Wick polynomials and time-evolution of cumulants

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Abstract

We show how Wick polynomials of random variables can be defined combinatorially as the unique choice which removes all "internal contractions" from the related cumulant expansions, also in a non-Gaussian case. We discuss how an expansion in terms of the Wick polynomials can be used for derivation of a hierarchy of equations for the time-evolution of cumulants. These methods are then applied to simplify the formal derivation of the Boltzmann-Peierls equation in the kinetic scaling limit of the discrete nonlinear Schrödinger equation (DNLS) with suitable random initial data. We also present a reformulation of the standard perturbation expansion using cumulants which could simplify the problem of a rigorous derivation of the Boltzmann-Peierls equation by separating the analysis of the solutions to the Boltzmann-Peierls equation from the analysis of the corrections. This latter scheme is general and not tied to the DNLS evolution equations.

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1 Introduction

Wick polynomials, also called Wick products, arose first in quantum field theory as a way of regularizing products of field operators. The principal goal there was to replace monomial products by polynomials with state dependent coefficients, chosen so as to remove singular terms appearing in the associated perturbation expansion.

The procedure can also be applied in more general probabilistic settings. The following definition is given in Wikipedia [1] and in the Encyclopedia of Mathematics [2]. Consider n (real) random variables y_j, $j = 1, 2, \ldots, n$, on some probability space $(\Omega, \mathscr{B}, \mu)$ and denote expectation over the probability measure μ by $\langle\cdot\rangle$. The Wick polynomial with powers $k_j \geq 0$, $j = 1, 2, \ldots, n$, are then defined recursively in the total degree $k_1 + k_2 + \cdots + k_n$ by the following conditions:

1. If $k_1 = k_2 = \cdots = k_n = 0$, set $:y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}: = 1$.
2. If the total degree is greater than zero, require that $\langle :y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}: \rangle = 0$.
3. For all j, require that the (algebraic) derivatives of the Wick polynomials satisfy

$$\partial_{y_j} :y_1^{k_1} \cdots y_j^{k_j} \cdots y_n^{k_n}: = k_j :y_1^{k_1} \cdots y_j^{k_j - 1} \cdots y_n^{k_n}: . \tag{1.1}$$

These conditions have a unique solution for which $:y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}:$ is a polynomial of total degree $k_1 + k_2 + \cdots + k_n$ in the variables y_j. (The uniqueness is algebraic, not only almost everywhere as random variables. That is, the conditions fix all coefficients of the polynomials. This can be seen by induction in the order $|I|$: the requirement in item 3 fixes all new coefficients apart from the constant, which is then fixed by the vanishing of the expectation value in item 2.) The coefficients are polynomials of expectations of the random variables y_j, and hence depend on the measure μ. The first order polynomial is obtained by simply centering the variable, $:y_1: = y_1 - \langle y_1 \rangle$, but already at second order more complex structures appear, $:y_1 y_2: = y_1 y_2 - \langle y_1 \rangle y_2 - \langle y_2 \rangle y_1 - \langle y_1 y_2 \rangle + 2\langle y_1 \rangle \langle y_2 \rangle$.

If the random variables have joint exponential moments, i.e., if there is $\beta > 0$ such that $\langle \mathrm{e}^{\beta \sum_j |y_j|} \rangle < \infty$, the Wick polynomials can also be obtained by differentiating a fairly simple generating function. It can then be defined for $\lambda \in \mathbb{R}^n$, such that $|\lambda_j| < \beta$ for all j, by

$$G_{\mathrm{w}}(\lambda; y_1, \ldots, y_n) = \frac{\exp\left(\sum_{i=1}^n \lambda_i y_i\right)}{\langle \exp\left(\sum_{i=1}^n \lambda_i y_i\right) \rangle} , \tag{1.2}$$

and then for all $k_j \geq 0$, $j = 1, 2, \ldots, n$,

$$:y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}: = \partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_n}^{k_n} G_{\mathrm{w}}(\lambda; y_1, \ldots, y_n)\Big|_{\lambda = 0} . \tag{1.3}$$

The generating function $G_{\mathrm{w}}(\lambda; y_1, \ldots, y_n)$ is also called "Wick exponential" and often denoted by "$:\exp\left(\sum_{i=1}^n \lambda_i y_i\right):$". For a derivation and basic properties of such Wick polynomials, see [3].

The Wick polynomials become particularly simple to use if the joint measure of y is Gaussian. Defining the covariance matrix by $C_{j'j} := \mathrm{Cov}(y_{j'}, y_j)$, a Gaussian measure has $\langle \exp(\lambda \cdot y) \rangle = \mathrm{e}^{\lambda \cdot \langle y \rangle + \lambda \cdot C\lambda/2}$. Therefore, the generating function of the Wick polynomials then reads simply $G_{\mathrm{w}}(\lambda; y) = \exp[\lambda \cdot (y - \langle y \rangle) - \lambda \cdot C\lambda/2]$, and the resulting Wick polynomials are closely related to Hermite polynomials. This is the setting encountered in the original problem of renormalization of quantum field theories (the "unperturbed measures" concern free fields and hence are Gaussian).

In the Gaussian case, one can also identify the Wick polynomials as arising from an orthogonalization procedure. Wiener chaos expansion and Malliavin calculus used for stochastic differential equations can be viewed as applications of such orthogonal projection techniques. [4]

In the non-Gaussian case, there are far fewer examples of applications of Wick polynomial techniques. The computations become then more involved. For instance, there is no explicit

formula for the generating function unless the inverse of the moment generating function happens to be known explicitly. In addition, then the polynomials typically no longer form an orthogonal set in $L^2(\mu)$.

The goals of the present contribution are two-fold. In the first part, we show that Wick polynomials have a natural combinatorial definition, closely connected to cumulants and the related cluster expansions of correlation functions. We also rederive their main properties without resorting to the generating function, hence without assuming Gaussianity or the existence of exponential moments.

In the second part, we show how Wick polynomial expansions may be used in the analysis of stochastic processes. In particular, the goal there is to apply the expansion to study the time-evolution of the cumulants, i.e., of the connected correlation functions, of the process. We will explain there why often it is cumulants, and not moments, which should be used as dynamical variables. For simplicity, we consider here only processes whose dynamics are deterministic and given by a differential equation, such as Hamiltonian evolution in classical particle systems. The randomness enters via the initial state. However, generalization to Markovian stochastic dynamics should be straightforward, for instance, if the generator of the process maps polynomials to polynomials.

In the general setup, the best one can hope for are recursion relations leading to an infinite hierarchy of equations connecting the evolution of the cumulants. We explain in Section 4 what immediate constructions are available for hierarchical study of the evolution of cumulants and Wick polynomials.

We give more explicit applications in section 5 where we study the evolution on a lattice of particles following the discrete nonlinear Schrödinger (DNLS) equation with random initial data. In particular, our goal is to show how the Wick polynomial expansion of the dynamics greatly simplifies the (still only formal) derivation of the related Boltzmann-Peierls equation. This case is one of the few examples of nonlinear Hamiltonian evolution where a rigorous analysis of the related perturbation expansion has been possible so far: in [7] it was proven that if the initial measure is a stationary Gibbs measure, then the time-correlations of the field follow an evolution equation derived using a perturbation expansion analogous to the one needed for the Boltzmann-Peierls equation.

An ultimate goal of the present reformulation of the evolution problem would be to complete the rigorous derivation, and hence give a region of validity, of the Boltzmann-Peierls equation. We show how the Wick polynomial expansion could help in this goal by separating the problem of solving the effect of the Boltzmann-Peierls evolution from the estimation of the corrections arising from the wave nature of the microscopic evolution, such as constructive interference. For the DNLS evolution the Wick polynomial expansion coincides with what was called "pair truncation" in [7]. In fact, the present work arose from an attempt to generalize this construction to other polynomial potentials, which we later realized to coincide with Wick polynomial expansions.

It should already be apparent from the above example that in order to use the Wick polynomials some care is needed in the choice of notations to avoid being overcome by lengthy formulae and intractable combinatorial estimates. We begin by explaining our choices in detail in Section 2. The first part containing the combinatorial definition and properties of Wick polynomials is given in Section 3. The second part discussing the use of Wick polynomial expansions for the study of evolution of cumulants begins in Section 4. We conclude it with the specific application to DNLS dynamics in Section 5. Some comments and possible further directions are discussed in Section 6.

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2 Setup and notations

We consider a collection y_j, $j \in J$ where J is some fixed nonempty index set, of real or complex random variables on some probability space $(\Omega, \mathscr{B}, \mu)$. If y_j are complex, we assume that the collection is closed under complex conjugation, i.e., that to every j there is $j' \in J$ such that $y_{j'} = y_j^*$.

Expectation over the probability measure μ will be denoted by $\mathbb{E}$ or $\langle\cdot\rangle$. In case the underlying measure needs to be identified, we denote the expectation by $\mathbb{E}_\mu$ or $\langle\cdot\rangle_\mu$. We use sequences of indices, $I = (i_1, i_2, \ldots, i_n) \in J^n$, to label monomials of the above random variables, with the following shorthand notation

$$y^I = y_{i_1} y_{i_2} \cdots y_{i_n} = \prod_{k=1}^{n} y_{i_k} \,. \tag{2.1}$$

We also set $y^\emptyset := 1$ if I is the empty sequence. Since all y_j commute with each other, we have $y^I = y^{I'}$ for any two sequences I, I' which differ by a permutation.

We will need to operate not only with such sequences but also with their subsequences and "partitions". This will be done by choosing a distinct label for each member of the sequence and collecting these into a set. How the labelling is done is not important, as long as one takes care when combining two "labelled" sets. We rely here on the following standard conventions: any sequence (i_k) can be uniquely identified with the function $k \mapsto i_k$ which itself is uniquely determined by its graph, the subset $\{(k, i_k) \,|\, k = 1, 2, \ldots, n\}$ of $\mathbb{N} \times J$. We consider subsequences to be subsets of the graph of the sequence. Partitions of the sequence then correspond to partitions of its graph which can be understood as partitions into nonempty subsequences.

Mathematically, this leads to the following structure. Finite (sub)sequences of indices are now uniquely labelled by the collection $\mathscr{I}$, which consists of those finite subsets $A \subset \mathbb{N} \times J$ with the property that if $(n, j), (n', j') \in A$ and $(n, j) \neq (n', j')$ then $n \neq n'$. We also allow the sequence to be empty which is identified with $\emptyset \in \mathscr{I}$. For nonempty sets, the natural number in the first component serves as a distinct label for each member in A. In addition, we can use the order of the natural numbers to collapse any $A \in \mathscr{I}$ back to a sequence $\check{A}$ in J: Given $A \in \mathscr{I}$ with $n > 0$ elements, there is a unique bijection $g : \{1, 2, \ldots, n\} \to A$ such that its first component is increasing, $g(k)_1 < g(k')_1$ for all $k < k'$. Using this g, we define $\check{A}_k := g(k)_2 \in J$ for $k = 1, 2, \ldots, n$.

To each finite sequence $I = (i_k)$ of n elements in J, we assign $\tilde{I} := \{(k, i_k) \,|\, k = 1, 2, \ldots, n\}$ as the set of labels. Obviously, then $\tilde{I}$ and any of its subsets belong to $\mathscr{I}$. The following list summarizes some basic notations and definitions which will be used later without further remark.

1. If I is a sequence, and a set is needed by the notation, the set is chosen to be $\tilde{I}$. For instance, the notation "$A \subset I$" means $A \subset \tilde{I}$.

2. The notation $\mathcal{P}(E)$ denotes the collection of partitions of the a set $E \in \mathscr{I}$. If I is a sequence, $\mathcal{P}(I) := \mathcal{P}(\tilde{I})$.

3. If $A \in \mathscr{I}$ and it is used in a place of a sequence, the formula always refers to the collapsed sequence $\check{A}$ obtained via the increasing bijection g above. For instance, then $y^A := y^{\check{A}} = \prod_{k=1}^{n} y_{g(k)_2}$. (Note that if I is a sequence, then $y^{\tilde{I}} = y^I$ in agreement with (2.1).)

4. If $A \in \mathscr{I}$, we denote the corresponding sequence of random variables by $y_A := (y_{\check{A}_k})_{k=1}^{|A|}$.

5. If $m \in \mathbb{N}$ and $A \in \mathscr{I}$, the notation $\widehat{A}^{(m)}$ refers to a set where any element with label m is cancelled, i.e., $\widehat{A}^{(m)} := \{(k, i_k) \in A \,|\, k \neq m\}$. Note that it is possible that $\widehat{A}^{(m)} = A$. If I is a sequence and $m \leq |I|$, $\widehat{I}^{(m)}$ corresponds to a sequence which is obtained by removing the m:th member from I.

6. If I, I' are two sequences, they can be merged into a new sequence $(i_1, \ldots, i_{|I|}, i'_1, \ldots, i'_{|I'|})$ which we denote by $I + I'$. If $A, B \in \mathscr{I}$, we take $A + B := \check{A} + \check{B}$. For a merged sequence,

the notation "$I \subset I + I'$" always refers to the collection of the labels of the first $|I|$ members and analogously "$I' \subset I + I'$" refers to the collection of the last $|I'|$ members. The merge operation is clearly associative, and we hence drop parentheses when it is applied iteratively; for instance, $I + I' + I''$ is a sequence of length $|I| + |I'| + |I''|$.

7. To avoid separate treatment of expressions involving empty sets and conditions, we employ here the following standard conventions: if the condition P is false, we define

$$\sum_P (\cdots) := 0, \quad \prod_P (\cdots) := 1, \quad \text{and set also} \quad \mathcal{P}(\emptyset) := \{\emptyset\}\,. \tag{2.2}$$

Similarly to the moments, to any $I \in \mathscr{I}$ we denote the corresponding cumulant by one of the following alternative notations

$$\kappa[y_I] = \kappa_\mu[y_I] = \mathbb{E}[y_{i_1}; y_{i_2}; \cdots; y_{i_n}] = \kappa(y_{i_1}, y_{i_2}, \cdots, y_{i_n})\,. \tag{2.3}$$

The corresponding Wick polynomial is denoted by

$$:y_{i_1} y_{i_2} \cdots y_{i_n}: = :y^I: = :y^I:_\mu\ . \tag{2.4}$$

Note that this notation is slightly formal, since the result is not a function only of the power y^I but depends on all subpowers, y^A, $A \subset I$, as well. It also requires that one carefully defines which random variables are being "Wick contracted". We will use parentheses for this purpose, if necessary. For instance, ":(y^I):" means $y^I - \mathbb{E}[y^I]$ which usually differs from $:y^I:$.

As an application of the above definitions, let us point out that the earlier defining Wick polynomial condition (1.1) is equivalent to the requirement that for every nonempty sequence I and any $j \in J$ we should have

$$\partial_{y_j} :y^I: = \sum_{k=1}^{|I|} \mathbb{1}(i_k = j)\, :y^{\widehat{I^{(k)}}}:\ . \tag{2.5}$$

Here, and in the following, $\mathbb{1}$ denotes the generic characteristic function: $\mathbb{1}(P) = 1$ if the condition P is true, and otherwise $\mathbb{1}(P) = 0$.

We recall that, if the random variables y_j, $j = 1, 2, \ldots, n$, have joint exponential moments, then moments, cumulants and Wick polynomials can be generated by differentiation of their respective generating functions which are

$$G_{\mathrm{m}}(\lambda) := \mathbb{E}[\mathrm{e}^{\lambda \cdot x}]\,, \quad g_{\mathrm{c}}(\lambda) := \ln G_{\mathrm{m}}(\lambda) \quad \text{and} \quad G_{\mathrm{w}}(\lambda; y) := \frac{\mathrm{e}^{\lambda \cdot y}}{\mathbb{E}[\mathrm{e}^{\lambda \cdot x}]} = \mathrm{e}^{\lambda \cdot y - g_{\mathrm{c}}(\lambda)}\,. \tag{2.6}$$

Here $\lambda \cdot x := \sum_{i=1}^n \lambda_i x_i$ (for the sake of clarity we have denoted the integrated random variable by "x" instead of "y") and the "generation" happens by evaluation of the I:th derivative at zero, i.e.,

$$\mathbb{E}[y^I] = \partial_\lambda^I G_{\mathrm{m}}(0)\,, \quad \kappa[y_I] = \partial_\lambda^I g_{\mathrm{c}}(0) \quad \text{and} \quad :y^I: = \partial_\lambda^I G_{\mathrm{w}}(0; y)\,, \tag{2.7}$$

where "∂_λ^I" is a shorthand notation for $\partial_{\lambda_{i_1}} \partial_{\lambda_{i_2}} \cdots \partial_{\lambda_{i_n}}$.

As a side remark, let us also recall that it is possible to replace the above definitions of generating functions by parametrizations which do not require the existence of any moments and hence work for arbitrary Borel probability measures μ. If all y_j are real, then replacing the exponential $\mathrm{e}^{\lambda \cdot y}$ by $\mathrm{e}^{\mathrm{i}\lambda \cdot y}$ yields an $L^1(\mu)$ function for all $\lambda \in \mathbb{R}^n$. If $y \in \mathbb{C}^n$, the same is achieved by using $\mathrm{e}^{\mathrm{i}(\lambda^* \cdot y + \lambda \cdot y^*)/2}$ and $\lambda \in \mathbb{C}^n$: in this case, differentiation with respect to $\operatorname{Re} \lambda_j$ generates "$\mathrm{i} \operatorname{Re} y_j$" and with respect to $\operatorname{Im} \lambda_j$ generates "$\mathrm{i} \operatorname{Im} y_j$". Naturally, without absolute integrability of the moments it is not guaranteed that any of the derivative exist. However, it might nevertheless be useful to inspect the time evolution of the generating function, in particular, if the time evolution is regularizing and improves the integrability of the moments.

3 Combinatorial definition and properties of the Wick polynomials

Let us first recall the "moments-to-cumulants" formula which holds for any $I \in \mathscr{I}$ as long as all moments y^A, $A \subset I$, belong to $L^1(\mu)$:

$$\mathbb{E}[y^I] = \sum_{\pi \in \mathcal{P}(I)} \prod_{A \in \pi} \kappa[y_A]\,, \tag{3.1}$$

where $\mathcal{P}(I)$ denotes the collection of partitions of the set I. (Or to be precise, of $\tilde{I}$. Here it is important to assign a distinct label to each random variable in the power y^I to get the combinatorics correctly.) For a partition $\pi \in \mathcal{P}(I)$, let us call the subsets $A \in \pi$ *clusters* or *blocks*. Let us also recall that the cumulants are multilinear, i.e., they are separately linear in each of the variables y_j, $j \in I$. These results are discussed, for instance, in [5] and [6] and also briefly in Appendix A here.

Let us point out that by the conventions adopted here, (3.1) is indeed valid also for the empty sequence $I = \emptyset$. Then the sum over partitions is not empty since it contains $\pi = \emptyset$. However, the corresponding product is empty since there is no A with $A \in \pi$. Therefore, the right hand side of (3.1) evaluates to one which agrees with our definition of $\mathbb{E}[y^\emptyset]$.

We next show that it is possible to choose a subset of the indices and remove all its "internal clusters" from the moments-to-cumulants formula by replacing the corresponding power with a polynomial of the same order. This will be achieved by using the following recursive definition.

Definition 3.1 *Suppose that $I_0 \in \mathscr{I}$ is such that $\mathbb{E}[|y^I|] < \infty$ for all $I \subset I_0$. We define polynomials $\mathcal{W}[y^I] := \sum_{E \subset I} c_E[y^I]\, y^E$ for $I \subset I_0$ inductively in $|I|$ using the following rule: set $\mathcal{W}[y^\emptyset] := 1$, and for $I \neq \emptyset$ use*

$$\mathcal{W}[y^I] := y^I - \sum_{\emptyset \neq E \subset I} \mathbb{E}[y^E]\, \mathcal{W}[y^{I \setminus E}]\,. \tag{3.2}$$

The definition makes sense since the $\mathcal{W}$-terms on the right hand side all have an order lower than $|I|$. It also implies that indeed each $\mathcal{W}[y^I]$ is a polynomial of order $|I|$, with only the term y^I being of the highest order. It is also straightforward to prove by induction that the coefficients $c_E[y^I]$ can be chosen so that they only depend on $\mathbb{E}[y^A]$ with $A \subset I$. Therefore, the definition of $\mathcal{W}[y^I]$ is independent of I_0 in the following sense: if $I_0, I_0' \in \mathscr{I}$ are such that $\mathbb{E}[|y^I|] < \infty$ for every $I \subset I_0$ and $I \subset I_0'$, then for all $I \subset I_0 \cap I_0'$ we have $\mathcal{W}[y^I; I_0] = \mathcal{W}[y^I; I_0']$. In Appendix A we explain how cumulants can also be defined via a similar recursive construction.

The following theorem shows that these polynomials indeed have the promised truncated moments-to-cumulants expansion. We also see that the polynomials are essentially uniquely defined by this property. What is perhaps surprising is that the coefficients of the polynomial can be chosen depending only on the moments of its constituent random variables. This implies that the same polynomial can be used for many different probability distributions, as long as the marginal distributions for the constituent random variables are the same.

Theorem 3.2 *Assume that the measure μ has all moments of order N, i.e., suppose that $\mathbb{E}[|y^I|] < \infty$ for all $I \in \mathscr{I}$ with $|I| \leq N$. Use Definition 3.1 to define $\mathcal{W}[y^I]$ for every such I.*

Then replacing y^I by $\mathcal{W}[y^I]$ removes all terms with clusters internal to I: the following truncated moments-to-cumulants formula holds for every $I' \in \mathscr{I}$ with $|I'| + |I| \leq N$

$$\mathbb{E}\Big[\mathcal{W}[y^I] y^{I'}\Big] = \sum_{\pi \in \mathcal{P}(I + I')} \mathbb{1}(A \cap I' \neq \emptyset \text{ for all } A \in \pi) \prod_{A \in \pi} \kappa[y_A]\,. \tag{3.3}$$

In particular, $\mathbb{E}\big[\mathcal{W}[y^I]\big] = 0$ if $I \neq \emptyset$.

In addition, if $I \in \mathscr{I}$ with $|I| \leq N/2$ and $\mathcal{W}'$ is a polynomial of order at most $|I|$ such that (3.3) holds for all I' with $|I'| \leq N - |I|$, then $\mathcal{W}'$ is μ-almost surely equal to $\mathcal{W}[y^I]$.

Corollary 3.3 *Assume that $\mathbb{E}[|y^I|] < \infty$ for all $I \in \mathscr{I}$. Then $\mathcal{W}[y^I]$ are μ-almost surely unique polynomials of order $|I|$ such that (3.3) holds for every $I' \in \mathscr{I}$.*

Proof: We make an induction in $|I|$. By (3.1), the claim is true for $|I| = 0$ since then $I = \emptyset$ and thus $\mathcal{W}[y^I] = 1$.

Assume then that $I \neq \emptyset$ and that the claim is true for sets of size less than $|I|$. Consider an arbitrary $I' \in \mathscr{I}$ such that $|I'| + |I| \leq N$. For $E \subset I$, denote $E^c := (I + I') \setminus E$. Given a partition π of $I + I'$, we can define

$$\pi_1 := \{A_1 \in \pi \,|\, A_1 \cap I' \neq \emptyset\}$$

and

$$\pi_0 := \pi \backslash \pi_1 \,.$$

Then $E := \cup \pi_0 \subset I$ and $\pi_0 \in \mathcal{P}(E)$, $\pi_1 \in \mathcal{P}(E^c)$ (also whenever E or E^c happens to be empty). Once π is fixed, the decomposition $\pi = \pi_0 \cup \pi_1$ is unique and we thus find that $1 = \sum_{E \subset I} \sum_{\pi_0 \in \mathcal{P}(E)} \sum_{\pi_1 \in \mathcal{P}(E^c)} \mathbb{1}(\pi = \pi_0 \cup \pi_1) \prod_{A_1 \in \pi_1} \mathbb{1}(A_1 \cap I' \neq \emptyset)$. Using this in the standard moments-to-cumulants formula shows that

$$\begin{aligned}
\mathbb{E}\Big[y^I y^{I'}\Big] = \mathbb{E}\Big[y^{I+I'}\Big] &= \sum_{\pi \in \mathcal{P}(I+I')} \prod_{A \in \pi} \kappa[y_A] \\
&= \sum_{\pi \in \mathcal{P}(I+I')} \sum_{E \subset I} \sum_{\pi_0 \in \mathcal{P}(E)} \sum_{\pi_1 \in \mathcal{P}(E^c)} \mathbb{1}(\pi = \pi_0 \cup \pi_1) \prod_{A_0 \in \pi_0} \kappa[y_{A_0}] \prod_{A_1 \in \pi_1} (\kappa[y_{A_1}] \mathbb{1}(A_1 \cap I' \neq \emptyset)) \\
&= \sum_{E \subset I} \sum_{\pi_0 \in \mathcal{P}(E)} \sum_{\pi_1 \in \mathcal{P}(E^c)} \prod_{A_0 \in \pi_0} \kappa[y_{A_0}] \prod_{A_1 \in \pi_1} (\kappa[y_{A_1}] \mathbb{1}(A_1 \cap I' \neq \emptyset)) \\
&= \sum_{\pi \in \mathcal{P}(I+I')} \mathbb{1}(A \cap I' \neq \emptyset \; \forall A \in \pi) \prod_{A \in \pi} \kappa[y_A] \\
&\quad + \sum_{\emptyset \neq E \subset I} \sum_{\pi_0 \in \mathcal{P}(E)} \prod_{A_0 \in \pi_0} \kappa[y_{A_0}] \sum_{\pi_1 \in \mathcal{P}(E^c)} \mathbb{1}(A \cap I' \neq \emptyset \; \forall A \in \pi_1) \prod_{A_1 \in \pi_1} \kappa[y_{A_1}] \\
&= \sum_{\pi \in \mathcal{P}(I+I')} \mathbb{1}(A \cap I' \neq \emptyset \; \forall A \in \pi) \prod_{A \in \pi} \kappa[y_A] + \sum_{\emptyset \neq E \subset I} \mathbb{E}[y^E] \, \mathbb{E}\Big[\mathcal{W}[y^{I \backslash E}] y^{I'}\Big] \,.
\end{aligned}$$

where in the last step we used the moments-to-cumulants formula and the induction hypothesis (note that E^c collapses to the sequence $(I \setminus E) + I'$). Hence, by the definition (3.2) equation (3.3) holds for this I. This completes the induction step and shows that (3.3) is valid for all I, I' with $|I| + |I'| \leq N$. If $I \neq \emptyset$ and $I' = \emptyset$, we have $I + I' \neq \emptyset$ so that for any $\pi \in \mathcal{P}(I + I')$ there is some $A \in \pi$ and then obviously $A \cap I' = \emptyset$. Thus (3.3) implies that $\mathbb{E}\big[\mathcal{W}[y^I]\big] = 0$ for $I \neq \emptyset$.

To prove uniqueness, suppose that $I \in \mathscr{I}$ with $|I| \leq N/2$ and $\mathcal{W}'$ is a polynomial of order at most $|I|$ such that (3.3) holds for all I' with $|I'| \leq N - |I|$. Then $P_I := \mathcal{W}' - \mathcal{W}[y^I]$ is a polynomial of order at most $|I|$ and $\mathbb{E}[P_I y^{I'}] = 0$ for all I' with $|I'| \leq N/2$. Since the collection of random variables is assumed to be closed under complex conjugation, this implies that also $\mathbb{E}[P_I (y^{I'})^*] = 0$ whenever $|I'| \leq N/2$. Thus we can take a linear combination of such equations and conclude that $\mathbb{E}\big[|P_I|^2\big] = 0$. This implies that $P_I = 0$ almost surely, i.e., that $\mathcal{W}' = \mathcal{W}[y^I]$ almost surely.

This concludes the proof of the Theorem. The Corollary is then an immediate consequence. □

In fact, the polynomials given by Definition 3.1 are equal to the standard Wick polynomials.

Proposition 3.4 *Suppose that $I_0 \in \mathscr{I}$ is such that $\mathbb{E}[|y^I|] < \infty$ for all $I \subset I_0$. Then $\mathcal{W}[y^I] = :y^I:$ for every $I \subset I_0$.*

Proof: If $I = \emptyset$, we have $\mathcal{W}[y^I] = 1 = :y^I:$, and else by Theorem 3.2 we have $\mathbb{E}[\mathcal{W}[y^I]] = 0$. Therefore, to prove $\mathcal{W}[y^I] = :y^I:$ it suffices to check that (2.5) holds when the Wick polynomials are replaced by $\mathcal{W}$-polynomials. We do this by induction over $|I|$. Firstly, if $I = \emptyset$, we have

$\partial_{y_j} \mathcal{W}[y^I] = 0$, as required. Assume then that $I \neq \emptyset$ and that the claim is true for sets of size less than $|I|$. For every $j \in J$ and nonempty $E \subset I$, the induction assumption implies that

$$\partial_{y_j} \mathcal{W}[y^{I\setminus E}] = \sum_{k=1}^{|I|} \mathbb{1}(i_k = j)\, \mathbb{1}((k, i_k) \notin E)\, \mathcal{W}[y^{\widehat{I}^{(k)}\setminus E}]\,. \tag{3.4}$$

(The second characteristic function allows keeping the labeling inherited from I by adding zero terms into the sum for the "missing" labels.) Since $\mathcal{W}[y^I]$ satisfies (3.2), we thus find that its algebraic derivatives satisfy an equality

$$\begin{aligned}
\partial_{y_j} \mathcal{W}[y^I] &= \sum_{k=1}^{|I|} \mathbb{1}(i_k = j) \Big[y^{\widehat{I}^{(k)}} - \sum_{\emptyset \neq E \subset I} \mathbb{E}[y^E]\, \mathbb{1}((k, i_k) \notin E)\, \mathcal{W}[y^{\widehat{I}^{(k)}\setminus E}] \Big] \\
&= \sum_{k=1}^{|I|} \mathbb{1}(i_k = j) \Big[y^{I'} - \sum_{\emptyset \neq E \subset I'} \mathbb{E}[y^E] \mathcal{W}[y^{I'\setminus E}] \Big]_{I' = \widehat{I}^{(k)}} \\
&= \sum_{k=1}^{|I|} \mathbb{1}(i_k = j) \mathcal{W}[y^{\widehat{I}^{(k)}}]\,,
\end{aligned} \tag{3.5}$$

where in the last equality we have applied (3.2). This proves that also $\mathcal{W}[y^I]$ satisfies (2.5) and hence completes the induction step. Therefore, the polynomials $\mathcal{W}[y^I]$ satisfy the defining properties of Wick polynomials and thus $\mathcal{W}[y^I] = :y^I:$. □

3.1 Basic properties of the Wick polynomials

In this section, we assume that there is $I_0 \in \mathscr{I}$ is such that $\mathbb{E}[|y^I|] < \infty$ for all $I \subset I_0$. This guarantees the existence of the Wick polynomials $\mathcal{W}[y^I]$ for all $I \subset I_0$, and allows using the results from the previous section. In particular, by Theorem 3.4 these are equal to the standard Wick polynomials and from now on we will use the standard notation $:y^I:$ for them.

The next Proposition collects some of the most important properties of Wick polynomials.

Proposition 3.5 *The following statements hold for any $I \subset I_0$:*

1.

$$y^I = \sum_{U \subset I} :y^U: \mathbb{E}[y^{I\setminus U}] = \sum_{U \subset I} :y^U: \sum_{\pi \in \mathcal{P}(I\setminus U)} \prod_{A \in \pi} \kappa[y_A]\,. \tag{3.6}$$

2. Wick polynomials are permutation invariant: if I' is a permutation of I, then $:y^{I'}: = :y^I:$.

3.

$$:y^I: = \sum_{U \subset I} y^U \sum_{\pi \in \mathcal{P}(I\setminus U)} (-1)^{|\pi|} \prod_{A \in \pi} \kappa[y_A]\,. \tag{3.7}$$

4. If $I' := \widehat{I}^{(1)}$ denotes the sequence obtained by cancelling the first element of I,

$$:y^I: = y_{i_1} :y^{I'}: \; - \sum_{(1,i_1) \in V \subset I} \kappa[y_V] :y^{I\setminus V}: = y_{i_1} :y^{I'}: \; - \sum_{U \subset I'} \kappa[y_{(i_1)+U}] :y^{I'\setminus U}:\,. \tag{3.8}$$

Proof: Item 1: The first equality in (3.6) follows directly from the definition (3.2) since then

$$y^I = \sum_{E \subset I} \mathbb{E}[y^E] :y^{I\setminus E}: = \sum_{U \subset I} \mathbb{E}[y^{I\setminus U}] :y^U:\,. \tag{3.9}$$

The second equality follows then by using the moments-to-cumulants expansion.

Item 2: The permutation invariance of the Wick polynomials follows using straightforward induction in the definition (3.2) since the random variables commute and hence the powers y^E are always permutation invariant.

Items 3 and 4: Let us first define $\widetilde{\mathcal{W}}[y^I]$ by setting it equal to the right hand side of (3.7) for any $I \subset I_0$. If $I = \emptyset$, we have $\widetilde{\mathcal{W}}[y^I] = 1 = :y^I:$. Suppose $I \neq \emptyset$. Since $\sum_{\pi\in\mathcal{P}(\emptyset)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A] = 1$, the definition yields a polynomial of order $|I|$ in y. Our goal is to prove that

$$\partial_{y_j}\widetilde{\mathcal{W}}[y^I] = \sum_{k=1}^{|I|}\mathbb{1}(i_k = j)\widetilde{\mathcal{W}}\big[y^{\widehat{I}^{(k)}}\big]\,, \tag{3.10}$$

$$\widetilde{\mathcal{W}}\big[y^I\big] = y_{i_1}\widetilde{\mathcal{W}}\big[y^{\widehat{I}^{(1)}}\big] - \kappa[y_I] - \sum_{(1,i_1)\in V\subsetneq I}\kappa[y_V]\widetilde{\mathcal{W}}\big[y^{I\setminus V}\big]\,. \tag{3.11}$$

Then the claim $\widetilde{\mathcal{W}}[y^I] = :y^I:$ follows by straightforward induction in $|I|$: Case $|I| = 0$ was proven above. Suppose $I \neq \emptyset$ and that $\widetilde{\mathcal{W}}[y^{I'}] = :y^{I'}:$ whenever $|I'| < |I|$. Then the induction assumption and Theorem 3.2 can be used to evaluate the expectation of the right hand side of (3.11), implying $\mathbb{E}\big[\widetilde{\mathcal{W}}[y^I]\big] = 0$. By (3.10), the polynomial $\widetilde{\mathcal{W}}[y^I]$ also satisfies the third defining condition of the Wick polynomials, equation (2.5). Hence, $\widetilde{\mathcal{W}}[y^I] = :y^I:$ which completes the induction step. Then (3.11) implies the first identity in (3.8) and the second identity is found by a relabeling of the summation variable. Hence, also item 4 follows.

To prove (3.10), consider some $I \neq \emptyset$. In the definition of $\widetilde{\mathcal{W}}[y^I]$, we can express the derivatives of y^U, $U \subset I$, as in (3.4). This shows that

$$\begin{aligned}
\partial_{y_j}\widetilde{\mathcal{W}}[y^I] &= \sum_{U\subset I}\sum_{k=1}^{|I|}\mathbb{1}(i_k = j)\,\mathbb{1}((k,i_k)\in U)y^{\widehat{U}^{(k)}}\sum_{\pi\in\mathcal{P}(I\setminus U)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A] \\
&= \sum_{k=1}^{|I|}\mathbb{1}(i_k = j)\sum_{V\subset\widehat{I}^{(k)}}y^V\sum_{\pi\in\mathcal{P}(\widehat{I}^{(k)}\setminus V)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A] \\
&= \sum_{k=1}^{|I|}\mathbb{1}(i_k = j)\widetilde{\mathcal{W}}\big[y^{\widehat{I}^{(k)}}\big]\,.
\end{aligned} \tag{3.12}$$

Therefore, (3.10) holds.

To prove (3.11), denote $x := (1, i_1)$ and $I' := \widehat{I}^{(1)}$. We first split the definition into two parts as follows:

$$\widetilde{\mathcal{W}}\big[y^I\big] = \sum_{U\subset I}\mathbb{1}(x\in U)y^U\sum_{\pi\in\mathcal{P}(I\setminus U)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A] + \sum_{U\subset I}\mathbb{1}(x\notin U)y^U\sum_{\pi\in\mathcal{P}(I\setminus U)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A]\,. \tag{3.13}$$

Following a reasoning similar to (3.12), we find that the first term in the sum on the right hand side is equal to $y_{i_1}\widetilde{\mathcal{W}}\big[y^{I'}\big]$. The second term is equal to

$$\begin{aligned}
&\sum_{U\subset I}\mathbb{1}(x\notin U)y^U\sum_{\pi\in\mathcal{P}(I\setminus U)}(-1)^{|\pi|}\Big(\prod_{x\notin A\in\pi}\kappa[y_A]\Big)\kappa[y_V]\big|_{x\in V\in\pi} \\
&\quad= -\sum_{U\subset I}\mathbb{1}(x\notin U)y^U\sum_{x\in V\subset I\setminus U}\kappa[y_V]\sum_{\pi\in\mathcal{P}((I\setminus U)\setminus V)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A] \\
&\quad= -\sum_{x\in V\subset I}\kappa[y_V]\sum_{U\subset I\setminus V}y^U\sum_{\pi\in\mathcal{P}((I\setminus V)\setminus U)}(-1)^{|\pi|}\prod_{A\in\pi}\kappa[y_A] \\
&\quad= -\sum_{x\in V\subset I}\kappa[y_V]\widetilde{\mathcal{W}}\big[y^{I\setminus V}\big]\,.
\end{aligned} \tag{3.14}$$

Therefore, (3.13) implies that also (3.11) holds. This completes the proof of the Theorem. □

Example 3.6 Written is terms of cumulants, the Wick polynomials of lowest order are

$$\begin{aligned}
:y: &= y - \kappa(y)\,,\\
:y_1y_2: &= y_1y_2 - \kappa(y_1,y_2) - \kappa(y_1)y_2 - \kappa(y_2)y_1 + \kappa(y_1)\kappa(y_2)\,,\\
:y_1y_2y_3: &= y_1y_2y_3 - \kappa(y_1,y_2,y_3) + \kappa(y_1,y_2)\kappa(y_3) + \kappa(y_1,y_3)\kappa(y_2) + \kappa(y_2,y_3)\kappa(y_1)\\
&\quad -\kappa(y_1)\kappa(y_2)\kappa(y_3) - \kappa(y_1,y_2)y_3 - \kappa(y_1,y_3)y_2 - \kappa(y_2,y_3)y_1\\
&\quad +\kappa(y_1)\kappa(y_2)y_3 + \kappa(y_1)\kappa(y_3)y_2 + \kappa(y_2)\kappa(y_3)y_1\\
&\quad -\kappa(y_1)y_2y_3 - \kappa(y_2)y_1y_3 - \kappa(y_3)y_1y_2\,.
\end{aligned}$$

Proposition 3.7 *The Wick polynomials are multilinear, i.e., if α, β are constants such that $y_j = \alpha y_i + \beta y_{i'}$ for some $j, i, i' \in J$, then, whenever I and k are such that $i_k = j$, we have*

$$:y^I: = \alpha\, :y^{\widehat{I^{(k)}}+(i)}: + \beta\, :y^{\widehat{I^{(k)}}+(i')}:\,.$$

Proof: The claim follows using multilinearity of cumulants in the representation formula (3.7). □

The following result extends the earlier theorem and shows that multiple application of Wick polynomial replacements continues to simplify the moments-to-cumulants formula by removing all terms with any internal clusters.

Proposition 3.8 *Assume that the measure μ has all moments of order N, i.e., suppose that $\mathbb{E}[|y^I|] < \infty$ for all $I \in \mathscr{I}$ with $|I| \le N$. Suppose $L \ge 1$ is given and consider a collection of $L+1$ index sequences $J', J_\ell \in \mathscr{I}$, $\ell = 1, \dots, L$, such that $|J'| + \sum_\ell |J_\ell| \le N$. Then for $I := \sum_{\ell=1}^L J_\ell + J'$ (with the implicit identification of J_ℓ and J' with the set of its labels in I) we have*

$$\mathbb{E}\Big[\prod_{\ell=1}^{L} :y^{J_\ell}: y^{J'}\Big] = \sum_{\pi\in\mathcal{P}(I)} \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A \not\subset J_\ell\ \forall \ell))\ . \tag{3.15}$$

Proof: We proceed via a double induction: the first induction is over L and the second induction is over $|J_L|$. The case $L = 1$ follows directly from Theorem 3.2. Now we assume as induction hypothesis of the first level induction that $L \ge 2$ and

$$\mathbb{E}\Big[\prod_{\ell=1}^{L-1} :y^{J_\ell}: y^{J'}\Big] = \sum_{\pi\in\mathcal{P}(\sum_{\ell=1}^{L-1} J_\ell + J')} \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A \not\subset J_\ell\ \forall \ell < L))\ . \tag{3.16}$$

Then we consider the second induction over $|J_L| =: m$. For $m = 0$, we have $:y^{J_L}: = 1$ and thus then the induction hypothesis (3.16) directly implies (3.15). As induction step of the second level hypothesis we take that, for fixed L, equation (3.15) holds for all $|J_L| < m$. Then, if $|J_L| = m$, we can use (3.2), (3.16) and the second level induction hypothesis to justify the following argument

analogous to the one used in the proof of Theorem 3.2

$$\begin{aligned}
\mathbb{E}\Big[\prod_{\ell=1}^{L} :y^{J_\ell}: y^{J'}\Big] &= \mathbb{E}\Big[\prod_{\ell=1}^{L-1} :y^{J_\ell}: y^{J_L+J'}\Big] - \sum_{\emptyset\neq E\subset J_L} \mathbb{E}[y^E]\,\mathbb{E}\Big[\prod_{\ell=1}^{L-1} :y^{J_\ell}: :y^{J_L\setminus E}: y^{J'}\Big] \\
&= \sum_{\pi\in\mathcal{P}(\sum_{\ell=1}^{L-1} J_\ell + J_L + J')} \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A\not\subset J_\ell\ \forall \ell < L)) \\
&\quad - \sum_{\emptyset\neq E\subset J_L} \sum_{\pi_0\in\mathcal{P}(E)} \prod_{A\in\pi_0} \kappa[y_A] \sum_{\pi_1\in\mathcal{P}(\sum_{\ell=1}^{L-1} J_\ell + (J_L\setminus E) + J')} \prod_{A'\in\pi_1} \Big(\kappa[y_{A'}] \\
&\qquad \times \mathbb{1}(A\not\subset J_\ell\ \forall \ell < L)\mathbb{1}(A'\not\subset J_L\setminus E)\Big) \\
&= \sum_{\pi\in\mathcal{P}(I)} \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A\not\subset J_\ell\ \forall \ell)) \\
&\quad + \sum_{\pi\in\mathcal{P}(I)} \mathbb{1}(\exists A\in\pi \text{ s.t. } A\subset J_L) \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A\not\subset J_\ell\ \forall \ell < L)) \\
&\quad - \sum_{\pi\in\mathcal{P}(I)} \mathbb{1}(\cup\{A\in\pi\,|\,A\subset J_L\}\neq\emptyset) \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A\not\subset J_\ell\ \forall \ell < L)) \\
&= \sum_{\pi\in\mathcal{P}(I)} \prod_{A\in\pi} (\kappa[y_A]\mathbb{1}(A\not\subset J_\ell\ \forall \ell))\ . \qquad (3.17)
\end{aligned}$$

This completes the induction step and hence also the proof. □

4 Cumulants and Wick polynomials as dynamical variables

To study the state of a random system, it is often better to use the cumulants rather than the moments of the random variables. For instance, if y, z are independent random variables we have $\mathbb{E}[y^n z^m] = \mathbb{E}[y^n]\mathbb{E}[z^m]$, which is typically nonzero, whereas the corresponding cumulant is zero whenever both $n, m \neq 0$. Hence, for systems where two regions become "asymptotically independent" (for instance, for sufficiently mixing stochastic processes), it is the cumulants, not moments, which will vanish in the limit.

To have a concrete example, let us consider a random lattice field y_x, $x \in \mathbb{Z}^d$, which is spatially sufficiently strongly mixing. Then, for instance, $\kappa(y_0, y_I) \to 0$ if the distance of the index set $I \subset \mathbb{Z}^d$ from the origin becomes unbounded. Often in the applications the mixing is so strong that the cumulants, in this case also called connected correlation functions, become "ℓ_1-clustering": for any order n one requires that $\sup_{x\in\mathbb{Z}^d} \sum_{I\in(\mathbb{Z}^d)^{n-1}} |\kappa(y_x, y_I)| < \infty$. Naturally, such a property should then not be expected to hold for standard correlation functions $\mathbb{E}[y^I]$, apart from some degenerate examples.

In addition to being mixing, the random fields found in the applications are often *translation invariant*. This means in particular that all moments $\mathbb{E}[y^I]$ remain invariant if every index in I is translated by a fixed amount, i.e., $\mathbb{E}[y^{I(x)}] = \mathbb{E}[y^I]$ for every $x \in \mathbb{Z}^d$ if we set $I(x)_\ell := i_\ell - x$. If the system is both ℓ_1-clustering and translation invariant, the cumulants of the Fourier transformed field $\widehat{y}_k := \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}2\pi k\cdot x} y_x$, k indexed by the d-torus $\mathbb{T}^d$, satisfy

$$\kappa[\widehat{y}_{(k_1,k_2,\ldots,k_n)}] = \delta\Big(\sum_{\ell=1}^{n} k_\ell\Big)\widehat{F}_n(k_1, k_2, \ldots, k_n)\,, \qquad (4.1)$$

where "δ" denotes the Dirac delta distribution and the arithmetic on $\mathbb{T}^d$ is defined via periodic identification. Here $\widehat{F}_n$ denotes the Fourier transform of $F_n(X) := \mathbb{1}(X_1 = 0)\kappa(y_X)$, $X \in (\mathbb{Z}^d)^n$, and for ℓ_1-clustering measures $\widehat{F}_n$ is a uniformly bounded continuous function of k. Therefore, although the cumulants are singular, their singularity structure is simple, entirely encoded in the

δ-multiplier. In contrast, by the moments-to-cumulants formula, then for $I := (1, 2, \dots, n)$ and any $k \in (\mathbb{T}^d)^n$

$$\mathbb{E}[\widehat{y}^{k_I}] = \sum_{\pi \in \mathcal{P}(I)} \prod_{A \in \pi} \left(\delta\Big(\sum_{j \in A} k_j\Big) \widehat{F}_{|A|}(k_A) \right), \tag{4.2}$$

which has ever more complicated singularity structure as the order of the moment is increased. (The above discussion can be made mathematically rigorous by replacing the infinite lattice by a periodic d-dimensional lattice. See [7] for details.)

Hence, for stochastic processes which lead towards a state which is mixing and translation invariant, it seems better to focus on the time-evolution of cumulants instead of the corresponding moments. However, it is not immediately clear how to avoid resorting to the moments as a middle step. It turns out that using Wick polynomials instead of "bare" monomials to define the time-evolution helps in achieving this goal. Recall that any monomial term y^I can always be expanded in terms of Wick polynomials using (3.6), albeit with state-dependent coefficients.

To have a concrete example how this could work in practice, we consider in the following the case of deterministic evolution with random initial data. Explicitly, we assume that the system is described by random variables $y_j(t)$, where j belongs to a fixed (finite) index set J, and $t \geq 0$ denotes time. The initial values $y_j(0)$ are random with a joint distribution μ_0, and for each realization of $y(0)$ we assume that the values at later times $t > 0$ are determined from the solution to the differential equation

$$\partial_t y_j(t) = \sum_{I \in \mathscr{I}_j} M_j^I(t) \, {:}y(t)^I{:}\,, \tag{4.3}$$

where the functions $M_j^I(t)$ are "interaction amplitudes" from the I:th Wick polynomial of $y(t)$ to $y_j(t)$. For each $j \in J$, the set $\mathscr{I}_j$ collects those $I \in \mathscr{I}$ which have a nonzero amplitude, i.e., $M_j^I(t) \neq 0$ for some $t > 0$. For simplicity, we assume here that $\mathscr{I}_j$ is finite and that the amplitudes $M_j^I(t)$ are some fixed functions of time. (They typically might depend on the cumulants of $y(t)$, but this is not relevant for the discussion below: it suffices that they are not random variables.)

We present a concrete example of such a dynamical system in Appendix B where we show how the evolution of N classical particles interacting via a polynomial interaction potential can be described by a system of this type assuming a known random distribution of initial positions and momenta. Another explicit example is given in Section 5.

The usefulness of representing the dynamics in the form (4.3) becomes apparent when looking at the evolution of cumulants. To avoid technical complications, let us suppose that the joint exponential moments of $y(t)$ exist and are continuously differentiable and uniformly bounded functions of t. This will simplify the discussion since it allows using the generating functions defined earlier in (2.6). With the shorthand notation $\lambda \cdot y := \sum_{j \in J} \lambda_j y_j$, the cumulant generating function of $y(t)$, t fixed, is $g_t(\lambda) = \ln\langle \mathrm{e}^{\lambda \cdot y(t)} \rangle$, and the Wick polynomial generating function is $G(\lambda; y(t)) = \frac{\mathrm{e}^{\lambda \cdot y(t)}}{\mathbb{E}[\mathrm{e}^{\lambda \cdot y(t)}]} = \mathrm{e}^{\lambda \cdot y(t) - g_t(\lambda)}$. The time-evolution of the cumulant generating function is connected to the Wick polynomial generating function by

$$\partial_t g_t(\lambda) = \langle \mathrm{e}^{\lambda \cdot y(t)} \rangle^{-1} \langle \lambda \cdot \partial_t y(t) \, \mathrm{e}^{\lambda \cdot y(t)} \rangle = \langle \lambda \cdot \partial_t y(t) \, G(\lambda; y(t)) \rangle \,. \tag{4.4}$$

Therefore, for any $I' \neq \emptyset$ (using the slightly symbolic notations "$I \setminus i$" and "y_i" instead of $\widehat{I}^{(k)}$ and y_{i_k} when $i = (k, i_k) \in I$)

$$\partial_t \kappa[y(t)_{I'}] = \partial_\lambda^{I'} \partial_t g_t(\lambda)\big|_{\lambda = 0} = \sum_{i \in I'} \langle \partial_t y_i(t) \, {:}y(t)^{I' \setminus i}{:} \rangle \,. \tag{4.5}$$

Hence, if the evolution satisfies (4.3), we obtain

$$\partial_t \kappa[y(t)_{I'}] = \sum_{i \in I'} \sum_{I \in \mathscr{I}_i} M_i^I(t) \langle {:}y(t)^I{:} \, {:}y(t)^{I' \setminus i}{:} \rangle \,. \tag{4.6}$$

In this case, determining the evolution of expectation values of all multiplications of two Wick products, $\langle :y(t)^{I_1}: :y(t)^{I_2}: \rangle$, where both I_1, I_2 are non-empty, would also yield a solution to the evolution of cumulants.

We can now obtain a closed cumulant evolution hierarchy using (4.6) and Theorems 3.2 and 3.8. First, note that for any $I \neq \emptyset$ and $j \in J$ we have

$$\langle :y(t)^I: :y_j(t): \rangle = \langle :y(t)^I: y_j(t) \rangle = \kappa[y(t)_{I+(j)}] \,, \tag{4.7}$$

since in this case there is exactly one non-internal cluster, the entire set $I + (j)$. In addition, if $I = \emptyset$, we clearly have $\langle :y(t)^I: :y(t)^{I'}: \rangle = \mathbb{1}(I' = \emptyset)$. Therefore, the first two cumulants satisfy, for arbitrary $j, j' \in J$,

$$\partial_t \kappa[y_j(t)] = \mathbb{1}(\emptyset \in \mathscr{I}_j) M_j^{\emptyset}(t) \,, \tag{4.8}$$

$$\partial_t \kappa[y(t)_{(j,j')}] = \sum_{\emptyset \neq I \in \mathscr{I}_j} M_j^I(t) \kappa[y(t)_{I+(j')}] + \sum_{\emptyset \neq I \in \mathscr{I}_{j'}} M_{j'}^I(t) \kappa[y(t)_{I+(j)}] \,. \tag{4.9}$$

For higher order cumulants, with $|I'| \geq 3$, the equation typically becomes nonlinear; we then have

$$\partial_t \kappa[y(t)_{I'}] = \sum_{i \in I'} \sum_{j \in J} \mathbb{1}((j) \in \mathscr{I}_i) M_i^{(j)}(t) \kappa[y(t)_{(j)+(I' \setminus i)}] + \sum_{i \in I'} \sum_{I \in \mathscr{I}_i, |I| \geq 2} M_i^I(t) \langle :y(t)^I: :y(t)^{I' \setminus i}: \rangle \,. \tag{4.10}$$

We have separated here the terms with $|I| = 1$ to show how they operate linearly on the cumulants of order $|I'|$ (note that $\kappa[y(t)_{(j)+(I' \setminus i)}] = \kappa[y(t)_{I''}]$ for the sequence I'' which is obtained from I' by replacing i with j). In the final sum, both $|I|$ and $|I' \setminus i|$ are greater than one, so it has a cumulant expansion

$$\langle :y(t)^I: :y(t)^{I' \setminus i}: \rangle = \sum_{\pi \in \mathcal{P}(I+(I' \setminus i))} \prod_{A \in \pi} (\kappa[y(t)_A] \mathbb{1}(A \cap I \neq \emptyset, \ A \cap (I' \setminus i) \neq \emptyset)) \,, \tag{4.11}$$

i.e., all clusters have to contain at least one element from both sequences. In particular, it cannot contain any singlets, i.e., it does not depend on any of $\kappa[y_j(t)]$, $j \in J$. Let us also point out that since in these terms $|I| + |I' \setminus i| > |I'|$, any linear term is necessarily of higher order. In particular, this means that lower order cumulants can appear only in nonlinear combinations.

Instead of studying the full cumulant hierarchy, one can also use evolution estimates for the Wick polynomials. The situation often encountered in the applications is that the properties of the initial measure are fairly well known, whereas very little a priori control exists for the time-evolved measure. In such a case, one can use the above result and obtain a perturbation expansion by applying the fundamental theorem of calculus. With the shorthand notation $y := y(0)$ we have

$$\begin{aligned} \kappa[y(t)_{I'}] &= \kappa[y_{I'}] + \int_0^t ds \sum_{i \in I'} \sum_{I \in \mathscr{I}_i} M_i^I(s) \langle :y(s)^I: :y(s)^{I' \setminus i}: \rangle \\ &= \kappa[y_{I'}] + \sum_{i \in I'} \sum_{I \in \mathscr{I}_i} \langle :y^I: :y^{I' \setminus i}: \rangle \int_0^t ds M_i^I(s) \\ &\quad + \sum_{i \in I'} \sum_{I \in \mathscr{I}_i} \int_0^t ds' \partial_{s'} \langle :y(s')^I: :y(s')^{I' \setminus i}: \rangle \int_{s'}^t ds M_i^I(s) \,, \end{aligned} \tag{4.12}$$

where we have applied Fubini's theorem to the final integral. This type of expansion could be helpful if the coefficients $\int_0^t ds M_i^I(s)$ behave better than $M_i^I(t)$, such as in the presence of fast oscillations. Further iterations of this procedure, using either the above cumulant hierarchy or any of the Wick polynomial hierarchies below, would then yield an expansion of $\kappa[y(t)_{I'}]$ in terms of the expectations at time $t = 0$ and the time dependent amplitudes $M^I(t)$. This is particularly useful if all $M^I(t)$ are small, since each iteration adds one more such factor.

Let us conclude this section by deriving recursion formulae for the products of Wick polynomials. As mentioned earlier, these could then be used instead of the direct cumulant hierarchy to study the time-evolution of the cumulants. For this, it would suffice to study $\langle :y(s')^I: :y(s')^{I'\setminus i}:\rangle$ appearing in (4.6), but typically the products of two terms do not satisfy a closed evolution equation and a full hierarchy will be needed. Let us begin with the evolution equation for $:y(t)^I:$. For any deterministic evolution process, we can obtain a fairly compact evolution equation by treating the time-derivative $\partial_t y_j$ as a new random variable:

$$\partial_t :y(t)^I: = \partial_\lambda^I \partial_t G(\lambda; y(t))\big|_{\lambda=0} = \sum_{i\in I} :(\partial_t y_i(t)) y(t)^{I\setminus i}: \,. \tag{4.13}$$

The form is analogous to the standard Leibniz rule. For products of Wick polynomials, we thus have

$$\partial_t \prod_{k=1}^n :y(t)^{I_k}: = \sum_{k=1}^n \sum_{i\in I_k} :(\partial_t y_i(t)) y(t)^{I_k\setminus i}: \prod_{k'\neq k} :y(t)^{I_{k'}}: \,. \tag{4.14}$$

Assuming (4.3) and using multilinearity, we then obtain the following equation involving "nested Wick products":

$$\partial_t \mathbb{E}\Big[\prod_{k=1}^n :y(t)^{I_k}: \Big] = \sum_{k=1}^n \sum_{i\in I_k} \sum_{I\in\mathscr{I}_i} M_i^I(t) \mathbb{E}\Big[:(:y(t)^I:) y(t)^{I_k\setminus i}: \prod_{k'\neq k} :y(t)^{I_{k'}}: \Big]. \tag{4.15}$$

The formula (4.15) is appealing in its simplicity but it does not directly lead to closed hierarchy of equations. This can be achieved by expanding the nested product in terms of cumulants and Wick products. For this, let us note that by (3.8) and the observation made after (4.7), we have for any I'

$$:(\partial_t y_i(t)) y(t)^{I'}: = \partial_t y_i(t) :y(t)^{I'}: \, - \sum_{U\subset I'} \mathbb{E}[\partial_t y_i(t) :y(t)^U:] :y(t)^{I'\setminus U}: \,. \tag{4.16}$$

Therefore, whenever (4.3) holds, we find that

$$\begin{aligned} \partial_t \mathbb{E}\Big[\prod_{k=1}^n :y(t)^{I_k}: \Big] = \sum_{k=1}^n \sum_{i\in I_k} \sum_{I_{n+1}\in\mathscr{I}_i} M_i^{I_{n+1}}(t) \Big\{ \mathbb{E}\Big[:y(t)^{I_k\setminus i}: \prod_{k'=1;k'\neq k}^{n+1} :y(t)^{I_{k'}}: \Big] \\ - \sum_{U\subset I_k\setminus i} \mathbb{E}\Big[:y(t)^{(I_k\setminus i)\setminus U}: \prod_{k'=1;k'\neq k}^{n} :y(t)^{I_{k'}}: \Big] \, \mathbb{E}\Big[:y(t)^{I_{n+1}}: :y(t)^U: \Big] \Big\} \end{aligned} \tag{4.17}$$

This forms a closed hierarchy of evolution equations for the collection of all expectation values of the type $\mathbb{E}\Big[\prod_{k=1}^n :y(t)^{I_k}:\Big]$.

A second alternative for the hierarchy follows from the observation that if $y(t)$ and $z(t)$ are two processes which start with independent, identically distributed initial data, then at any later moment they are also independent and identically distributed and hence

$$\begin{aligned} \partial_t G(\lambda; y(t)) &= G(\lambda; y(t)) \left(\lambda \cdot \partial_t y(t) - \mathbb{E}_z[\lambda \cdot \partial_t z(t) \, G(\lambda; z(t))] \right) \\ &= \mathbb{E}_z \left[G(\lambda; y(t)) G(\lambda; z(t)) \lambda \cdot (\partial_t y(t) - \partial_t z(t)) \right] \end{aligned} \tag{4.18}$$

where in the second equality we have used $\mathbb{E}_z[G(\lambda; z(t))] = 1$. Consider then the product measure for the processes y, z and let G' denote the corresponding Wick polynomial generating function. Since by Fubini's theorem then $\mathbb{E}_{y,z}[\mathrm{e}^{\lambda\cdot(y(t)+z(t))}] = \mathbb{E}_y[\mathrm{e}^{\lambda\cdot y(t)}]^2$, now $G(\lambda; y(t))G(\lambda; z(t)) = G'(\lambda; y(t) + z(t))$. Hence, for dynamics satisfying (4.3)

$$\partial_t \mathbb{E}_y \left[\prod_{k=1}^n :y(t)^{I_k}: \right] = \sum_{k=1}^n \sum_{i\in I_k} \sum_{I\in\mathscr{I}_i} M_i^I(t) \mathbb{E}_{y,z} \Big[(:y(t)^I: - :z(t)^I:) \;\; :(y(t)+z(t))^{I_k\setminus i}: \prod_{k'\neq k} :y(t)^{I_{k'}}: \Big] . \tag{4.19}$$

Let us point out that the earlier expression in (4.17) follows from the above one if we expand the power $(y(t)+z(t))^{I_k\setminus i}$ and then use the fact that the joint measure is a product measure. The formula does not yet yield a closed hierarchy but the following generalization does so: if $z_{k,\ell}(t)$ are processes such that their joint initial distribution is given, then

$$\partial_t \mathbb{E}\Big[\prod_{k=1}^{n} :\Big(\sum_{\ell} z_{k,\ell}(t)\Big)^{I_k}:\Big] = \sum_{k=1}^{n}\sum_{i\in I_k}\sum_{I\in\mathscr{I}_i}\sum_{\ell} M_i^I(t)$$
$$\times\, \mathbb{E}'_k\Big[(:z_{k,\ell}(t)^I: - :z'_{k,\ell}(t)^I:)\ :\Big(\sum_{\ell'}(z_{k,\ell'}(t)+z'_{k,\ell'}(t))\Big)^{I_k\setminus i}: \prod_{k'\neq k} :\Big(\sum_{\ell'} z_{k',\ell'}(t)\Big)^{I_{k'}}:\Big]\,, \quad (4.20)$$

where $\mathbb{E}'_k$ refers to a measure where $z_{k,\ell}$ for each ℓ has been independently duplicated in the $z'_{k,\ell}$-process.

A possible benefit of this formulation could be when $z_{k,\ell}(t)$ have mean zero and are independent for all ℓ. Then the central limit theorem governs the behavior of $\sum_\ell z_{k,\ell}$ when there are many terms in the sum. Therefore, it could be of help in controlling the otherwise difficult case where one has performed many iterations starting from (4.19).

5 Kinetic theory of the discrete NLS equation revisited

In this section we apply the previous Wick polynomial techniques to the discrete nonlinear Schrödinger equation. This example is chosen since it has a particular simple, but nontrivial, Wick expansion of the evolution equation. In addition, we can then rely on the rigorous results and known properties from an earlier work on the model [7]. We focus on the kinetic theory of the model on the "kinetic" time-scale which is $O(\lambda^{-2})$ if the nonlinear coupling λ is taken to zero.

We begin by going through the derivation of the Boltzmann-Peierls equation in the spatially homogeneous case, and show how the task is simplified by using the Wick expanded dynamics and the cumulant hierarchy, as explained in Section 4. We only consider terms which would be present in the simplified case of Gaussian initial data. We give an example in Appendix C which highlights the mechanism leading to suppression of the additional effects of non-Gaussian initial data in the kinetic scaling limit.

For this particular setup, it is easy to find dynamical variables whose evolution equation does not have a linear part. This is an important simplification since it negates a term which becomes rapidly oscillating on the kinetic time-scale, having a divergent frequency $O(\lambda^{-1})$ in the kinetic scaling limit. The effect becomes apparent when looking at field time-correlations instead of the evolution of equal time cumulants. We discuss the issue in more detail in Section 6.1.

In [7], the initial data is taken to be given by a thermal Gibbs measure which is stationary both in space and in time. We do not assume the initial data to be time-stationary here, but the computations in this section require space translation invariance. The spatially inhomogeneous case is technically substantially more complicated, and we discuss it only briefly in Section 6.1.

The results in this section are derived in the spirit of standard perturbation theory and focus solely on evolution on short kinetic time scales, $t=\tau\lambda^{-2}$ with $0<\tau\ll 1$. It is however possible to apply the cumulant hierarchy differently, leading to equations which do not require taking a scaling limit. We conclude the study of the DNLS model in Section 6.2 by proposing a reformulation of the problem which leads to Boltzmann type evolution equations which could be accurate also for times longer than $O(\lambda^{-2})$. The discussion is not completely mathematically rigorous, but we propose a scheme which could be used to this end under some natural conditions about the time-evolved state.

The discrete NLS equation on the lattice $\mathbb{Z}^d$ deals with functions $\psi:\mathbb{R}\times\mathbb{Z}^d\to\mathbb{C}$ which satisfy

$$\mathrm{i}\partial_t\psi_t(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^d}\alpha(\mathbf{x}-\mathbf{y})\psi_t(\mathbf{y}) + \lambda|\psi_t(\mathbf{x})|^2\psi_t(\mathbf{x})\,. \quad (5.1)$$

Here the function $\alpha : \mathbb{Z}^d \to \mathbb{R}$ is called the hopping amplitude and we assume that it is symmetric, $\alpha(\mathbf{x}) = \alpha(-\mathbf{x})$, and exponentially decreasing. The parameter $\lambda > 0$ is considered to be small, and in the kinetic scaling limit we take $\lambda \to 0$ and $t = \tau\lambda^{-2}$ with $\tau > 0$ fixed. The initial field ψ_0 is assumed to be random, bounded on finite subsets of the lattice, and to have an ℓ_1-clustering distribution. We aim at controlling the moments of the random variables $\psi_t(\mathbf{x})$ and $\psi_t(\mathbf{x})^*$ which we label using $\psi_t(\mathbf{x}, +1) := \psi_t(\mathbf{x})$ and $\psi_t(\mathbf{x}, -1) := \psi_t(\mathbf{x})^*$.

Since we do not assume that ψ_0 is ℓ_2-summable, even the (almost sure) existence of solutions to (5.1) becomes an issue. To our knowledge, it has not been proven for the above setup, and most likely, some additional assumptions about the increase of the values of the initial field at infinity are needed for proper existence theory. However, these problems can be easily avoided by replacing the infinite lattice $\mathbb{Z}^d$ by a finite lattice with periodic boundary conditions: see [7] for details. This would merely result in replacing the lattice, the Fourier space and transform, and the associated δ-functions by their finite lattice counterparts. Since even then the final limits cannot be rigorously controlled, we opt here for some additional formality in the discussion, but with less complicated formulae to deal with.

For technical simplicity, here we also only consider initial data which are "gauge invariant": we will always suppose $\psi_0(\mathbf{x})$ has the same distribution as $\mathrm{e}^{\mathrm{i}\theta}\psi_0(\mathbf{x})$ for any $\theta \in [0, 2\pi]$. In fact, this transformation commutes with the time evolution, i.e., if the initial field is changed from ψ_0 to $\mathrm{e}^{\mathrm{i}\theta}\psi_0$, the time-evolved field will change from ψ_t to $\mathrm{e}^{\mathrm{i}\theta}\psi_t$. In particular, also the field ψ_t will then be gauge invariant. The main reason for insisting on this assumption is that it will automatically force many cumulants to be zero and hence simplify the combinatorics associated with the cumulant hierarchy. Gauge invariance implies that a moment is zero unless it has the same number of ψ and ψ^* factors, even when the fields are evaluated at different times. Hence, it implies that every odd moment of the fields is zero and hence also every odd cumulant. Similarly, we see that even cumulants are also zero if they concern a different number of ψ^* and ψ variables.

For instance, we find using (3.7) that for any gauge invariant state and any $a_j := \psi_{t_j}(\mathbf{x}_j, \sigma_j)$, $j = 1, 2, 3$,

$$:a_1a_2a_3: = a_1a_2a_3 - \mathbb{E}[a_1a_2]a_3 - \mathbb{E}[a_1a_3]a_2 - \mathbb{E}[a_2a_3]a_1\,. \tag{5.2}$$

This is the definition of the "pairing truncation operation" $\widehat{\mathcal{P}}$ given in Lemma 3.2 of [7]. Applying the truncation operation in the evolution equation was one of the key changes to the standard perturbation theory which allowed the rigorous analysis in [7]. With the benefit of hindsight, we can now identify it as a Wick contraction of the random variables.

Under the above assumptions and using (3.6), we find that (5.1) is equivalent to the following Wick contracted evolution equation

$$\mathrm{i}\partial_t\psi_t(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^d} \alpha(\mathbf{x}-\mathbf{y})\,:\psi_t(\mathbf{y}): + 2\lambda\mathbb{E}[\psi_t(\mathbf{x})^*\psi_t(\mathbf{x})]\,:\psi_t(\mathbf{x}): + \lambda\,:\psi_t(\mathbf{x})^*\psi_t(\mathbf{x})\psi_t(\mathbf{x}):\,. \tag{5.3}$$

Hence, the random variables $\psi_t(\mathbf{x}, \sigma)$, $\sigma = \pm 1$, satisfy an evolution equation of a form required in the previous section, in (4.3),

$$\mathrm{i}\sigma\partial_t\psi_t(\mathbf{x}, \sigma) = \sum_{\mathbf{y}\in\mathbb{Z}^d} \alpha(\mathbf{x}-\mathbf{y})\,:\psi_t(\mathbf{y}, \sigma): + \lambda R_t(\mathbf{x})\,:\psi_t(\mathbf{x}, \sigma): + \lambda\,:\psi_t(\mathbf{x}, -1)\psi_t(\mathbf{x}, \sigma)\psi_t(\mathbf{x}, 1):\,, \tag{5.4}$$

where we have defined $R_t(x) := 2\mathbb{E}[|\psi_t(\mathbf{x})|^2] \geq 0$, which is also equal to $2\kappa(\psi_t(\mathbf{x}, -1), \psi_t(\mathbf{x}, 1))$.

5.1 Translation invariant initial measures

The evolution problem simplifies significantly, if we assume that the initial data is not only gauge, but also translation invariant. Since also spatial translations commute with the time evolution, we can use the earlier results for the cumulants of Fourier transforms of the random field. In particular, then for any $t \geq 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$,

$$\mathbb{E}[\psi_t(\mathbf{x})^*\psi_t(\mathbf{y})] = \mathbb{E}[\psi_t(\mathbf{x}-\mathbf{y})^*\psi_t(0)]\,. \tag{5.5}$$

This implies that $R_t(\mathbf{x}) = R_t(0) =: R_t$ for all t, and therefore the evolution equation (5.4) for translation and gauge invariant initial data can be written as

$$\mathrm{i}\sigma\partial_t\psi_t(\mathbf{x},\sigma) = \sum_{\mathbf{y}\in\mathbb{Z}^d} \alpha_t^\lambda(\mathbf{x}-\mathbf{y}) :\psi_t(\mathbf{y},\sigma): + \lambda :\psi_t(\mathbf{x},-1)\psi_t(\mathbf{x},\sigma)\psi_t(\mathbf{x},1): , \tag{5.6}$$

where $\alpha_t^\lambda(\mathbf{x}) := \alpha(\mathbf{x}) + \lambda\mathbb{1}(\mathbf{x}=\mathbf{0})R_t$.

Using multilinearity of the Wick polynomials, we thus find the following evolution equation for the Fourier transformed fields $\widehat{\psi}_t(\mathbf{k},\sigma) := \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}2\pi\mathbf{k}\cdot\mathbf{x}}\psi_t(\mathbf{x},\sigma)$,

$$\begin{aligned}\partial_t\widehat{\psi}_t(\mathbf{k},\sigma) &= -\mathrm{i}\sigma\omega_t^\lambda(\mathbf{k}) :\widehat{\psi}_t(\mathbf{k},\sigma): \\ &\quad - \mathrm{i}\sigma\lambda\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) :\widehat{\psi}_t(\mathbf{k}_1,-1)\widehat{\psi}_t(\mathbf{k}_2,\sigma)\widehat{\psi}_t(\mathbf{k}_3,1): ,\end{aligned} \tag{5.7}$$

where

$$\omega_t^\lambda(\mathbf{k}) := \widehat{\alpha}_t^\lambda(\mathbf{k}) = \widehat{\alpha}(\mathbf{k}) + \lambda R_t \,. \tag{5.8}$$

For later use, let us point out that the definitions of the random fields imply the following rule for complex conjugation of the Fourier transformed fields: $\widehat{\psi}_t(\mathbf{k},\sigma)^* = \widehat{\psi}_t(-\mathbf{k},-\sigma)$. In addition, the assumed symmetry of α implies the symmetry $\omega_t^\lambda(-\mathbf{k}) = \omega_t^\lambda(\mathbf{k})$.

We recall that in the present translation invariant setting, the n:th cumulants satisfy for $k \in (\mathbb{T}^d)^n$, $\sigma \in \{-1,1\}^n$

$$\kappa(\widehat{\psi}_t(\mathbf{k}_1,\sigma_1),\dots,\widehat{\psi}_t(\mathbf{k}_n,\sigma_n)) = \delta\Big(\sum_{\ell=1}^n \mathbf{k}_\ell\Big)\widehat{F}_n(k,\sigma;t)\,, \tag{5.9}$$

where $F_n(x,\sigma;t) := \mathbb{1}(\mathbf{x}_1 = 0)\kappa(\psi_t(\mathbf{x}_1,\sigma_1),\dots,\psi_t(\mathbf{x}_n,\sigma_n))$ is identically zero unless $\sum_\ell \sigma_\ell = 0$. We are now mainly interested in the evolution of the lowest nonzero cumulants, i.e., of $F_2(\mathbf{x},(-1,1))$. We denote its Fourier transform by W; more precisely, we set

$$W_t^\lambda(\mathbf{k}) := \sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}2\pi\mathbf{k}\cdot\mathbf{x}}\kappa(\psi_t(0)^*,\psi_t(\mathbf{x})) = \sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}2\pi\mathbf{k}\cdot\mathbf{x}}\mathbb{E}[\psi_t(0)^*\psi_t(\mathbf{x})]\,. \tag{5.10}$$

It follows that $\widehat{F}_2((\mathbf{k}_1,\mathbf{k}_2),(-1,1);t) = W_t^\lambda(\mathbf{k}_2)$ and $\widehat{F}_2((\mathbf{k}_1,\mathbf{k}_2),(1,-1);t) = W_t^\lambda(-\mathbf{k}_2)$. Therefore, we have the following general rule for second order cumulants:

$$\kappa(\widehat{\psi}_t(\mathbf{k}_1,\sigma_1),\widehat{\psi}_t(\mathbf{k}_2,\sigma_2)) = \delta(\mathbf{k}_1+\mathbf{k}_2)\mathbb{1}(\sigma_1+\sigma_2=0)W_t^\lambda(\sigma_2\mathbf{k}_2)\,. \tag{5.11}$$

Therefore, to study the evolution of all second moments in this systems, it suffices to study the function W_t^λ. In particular, clearly $R_t = 2\int_{\mathbb{T}^d} d\mathbf{k}\, W_t^\lambda(\mathbf{k})$. Note that by translation invariance, W_t^λ is always real valued.

5.1.1 Heuristic derivation of the Boltzmann-Peierls equation

After these preliminaries, we are ready for an application of the cumulant hierarchy to study the Fourier transformed fields. Our first goal is to justify the Boltzmann-Peierls equation which has been conjectured before, based on perturbation expansions in [7]. The conjecture says that in the kinetic scaling limit the function W should converge: it is assumed that there exists a limit

$$W_\tau(\mathbf{k}) := \lim_{\lambda\to 0} W_{\tau\lambda^{-2}}^\lambda(\mathbf{k})\,. \tag{5.12}$$

In addition, the analysis of the perturbation series suggests that the limit satisfies the following homogeneous Boltzmann-Peierls equation:

$$\partial_t W_t(\mathbf{k}) = \mathcal{C}(W_t(\cdot))(\mathbf{k})\,, \tag{5.13}$$

with the collision operator

$$\mathcal{C}(W(\cdot))(\mathbf{k}) = 4\pi \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega+\omega_1-\omega_2-\omega_3)$$
$$\times [W(\mathbf{k}_1)W(\mathbf{k}_2)W(\mathbf{k}_3) + W(\mathbf{k})W(\mathbf{k}_2)W(\mathbf{k}_3) - W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_3) - W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_2)] \tag{5.14}$$

where $\omega_i := \widehat{\alpha}(\mathbf{k}_i)$ and $\omega := \widehat{\alpha}(\mathbf{k})$.

In fact, a lucky accident hides the fact that our present random fields are actually ill suited for taking of the scaling limit: it is clear from the linear part in (5.7) that they are highly oscillatory, and only observables where these oscillations cancel out, can be hoped to have a (nonzero) limiting value in the kinetic scaling limit. Fortunately, there is a simple "renormalization" which cancels these fast oscillations. If we define a new random field by the formula

$$a_t(\mathbf{k},\sigma) = \widehat{\psi}_t(\mathbf{k},\sigma)\exp\left(\mathrm{i}\sigma\int_0^t ds\,\omega_s^\lambda(\mathbf{k})\right), \tag{5.15}$$

then it clearly satisfies an equation without a linear term. Explicitly, then

$$\partial_t a_t(\mathbf{k}_1,\sigma) = -\mathrm{i}\sigma\lambda\int_{(\mathbb{T}^d)^3} d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$
$$\times\, \mathrm{e}^{\mathrm{i}t(\sigma\omega_1+\omega_2-\sigma\omega_3-\omega_4)} :a_t(\mathbf{k}_2,-1)a_t(\mathbf{k}_3,\sigma)a_t(\mathbf{k}_4,1): \,. \tag{5.16}$$

Note that due to the alternating signs, the time dependent terms cancel each other out in the oscillatory phase term inside the integral. In fact, the same happens in the second order cumulants, as can be checked by using (5.11) and the symmetry of ω_s^λ: we then find that

$$\kappa(a_t(\mathbf{k}_1,\sigma_1),a_t(\mathbf{k}_2,\sigma_2)) = \delta(\mathbf{k}_1+\mathbf{k}_2)\mathbb{1}(\sigma_1+\sigma_2=0)W_t^\lambda(\sigma_2\mathbf{k}_2)\,. \tag{5.17}$$

It is clear that multiplication with a nonrandom term as in (5.15) does not spoil the gauge invariance of the field so we can rely on it also when working with the cumulants of the a-fields.

We can now study the evolution of $W_t^\lambda(\mathbf{k})$ by employing the expansion given in (4.12) to the cumulant $\kappa(a_t(\mathbf{k}',\sigma'),a_t(\mathbf{k},\sigma))$. We then find using any $\sigma' = -\sigma$ that

$$\begin{aligned}
&\delta(\mathbf{k}'+\mathbf{k})(W_t^\lambda(\sigma\mathbf{k}) - W_0^\lambda(\sigma\mathbf{k}))\\
&= -\mathrm{i}\lambda\sigma\int_0^t ds\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}s(\sigma\omega+\omega_1-\sigma\omega_2-\omega_3)}\kappa[(a_0)_I]\\
&\quad - \mathrm{i}\lambda\sigma'\int_0^t ds\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}s(\sigma'\omega'+\omega_1-\sigma'\omega_2-\omega_3)}\kappa[(a_0)_{I'}]\\
&\quad - \lambda^2\sigma\sum_{\ell\in I}\sigma_\ell\int_0^t ds\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}s(\sigma\omega+\omega_1-\sigma\omega_2-\omega_3)}\\
&\qquad \times\int_0^s ds'\int_{(\mathbb{T}^d)^3} d\mathbf{k}_4 d\mathbf{k}_5 d\mathbf{k}_6 \delta(\mathbf{k}_\ell-\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)\mathrm{e}^{\mathrm{i}s'(\sigma_\ell\omega_\ell+\omega_4-\sigma_\ell\omega_5-\omega_6)}\mathbb{E}\big[:a_{s'}^{J_\ell}::a_{s'}^{\widehat{I}^{(\ell)}}:\big]\\
&\quad - \lambda^2\sigma'\sum_{\ell\in I'}\sigma_\ell\int_0^t ds\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}s(\sigma'\omega'+\omega_1-\sigma'\omega_2-\omega_3)}\\
&\qquad \times\int_0^s ds'\int_{(\mathbb{T}^d)^3} d\mathbf{k}_4 d\mathbf{k}_5 d\mathbf{k}_6 \delta(\mathbf{k}_\ell-\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)\mathrm{e}^{\mathrm{i}s'(\sigma_\ell\omega_\ell+\omega_4-\sigma_\ell\omega_5-\omega_6)}\mathbb{E}\big[:a_{s'}^{J_\ell}::a_{s'}^{\widehat{I}^{(\ell)}}:\big]\,,
\end{aligned} \tag{5.18}$$

where

$$I = ((\mathbf{k}_1,-1),(\mathbf{k}_2,\sigma),(\mathbf{k}_3,1),(\mathbf{k}',\sigma'))\,, \tag{5.19}$$
$$I' = ((\mathbf{k}_1,-1),(\mathbf{k}_2,\sigma'),(\mathbf{k}_3,1),(\mathbf{k},\sigma))\,, \tag{5.20}$$
$$J_\ell = ((\mathbf{k}_4,-1),(\mathbf{k}_5,\sigma_\ell),(\mathbf{k}_6,1))\,. \tag{5.21}$$

Following the standard perturbation recipe, we next apply the cumulant hierarchy to the terms depending on $a_{s'}$ in (5.18). This results in a sum of two terms: one, in which every $a_{s'}$ has been replaced by a_0, plus a "correction" which we denote by $\delta(\mathbf{k}'+\mathbf{k})\mathcal{R}_3(\sigma\mathbf{k},t)$. Further iterations of the perturbation expansion and a careful study of the oscillations of the term by term expansion as in [7] leads us to the conjecture that $\mathcal{R}_3(\mathbf{k},\tau\lambda^{-2})$ should converge in the kinetic scaling limit, as $\lambda\to 0$, at least for sufficiently nondegenerate dispersion relations and for large enough dimension d. In addition, the analysis indicates that the limit value is $O(\tau^2)$, which is negligible compared to the contribution from the other terms following from (5.18). However, the term by term analysis does not suffice to actually prove the claim since the method which was used to rigorously control the convergence of the perturbation expansion in [7] was based on time stationarity of the initial state. This assumption cannot be made here since we are interested in nontrivial time evolution effects. Instead of going into the details of the above argument, we discuss a less technically involved motivation for the claim in Section 6.2.

Thus now we need to evaluate expectations of the form $\mathbb{E}[:a_1a_2a_3::a_4a_5a_6:]$ where each a_i stands for one of the field variables. The cumulant expansion in Theorem 3.8 and the vanishing of the third order cumulants imply

$$\begin{aligned}\mathbb{E}[:a_1a_2a_3::a_4a_5a_6:] &= \kappa(a_1,a_4)\kappa(a_2,a_5)\kappa(a_3,a_6)+\kappa(a_1,a_4)\kappa(a_2,a_6)\kappa(a_3,a_5)\\ &\quad+\kappa(a_1,a_5)\kappa(a_2,a_4)\kappa(a_3,a_6)+\kappa(a_1,a_5)\kappa(a_2,a_6)\kappa(a_3,a_4)\\ &\quad+\kappa(a_1,a_6)\kappa(a_2,a_4)\kappa(a_3,a_5)+\kappa(a_1,a_6)\kappa(a_2,a_5)\kappa(a_3,a_4)\\ &\quad+\kappa(a_1,a_2,a_3,a_4,a_5,a_6)+\text{“}9\times\kappa_2\kappa_4\text{”}\end{aligned}\tag{5.22}$$

where the last contribution denotes a sum of the nine terms consisting of a product of a second order cumulant and a fourth order cumulant. Naturally, also some of the above terms can be zero because of the gauge invariance constraints.

To better work with the expressions arising from (5.18), let us next introduce a few shorthand notations. We denote

$$\begin{aligned} W(\mathbf{k}) &:= W_0^\lambda(\mathbf{k})\,,\\ \int d\mathbf{k}_{12\cdots n} &:= \int_{(\mathbb{T}^d)^n} d\mathbf{k}_1 d\mathbf{k}_2\cdots d\mathbf{k}_n\,,\\ \delta(\mathbf{k}-\mathbf{k}_{ijk}) &:= \delta(\mathbf{k}-\mathbf{k}_i-\mathbf{k}_j-\mathbf{k}_k)\,,\\ \Omega^{++}_{--} &:= \omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)-\omega(\mathbf{k})\,,\\ \Omega^{+-}_{-+} &:= \omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)+\omega(\mathbf{k})\,. \end{aligned}\tag{5.23}$$

We also choose $\sigma=1,\sigma'=-1$ and we will only consider the pairing contractions (i.e., the Gaussian contractions) in the expansion (5.22). In fact, all terms arising from the non-pairing contractions are typically negligible in the kinetic scaling limit of the present type. As an example, in Appendix C we show how the first order terms in (5.18) vanish in the kinetic limit by assuming sufficient regularity of the dispersion relation ω and the ℓ_1-clustering property of the fourth order cumulants. As explained in [7], the contribution from the non-pairing terms in (5.22) can be controlled by similar techniques but we will skip this more involved analysis here.

Hence, after integrating out the variables $\mathbf{k}_i$, $i=4,5,6$, the fourth term in (5.18) gives

$$\begin{aligned}&2\lambda^2\delta(\mathbf{k}+\mathbf{k}')\bigg[\int d\mathbf{k}_{123}\delta(\mathbf{k}-\mathbf{k}_{123})W(\mathbf{k})W(\mathbf{k}_2)W(\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{+-}_{-+}}\\ &\quad-\int d\mathbf{k}_{123}\delta(\mathbf{k}-\mathbf{k}_{123})W(\mathbf{k})W(-\mathbf{k}_1)W(\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{+-}_{-+}}\\ &\quad-\int d\mathbf{k}_{123}\delta(\mathbf{k}-\mathbf{k}_{123})W(\mathbf{k})W(-\mathbf{k}_1)W(\mathbf{k}_2)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{+-}_{-+}}\\ &\quad+\int d\mathbf{k}_{123}\delta(\mathbf{k}-\mathbf{k}_{123})W(-\mathbf{k}_1)W(\mathbf{k}_2)W(\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{+-}_{-+}}\bigg]+\text{NPC}\end{aligned}\tag{5.24}$$

where NPC stands for "non-pairing contraction terms". We proceed in the same way for the fifth term in (5.18) yielding

$$
\begin{aligned}
&2\lambda^2\delta(\mathbf{k}+\mathbf{k}')\Big[\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_{123})W(\mathbf{k})W(-\mathbf{k}_1)W(-\mathbf{k}_2)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{++}_{--}}\\
&\quad-\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_{123})W(\mathbf{k})W(-\mathbf{k}_1)W(\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{++}_{--}}\\
&\quad-\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_{123})W(\mathbf{k})W(-\mathbf{k}_2)W(\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{++}_{--}}\\
&\quad+\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_{123})W(-\mathbf{k}_1)W(-\mathbf{k}_2)W(\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')\Omega^{++}_{--}}\Big]+\mathrm{NPC}\,.
\end{aligned}
\tag{5.25}
$$

By changing integration variables so that $\mathbf{k}_1\to-\mathbf{k}_1$ in (5.24) and $\mathbf{k}_1\to-\mathbf{k}_3$, $\mathbf{k}_2\to-\mathbf{k}_2$, $\mathbf{k}_3\to\mathbf{k}_1$ in (5.25), we obtain

$$
\begin{aligned}
&W^\lambda_t(\mathbf{k})-W(\mathbf{k})-(\mathcal{R}_3(\mathbf{k},t)+\mathrm{NPC})\\
&\quad=2\lambda^2\int_{(\mathbb{T}^d)^3}d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{\mathrm{i}(s-s')(\omega_1-\omega_2-\omega_3+\omega)}\\
&\qquad\times[W(\mathbf{k})W(\mathbf{k}_2)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_2)+W(\mathbf{k}_1)W(\mathbf{k}_2)W(\mathbf{k}_3)]\\
&\qquad+2\lambda^2\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\int_0^t ds\int_0^s ds'\mathrm{e}^{-\mathrm{i}(s-s')(\omega_1-\omega_2-\omega_3+\omega)}\\
&\qquad\times[W(\mathbf{k})W(\mathbf{k}_2)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_2)+W(\mathbf{k}_1)W(\mathbf{k}_2)W(\mathbf{k}_3)]\\
&\quad=2\lambda^2\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\int_0^t ds\int_{-s}^s dr\,\mathrm{e}^{\mathrm{i}r(\omega_1-\omega_2-\omega_3+\omega)}\\
&\qquad\times[W(\mathbf{k}_1)W(\mathbf{k}_2)W(\mathbf{k}_3)+W(\mathbf{k})W(\mathbf{k}_2)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_2)]\,.
\end{aligned}
\tag{5.26}
$$

Note that for any $\Omega\in\mathbb{R}$ we have $\lambda^2\int_0^t ds\int_{|r|\le s}dr\,\mathrm{e}^{\mathrm{i}r\Omega}=\int_{|r|\le t}dr\,\mathrm{e}^{\mathrm{i}r\Omega}(\lambda^2t-\lambda^2|r|)$. By setting $t=\tau\lambda^{-2}$ and taking $\lambda\to0$, this expression formally converges to $\tau\int_{-\infty}^{\infty}dr\,\mathrm{e}^{\mathrm{i}r\Omega}=\tau2\pi\delta(\Omega)$. Therefore, doing this in (5.26) yields the conjecture that

$$
\begin{aligned}
&W_\tau(\mathbf{k})-W(\mathbf{k})-O(\tau^2)=\tau4\pi\int d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega+\omega_1-\omega_2-\omega_3)\\
&\quad\times[W(\mathbf{k}_1)W(\mathbf{k}_2)W(\mathbf{k}_3)+W(\mathbf{k})W(\mathbf{k}_2)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_3)-W(\mathbf{k})W(\mathbf{k}_1)W(\mathbf{k}_2)]\,.
\end{aligned}
\tag{5.27}
$$

Since here $W(\mathbf{k})=W_0(\mathbf{k})$, if we divide the left hand side by τ and then take $\tau\to0$, it converges to $\partial_\tau W_\tau(\mathbf{k})$ at $\tau=0$. Dividing the right hand side of (5.27) by τ yields $\mathcal{C}(W_0(\cdot))(\mathbf{k})$, as defined in (5.14). Therefore, the Boltzmann-Peierls equation (5.13) should hold at $\tau=0$. Assuming that the state of the original system remains so regular that the estimates leading to the conjecture continue to hold, we thus find that the Boltzmann-Peierls equation should be valid for the limit of $W^\lambda_{\tau\lambda^{-2}}$ also at later times τ, as was claimed in the beginning of the subsection.

5.1.2 Decay of field time-correlations

As a second example of how the standard perturbation expansion works for the cumulants, we consider a kinetic scaling limit of time-correlations. In particular, our goal is to show how the main results proven in [7] relate to the present cumulant hierarchy expansions.

The notation "$\hat{a}_t$" was used in [7] to define the finite periodic lattice analogue of the present a_t-field. (One can compare the definition of "$\hat{a}_t$" in (3.9) and its evolution equation in (3.10) in [7] to those given for a_t here.) Translated to the present infinite lattice setup, the main theorem of [7]

(Theorem 2.4) leads to the following conjecture about the decay of time correlations of a_t: start the system from an ℓ_1-clustering equilibrium Gibbs state. Then there is a continuous function $A_t^\lambda(\mathbf{k})$ such that $\mathbb{E}[a_0(\mathbf{k}',-1)a_t(\mathbf{k},1)] = \delta(\mathbf{k}'+\mathbf{k})A_t^\lambda(\mathbf{k})$. The conjecture is that the kinetic scaling limit of A^λ exists and its decay is governed by the "loss term" of the Boltzmann-Peierls equation (5.13) evaluated at the corresponding limit equilibrium covariance function $W^{\mathrm{eql}}(\mathbf{k}) = \beta^{-1}/(\omega(\mathbf{k})-\mu)$ where $\beta > 0$ and $\mu \in \mathbb{R}$ are parameters determined by the equilibrium state. (Such functions W^{eql} are indeed stationary solutions of (5.13).) More precisely, [7, Theorem 2.4] is consistent with the conjecture that

$$\lim_{\lambda\to 0} A^\lambda_{\tau\lambda^{-2}}(\mathbf{k}) = W^{\mathrm{eql}}(\mathbf{k})\mathrm{e}^{-\tau\Gamma(W^{\mathrm{eql}}(\cdot))(\mathbf{k})}\,, \tag{5.28}$$

where

$$\begin{aligned}\Gamma(W(\cdot))(\mathbf{k}) = -2\int_0^\infty \mathrm{d}r \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}r(\omega_1-\omega_2-\omega_3+\omega)}\\ \times\left[W(\mathbf{k}_2)W(\mathbf{k}_3) - W(\mathbf{k}_1)W(\mathbf{k}_3) - W(\mathbf{k}_1)W(\mathbf{k}_2)\right].\end{aligned} \tag{5.29}$$

Instead of assuming that the system starts from an equilibrium state, let us consider more general states which we assume to be gauge and translation invariant and ℓ_1-clustering. We can immediately use the results derived in Section 4 if we consider the "a_0" term to be a new field which has trivial time evolution with zero amplitudes, i.e., the corresponding $\mathscr{I}_j$-set is empty. The net effect of this change is that more than half of the terms analyzed in the previous section will be absent. For instance, the expansion of the time correlation using (4.13) reads

$$\begin{aligned}&\kappa[a_0(\mathbf{k}',-1), a_t(\mathbf{k},1)]\\ &\quad= \kappa[a_0(\mathbf{k}',-1), a_0(\mathbf{k},1)] - \mathrm{i}\lambda\int_0^t ds \int d\mathbf{k}_{123}\delta(\mathbf{k}-\mathbf{k}_{123})\mathrm{e}^{\mathrm{i}s(\omega+\omega_1-\omega_2-\omega_3)}\kappa[(a_0)_{I'+(\mathbf{k}',-1)}]\\ &\qquad - \lambda^2\sum_{\ell\in I'}\sigma_\ell\int_0^t ds\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}s(\omega+\omega_1-\omega_2-\omega_3)}\\ &\qquad\quad\times\int_0^s ds'\int d\mathbf{k}_4 d\mathbf{k}_5 d\mathbf{k}_6\delta(\mathbf{k}_\ell-\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)\mathrm{e}^{\mathrm{i}s'(\sigma_\ell\omega_\ell+\omega_4-\sigma_\ell\omega_5-\omega_6)}\mathbb{E}\big[:a_{s'}^{J_\ell}::a_0(\mathbf{k}',-1)a_{s'}^{\widehat{I'}^{(\ell)}}:\big]\end{aligned} \tag{5.30}$$

where $I' = ((\mathbf{k}_1,-1),(\mathbf{k}_2,1),(\mathbf{k}_3,1))$ and $J_\ell = ((\mathbf{k}_4,-1),(\mathbf{k}_5,\sigma_\ell),(\mathbf{k}_6,1))$.

As in Section 5.1.1, we now assume that only pairings contribute in the kinetic scaling limit. When applying (5.22) to expand $\mathbb{E}[:a_{s'}^{J_\ell}::a_0(\mathbf{k}',-1)a_{s'}^{\widehat{I'}^{(\ell)}}:]$, we note that every pairing term results in a product containing a factor $A^\lambda_{s'}(-\mathbf{k}')$ and a product of two $W^\lambda_{s'}$-terms. The rest of the structure is identical to the one considered earlier, some of the terms are merely missing now. We then use the perturbation expansion to the product once more. This produces a term where s' is set to 0, and a remainder which we assume to be negligible as before. The rest of the computation is essentially the same as in Section 5.1.1, yielding

$$\begin{aligned}&A^\lambda_{\tau\lambda^{-2}}(\mathbf{k}) - A^\lambda_0(\mathbf{k}) - (\text{terms higher order in } \lambda \text{ or } \tau)\\ &\quad= 2\lambda^2\int_{(\mathbb{T}^d)^3} d\mathbf{k}_{123}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\int_0^{\tau\lambda^{-2}} ds\int_0^s dr\,\mathrm{e}^{\mathrm{i}r(\omega_1-\omega_2-\omega_3+\omega)}\\ &\qquad\times\left[A^\lambda_0(\mathbf{k})W^\lambda_0(\mathbf{k}_2)W^\lambda_0(\mathbf{k}_3) - A^\lambda_0(\mathbf{k})W^\lambda_0(\mathbf{k}_1)W^\lambda_0(\mathbf{k}_3) - A^\lambda_0(\mathbf{k})W^\lambda_0(\mathbf{k}_1)W^\lambda_0(\mathbf{k}_2)\right].\end{aligned} \tag{5.31}$$

Hence, if we divide the equation by τ and then take $\lambda\to 0$, followed by $\tau\to 0$, we find that $A_\tau(\mathbf{k}) := \lim_{\lambda\to 0} A^\lambda_{\tau\lambda^{-2}}(\mathbf{k})$ should satisfy at $\tau = 0$

$$\partial_\tau A_\tau(\mathbf{k}) = -A_\tau(\mathbf{k})\Gamma(W_\tau(\cdot))(\mathbf{k})\,, \tag{5.32}$$

where Γ has been defined in (5.29). As before, the conjecture is that this equation continues to hold for other values $\tau > 0$, as well.

Once W_τ is given, equation (5.32) is straightforward to solve. Since $A_0^\lambda(\mathbf{k}) = W_0(\mathbf{k})$, the solution reads

$$A_\tau(\mathbf{k}) = W_0(\mathbf{k})\mathrm{e}^{-\int_0^\tau \mathrm{d}s\,\Gamma(W_s(\cdot))(\mathbf{k})}\,. \tag{5.33}$$

If the system is started in an equilibrium state with $W_0 = W^{\mathrm{eql}}$, we have $W_s = W^{\mathrm{eql}}$ for all s. Thus (5.33) implies (5.28) in this special case.

6 Discussion about further applications

6.1 Limitations of the direct renormalization procedure: inhomogeneous DNLS

The field renormalization used with the translational invariant data greatly simplified the evolution equation by removing the linear term. The renormalization procedure, given in (5.15), was a simple multiplication by a time and k-dependent function and the time-dependent first order terms had no effect in the interaction term. Unfortunately, this case is atypical: most commonly, the necessary renormalization is not a multiplication operator and the first order terms will also affect the oscillatory phase terms arising from the harmonic evolution. In fact, this happens also for the DNLS model as soon as we drop the requirement that the initial data is translation invariant. To explain the changes needed in the renormalization procedure, we discuss in this subsection the DNLS model with inhomogeneous initial data in some more detail.

Before considering the inhomogeneous case, let us begin with an example which emphasizes the importance of the field renormalization even for translation invariant initial data if one considers taking kinetic scaling limits of all field observables. We inspect the time correlation of the "bare" $\widehat{\psi}$-fields, i.e., $\mathbb{E}[\widehat{\psi}_0(\mathbf{k}',-1)\widehat{\psi}_t(\mathbf{k},1)]$ assuming spatially homogeneous, gauge invariant initial data. Then there exists a function $\Psi_t(\mathbf{k})$ such that $\mathbb{E}[\widehat{\psi}_0(\mathbf{k}',-1)\widehat{\psi}_t(\mathbf{k},1)] = \delta(\mathbf{k}'+\mathbf{k})\Psi_t(\mathbf{k})$. By using (5.7) in (4.6), we find the following evolution equation for Ψ_t:

$$\begin{aligned}\delta(\mathbf{k}'+\mathbf{k})\Psi_t(\mathbf{k}) &= \kappa(\widehat{\psi}_0(\mathbf{k}',-1),\widehat{\psi}_t(\mathbf{k},1)) = \delta(\mathbf{k}'+\mathbf{k})\Psi_0(\mathbf{k}) - \mathrm{i}\delta(\mathbf{k}'+\mathbf{k})\int_0^t ds\,\omega_s^\lambda(\mathbf{k})\Psi_s(\mathbf{k})\\ &- \mathrm{i}\lambda\int_0^t ds\int_{\mathbb{T}^{3d}} d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathbb{E}[:\widehat{\psi}_0(\mathbf{k}',-1)::\widehat{\psi}_t(\mathbf{k}_1,-1)\widehat{\psi}_t(\mathbf{k}_2,1)\widehat{\psi}_t(\mathbf{k}_3,1):]\,.\end{aligned} \tag{6.1}$$

The linear equation associated to (6.1) thus has the form

$$f_t(\mathbf{k}) = f_0(\mathbf{k}) - \mathrm{i}\int_0^t ds\,\omega_s^\lambda(\mathbf{k})f_s(\mathbf{k})\,, \tag{6.2}$$

which is solved by $f_t(\mathbf{k}) = U_t(\mathbf{k})f_0(\mathbf{k})$ where $U_t(\mathbf{k}) = \exp(-\mathrm{i}\int_0^t ds\,\omega_s^\lambda(\mathbf{k}))$. We recall that $\omega_s^\lambda(\mathbf{k}) = \omega(\mathbf{k}) + \lambda R_s$, and thus at a kinetic time scale, with $t = \tau\lambda^{-2}$, we have $\int_0^t ds\,\omega_s^\lambda(\mathbf{k}) = \tau\lambda^{-2}\omega(\mathbf{k}) + O(\lambda^{-1})$. Therefore, U_t has unbounded oscillations in the kinetic scaling limit. Also, we find that even though the effect of the first order term proportional to R_s is subdominant, it is still rapidly oscillating on the kinetic time-scale and should not be "expanded" in any perturbative treatment of the problem.

We could solve the problem with these unbounded oscillations by considering instead of $\widehat{\psi}_t$ the renormalized field $a_t = U_t^{-1}\widehat{\psi}_t$. In fact, by the results of Section 5.1.2 and using $a_0 = \widehat{\psi}_0$ we find that

$$\delta(\mathbf{k}+\mathbf{k}')A_t^\lambda(\mathbf{k}) = \mathbb{E}[a_0(\mathbf{k}',-1)a_t(\mathbf{k},1)] = U_t(\mathbf{k})\mathbb{E}\Big[\widehat{\psi}_0(\mathbf{k}',-1)\widehat{\psi}_t(\mathbf{k},1)\Big] = \delta(\mathbf{k}+\mathbf{k}')U_t(\mathbf{k})\Psi_t(\mathbf{k})\,. \tag{6.3}$$

We have argued in Section 5.1.2 that the kinetic scaling limit of A_t^λ exists. Then $\Psi_{\tau\lambda^{-2}}(\mathbf{k})$ *cannot* have a convergent limit as $\lambda \to 0$; instead, it has fast oscillations proportional to $U^{-1}_{\tau\lambda^{-2}}(\mathbf{k})$. Let us also once more stress that the "zeroth order renormalization", i.e., countering the free evolution, does not remove all of the unbounded oscillations but still leaves those resulting from the R_t term.

However, the above renormalization procedure cannot be straightforwardly extended to more complicated cases. Consider next the DNLS model with inhomogeneous initial data. As in section 5.1.1, our goal is to find the right observable which satisfies the Boltzmann equation in the kinetic scaling limit. The evolution equation for the bare field $\widehat{\psi}$ reads

$$\begin{aligned}\partial_t\widehat{\psi}_t(\mathbf{k},\sigma) &= -\mathrm{i}\sigma\omega(\mathbf{k})\widehat{\psi}_t(\mathbf{k},\sigma)\\ &\quad - 2\mathrm{i}\sigma\lambda\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\kappa(\widehat{\psi}_t(\mathbf{k}_1,-1),\widehat{\psi}_t(\mathbf{k}_3,1))\widehat{\psi}_t(\mathbf{k}_2,\sigma)\\ &\quad - \mathrm{i}\sigma\lambda\int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,{:}\widehat{\psi}_t(\mathbf{k}_1,-1)\widehat{\psi}_t(\mathbf{k}_2,\sigma)\widehat{\psi}_t(\mathbf{k}_3,1){:}\,. \end{aligned} \tag{6.4}$$

By following the same strategy as for the homogeneous case, let be U_t, $t \geq 0$, denote the family of linear operators which solves the linear part of (6.4): we suppose that it solves the operator equation $\partial_t U_t = -\mathrm{i}H_t U_t$ where $(H_t f)(\mathbf{k},\sigma) = \sigma\omega(\mathbf{k})f(\mathbf{k},\sigma) + 2\sigma\lambda\int_{\mathbb{T}^d} d\mathbf{k}' K_t(\mathbf{k}-\mathbf{k}')f(\mathbf{k}',\sigma)$. The time-dependent convolution kernel $K_t(\mathbf{k})$ should be equal to $\int_{\mathbb{T}^d} d\mathbf{k}_1\kappa(\widehat{\psi}_t(\mathbf{k}_1,-1),\widehat{\psi}_t(\mathbf{k}-\mathbf{k}_1,1))$. This can be done either by first solving the implicit equation for the above integral over the cumulant, or by leaving K_t arbitrary and fixing it by some minimization procedure at the end. If such a family U_t can be found, we may define as before $\tilde{a}_t(\mathbf{k},\sigma) = (U_t^{-1}\widehat{\psi}_t)(\mathbf{k},\sigma)$ and, since $\partial_t U_t^{-1} = -U_t^{-1}(\partial_t U_t)U_t^{-1} = \mathrm{i}U_t^{-1}H_t$, it then satisfies an evolution equation

$$\begin{aligned}\partial_t\tilde{a}_t(\mathbf{k},\sigma) &= -\mathrm{i}\sigma\lambda\int_{(\mathbb{T}^d)^3} d^3\mathbf{k}\int_{\mathbb{T}^d} d\mathbf{k}''\int_{(\mathbb{T}^d)^3} d^3\mathbf{k}'\,\delta(\mathbf{k}''-\mathbf{k}_1'-\mathbf{k}_2'-\mathbf{k}_3')\\ &\quad\times\tilde{u}_t(\mathbf{k},\mathbf{k}'',\sigma)u_t(\mathbf{k}_1',\mathbf{k}_1,-1)u_t(\mathbf{k}_2',\mathbf{k}_2,\sigma)u_t(\mathbf{k}_3',\mathbf{k}_3,1)\,{:}\,\tilde{a}_t(\mathbf{k}_1,-1)\tilde{a}_t(\mathbf{k}_2,\sigma)\tilde{a}_t(\mathbf{k}_3,1){:}\,, \end{aligned} \tag{6.5}$$

where $u_t(\mathbf{k},\mathbf{k}',\sigma)$ and $\tilde{u}_t(\mathbf{k},\mathbf{k}',\sigma)$ denote the formal integral kernels of the operators U_t and U_t^{-1}, respectively. (Note that the operators U_t are diagonal in σ but not any more in $\mathbf{k}$.)

Apart from some special cases it seems difficult to gain sufficient control over the operators U_t to consider taking a kinetic limit using the observables $\tilde{a}_t$, unlike with the explicit phases factors which appeared in the spatially homogeneous case. Even though U_t approach the same multiplication operator as before when $\lambda \to 0$ for a fixed t, it is not clear that the corrections do not contribute in the limit, since we need to consider $t = O(\lambda^{-2})$. Thus, although the cumulant expansion of $\tilde{a}_t$-fields is simpler than that of ψ_t-fields, to control the kinetic scaling limit looks intractable. Hence, new approaches for the study of the kinetic time scales are called for.

6.2 Kinetic theory beyond kinetic time-scales?

In this section we propose a new approach to the problem when a renormalization scheme with a convergent kinetic scaling limit cannot be found or controlled. The approach does not require taking $\lambda \to 0$, and, if successful, it may also yield estimates which are valid beyond the standard kinetic time scales which was $O(\lambda^{-2})$ in the above DNLS case.

The main idea can be summarized in the following simple observation. Suppose f_t is a solution to the equation

$$f_t = f_0 + R_t + \int_0^t \mathrm{d}s\, F_{s,t}[f_s] \tag{6.6}$$

where $R_t = O(\varepsilon)$ uniformly in t and $F_{s,t}$, $0 \leq s \leq t$, is an explicit, but possibly nonlinear functional of f_s. Suppose furthermore that there is another, "simpler", functional $\Phi_{s,t}$ such that $F_{s,t} = \Phi_{s,t} + O(\varepsilon(1+|t-s|)^{-p})$ with $p > 1$; by simpler we mean that the evolution problem

$$\varphi_t = S_t + \int_0^t \mathrm{d}s\,\Phi_{s,t}[\varphi_s]\,, \tag{6.7}$$

for any bounded "source term" S_t, is easier to study than (6.6). Under these assumptions, any solution to (6.6) satisfies

$$f_t = f_0 + \rho_t + \int_0^t \mathrm{d}s\, \Phi_{s,t}[f_s]\,, \tag{6.8}$$

where $\rho_t := R_t + \int_0^t \mathrm{d}s\, (F_{s,t}[f_s] - \Phi_{s,t}[f_s])$ is $O(\varepsilon)$ uniformly in t. Therefore, if we could prove that (6.7) is stable under perturbations of the source term S_t, we may conclude that the solution φ_t to

$$\varphi_t = f_0 + \int_0^t \mathrm{d}s\, \Phi_{s,t}[\varphi_s] \tag{6.9}$$

then approximates the "true" solution f_t with an error which is $O(\varepsilon)$ *uniformly in time*.

To have a concrete example, consider again the DNLS with gauge invariant initial data. We sketch below two conjectures about this system. The details of the conjectures should not be taken too seriously: they should be considered more as examples of what could happen in general rather than as specific conjectures about the behavior of the DNLS system. For this reason, the discussion will be kept on a very loose level; in particular, we do not wish to make any specific claims about what kind of metrics should be used for studying the uniform boundedness in time.

Let us begin with the case of inhomogeneous, gauge invariant initial data. We move to slowly varying fields by cancelling the free evolution term. This renormalization leads to equations which are almost identical to those in the homogeneous case. Namely, for the field $b_t(\mathbf{k},\sigma) := e^{\mathrm{i}\sigma\omega(\mathbf{k})t}\hat{\psi}_t(\mathbf{k},\sigma)$ we have

$$\begin{aligned}\partial_t b_t(\mathbf{k},\sigma) = {}& -2\mathrm{i}\sigma\lambda \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}t(\sigma\omega+\omega_1-\sigma\omega_2-\omega_3)} B_t(\mathbf{k}_1,\mathbf{k}_3)\, {:}b_t(\mathbf{k}_2,\sigma){:} \\ & - \mathrm{i}\sigma\lambda \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}t(\sigma\omega+\omega_1-\sigma\omega_2-\omega_3)} {:}b_t(\mathbf{k}_1,-1)b_t(\mathbf{k}_2,\sigma)b_t(\mathbf{k}_3,1){:}\end{aligned} \tag{6.10}$$

where $B_t(\mathbf{k}',\mathbf{k}) = \kappa(b_t(\mathbf{k}',-1), b_t(\mathbf{k},1))$. By the gauge invariance, $\kappa(b_t(\mathbf{k}',\sigma'), b_t(\mathbf{k},\sigma)) = 0$, unless $\sigma'+\sigma = 0$. Hence, there is only one other nonzero second order cumulant, $\kappa(b_t(\mathbf{k}',1), b_t(\mathbf{k},-1)) = B_t(\mathbf{k},\mathbf{k}') = B_t(-\mathbf{k}',-\mathbf{k})^*$.

To study the fourth order cumulants, it suffices to concentrate on the function

$$D_t(k) := \kappa(b_t(\mathbf{k}_1,-1), b_t(\mathbf{k}_2,-1), b_t(\mathbf{k}_3,1), b_t(\mathbf{k}_4,1))\,,\ k \in (\mathbb{T}^d)^4\,.$$

Then the first two equations in the cumulant hierarchy are equivalent to

$$\begin{aligned}\partial_t B_t(\mathbf{k}',\mathbf{k}) = {}& -2\mathrm{i}\lambda \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}t(\omega+\omega_1-\omega_2-\omega_3)} B_t(\mathbf{k}_1,\mathbf{k}_3)B_t(\mathbf{k}_2,\mathbf{k}') \\ & + 2\mathrm{i}\lambda \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}t(-\omega'+\omega_1+\omega_2-\omega_3)} B_t(\mathbf{k}_1,\mathbf{k}_3)B_t(\mathbf{k},\mathbf{k}_2) \\ & - \mathrm{i}\lambda \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}t(\omega+\omega_1-\omega_2-\omega_3)} D_t(\mathbf{k}',\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \\ & + \mathrm{i}\lambda \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{e}^{\mathrm{i}t(-\omega'+\omega_1+\omega_2-\omega_3)} D_t(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})\,,\end{aligned} \tag{6.11}$$

$$\begin{aligned}\partial_t D_t(k) = {}& -\mathrm{i}2\lambda \sum_{\ell=1}^4 \sigma_\ell \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1' d\mathbf{k}_2' d\mathbf{k}_3' \delta(\mathbf{k}_\ell-\mathbf{k}_1'-\mathbf{k}_2'-\mathbf{k}_3')\mathrm{e}^{\mathrm{i}t(\sigma_\ell\omega_\ell+\omega_1'-\sigma_\ell\omega_2'-\omega_3')} \\ & \times B_t(\mathbf{k}_1',\mathbf{k}_3')D_t(\text{“replace } \mathbf{k}_\ell \text{ by } \mathbf{k}_2'\text{”}) \\ & - \mathrm{i}\lambda \sum_{\ell=1}^4 \sigma_\ell \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1' d\mathbf{k}_2' d\mathbf{k}_3' \delta(\mathbf{k}_\ell-\mathbf{k}_1'-\mathbf{k}_2'-\mathbf{k}_3')\mathrm{e}^{\mathrm{i}t(\sigma_\ell\omega_\ell+\omega_1'-\sigma_\ell\omega_2'-\omega_3')} \\ & \times (\text{“}(1\times\kappa_6) + (5\times B_tD_t) + (6\times B_tB_tB_t)\text{”})\,,\end{aligned} \tag{6.12}$$

where in the second formula, $\sigma := (-1,-1,1,1)$ and on the last line we have applied (5.22) and merely denoted how many nonzero terms each type of partition can have. Let us point out that the first two terms in (6.11) cancel each other out if the state is spatially homogeneous, since then $B_t(\mathbf{k}',\mathbf{k}) \propto \delta(\mathbf{k}'+\mathbf{k})$. For an inhomogeneous state, however, the cancellation need not be exact, and since this term is then not $O(\lambda^2)$, it will likely prevent taking of the kinetic scaling limit of B_t directly.

If we now integrate the B equation as in (4.12), we find that

$$\begin{aligned} B_t(\mathbf{k}',\mathbf{k}) &= B_0(\mathbf{k}',\mathbf{k}) - \mathrm{i}\lambda \int_0^t \mathrm{d}s \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \mathrm{e}^{\mathrm{i}s(\omega+\omega_1-\omega_2-\omega_3)} \\ &\quad \times (2B_s(\mathbf{k}_1,\mathbf{k}_3)B_s(\mathbf{k}_2,\mathbf{k}') + D_0(\mathbf{k}',\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)) \\ &\quad + \mathrm{i}\lambda \int_0^t \mathrm{d}s \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \mathrm{e}^{\mathrm{i}s(-\omega'+\omega_1+\omega_2-\omega_3)} \\ &\quad \times (2B_s(\mathbf{k}_1,\mathbf{k}_3)B_s(\mathbf{k},\mathbf{k}_2) + D_0(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})) \\ &\quad + \lambda^2 \int_0^t \mathrm{d}s' \int_{s'}^t \mathrm{d}s \sum (\cdots) \,. \end{aligned} \tag{6.13}$$

In the final sum, each term depends on s only via the oscillatory phases. These can be collected together and they have a structure

$$\mathrm{e}^{\mathrm{i}s(\sigma_0(\omega_0-\omega_2)+\omega_1-\omega_3)} \mathrm{e}^{\mathrm{i}s'(\tilde{\sigma}_\ell\tilde{\omega}_\ell+\omega_1'-\tilde{\sigma}_\ell\omega_2'-\omega_3')} \,, \tag{6.14}$$

where $(\sigma_0,\mathbf{k}_0)$ is equal to $(1,\mathbf{k})$, if the term arises from the second last term in (6.11) and it is equal $(-1,\mathbf{k}')$ if it arises from the last term. The pair $(\tilde{\sigma}_\ell,\tilde{\omega}_\ell)$ comes from arguments of the corresponding "D_t-term" and thus depends also on this choice. Therefore, the s-integral over the oscillatory phase can be computed explicitly and to each of the terms in the sum it will produce a factor

$$\mathrm{e}^{\mathrm{i}s'(\sigma_0(\omega_0-\omega_2)+\omega_1-\omega_3+\tilde{\sigma}_\ell\tilde{\omega}_\ell+\omega_1'-\sigma_\ell\omega_2'-\omega_3')} \int_0^{t-s'} \mathrm{d}r \, \mathrm{e}^{\mathrm{i}r(\sigma_0(\omega_0-\omega_2)+\omega_1-\omega_3)} \,. \tag{6.15}$$

It is difficult to go further in the analysis of the oscillatory phases without resorting to graph theory, and we will not pursue it here. However, already the simple example given in Appendix C shows that, if the state is ℓ_1-clustering, the oscillations may result in time-integrals which are absolutely convergent over $[0,\infty)$. For instance, this explicit example implies that if the initial state is homogeneous and ℓ_1-clustering, then the two terms depending on D_0 in (6.13) are $O(\lambda)$ *uniformly in time*. If we assume that similar bounds are valid for inhomogeneous states and every term containing κ_6, the two-component field $f_t = (B_t, D_t)$ behaves as the model considered in the beginning of the section.

Therefore, assuming uniform boundedness of κ_6 allows using the principles described in the beginning of this section and results in a conjecture about the evolution of the cumulants. Suppose that there is a metric for the cumulants, similar or given by ℓ_1-clustering estimates, such that the following results hold for sufficiently regular dispersion relations ω and initial data:

1. Suppose that the contribution from $\kappa_6(t)$ is uniformly bounded in time, with a bound $O(\lambda^{1-q})$ where $0 \le q \le 1$.

2. Consider the evolution equation obtained for (B_t, D_t) by the standard "closure relation", i.e., by setting $\kappa_6 \to 0$ in (6.12). Assume that this equation is stable under all uniformly bounded *time-dependent* perturbations of the source term.

If both of the above hold, then the solutions to the closure evolution equation for f_t remain close to (B_t, D_t) uniformly in t. In particular, the difference in the first (covariance) component is always $O(\lambda^{2-q})$. This would validate the closure equations as good approximations even to $t \to \infty$ asymptotic behavior of the covariance function for all sufficiently small λ. Note that no scaling

limit needs to be taken; in particular, it is *not* claimed that the kinetic scaling limit of B_t or D_t would exist.

Another application can be obtained for a one-component case with $f_t = W_t^\lambda$ as follows: Consider the spatially homogeneous case and the exact evolution equation for W_t^λ obtained from (5.18) by "dividing out" $\delta(\mathbf{k}'+\mathbf{k})$ from both sides. Take R_t to include all terms which contain either κ_4 or κ_6. Then the remaining pairing terms yield an explicit definition for $F_{s,t}$ such that (6.6) holds. (In fact, then $F_{s,t}[W]$ is equal to the right hand side of (5.26).) Next choose "$\Phi_{s,t}$" equal to the Boltzmann collision operator $\lambda^2\mathcal{C}$ with $\mathcal{C}$ defined in (5.14). If W_t comes from an ℓ_1-clustering state and the free evolution is sufficiently dispersive, then $F_{s,t}[W_s] = \mathcal{C}[W_s] + O(\lambda^2(1+|t-s|)^{-p})$, with $p > 1$. (For instance, the estimates given in [7], in Proposition 7.4 and in the Appendix, prove the bound with $p = 3d/7 - 1$ for a nearest neighbor dispersion relation—the computation is essentially the same as in Appendix C below. Using the notations defined in (C.3), the result also allows to quantify the dependence on the ℓ_1-clustering assumption: the bound is proportional to $\|\kappa_2(s)\|_1^3\lambda^2(1+|t-s|)^{-p}$. Hence, then $p > 1$ at least if $d \geq 5$. However, more careful estimates or the addition of next to nearest neighbor hopping could improve the bound.)

Whenever this is the case, we obtain a second conjecture about the homogeneous DNLS equation. Suppose that there is a metric for the cumulants such that the following results hold for sufficiently regular ω and initial data:

1. Suppose that the cumulants remain uniformly bounded in this metric, with an upper bound which implies that $\|\kappa_2\|_1 = O(1)$ and that all higher order cumulants have ℓ_1-clustering norm which is $O(\lambda^{-q})$ for some $0 \leq q < 2$.

2. Assume that the corresponding Boltzmann-Peierls equation is stable under all uniformly bounded *time-dependent* perturbations of the source term.

If both of the above hold, then the solutions W_τ to the Boltzmann-Peierls equation with initial data W_0^λ are $O(\lambda^{2-q})$ close to $W^\lambda_{\tau\lambda^{-2}}$ uniformly in τ. In particular, any stationary limit $\lim_{t\to\infty} W_t^\lambda$ can differ from the limit of the solution to the Boltzmann-Peierls equation only by $O(\lambda^{2-q})$.

The main benefit from using the Boltzmann-Peierls equation instead of the closure hierarchy concerns the second assumption: the homogeneous Boltzmann-Peierls equations enjoy many simplifying properties and a priori estimates, see for instance [9]. For example, they typically have an entropy functional and an associated "H-theorem" which allow to classify all stationary solutions to the equation. There are also many techniques developed to control the convergence towards the stationary solution.

As a final example, let us remark that even when total uniformity in time cannot be achieved, it might be possible to go beyond the kinetic time-scales using the above methods. Consider $F_{s,t}[W_t] = \mathcal{C}[W_t] + O(\lambda^2(1+|t-s|)^{-p})$ for some $0 < p < 1$. Then the correction is not integrable and the perturbation ρ_t to the source term is $O(\lambda^2 t^{1-p})$. This remains $O(\lambda^\varepsilon)$ for all $t = O(\lambda^{-(2-\varepsilon)/(1-p)})$. Hence, for instance, the earlier nearest neighbor estimate with $d = 3$ would imply that the corrections to the Boltzmann-Peierls equation remain small for $t = O(\lambda^{-2-4/5+\delta})$, that is, even for times much longer than the ones implied by the kinetic scaling limit.

A Combinatorial definition of cumulants

It is shown in [8] that cumulants are connected to moments via a formula which is very similar to the definition we used here for the Wick polynomials: if $I \neq \emptyset$, for any $x \in I$ we have

$$\mathbb{E}[y^I] = \sum_{E : x \in E \subset I} \mathbb{E}[y^{I\setminus E}]\kappa[y_E]\,. \tag{A.1}$$

(The formula follows straightforwardly from the identity $\partial_x G_\mathrm{m} = G_\mathrm{m}\partial_x g_\mathrm{c}$.) In fact, this formula allows a definition of cumulants which does not rely on differentiation or on the existence of exponential moments. Namely, if $I_0 \in \mathscr{I}$ is such that $\mathbb{E}[|y^I|] < \infty$ for all $I \subset I_0$, then to each $I \subset I_0$, $I \neq \emptyset$, we can associate a number $\kappa[y_I]$ by requiring that $\kappa[y_I] = \mathbb{E}[y^I] - \sum_{E : x \in E \subsetneq I} \mathbb{E}[y^{I\setminus E}]\kappa[y_E]$

with $x = (1, i_1)$. The definition is used inductively in $|I| \geq 1$ and it has a unique solution. (Note that the empty cumulant $\kappa[y_\emptyset]$ never appears in the moments-to-cumulants formula, and for our purposes it can be left undefined. To be consistent with the derivatives of the generating function, we may for instance set $\kappa[y_\emptyset] := 0$.) Therefore, these numbers have to coincide with the standard cumulants in the case when exponential moments exist and hence (A.1) holds.

The following known properties of cumulants can then be derived directly from the above definition using induction in $|I|$ and applying techniques similar to what we have used for Wick polynomials in Section 3.1:

1. The cumulants are multilinear, in the same manner as was stated for Wick polynomials in Proposition 3.7.
2. The moments-to-cumulants expansion (3.1) holds.
3. The cumulants are permutation invariant: if I' is a permutation of I, then $\kappa[y_{I'}] = \kappa[y_I]$.
4. If joint exponential moments exist, then $\kappa[y_I] = \partial_\lambda^I g_{\mathrm{c}}(0)$ with $g_{\mathrm{c}}(\lambda) := \ln \mathbb{E}[\mathrm{e}^{\lambda \cdot y}]$.

However, let us skip the proofs here. In the text, we assume these results to be known and refer to the references for details of their proofs.

B Classical particle system with random initial data

Consider the evolution of N classical particles interacting via a polynomial interaction potential, with the initial data given by some random probability measure. We show here how it can be recast in the form of the evolution equation discussed in Section 4.

We consider the random variables y_j, indexed by $J = \{(i, n)\}_{i,n}$, where n is one of the N different particle labels and $i = 1, 2$ differentiates between the particle position and momentum: we define $y_{(1,n)}(t) := q_n(t)$ and $y_{(2,n)}(t) := p_n(t)$. If all particles move in $\mathbb{R}$, have the same mass, and have only pair interactions via the potential $V(q) := \sum_{n',n;n' \neq n} \lambda_{nn'} \frac{1}{2a}(q_n - q_{n'})^a$, $a \geq 2$ even and $\lambda_{n'n} = \lambda_{nn'}$, then we have $\partial_t q_n(t) = p_n(t)$ and

$$\partial_t p_n(t) = -\sum_{n' \neq n} \lambda_{nn'} (q_n(t) - q_{n'}(t))^{a-1} . \tag{B.1}$$

Here $(q_n - q_{n'})^{a-1} = \sum_{k=0}^{a-1} (-1)^{a-1-k} \binom{a-1}{k} q_n^k q_{n'}^{a-1-k}$, and if we define $I_{n,n',k}$ as a sequence of length $a-1$ containing first k repetitions of $(1, n)$ and then $a-1-k$ repetitions of $(1, n')$, then by (3.6) we have $q_n(t)^k q_{n'}(t)^{a-1-k} = \sum_{V \subset I_{n,n',k}} {:}y(t)^V{:}\, \mathbb{E}(y(t)^{I_{n,n',k} \setminus V})$. Define thus as the collection $\mathscr{I}_{(2,n)}$ all such sequences U which contain k_1 repetitions of $(1, n)$ followed by k_2 repetitions of $(1, n')$ where $n' \neq n$, $k_1, k_2 \geq 0$, and $k_1 + k_2 \leq a - 1$. Set also for each $U \in \mathscr{I}_{(2,n)}$

$$M^U_{(2,n)}(t) := \lambda_{nn'} \sum_{k=k_1}^{a-1-k_2} \binom{a-1}{k} (-1)^{a-k} \sum_{V \subset I_{n,n',k}} \mathbb{E}(y(t)^{I_{n,n',k} \setminus V}) \mathbb{1}(V = U) . \tag{B.2}$$

Therefore, (4.3) holds for all $j \in J$ after we also define $\mathscr{I}_{(1,n)} := \{\emptyset, ((2, n))\}$ and set $M^\emptyset_{(1,n)}(t) := \mathbb{E}(y_{(2,n)}(t))$ and $M^U_{(1,n)}(t) := 1$ if $U = ((2, n))$.

C Estimation of the first order non-pairing contributions to (5.18)

In this appendix, we show how to estimate the first order non-pairing contraction terms in (5.18). To this end we need to make an assumption on the dispersion relation $\omega(\mathbf{k})$. Let us consider the

so called free propagator

$$p_t(\mathbf{x}) = \int_{\mathbb{T}^d} d\mathbf{k} \mathrm{e}^{\mathrm{i}2\pi\mathbf{x}\cdot\mathbf{k}} \mathrm{e}^{-\mathrm{i}t\omega(\mathbf{k})} \,. \tag{C.1}$$

As in the assumption "(DR2)" in [7], we now suppose that there are $C, \delta > 0$ such that for all $t \in \mathbb{R}$,

$$\|p_t\|_3^3 = \sum_{\mathbf{x}\in\mathbb{Z}^d} |p_t(\mathbf{x})|^3 \leq C(1+t^2)^{-(1+\delta)/2} \,. \tag{C.2}$$

Furthermore, we assume also the already mentioned ℓ_1-clustering property (see section 4) which we slightly rephrase as follows for each cumulant of order n: we require that

$$\|\kappa_n\|_1 := \sup_{\sigma\in\{\pm1\}^n} \sum_{x\in(\mathbb{Z}^d)^n} \mathbb{1}(\mathbf{x}_1 = 0) \big|\kappa(\psi(\mathbf{x}_1,\sigma_1),\ldots,\psi(\mathbf{x}_n,\sigma_n))\big| < \infty \,. \tag{C.3}$$

We recall that the physical meaning of this condition is that the cumulants decay fast enough in space so that they are summable, once the translational invariance is taken into account.

We recall that the first order non-pairing contributions in (5.18) are

$$\begin{aligned} &-\mathrm{i}\lambda\sigma \int_0^t ds \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \mathrm{e}^{\mathrm{i}s(\sigma\omega+\omega_1-\sigma\omega_2-\omega_3)} \\ &\quad\times \kappa[a(\mathbf{k}_1,-1); a(\mathbf{k}_2,\sigma); a(\mathbf{k}_3,1); a(\mathbf{k}',\sigma')] \end{aligned} \tag{C.4}$$

and a term which is obtained from (C.4) by swapping $(\mathbf{k},\sigma) \leftrightarrow (\mathbf{k}',\sigma')$. As stated in (4.1), by translation invariance we have

$$\kappa[a(\mathbf{k}_1,-1); a(\mathbf{k}_2,\sigma); a(\mathbf{k}_3,1); a(\mathbf{k}',\sigma')] = \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}') \widehat{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}',\sigma,\sigma') \tag{C.5}$$

where $F(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\sigma,\sigma') = \mathbb{1}(\mathbf{x}_1=0)\kappa[a(\mathbf{x}_1,-1); a(\mathbf{x}_2,\sigma); a(\mathbf{x}_3,1); a(\mathbf{x}_4,\sigma')]$. Clearly, $\|F\|_1 \leq \|\kappa_4\|_1 < \infty$ by the assumed ℓ_1-clustering.

Therefore, the term in (C.4) is bounded by

$$\begin{aligned} &\lambda \Big| \int_0^t ds \int_{(\mathbb{T}^d)^3} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \mathrm{e}^{\mathrm{i}s(\sigma\omega+\omega_1-\sigma\omega_2-\omega_3)} \\ &\qquad\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}') \widehat{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}',\sigma,\sigma') \Big| \\ &\leq \lambda\delta(\mathbf{k}+\mathbf{k}') \int_0^t ds \Big| \int_{(\mathbb{T}^d)^2} d\mathbf{k}_1 d\mathbf{k}_2 \; \mathrm{e}^{\mathrm{i}s(\omega_1-\sigma\omega_2-\omega_3)} \widehat{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}',\sigma,\sigma') \Big|_{\mathbf{k}_3=\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \Big| \\ &\leq \lambda\delta(\mathbf{k}+\mathbf{k}') \int_0^t ds \sum_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4} |F(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\sigma,\sigma')| \\ &\qquad\times \Big| \sum_{\mathbf{y}} \mathrm{e}^{-\mathrm{i}2\pi\mathbf{k}\cdot(\mathbf{y}-\mathbf{x}_3)} p_{-s}(\mathbf{y}) p_{\sigma s}(\mathbf{y}-\mathbf{x}_2) p_s(\mathbf{y}-\mathbf{x}_3) \Big| \\ &\leq \lambda\delta(\mathbf{k}+\mathbf{k}')\|\kappa_4\|_1 \int_0^t ds \, \|p_s\|_3^3 \leq \lambda C\delta(\mathbf{k}+\mathbf{k}')\|\kappa_4\|_1 \int_0^t ds (1+s^2)^{-(1+\delta)/2} \\ &\leq \lambda C'\delta(\mathbf{k}+\mathbf{k}')\|\kappa_4\|_1 \end{aligned} \tag{C.6}$$

where C' is a constant which depends only on C and δ. We have used the inverse Fourier transform of $\widehat{F}$ and (C.1) in the second inequality and Hölder's inequality in the third one. Since the bound in invariant under the swap $(\mathbf{k},\sigma) \leftrightarrow (\mathbf{k}',\sigma')$, it bounds also the second non-pairing contribution in (5.18). Therefore, we see that the first order contributions are $O(\lambda)$ uniformly in t.

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