|$$

where π ranges over all bounded $*$-representations of $\mathcal{C}_c(\mathcal{G})$. One can similarly define the reduced C^*-algebra $C^*_{\mathrm{red}}(\mathcal{G})$ by restricting the above supremum to the family of *regular* representations $\{\Lambda_x \mid x \in \mathcal{G}^{(0)}\}$, where we define

$$(\forall x \in \mathcal{G}^{(0)}) \quad \Lambda_x \colon \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}, \lambda^x)), \quad \Lambda_x(\varphi)\psi := \varphi * \psi.$$

There is a canonical surjective $*$-homomorphism $C^*(\mathcal{G}) \to C^*_{\mathrm{red}}(\mathcal{G})$. and if it is also injective then the groupoid $\mathcal{G}$ is called *metrically amenable*.

Remark 2.1. Any locally closed subset (i.e., a difference of two open subsets) of a locally compact space is in turn locally compact with its relative topology. Then any $\mathcal{G}$-invariant locally closed subset $A \subseteq \mathcal{G}^{(0)}$ gives rise to a locally compact groupoid $\mathcal{G}_A$, its corresponding *reduced* groupoid, with its set of objects A and its set of morphisms $\mathcal{G}_A := d^{-1}(A)$. Then $\mathcal{G}_A$ has a left Haar system λ_A obtained by restricting the Haar system λ of $\mathcal{G}$ to $\mathcal{G}_A$, since the Tietze extension theorem implies that every function in $\mathcal{C}_c(\mathcal{G}_A)$ extends to a function in $\mathcal{C}_c(\mathcal{G})$, using the fact that $\mathcal{G}_A$ is a locally closed subset of $\mathcal{G}$. In particular, we can construct the corresponding C^*-algebra $C^*(\mathcal{G}_A)$ and reduced C^*-algebra $C^*_{\mathrm{red}}(\mathcal{G}_A)$.

If the above subset subset $A \subseteq \mathcal{G}^{(0)}$ is closed, then the subset $d^{-1}(A) \subseteq \mathcal{G}$ is also closed, so the restriction map $\mathcal{C}_c(\mathcal{G}) \to \mathcal{C}_c(d^{-1}(A))$ is well defined, and it extends by continuity both to a $*$-homomorphism $\mathcal{R}_A \colon C^*(\mathcal{G}) \to C^*(\mathcal{G}_A)$ and to a $*$-homomorphism of the reduced C^*-algebras $(\mathcal{R}_A)_{\mathrm{red}} \colon C^*_{\mathrm{red}}(\mathcal{G}) \to C^*_{\mathrm{red}}(\mathcal{G}_A)$ which are related by the commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{G}) & \xrightarrow{\mathcal{R}_A} & C^*(\mathcal{G}_A) \\ \downarrow & & \downarrow \\ C^*_{\mathrm{red}}(\mathcal{G}) & \xrightarrow{(\mathcal{R}_A)_{\mathrm{red}}} & C^*_{\mathrm{red}}(\mathcal{G}_A) \end{array} \tag{2.1}$$

where the vertical arrows are the natural quotient homomorphisms.

For later use, we record the following basic facts on reductions of a groupoid to open or closed invariant subsets of the space of units.

Proposition 2.2. *If $\mathcal{G}$ is any locally compact groupoid with a left Haar system, then the following assertions hold:*

(i) There exists a bijective correspondence $U \longleftrightarrow I(U)$ between $\mathcal{G}$-invariant open subsets $U \subseteq G^{(0)}$ and some closed two-sided ideals of $C^*(\mathcal{G})$ such that for every U with its complement $F := G^{(0)} \setminus U$ one has a short exact sequence

$$0 \to I(U) \to C^*(\mathcal{G}) \xrightarrow{\mathcal{R}_Y} C^*(\mathcal{G}_F) \to 0$$

and a natural $*$-isomorphism $I(U) \simeq C^*(\mathcal{G}_U)$.

(ii) For every U and F as above one has the short exact sequence of multiplier algebras

$$0 \to \operatorname{Ker} \mathcal{R}^{**}_{\partial U} \to M(C^*(\mathcal{G})) \xrightarrow{\mathcal{R}^{**}_F} M(C^*(\mathcal{G}_F)) \to 0.$$

(iii) If U and F as above have the additional property that the canonical quotient map $C^*(\mathcal{G}_F) \to C^*_{\rm red}(\mathcal{G}_F)$ is an isomorphism, then one has the short exact sequence of reduced C^*-algebras

$$0 \to C^*_{\rm red}(\mathcal{G}_U) \to C^*_{\rm red}(\mathcal{G}) \xrightarrow{(\mathcal{R}_F)_{\rm red}} C^*_{\rm red}(\mathcal{G}_F) \to 0.$$

Proof. The first assertion is well known, see for instance [MRW96, Lemma 2.1].

For the second assertion first recall from [We93, Subsect. 1.5 and Rem. 2.2.3] that if $\mathcal{A}$ is any C^*-algebra, then its multiplier algebra can be identified as a C^*-subalgebra of the universal enveloping von Neumann algebra $\mathcal{A}^{**}$ as $M(\mathcal{A}) = \{a \in \mathcal{A}^{**} \mid a\mathcal{A} + \mathcal{A}a \subseteq \mathcal{A}\}$. Then the conclusion follows by [We93, Th. 2.3.9], which says that if $\mathcal{A}_1 \to \mathcal{A}_2 \to 0$ is an exact sequence of σ-unital C^*-algebras, then the corresponding sequence of multiplier algebras $M(\mathcal{A}_1) \to M(\mathcal{A}_2) \to 0$ is also exact, and every separable C^*-algebra is σ-unital. In particular, this is the case for $C^*_{\rm red}(\mathcal{G})$, which is separable as a quotient of the separable C^*-algebra $C^*(\mathcal{G})$.

The third assertion was noted in [Re91, Rem. 4.1]. □

Remark 2.3. Let $\mathcal{G}$ be a locally compact groupoid with a Haar system. Fix any $x \in \mathcal{G}^{(0)}$ for which the corresponding $\mathcal{G}$-orbit $U := \mathcal{G}.x \subseteq \mathcal{G}^{(0)}$ is a locally closed set. Then there exist a positive measure μ on U and a $*$-isomorphism $C^*(\mathcal{G}_U) \simeq C^*(\mathcal{G}(x)) \otimes \mathcal{K}(L^2(U,\mu))$. (See [MRW87, Th. 3.1] and [Bu03, Th. 7].)

Now we can draw the following corollary of Proposition 2.2, which can be illustrated for instance by the Toeplitz algebra.

Corollary 2.4. *If $\mathcal{G}$ is a locally compact groupoid with a left Haar system for which there exists an open principal orbit $U \subseteq \mathcal{G}^{(0)}$, then there exists a positive measure μ on U such that $C^*(\mathcal{G})$ has a closed ideal isomorphic to $\mathcal{K}(L^2(U,\mu))$ and there exists a short exact sequence*

$$0 \to \mathcal{K}(L^2(U,\mu)) \to C^*(\mathcal{G}) \xrightarrow{\mathcal{R}_Y} C^*(\mathcal{G}_F) \to 0$$

where $F = \mathcal{G}^{(0)} \setminus U$.

Proof. We first note that the hypothesis that U is an principal orbit equivalent to the fact that $\mathcal{G}_U$ is a pair groupoid, and in particular $\mathcal{G}(x) = \{\mathbf{1}\}$ for any $x \in U$. Then Remark 2.3 implies $C^*(\mathcal{G}_U) \simeq \mathcal{K}(L^2(U,\mu))$ for a suitable measure μ on U. Now the conclusion follows by Proposition 2.2(i). □

2.2. **Open dense orbits.** We now note a property of the ideals $I(U)$ from Proposition 2.2. This is a very special instance of [CAR13, Thm. 6.1], with a stronger conclusion.

Proposition 2.5. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be any locally compact groupoid, which has a Haar system, and whose orbits are locally closed. Then $C^*(\mathcal{G}) \simeq \mathcal{K}$ if and only if $\mathcal{G}$ is a pair groupoid.*

Proof. If $\mathcal{G}$ is a pair groupoid, then clearly $C^*(\mathcal{G}) \simeq \mathcal{K}$ by Remark 2.3.

Assume now that $C^*(\mathcal{G}) \simeq \mathcal{K}$. We first prove that $\mathcal{G}$ is transitive. The orbit space $\mathcal{G}^{(0)}/\mathcal{G}$ is a T_0 space by [Ra90, Th. 2.1((4) $\Leftrightarrow$ (5))]. If there exist two distinct points $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathcal{G}^{(0)}/\mathcal{G}$, then by the T_0 property, we may assume that there is an open neighbourhood $V \subset \mathcal{G}^{(0)}/\mathcal{G}$ of $\mathcal{O}_1$ with $\mathcal{O}_2 \notin V$. Denote by $q\colon \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}/\mathcal{G}$ the quotient map. Then $U = q^{-1}(V)$ is a non-empty open subset on $\mathcal{G}^{(0)}$, different from $\mathcal{G}^{(0)}$, and $\mathcal{G}$-invariant. It follows, by the bijective correspondence in Proposition 2.2 (i), that $I(U)$ is a nontrivial closed ideal of $C^*(\mathcal{G})$. This is impossible since $C^*(\mathcal{G}) \simeq \mathcal{K}$. Therefore $\mathcal{G}$ is transitive.

By Remark 2.3 we have now that, for any $x \in \mathcal{G}^{(0)}$, $C^*(\mathcal{G}) \simeq C^*(\mathcal{G}(x)) \otimes \mathcal{K}$. If $\mathcal{G}(x) \neq \{1\}$, then by Gelfand-Raikov theorem there is an irreducible representation π of $\mathcal{G}(x)$ different from the trivial representation τ. Thus $\pi \otimes \mathrm{id}$ and $\tau \otimes \mathrm{id}$ are two non-equivalent irreducible representations of $C^*(\mathcal{G}(x)) \otimes \mathcal{K}$, which is a contradiction with the assumption $C^*(\mathcal{G}) \simeq \mathcal{K}$. Hence we must have $\mathcal{G}(x) = \{1\}$, and this finishes the proof. $\square$

Lemma 2.6. *If $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is any locally compact groupoid with a left Haar system. Assume that for $x_0 \in \mathcal{G}^{(0)}$ the regular representation $\Lambda_{x_0}\colon C^*(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_{x_0}))$ is injective. Then the $\mathcal{G}$-orbit of x_0 is dense in $\mathcal{G}^{(0)}$.*

Proof. We will actually prove a stronger fact, namely that if the regular representation $\Lambda_{x_0}|_{\mathcal{C}_c(\mathcal{G})}$ is injective, then the orbit $\mathcal{G}.x_0$ is dense in $\mathcal{G}^{(0)}$. To this end we assume that there exists an open nonempty set $U \subseteq \mathcal{G}^{(0)}$ with $U \cap \mathcal{G}.x_0 = \emptyset$, and we will show that this leads to a contradiction. Specifically, since U is open and nonempty, then $V := d^{-1}(U)$ ($\supseteq U$) is an open nonempty subset of $\mathcal{G}$. Since $\mathcal{G}$ is locally compact, it then easily follows by Urysohn's lemma that there exists $\varphi \in \mathcal{C}_c(\mathcal{G}) \setminus \{0\}$ with $\operatorname{supp}\varphi \subseteq V$. In particular, $\varphi(k) = 0$ if $k \in \mathcal{G} \setminus V = d^{-1}(\mathcal{G}^{(0)} \setminus U) \supseteq d^{-1}(\mathcal{G}.x_0)$. Then for every $\psi \in \mathcal{C}_c(\mathcal{G}_{x_0})$ and $g \in \mathcal{G}$ we have

$$(\Lambda_{x_0}(\varphi)\psi)(g) = \int\limits_{\mathcal{G}_{x_0}} \varphi(gh^{-1})\psi(h)dh = 0$$

because here we have $d(gh^{-1}) = r(h) \in r(\mathcal{G}_{x_0}) = \mathcal{G}.x_0$, hence $\varphi(gh^{-1}) = 0$. Since $\mathcal{C}_c(\mathcal{G}_{x_0})$ is dense in $L^2(\mathcal{G}_{x_0})$, it then follows $\varphi \in \operatorname{Ker}(\Lambda_{x_0}|_{\mathcal{C}_c(\mathcal{G})}) \setminus \{0\}$, which is a contradiction with our assumption. $\square$

Proposition 2.7. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be any locally compact groupoid with a left Haar system. If $U \subseteq \mathcal{G}^{(0)}$ is any open $\mathcal{G}$-invariant set and $x_0 \in U$, then the following assertions hold:*

(i) For every $x \in \mathcal{G}^{(0)} \setminus U$ the ideal $C^*(\mathcal{G}_U)$ of $C^*(\mathcal{G})$ is contained in the kernel of the regular representation $\Lambda_x\colon C^*(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_x))$, and similarly, for the

reduced C^*-algebras, the ideal $C^*_{\rm red}(\mathcal{G}_U)$ of $C^*_{\rm red}(\mathcal{G})$ is contained in the kernel of the regular representation $\Lambda_x\colon C^*_{\rm red}(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_x))$.

(ii) *If U is an orbit of $\mathcal{G}$, then the regular representation $\Lambda_{x_0}\colon C^*_{\rm red}(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_{x_0}))$ is faithful if and only if U is dense in $\mathcal{G}^{(0)}$, for any $x_0 \in U$.*

Proof. Assertion (i) is a direct consequence of Lemma 2.6.

For Assertion (ii), first note that since $\mathcal{G}_U$ is transitive and $x_0 \in U$, it follows that $U = \mathcal{G}.x_0$. Now, if the representation Λ_{x_0} is faithful, then the set U is dense again as a consequence of Lemma 2.6. Conversely, assume that U is dense in M. Since the topology of U is second countable, we may select any sequence of points $x_1, x_2, \dots \in U$ which is dense in U. The corresponding infinite convex combination of Dirac measures $\nu := \sum\limits_{n\ge 1} \frac{1}{2^n}\delta_{x_n}$ is a measure on $\mathcal{G}^{(0)}$ with dense support, hence by [KS02, Cor. 2.4] the representation $\mathrm{Ind}_\nu := \int\limits_{\mathcal{G}^{(0)}}^{\oplus} \Lambda_x \mathrm{d}\nu(x)$ of $C^*_{\rm red}(\mathcal{G})$ is faithful. But for every $n \ge 1$ the representation Λ_{x_n} is unitarily equivalent to Λ_{x_0} because $x_n \in U = \mathcal{G}.x_0$, hence it follows that $\mathrm{Ker}\,\Lambda_{x_0} = \mathrm{Ker}\,(\mathrm{Ind}_\nu) = \{0\}$, and this completes the proof. $\square$

3. Group bundles

Definition 3.1. A *group bundle* is a locally compact groupoid $\mathcal{T} \rightrightarrows S$ whose range and domain maps are equal.

Remark 3.2. We recall that by definition the range and domain maps of any locally compact groupoid are assumed to be open maps. Then, as a direct consequence of [Re91, Lemma 1.3], any group bundle has a left Haar system.

We now briefly describe an important class of examples of group bundles, namely the principal bundles.

Example 3.3 (associated bundles of a principal bundle). Let H be any topological group and $q\colon P \to M$ be any principal H-bundle. This means that q is a locally trivial continuous surjective map and one has a continuous right group action

$$P \times H \to P, \quad (x, h) \mapsto xh$$

with the following properties:

- One has $q(xh) = q(x)$ for all $x \in P$ and $h \in H$ (that is, the above group action preserves the fibers of q).
- For every $x \in P$ the map $H \to q^{-1}(q(x))$, $h \mapsto xh$, is a homeomorphism.

Now let F be any H-space, that is, a topological space endowed with a continuous left group action $H \times F \to F$. Then the *associated fiber bundle* with fiber F is $Q\colon P \times_H F \to M$, defined as follows:

- The domain of Q is the quotient topological space of $P \times F$ by the equivalence relation
$$(xh, f) \sim (x, hf) \text{ for all } x \in P, f \in F, h \in H.$$
The equivalence class of any pair $(x, f) \in P \times F$ is denoted by $[(x, f)]$ and the quotient topology on $P \times_H F$ is the strongest topology for which the quotient map $P \times F \to P \times_H F$, $(x, f) \mapsto [(x, f)]$, is continuous.

- The map Q is defined by $[(x,f)] \mapsto q(x)$, and it is well defined since the fibers of q are preserved by the action of H on P.

As $q\colon P \to M$ is locally trivial with fiber H, one sees that $Q\colon P\times_H F \to M$ is locally trivial with fiber F. For instance, for the trivial principal bundle $M \times H \to M$ and any H-space F, the associated fiber bundles is the trivial bundle $(M \times H) \times_H F = M \times F \to M$.

Example 3.4 (locally trivial group bundles). If G is a locally compact group, its automorphism group $\mathrm{Aut}(G)$ is in turn a topological group with its compact-open topology, which may not be locally compact. For instance, if $\mathcal{V}$ is any infinite-dimensional vector space over an arbitrary field and G is the additive discrete group $(\mathcal{V}, +)$, then $\mathrm{Aut}(G)$ fails to be locally compact, as mentioned at the very beginning of [Ho52].

Now let G be a Lie group with its Lie algebra $\mathfrak{g} = T_{\mathbf{1}}G$.

(i) If G is connected, then the homomorphism of topological groups

$$\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g}), \quad \alpha \mapsto T_{\mathbf{1}}\alpha$$

is a homeomorphism onto a closed subgroup of the Lie group $\mathrm{Aut}(\mathfrak{g})$ by [Ho52, Th. 1]. Thus $\mathrm{Aut}(G)$ is topologically isomorphic to a Lie group, hence it is in turn a Lie group. If moreover G is simply connected, then the above homomorphism is surjective, hence it gives an isomorphism of Lie groups $\mathrm{Aut}(G) \simeq \mathrm{Aut}(\mathfrak{g})$.

(ii) If G is not necessarily connected, let $G_{\mathbf{1}}$ be the connected component of $\mathbf{1} \in G$, which is a (connected, open, and closed) normal subgroup of G. If the quotient group $G/G_{\mathbf{1}}$ is finitely generated (equivalently, if G is generated by some compact subset), then $\mathrm{Aut}(G)$ is a Lie group with countably many connected components by [Ho52, Th. 2].

Let G be any locally compact group with $\mathrm{Aut}(G)$ viewed as a topological group with its compact open topology. We will regard G as an $\mathrm{Aut}(G)$-space via the tautological action $\mathrm{Aut}(G) \times G \to G$, which is a continuous left action.

Now fix any subgroup $H \subseteq \mathrm{Aut}(G)$. For any principal H-bundle $q\colon P \to M$ we construct the associated bundle $Q\colon P \times_H G \to M$. This is a locally trivial bundle with fiber G and structure group H, hence a bundle of locally compact groups.

A bundle of locally compact groups with fiber G as above can be equivalently defined as a locally trivial bundle with fiber G, whose transition functions are continuous H-valued functions and for which the coordinate changes act by topological group automorphisms of the fiber G.

If G is a connected Lie group (or if it is compactly generated) so that $\mathrm{Aut}(G)$ is a Lie group, and if moreover q is a smooth principal bundle, then it follows that Q is a Lie group bundle.

3.1. **Duals of C^*-algebras of regular groupoids.** In this subsection we discuss a class of groupoids including the examples that motivated the present investigation, and for which the irreducible representations of their corresponding C^*-algebras can be described in terms of the representation theory of their isotropy groups.

We will use the following terminology which is inspired by (but slightly different from) [Go10] and [Go12]; see also [Ra90].

Definition 3.5. A locally compact groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ endowed with a left Haar system λ is called *regular* if it satisfies the following additional conditions:

(1) Every orbit of $\mathcal{G}$ is a locally closed subset of $\mathcal{G}^{(0)}$.
(2) The isotropy subgroupoid $\mathcal{G}(\cdot) := \bigsqcup\limits_{x\in\mathcal{G}^{(0)}} \mathcal{G}(x)$ is a group bundle.

In the above definition, the condition that $\mathcal{G}(\cdot)$ be a group bundle actually requires that the natural projection $\mathcal{G}(\cdot) \to \mathcal{G}^{(0)}$ is an open map, and this implies that $\mathcal{G}(\cdot)$ is a locally compact groupoid with a Haar system (see Remark 3.2).

Lemma 3.6. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a regular groupoid. The following assertions hold:*

(i) The C^*-algebra $C^*(\mathcal{G}(\cdot))$ is a $\mathcal{C}_0(\mathcal{G}^{(0)})$-algebra and is $\mathcal{C}_0(\mathcal{G}^{(0)})$-linearly $*$-isomorphic to the algebra of sections of an upper semicontinuous C^*-bundle over $\mathcal{G}^{(0)}$ whose fiber over any $x \in \mathcal{G}^{(0)}$ is $C^*(\mathcal{G}(x))$.

(ii) On the level of dual spaces of C^*-algebras one has

$$\widehat{C^*(\mathcal{G}(\cdot))} = \bigsqcup_{x\in\mathcal{G}^{(0)}} \widehat{C^*(\mathcal{G}(x))}$$

and there exists a continuous map $\widehat{C^*(\mathcal{G}(\cdot))} \to \mathcal{G}^{(0)}$ whose fiber over any $x \in \mathcal{G}^{(0)}$ is $\widehat{C^*(\mathcal{G}(x))}$.

Proof. For Assertion (i), let $(A, \mathcal{G}(\cdot), \alpha)$ be the groupoid dynamical system defined by the trivial action of $\mathcal{G}(\cdot)$ on the upper semicontinuous C^*-bundle with constant 1-dimensional fibers whose corresponding C^*-algebra of sections is $\mathcal{C}_0(\mathcal{G}^{(0)})$. Then $A \rtimes_\alpha \mathcal{G}(\cdot) = C^*(\mathcal{G}(\cdot))$, hence this is a $\mathcal{C}_0(\mathcal{G}^{(0)})$-algebra by [Go12, Prop. 1.2]. The second part of the assertion then follows by [Wi07, Th. C.26]. In order to describe the fibers of the corresponding upper semicontinuous C^*-bundle, we may use for every $x \in \mathcal{G}^{(0)}$ the short exact sequence

$$0 \to C^*(\mathcal{G}(\cdot)_{\mathcal{G}^{(0)}\setminus\{x\}}) \to C^*(\mathcal{G}(\cdot)) \xrightarrow{\mathcal{R}_x} C^*(\mathcal{G}(x)) \to 0$$

provided by Proposition 2.2(i) for $F = \{x\}$, which is a closed subset of $\mathcal{G}^{(0)}$ because $\mathcal{G}$ is Hausdorff.

Assertion (ii) then follows by [Wi07, Prop. C.5] (see also [Go12, Rem. 1.3]). $\square$

With the above lemma at hand, we can prove the following theorem, where we use the notation Ind for induced representations of groupoids (see for instance [IW09]).

Theorem 3.7. *For a regular groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$, define*

$$\Phi\colon \widehat{C^*(\mathcal{G}(\cdot))} \to \widehat{C^*(\mathcal{G})}, \quad \Phi([\pi]) := [\mathrm{Ind}_{\mathcal{G}(x)}^{\mathcal{G}} \pi],$$

where $[\pi] \in \widehat{\mathcal{G}(x)} \simeq \widehat{C^*(\mathcal{G}(x))} \subseteq \widehat{C^*(\mathcal{G}(\cdot))}$ for $x \in \mathcal{G}^{(0)}$. Then Φ is a continuous open surjective map, and it induces a homeomorphism

$$\widehat{C^*(\mathcal{G}(\cdot))}/\mathcal{G} \simeq \widehat{C^*(\mathcal{G})}$$

where the left-hand side is the quotient space of $\widehat{C^*(\mathcal{G}(\cdot))} \simeq \bigsqcup\limits_{x\in\mathcal{G}^{(0)}} \widehat{\mathcal{G}(x)}$ by the natural action of $\mathcal{G}$.

Proof. The definition of Φ is correct by [Go10, Prop. 4.13], where it was established that then every irreducible representation of some isotropy group of $\mathcal{G}$ induces an irreducible representation of $C^*(\mathcal{G})$. The assertion follows by Lemma 3.6 and [Go12, Th. 2.22]. $\square$

Remark 3.8. Let $p\colon A \to X$ be an open continuous and surjective mapping, where A and X are topological spaces. Define the equivalence relation

$$R = \{(a, b) \mid p(a) = p(b)\}.$$

Then it is easy to see that the map $\theta\colon A/R \to X$ defined by $\theta(p^{-1}(x)) = x$, is a homeomorphism, where A/R is endowed with the quotient topology. Thus, in particular, if X is locally compact and Hausdorff, then the space A/R is also locally compact and Hausdorff.

For a locally compact space T we denote by βT its Stone-Čech compactification.

Corollary 3.9. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a regular groupoid whose the isotropy groups $\mathcal{G}(x)$ are of type I and amenable. Assume in addition that the orbit space $\mathcal{G}^{(0)}/\mathcal{G}$ is locally compact and Hausdorff. Then $M(C^*(\mathcal{G}))$ is a $C(\beta(\mathcal{G}^{(0)}/\mathcal{G}))$-algebra.*

Proof. We first note that $C^*(\mathcal{G})$ is of type I, by [Cl07, Thm. 7.2]. From Lemma 3.6 we have that $C^*(\mathcal{G}(\cdot))$ is a $C_0(\mathcal{G}^{(0)})$-algebra, hence there is an continuous map $\widehat{C^*(\mathcal{G}(\cdot))} \to \mathcal{G}^{(0)}$. This map commutes with the natural actions of the groupoid $\mathcal{G}$ on its domain and range, hence it induces a continuous map $\widehat{C^*(\mathcal{G}(\cdot))}/\mathcal{G} \to \mathcal{G}^{(0)}/\mathcal{G}$. By Theorem 3.7 we obtain a continuous map $\widehat{C^*(\mathcal{G})} \to \mathcal{G}^{(0)}/\mathcal{G}$. This implies, by [AS11, Prop. 1.2], that $M(C^*(\mathcal{G}))$ is a $C(\beta(\mathcal{G}^{(0)}/\mathcal{G}))$-algebra. This concludes the proof. $\square$

4. Pullback of groupoids

A general method of constructing new examples of groupoids is the pullback. As we will see below, the operation of pullback by continuous open surjections preserves most of the properties that are relevant from the operator algebraic perspective, as for instance the topological properties of the orbits, the homeomorphism class of the orbit space, the isomorphism classes of isotropy groups, and existence of Haar systems.

Definition 4.1. Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a groupoid and $\theta\colon N \to \mathcal{G}^{(0)}$ be any map. The *pullback* of $\mathcal{G}$ by θ is the groupoid $\theta^{\downarrow\downarrow}(\mathcal{G}) \rightrightarrows N$ defined by

$$\theta^{\downarrow\downarrow}(\mathcal{G}) := \{(n_1, g, n_2) \in N \times \mathcal{G} \times N \mid g \in \mathcal{G}_{\theta(n_1)}^{\theta(n_2)}\}$$

with its domain/range maps defined by $d(n_1, g, n_2) = n_1$ and $r(n_1, g, n_2) = n_2$ for all $(n_1, g, n_2) \in \theta^{\downarrow\downarrow}(\mathcal{G})$. The middle projection $\Theta(n_1, g, n_2) = g$, defines a map $\Theta\colon \theta^{\downarrow\downarrow}(\mathcal{G}) \to \mathcal{G}$ which is a groupoid morphism.

If $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a topological groupoid, N a topological space and θ is continuous, then $\theta^{\downarrow\downarrow}(\mathcal{G}) \rightrightarrows N$ is a topological groupoid with its topology induced from $N \times \mathcal{G} \times N$.

We show in Proposition 4.8 below that pullbacks of group bundles by surjective open maps are regular groupoids.

Remark 4.2. The map Θ makes the following diagram commutative (in which the vertical arrows are either both domain maps or both range maps)

$$\begin{array}{ccc} \theta^{\downarrow\downarrow}(\mathcal{G}) & \xrightarrow{\Theta} & \mathcal{G} \\ \Big\downarrow & & \Big\downarrow \\ N & \xrightarrow{\theta} & \mathcal{G}^{(0)} \end{array}$$

and moreover the map Θ gives a bijection $(\theta^{\downarrow\downarrow}(\mathcal{G}))_{n_1}^{n_2} \to \mathcal{G}_{\theta(n_1)}^{\theta(n_2)}$, for all $n_1, n_2 \in N$.

In particular, for every $n_0 \in N$, one has the algebraic isomorphism of isotropy groups $(\theta^{\downarrow\downarrow}(\mathcal{G}))_{n_0}^{n_0} \to \mathcal{G}_{\theta(n_0)}^{\theta(n_0)}$, which is also an isomorphism of topological groups if $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a topological groupoid.

Remark 4.3. For every $n_0 \in N$, its corresponding d-fiber of $\theta^{\downarrow\downarrow}(\mathcal{G})$ can be described as

$$(\theta^{\downarrow\downarrow}(\mathcal{G}))_{n_0} = \{(n_0, g, n) \in \theta^{\downarrow\downarrow}(\mathcal{G}) \mid g \in \mathcal{G}_{\theta(n_0)},\ r(g) = \theta(n)\} \tag{4.1}$$

hence one has the commutative diagram

$$\begin{array}{ccc} (\theta^{\downarrow\downarrow}(\mathcal{G}))_{n_0} & \xrightarrow{\Theta} & \mathcal{G}_{\theta(n_0)} \\ {\scriptstyle r}\Big\downarrow & & \Big\downarrow{\scriptstyle r} \\ N & \xrightarrow{\theta} & \mathcal{G}^{(0)} \end{array}$$

Thus $(\theta^{\downarrow\downarrow}(\mathcal{G}))_{n_0}$ is in fact the fiber product $\mathcal{G}_{\theta(n_0)} \times_{\mathcal{G}^{(0)}} N$.

Remark 4.4. The $\theta^{\downarrow\downarrow}(\mathcal{G})$-orbit of the point $n_0 \in N$ is the inverse image through θ of the $\mathcal{G}$-orbit of the point $\theta(n_0) \in \mathcal{G}^{(0)}$, since, using (4.1), one obtains

$$(\theta^{\downarrow\downarrow}(\mathcal{G})).n_0 = r((\theta^{\downarrow\downarrow}(\mathcal{G}))_{n_0}) = \theta^{-1}(r(\mathcal{G}_{\theta(n_0)})) = \theta^{-1}(\mathcal{G}.\theta(n_0)). \tag{4.2}$$

This shows that if both $\mathcal{G}^{(0)}$ and N are topological spaces, θ is continuous, and the $\mathcal{G}$-orbit of the point $\theta(n_0)$ is an open/closed/locally closed subset of $\mathcal{G}^{(0)}$, then so is the $\theta^{\downarrow\downarrow}(\mathcal{G})$-orbit of the point n_0 in N.

Proposition 4.5. *The orbit spaces of the groupoids $\theta^{\downarrow\downarrow}(\mathcal{G})$ and $\mathcal{G}$ are related by*

$$\beta\colon N/\theta^{\downarrow\downarrow}(\mathcal{G}) \to \mathcal{G}^{(0)}/\mathcal{G}, \quad (\theta^{\downarrow\downarrow}(\mathcal{G})).n \mapsto \mathcal{G}.\theta(n)$$

which is a well-defined map and has the following properties:

(i) β is injective;
(ii) if the image of θ intersects every $\mathcal{G}$-orbit, then also β is surjective;
(iii) if $\mathcal{G}$ is a topological groupoid and the map θ is continuous, then β is continuous with respect to the quotient toplogies on the orbit spaces;
(iv) if $\mathcal{G}$ is a topological groupoid whose range map $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is an open map, and the map θ is continuous, open, and surjective, then β is a homeomorphism.

Proof. In fact, properties (i)–(ii) follow by (4.2). For proving property (iii), one needs the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\ \theta\ } & \mathcal{G}^{(0)} \\ \Big\downarrow & & \Big\downarrow \\ N/\theta^{\downarrow\downarrow}(\mathcal{G}) & \xrightarrow{\ \beta\ } & \mathcal{G}^{(0)}/\mathcal{G} \end{array}$$

whose vertical arrows are the quotient maps, and in which the map Θ is continuous. For property (iv) we also use the above diagram to check that if both θ and the quotient map $\mathcal{G}^{(0)} \to \mathcal{G}^{(0)}/\mathcal{G}$ are open, then β is an open map, and then the assertion follows using also properties (i)–(iii). It remains to note that since the range map $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is an open map, it follows that the quotient map $q\colon \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}/\mathcal{G}$ is always open, because for every open set $U \subseteq \mathcal{G}^{(0)}$ one has $q^{-1}(q(U)) = r(d^{-1}(U))$ which is open in $\mathcal{G}^{(0)}$, hence $q(U) \subseteq \mathcal{G}^{(0)}/\mathcal{G}$ is open by the definition of the quotient topology. □

Proposition 4.6. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be any locally compact groupoid with a left Haar system, and N be any second countable, locally compact topological space. If $\theta\colon N \to \mathcal{G}^{(0)}$ is any continuous open map and the image of θ intersects every $\mathcal{G}$-orbit, then $\theta^{\downarrow\downarrow}(\mathcal{G}) \rightrightarrows N$ is a locally compact groupoid with a left Haar system, and the C^*-algebras $C^*(\mathcal{G})$ and $C^*(\theta^{\downarrow\downarrow}(\mathcal{G}))$ are Morita equivalent.*

Proof. The fibered product

$$\mathcal{G} \times_{\mathcal{G}^{(0)}} N := \{(g,n) \in \mathcal{G} \times N \mid d(g) = \theta(n)\}$$

gives a Morita equivalence from the groupoid $\mathcal{G}$ to the groupoid $\theta^{\downarrow\downarrow}(\mathcal{G})$. See [Mr96, Cor. II.1.7] for some more details in this connection (and also [MM03, Ex. 5.10(4)] for the special case of Lie groupoids). The hypothesis that $\theta\colon N \to \mathcal{G}^{(0)}$ is an open map is needed in order to ensure that the map

$$\mathcal{G} \times_{\mathcal{G}^{(0)}} N \to \mathcal{G}^{(0)}, \quad (g,n) \mapsto r(g)$$

is open, hence the canonical groupoid morphism $\Theta\colon \theta^{\downarrow\downarrow}(\mathcal{G}) \to \mathcal{G}$ is an essential equivalence.

Now, since the groupoids $\mathcal{G}$ and $\theta^{\downarrow\downarrow}(\mathcal{G})$ are equivalent and $\mathcal{G}$ has a Haar system, it follows by [Wi15, Th. 2.1] that also $\theta^{\downarrow\downarrow}(\mathcal{G})$ has a Haar system, and then [MRW87, Th. 2.8] implies that the C^*-algebras $C^*(\mathcal{G})$ and $C^*(\theta^{\downarrow\downarrow}(\mathcal{G}))$ are Morita equivalent. □

Lemma 4.7. *Let $p\colon \mathcal{T} \to S$ be any group bundle and $\theta\colon \Xi \to S$ be any continuous map, and define $\theta^*(\mathcal{T}) := \{(\xi,t) \in \Xi \times \mathcal{T} \mid \theta(\xi) = p(t)\} \subseteq \Xi \times \mathcal{T}$ with its relative topology. Then $\theta^*(\mathcal{T}) \rightrightarrows \Xi$, $(\xi,t) \mapsto \xi$, has the canonical structure of a group bundle.*

Proof. It is clear that the fibers of $p_1\colon \theta^*(\mathcal{T}) \rightrightarrows \Xi$, $p_1(\xi,t) := \xi$, are locally compact groups, hence it remains to check that p_1 is an open map. To this end, for arbitrary open subsets $V \subseteq \Xi$ and $W \subseteq \mathcal{T}$ with $(V \times W) \cap \theta^*(p) \neq \emptyset$, we must prove that $p_1((V \times W) \cap \theta^*(p))$ is an open subset of Ξ. One has

$$p_1((V \times W) \cap \theta^*(p)) = \{\xi \in \Xi \mid (\exists t \in W)\ \theta(\xi) = p(t)\} = \theta^{-1}(p(W))$$

and this is an open subset of Ξ since θ is continuous and p is an open map. $\square$

Proposition 4.8. *Let $\mathcal{T} \to S$ be any group bundle and $\theta\colon N \to S$ be any surjective open map. Then $\theta^{\downarrow\downarrow}(\mathcal{T}) \rightrightarrows N$ is a regular groupoid.*

Proof. By Remark 3.2, the group bundle $\mathcal{T} \to S$ has a Haar system. It then follows by Proposition 4.6 that $\theta^{\downarrow\downarrow}(\mathcal{T}) \rightrightarrows N$ is a locally compact groupoid with a Haar system. Therefore, according to Definition 3.5, it remains to check that the isotropy subgroupoid of $\theta^{\downarrow\downarrow}(\mathcal{T}) \rightrightarrows N$ is a group bundle. But it is easily seen that the isotropy subgroupoid of $\theta^{\downarrow\downarrow}(\mathcal{T}) \rightrightarrows N$ is $\theta^*(\mathcal{T}) \rightrightarrows N$, which is a group bundle by Lemma 4.7. This completes the proof. $\square$

We now introduce a special class of groupoids which prove to be suitable for describing the coadjoint transformation groups associated to exponential solvable Lie groups. The importance of these groupoids stems from Propositions 4.6 and 4.8 and Theorem 3.7. Loosely speaking, these results together imply that the dual spaces of C^*-algebras of these groupoids can be computed from C^*-algebras of group bundles.

Definition 4.9. Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be any locally compact groupoid. We say that $\mathcal{G}$ is a *piecewise pullback of group bundles* with *pieces* V_k for $k = 1, \dots, n$ if the following conditions are satisfied.

(i) There exists an increasing family
$$\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = \mathcal{G}^{(0)}$$
where U_k are open $\mathcal{G}$-invariant subsets of $\mathcal{G}^{(0)}$ and
$$V_k = U_k \setminus U_{k-1}$$
for $k = 1, \dots, n$.

(ii) For $k = 1, \dots, n$, there exist an open continuous surjective map $\theta_k\colon V_k \to S_k$ and a group bundle $\mathcal{T}_k \to S_k$ for which one has an isomorphism of topological groupoids $\mathcal{G}_{V_k} \simeq \theta_k^{\downarrow\downarrow}(\mathcal{T}_k)$.

Remark 4.10. In Definition 4.9, the orbits of $\mathcal{G}$ are exactly the sets $\theta_k^{-1}(x)$ for $x \in S_k$ and $k = 1, \dots, n$. This shows that for every k, the orbits of the reduced groupoid $\mathcal{G}_{V_k}$ are closed, and moreover the orbit space of that groupoid is homeomorphic to S_k, by Remark 3.8 or Proposition 4.5.

We also note that $\mathcal{G}_{V_1}$ is a pair groupoid if and only if the set $\mathcal{T}_1$ is a singleton.

Theorem 4.11. *Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a piecewise pullback of group bundles that has a left Haar system. Then the following assertions hold:*

(i) The orbits of $\mathcal{G}$ are locally closed subsets of $\mathcal{G}^{(0)}$.

(ii) If the isotropy groups of $\mathcal{G}$ are amenable, then $\mathcal{G}$ is metrically amenable.

(iii) The C^*-algebra of $\mathcal{G}$ has a sequence of closed two-sided ideals
$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = C^*(\mathcal{G})$$
such that for $k = 1, \dots, n$ the subquotient $\mathcal{J}_k/\mathcal{J}_{k-1}$ is Morita equivalent to the C^*-algebra of sections of an upper semicontinuous C^*-bundle whose fibers are C^*-algebras of isotropy groups of $\mathcal{G}$, and every isotropy group of $\mathcal{G}$ occurs in this way for exactly one value of k.

(iv) For $k = 1, \dots, n$, the multiplier algebra of the C^*-algebra $\mathcal{J}_k/\mathcal{J}_{k-1}$ is a $C(\beta(V_k/\mathcal{G}))$-algebra, where V_k for $k = 1, \dots, n$ are the pieces of $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$.
(v) If all isotropy groups of $\mathcal{G}$ are amenable, the C^*-algebra of $\mathcal{G}$ is of type I if and only if all the isotropy groups of $\mathcal{G}$ are of type I.

Proof. Assertion (i) follows directly by Remark 4.10.

Furthermore, since the orbits of $\mathcal{G}$ are locally closed and $\mathcal{G}$ is locally compact and second countable, it follows by [Ra90, Th. 2.1((5)⇔(4))] that the orbit space of $\mathcal{G}$ is a topological space of type T_0. Then [SW13, Th. 4] implies that if the isotropy groups of $\mathcal{G}$ are amenable, then $\mathcal{G}$ is a measurewise amenable groupoid, hence it is also metrically amenable (see for instance [Re80, Sect. II.3]), and this concludes the proof of Assertion (ii).

For Assertion (iii), use the bijective correspondence from Proposition 2.2(i) to define $\mathcal{J}_k := I(U_k)$ for $k = 0, \dots, n$. Then one has the isomorphism of C^*-algebras $\mathcal{J}_k/\mathcal{J}_{k-1} \simeq C^*(\mathcal{G}_{V_k}^{V_k})$. On the other hand, one has an isomorphism of topological groupoids $\mathcal{G}_{V_k} \simeq \theta_k^{\downarrow\downarrow}(\mathcal{T}_k)$ by Definition 4.9, where $\theta_k \colon V_k \to S_k$ is an open continuous injection and $\mathcal{T}_k \rightrightarrows S_k$ is a group bundle. Hence Proposition 4.6 implies that the C^*-algebras $C^*(\mathcal{G}_{V_k})$ and $C^*(\mathcal{T}_k)$ are Morita equivalent.

For Assertion (iv) It follows by Corollary 3.9 that $M(C^*(\mathcal{G}_{V_k}))$ is a $C(\beta(V_k/\mathcal{G}))$-algebra, because the orbit space $V_k/\mathcal{G}$ is Hausdorff by Remark 4.10.

Finally, since $\mathcal{T}_k \rightrightarrows S_k$ is a a group bundle, it follows by Lemma 3.6 that $C^*(\mathcal{T}_k)$ is a $\mathcal{C}_0(S_k)$-algebra. This, along with [Cl07, Thm. 7.1], also implies Assertion (v), because if two C^*-algebras are Morita equivalent, then one of them is of type I if and only if the other one is. $\square$

5. Application to coadjoint dynamical systems

Let G be any exponential solvable Lie group with its Lie algebra $\mathfrak{g}$. One has the coadjoint action of G on $\mathfrak{g}^*$, and this allows us to define the semidirect product $\mathfrak{g}^* \rtimes G$. This is naturally diffeomorphic to the cotangent bundle T^*G, and thus the Lie group $\mathfrak{g}^* \rtimes G$ can be called the cotangent group of G. One has the short exact sequence of Lie groups

$$0 \to \mathfrak{g}^* \to \mathfrak{g}^* \rtimes G \to G \to \mathbf{1}$$

which leads to a short exact sequence of C^*-algebras

$$0 \to \mathcal{J} \to C^*(\mathfrak{g}^* \rtimes G) \to C^*(G) \to 0.$$

One also has $*$-isomorphisms

$$C^*(\mathfrak{g}^* \rtimes G) \simeq C^*(\mathfrak{g}^*) \rtimes G \simeq \mathcal{C}_0(\mathfrak{g}) \rtimes G$$

where we have used the same notation for semidirect products of groups and for crossed products of C^*-algebras acted on by groups. We are in a situation in which much information on representation theory of G is encoded in the transformation group defined by the coadjoint action $\mathrm{Ad}_G^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$, and the same is valid for the corresponding C^*-algebras. More precisely, the method of coadjoint orbits gives a homeomorphism $\mathfrak{g}^*/G \simeq \widehat{G}$ (see [LeLu94]), and the remark above shows that the transformation group C^*-algebra $C^*(\mathfrak{g}^* \rtimes G)$ may also hold an important role for understanding the group algebra $C^*(G)$.

5.1. **Main result.**

Lemma 5.1. *Let H be any locally compact group with its space of closed subgroups denoted by $\mathcal{S}$. Define $\mathcal{T} := \{(x, K) \in H \times \mathcal{S} \mid x \in K\} \subseteq H \times \mathcal{S}$ with its relative topology. Then the canonical projection $p\colon \mathcal{T} \to \mathcal{S}$, $p(x, K) := K$, is a group bundle.*

Proof. We recall from [Gl62] that $\mathcal{S}$ is a compact space and a basis of its topology consists of the sets

$$\mathcal{U}(C, \mathcal{F}) := \{K \in \mathcal{S} \mid K \cap C = \emptyset;\ (\forall A \in \mathcal{F})\ K \cap A \neq \emptyset\}$$

for all compact sets $C \subseteq H$ and all finite sets $\mathcal{F}$ of open subsets of H. We must prove that for any set $\mathcal{U}(C, \mathcal{F})$ as above and any open set $D \subseteq H$, if $(\mathcal{U}(C, \mathcal{F}) \times D) \cap \mathcal{T} \neq \emptyset$, then $p((\mathcal{U}(C, \mathcal{F}) \times D) \cap \mathcal{T})$ is an open subset of $\mathcal{S}$. Indeed, for any $K \in \mathcal{S}$, we have $K \in p((\mathcal{U}(C, \mathcal{F}) \times D) \cap \mathcal{T})$ if and only if $K \in \mathcal{U}(C, \mathcal{F})$ and there exists $x \in D \cap K$, since if that is the case, then $(K, x) \in (\mathcal{U}(C, \mathcal{F}) \times D) \cap \mathcal{T}$ and $K = p(K, x)$. Therefore

$$p((\mathcal{U}(C, \mathcal{F}) \times D) \cap \mathcal{T}) = \{K \in \mathcal{U}(C, \mathcal{F}) \mid K \cap D \neq \emptyset\} = \mathcal{U}(C, \mathcal{F} \cup \{D\})$$

which is an open subset of $\mathcal{S}$, and this concludes the proof. $\square$

Theorem 5.2. *The coadjoint dynamical system of any exponential solvable Lie group is a piecewise pullback of group bundles.*

Proof. Let G be an exponential solvable Lie group, with its coadjoint dynamical system $(G, \mathfrak{g}^*, \mathrm{Ad}^*_G)$. It is known from [Cu92] that there exists a partition $\mathfrak{g}^* = \bigsqcup_{k=1}^{n} A_k$ into G-invariant, Zariski-open subsets satisfying the following conditions for $k = 1, \dots, n$:

(1) The set $D_k := \bigsqcup_{j=1}^{k} A_j$ is open in $\mathfrak{g}^*$.
(2) The quotient map $q|_{A_k}\colon A_k \to A_k/\mathrm{Ad}^*_G$ admits a continuous cross-section $\gamma_k\colon A_k/\mathrm{Ad}^*_G \to A_k$, and we denote $\Xi_k := \gamma_k(A_k/\mathrm{Ad}^*_G) \subseteq \mathfrak{g}^*$.
(3) The map $\xi \mapsto G(\xi)$ is continuous from A_k into the space of closed subgroups of G.

In fact, the above property (3) is not explicitly mentioned in [Cu92], but it follows by straightforward arguments, using the fact that the dimension of coadjoint G-orbits contained in A_k is constant.

Then we may define the bundle of coadjoint isotropy groups $\Gamma_k := \bigsqcup\limits_{\xi \in \Xi_k} G(\xi)$, with its canonical projection $\Pi_k\colon \Gamma_k \to \Xi_k$, where $\Pi_k^{-1}(\xi) = G(\xi)$ for all $\xi \in \Xi_k$. It follows by Lemmas 5.1 and 4.7 that $\Pi_k\colon \Gamma_k \to \Xi_k$ is a group bundle, hence it has a Haar system.

Let $\mathcal{G}_k$ be the transformation-group groupoid defined by the coadjoint action of G on A_k. Hence one has $\mathcal{G}_k^{(0)} = A_k$, $\mathcal{G}_k = G \times A_k$, the domain map $d\colon G \times A_k \to A_k$, $d(g, \xi) := \xi$, and the range map $r\colon G \times A_k \to A_k$, $d(g, \xi) := \mathrm{Ad}^*_G(g)\xi$. Then the map $\sigma_k\colon A_k \to G \times A_k$, $\sigma_k(\xi) := (\mathbf{1}, \xi)$, satisfies $d \circ \sigma = \mathrm{id}$.

Then define $\theta := \gamma_k \circ q|_{A_k}\colon A_k \to \Xi_k$, and this is an open continuous surjective map. The pullback of Π_k by θ_k is

$$\theta_k^{\downarrow\downarrow}(\Pi_k) := \{(\zeta, h, \eta) \in A_k \times \Gamma_k \times A_k \mid \theta_k(\zeta) = \Pi_k(h) = \theta_k(\eta)\}.$$

The projections on the first and third coordinates define a groupoid $\theta_k^{\downarrow\downarrow}(\Pi_k) \rightrightarrows A_k$ and the map

$$\Phi_k \colon \mathcal{G}_k \to \theta_k^{\downarrow\downarrow}(\Pi_k), \quad \Phi_k(g) := (r(g), \sigma_k(r(g)) g \sigma_k(d(g))^{-1}, d(g))$$

is a groupoid isomorphism, with its inverse given by a formula similar to (1.2). These explicit formulas for Φ_k and Φ_k^{-1}, involving compositions of continuous maps, show that these maps are homeomorphisms, hence they are isomorphisms of topological groupoids. Thus the conditions of Definition 4.9, and this completes the proof. $\square$

Remark 5.3. In the above proof, in order to compute $\theta_k^{\downarrow\downarrow}(\Pi_k)$ more explicitly, note that for any $\xi, \eta \in A_k$ we have

$$\begin{aligned} \theta_k(\zeta) = \theta_k(\eta) &\iff q(\zeta) = q(\eta) \\ &\iff (\exists \mathcal{O} \in \mathfrak{g}^*/G)\ \zeta, \eta \in \mathcal{O} \subseteq A_k \\ &\iff (\exists \xi \in \Xi_k)\ \zeta, \eta \in \mathrm{Ad}_G^*(G)\xi \end{aligned}$$

and if this is the case then $(\zeta, h, \eta) \in \theta_k^{\downarrow\downarrow}(\Pi_k)$ if and only if $\Pi_k(h) = \xi$, that is, if and only if $h \in G(\xi)$. Therefore

$$\theta_k^{\downarrow\downarrow}(\Pi_k) = \bigsqcup_{\xi \in \Xi_k} \mathcal{O}_\xi \times G(\xi) \times \mathcal{O}_\xi$$

where $\mathcal{O}_\xi := \mathrm{Ad}_G^*(G)\xi \subseteq \mathfrak{g}^*$ is the coadjoint orbit of $\xi \in \Xi_k$.

5.2. **Some specific examples.** In the following example we use somewhat unusual notation in structure theory of semisimple Lie groups (see for instance [Kn02]), in the sense that we keep the notation G for a Borel subgroup, which is the solvable Lie group whose coadjoint orbits we wish to discuss here.

Example 5.4. Let $\mathfrak{s}$ be any complex semisimple Lie algebra, hence its Killing form

$$B_{\mathfrak{s}} \colon \mathfrak{s} \times \mathfrak{s} \to \mathbb{C}, \quad B_{\mathfrak{s}}(X, Y) := \mathrm{Tr}((\mathrm{ad}_{\mathfrak{s}} X)(\mathrm{ad}_{\mathfrak{s}} Y))$$

is nondegenerate, where for every $X \in \mathfrak{s}$ we define as usual $\mathrm{ad}_{\mathfrak{s}} X \colon \mathfrak{s} \to \mathfrak{s}$, $(\mathrm{ad}_{\mathfrak{s}} X)Y = [X, Y]$ for $X, Y \in \mathfrak{s}$. Then there exists a conjugate-linear mapping $\mathfrak{s} \to \mathfrak{s}$, $X \mapsto X^*$, such that for all $X, Y \in \mathfrak{s}$ we have $(X^*)^* = X$, $[X, Y]^* = [Y^*, X^*]$ and $B_{\mathfrak{s}}(X, X^*) \geq 0$. Such a mapping is unique up to an automorphism of $\mathfrak{s}$.

Select any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, that is, $\mathfrak{h}$ is a maximal abelian self-adjoint subalgebra of $\mathfrak{s}$. Denote the eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{s}$ by

$$\mathfrak{s}^\alpha := \{X \in \mathfrak{s} \mid (\forall H \in \mathfrak{h}) \quad [H, X] = \alpha(H)X\} \text{ for a linear functional } \alpha \colon \mathfrak{h} \to \mathbb{C}$$

and consider the corresponding set of roots $\Delta(\mathfrak{s}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq \{0\}\}$. Then $\mathfrak{h} = \mathfrak{s}^0 = \{X \in \mathfrak{s} \mid [\mathfrak{h}, X] = \{0\}\}$ and the root space decomposition of $\mathfrak{h}$ is given by

$$\mathfrak{s} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{s}, \mathfrak{h})} \mathfrak{s}^\alpha.$$

For every $\alpha \in \Delta(\mathfrak{s}, \mathfrak{h})$ we have $\dim \mathfrak{s}^\alpha = 1$ and we may choose $X_\alpha \in \mathfrak{s}^\alpha \setminus \{0\}$ such that $X_\alpha^* = X_{-\alpha}$. Let $\{\alpha_1, \dots, \alpha_r\} \subset \Delta(\mathfrak{s}, \mathfrak{h})$ be a system of simple roots, that is, a maximal subset with the property $\alpha_j + \alpha_k \notin \Delta(\mathfrak{s}, \mathfrak{h})$ for all $j, k = 1, \dots, r$. One has $r = \dim \mathfrak{h}$, and this called the rank of $\mathfrak{s}$.

If we also define $\Delta^+(\mathfrak{s},\mathfrak{h})$ as the subset of $\Delta(\mathfrak{s},\mathfrak{h})$ consisting of linear combinations of symple roots with nonnegative integer coefficients, then one has $\Delta(\mathfrak{s},\mathfrak{h}) = \Delta^+(\mathfrak{s},\mathfrak{h}) \sqcup (-\Delta^+(\mathfrak{s},\mathfrak{h}))$, hence the above root space decomposition of $\mathfrak{s}$ implies the triangular decomposition

$$\mathfrak{s} = \mathfrak{n}^- \dotplus \mathfrak{h} \dotplus \mathfrak{n}^+$$

where $\mathfrak{n}^\pm := \bigoplus\limits_{\pm\alpha\in\Delta^+(\mathfrak{s},\mathfrak{h})} \mathfrak{s}^\alpha$.

We are mainly interested in the Lie algebra

$$\mathfrak{g} := \mathfrak{h} \dotplus \mathfrak{n}^+ = \mathfrak{h} \ltimes \mathfrak{n}^+,$$

which is a Borel subalgebra of $\mathfrak{g}$. This is a solvable complex Lie algebra, and we denote by G its corresponding connected, simply connected Lie group, however it is not an exponential Lie group.

It follows by [Ko12, Th. 1.7] that the group G has a nonempty open coadjoint orbit if and only if the rank of $\mathfrak{s}$ is equal to the cardinality of a maximal set of strongly orthogonal roots. If this is the case, then the open orbit is unique hence it is dense in $\mathfrak{s}^*$. By Proposition 2.7, the regular representation of $C^*(\mathfrak{g}^* \rtimes G)$ corresponding to that open orbit is faithful. By [Ko12, Rem. 2.8], if $\mathfrak{s}$ is a simple Lie algebra, then it satisfies the above condition if and only if it is a classical Lie algebra of type $\mathfrak{so}(2\ell+1,\mathbb{C})$, $\mathfrak{sp}(2\ell,\mathbb{C})$, $\mathfrak{so}(2\ell,\mathbb{C})$ with ℓ even, or an exceptional Lie algebra of type G_2, F_4, E_7, or E_8.

We now give one of the simplest examples of Lie groups that fall under the hypotheses of Theorem 5.2. Many other examples of transformation-group groupoids can be constructed following this pattern. See for instance Example 5.6 below.

Example 5.5 (the $ax+b$-group). Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in M_2(\mathbb{R}) \mid a > 0 \right\}$$

which is a multiplicative group of matrices. For the tautological action of G on $\mathbb{R}^2$ denote by $G_{x,y}$ and $\mathcal{O}_{x,y}$ the isotropy group and the G-orbit of any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Then one has

- if $y \neq 0$, then $G_{x,y} = \{\mathbf{1}\}$ and $\mathcal{O}_{x,y} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \mid uy > 0 \right\}$;
- if $y = 0 \neq x$, then
$$G_{x,0} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\} \text{ and } \mathcal{O}_{x,0} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid ux > 0 \right\};$$
- if $y = x = 0$, then $G_{0,0} = G$ and $\mathcal{O}_{0,0} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Consequently, if we define

$$M_\pm := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \pm y \geq 0,\ x^2 + y^2 \neq 0 \right\} \supset \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \pm y > 0 \right\} =: U_\pm$$

then $M_\pm$ is acted on by the group G with dense open orbit $U_\pm$ and the isotropy groups at points of the boundary $F := M_\pm \setminus U_\pm$ are all isomorphic to $(\mathbb{R},+)$.

Hence the transformation-group groupoids $\mathcal{G}_\pm := G \ltimes M_\pm \rightrightarrows M_\pm$ have dense open orbits $U_\pm$ with $(\mathcal{G}_\pm)_{U_\pm}^{U_\pm} = U_\pm \times U_\pm$, and moreover $(\mathcal{G}_\pm)_F^F = \mathbb{R} \times F \to F$ is a trivial group bundle on $F = \mathbb{R} \setminus \{0\}$. We get thus immediately the following short exact sequence

$$0 \to \mathcal{K}(L^2(U_\pm)) \to C^*(\mathcal{G}_\pm) \to \mathcal{C}_0(\mathbb{R} \setminus \{0\}, \mathcal{C}_0(\mathbb{R})) \to 0.$$

In the following example we describe a more general situation, which recovers Example 5.5 for $\mathfrak{s} = \mathfrak{su}(1,1)$.

Example 5.6. Now let $\mathfrak{s} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra. Select any maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, consider the corresponding restricted-root spaces and define

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{s}, \mathfrak{a})} \mathfrak{s}^\alpha$$

with respect to a lexicographic ordering on $\mathfrak{a}^*$. Then one has the Iwasawa decomposition $\mathfrak{s} = \mathfrak{k} \dotplus \mathfrak{a} \dotplus \mathfrak{n}$.

We are mainly interested in the Lie algebra

$$\mathfrak{g} := \mathfrak{a} \dotplus \mathfrak{n} = \mathfrak{a} \ltimes \mathfrak{n},$$

which is a Borel subalgebra of $\mathfrak{s}$. This is a completely solvable Lie algebra, hence its corresponding connected, simply connected Lie group G is an exponential Lie group. Moreover, denoting $r := \dim \mathfrak{a}$, one can prove that if $\mathfrak{s}$ is of hermitian type, then G has 2^r open coadjoint orbits (see [RV76, Prop. 3.3.1] and [In01, Th. 1.3]).

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