Translator's Errata for INT, Ch. 9

IX.1. Footnote (†) added:

(†) *Note by the Translator*: Ch. IX of GT was translated from the 2nd edition of the French original, whereas the present chapter refers to the 3rd edition; references to items not in GT will be routed to TG.

IX.10. In the footnote (1), change the reference to TG, IX, §6, No. 10, Th. 6.

IX.18. Revise footnote 1 as follows, and indicate at the end that the footnote has been added by the Translator:

[1] The cited appendix on Lindelöf spaces does not appear in GT. Lindelöf spaces are defined in GT in Ch. I, §9, Exer. 14. Souslin spaces (and Lusin spaces) are defined in TG for Hausdorff spaces (TG, IX, §6, No. 2, Def. 2 and No. 4, Def. 7); in GT they are required to be metrizable (GT, IX, §6, No. 2, Def. 2 and No. 4, Def. 6). (*Transl.*)

IX.31, ℓ. 9. In the reference, replace GT by TG.

IX.31. Revise footnote (1) to the following:

(1) A capacity f on T is said to be right-continuous if, for every compact set K in T, $f(\mathrm{K}) = \inf_{\mathrm{U}} f(\mathrm{U})$ as U runs over the open sets $\mathrm{U} \supset \mathrm{K}$. In GT, a "capacity" is defined by three axioms (GT, IX, §6, No. 9, Def. 8). In TG, a function satisfying only the first two is called a capacity, but a right-continuous capacity also satisfies the third (TG, *loc. cit.*, *Remarque*). (*Transl.*)

IX.40. Revise footnote (1) as follows:

(1) If $f \cdot \varphi_{\mathrm{A}}$ is not universally measurable, cf. *Remark* 2) below. (*Transl.*)

QUESTION to the Author: Is it known that the set A is universally measurable, and, if so, can a reference be given here? If not, is there a better reference for filling the gap in the argument?

IX.40. Revise footnote (2) as follows:

(2) Cf. the footnote to *Remark* 1 of §1, No. 9. (*Transl.*)

IX.48. Revise footnote (2) to the following:

(2) In GT, every Souslin space has a countable base for open sets (GT, IX, §6, No. 2, Prop. 4), hence is Lindelöf (GT, I, §9, Exer. 14); but see the footnote on p. IX.18. (*Transl.*)

IX.48. In the line following the first display in the proof of Prop. 3, change the reference to TG, IX, §6, No. 10 and add to its right the following footnote (3):

(3) See footnote (1) on p. IX.31. (*Transl.*)

IX.49. Delete footnote (4) and revise footnote (3) to be the following new footnote (4):

(4) In GT, every Borel set in a Souslin space is a Souslin set (GT, IX, §6, No. 3, Prop. 11); but see the footnotes on pp. IX.18 and IX.31. (*Transl.*)

IX.63, ℓ. −3, −2. Return the reference to its form in the French original: *loc. cit.*, Cor. 1 of Th. 1.

IX.64. Add the following footnote (1) to the statement of Cor. 2:

(1) Cf. the footnote on p. IX.18. (*Transl.*)

IX.64. In the references in $\ell\ell$. 14, 18, 19, change GT to TG.

IX.64. In the references in $\ell\ell$. 18 and 19, change Prop. 12 to Prop. 11.

IX.73, ℓ. 5. Add "(*Transl.*)" to footnote (1):

(1) Also called a 'projective system'. (*Transl.*)

IX.87. Revise footnote (2) as follows:

(2) The term *espace mesuré* was used in the first edition of Ch. III (§2, No. 2, p. 52) for a space (locally compact, there) equipped with a measure. (*Transl.*)

IX.90, Footnote (3). QUESTION to the Author: Is this the correct interpretation of "place no weight at 0" (a term that appears not to have previously been defined)? The term also occurs in IX.91, ℓ. 13. (Of course, "place no mass at 0" has been defined in connection with discrete measures.) The term is never used again in INT; as it would be a nuisance to rearrange the index of terminology to accomodate it, the simplest strategy is to let it be defined in the footnote, and if the word "weight" is required for some other context, introduce it into the index of terminology for that context. Incidentally, an alternative to "weight" might be "burden" (possibly more consistent with the term encumbrance).

(3) That is, the measures on I that are concentrated on $\mathrm{T} = \mathrm{I} - \{0\}$. (*Transl.*)

IX.91, ℓ. 13. See the preceding QUESTION.

IX.91. "(*Transl.*)" added to footnote (4):

(4) In the cited Cor. 2, read 'second' (axiom of countability) instead of 'first'. (*Transl.*)

IX.93, ℓ. 6. In the reference to TVS, replace "No. 2, Th. 1" by "No. 1, *Scolium*".

IX.97. In the reference to TVS in line 4 of the *Remark*, replace "No. 2, Th. 1" by "No. 1, *Scolium*".

CHAPTER IX

Measures on Hausdorff topological spaces

If T *is a set, and* A *is a subset of* T, *we denote by* φ_{A} *the characteristic function of* A, *provided this does not lead to any confusion. The set* $\overline{\mathbf{R}}_{+}^{\mathrm{T}}$ *of numerical functions* $\geqslant 0$ *(finite or not) defined on* T *will be denoted by* $\mathscr{F}_{+}(\mathrm{T})$, *or simply* $\mathscr{F}_{+}$ *if there is no ambiguity as to* T; *this set will always be equipped with its natural order structure. Recall that the product of two elements of* $\mathscr{F}_{+}$ *is always defined, thanks to the convention* $0\cdot(+\infty)=0$. *If* A *is a subset of* T, *and* f *is a function defined on* T, *the restriction* $f\big|\mathrm{A}$ *of* f *to* A *may be denoted* f_{A} *in this chapter, if this creates no confusion; an analogous notation will be employed for induced measures. On the other hand, if* $f\in\mathscr{F}_{+}(\mathrm{A})$ *we shall denote by* f^{0} *the extension by* 0 *of* f *to* T, *that is, the function defined on* T *that coincides with* f *on* A *and with* 0 *on* $\mathrm{T}-\mathrm{A}$.

All topological spaces considered in this chapter are assumed to be Hausdorff, absent express mention to the contrary.[†] *From §1, No. 4 on, except for §5, all measures will be assumed to be positive, absent express mention to the contrary.*

§1. PREMEASURES AND MEASURES ON A TOPOLOGICAL SPACE

1. Encumbrances

Definition 1. — *Let* T *be a set. One calls encumbrance on* T *any mapping* p *of* $\mathscr{F}_{+}(\mathrm{T})$ *into* $\overline{\mathbf{R}}_{+}$ *that has the following properties*:

a) If f *and* g *are two elements of* $\mathscr{F}_{+}$ *such that* $f\leqslant g$, *then* $p(f)\leqslant p(g)$.

b) If f *is an element of* $\mathscr{F}_{+}$, *and* t *is a number* $\geqslant 0$, *then* $p(tf)=tp(f)$.

[†] *Note by the Translator*: Ch. IX of GT was translated from the 2nd edition of the French original, whereas the present chapter refers to the 3rd edition; references to items not in GT will be routed to TG.

of T (*loc. cit.*, Cor. 3), and the Souslin sets (Ch. IV, §5, No. 1, Cor. 2 of Prop. 3)[1]. The usual algebraic operations on numerical functions preserve measurability (Ch. IV, §5, No. 3), as do the operations of countable passage to the limit (*loc. cit.*, No. 4, Th. 2 and Cor. 1). The following property merits more explicit mention:

PROPOSITION 4. — *Let f be a positive function and $(g_n)_{n\geqslant 1}$ a sequence of μ-measurable positive functions on T. Setting $g = \sum_{n\geqslant 1} g_n$, one has*

$$\mu^\bullet(fg) = \sum_{n\geqslant 1} \mu^\bullet(fg_n)\,. \tag{4}$$

Set $h_n = \sum_{i=1}^{n} g_i$ for all $n \geqslant 1$. For every compact subset K of T,

$$\mu_{\mathrm{K}}^\bullet\big((fh_n)_{\mathrm{K}}\big) = \sum_{i=1}^{n} \mu_{\mathrm{K}}^\bullet\big((fg_i)_{\mathrm{K}}\big)$$

by Prop. 2 of Ch. V, §1, No. 1 applied to the compact space K. Passing to the limit with respect to the increasing directed set of compact subsets of T, one obtains

$$\mu^\bullet(fh_n) = \sum_{i=1}^{n} \mu^\bullet(fg_i)\,.$$

Now, fg is the limit of the increasing sequence $(fh_n)_{n\geqslant 1}$, whence $\mu^\bullet(fg) = \lim_{n\to\infty} \mu^\bullet(fh_n)$; the preceding formula then immediately implies (4).

COROLLARY. — *Let (A_n) be a sequence of pairwise disjoint measurable subsets, with union A. For every subset B of T,*

$$\mu^\bullet(\mathrm{A}\cap\mathrm{B}) = \sum_{n} \mu^\bullet(\mathrm{A}_n\cap\mathrm{B})$$

and in particular

$$\mu^\bullet(\mathrm{A}) = \sum_{n} \mu^\bullet(\mathrm{A}_n)\,.$$

Among the properties of measurable functions or sets that extend as above to Hausdorff spaces, we cite also Prop. 12 of Ch. IV, §5, No. 8 (μ-dense families of compact sets). Thus, a function f with values in a topological

[1] The proof of this corollary is valid without modification for Souslin sets in a nonmetrizable locally compact space (TG, IX, §6, No. 10, Th. 6).

moderated. The remarks following Def. 2 of Ch. V, §1, No. 2 can immediately be extended to the present context. In particular, the sum of a sequence of moderated positive functions is moderated.

Remarks. — 1) On a Lindelöf space T (TG, IX, Appendix I, Def. 1),[1] and in particular on a Souslin space (*ibid.*, Cor. of Prop. 1), every measure is moderated. For, the open sets of finite measure form a covering of T, from which one can extract a countable covering of T.

2) Beware, however, that the existence of a sequence of Borel sets of finite measure for μ, with union T, does not necessarily imply the existence of a sequence of *open* sets of finite measure with union T (in other words, does not imply that μ is moderated). See Exer. 8.

PROPOSITION 14. — *Let $f \in \mathscr{F}_+(\mathrm{T})$. If f is μ-moderated, then $\mu^*(f) = \mu^\bullet(f)$; if f is not μ-moderated, then $\mu^*(f) = +\infty$.*

If $\mu^*(f) < +\infty$, there exists a lower semi-continuous function $g \geqslant f$ such that $\mu^\bullet(g) < +\infty$. For every $n \in \mathbf{N}$, let G_n be the set of $t \in \mathrm{T}$ such that $g(t) > 1/n$; the set G_n is open, one has $\mu^\bullet(\mathrm{G}_n) \leqslant n\mu^\bullet(g) < +\infty$, and f is zero outside the union of the G_n: the function f is therefore moderated.

Next, let us show that μ^* and $\mu^\bullet$ have the same value for moderated functions. Since μ^* and $\mu^\bullet$ are encumbrances, it suffices to establish the relation $\mu^*(f) = \mu^\bullet(f)$ when f is a positive function, bounded above by a constant M, and zero outside an open set G of finite measure, which we shall now do.

The measure μ is the supremum, in $\mathscr{M}(\mathrm{T})$, of an increasing directed family $(\mu_i)_{i\in\mathrm{I}}$ of measures with compact support: this follows at once from Prop. 9 of No. 8. Let g be a lower semi-continuous function on T, between f and the lower semi-continuous function $\mathrm{M}\varphi_{\mathrm{G}}$. Set $\nu_i = \mu - \mu_i$; then $\mu^\bullet = \mu_i^\bullet + \nu_i^\bullet$ (No. 2, *Remark* 1), consequently

$$\begin{aligned}\mu^\bullet(g) - \mu^\bullet(f) &= \big(\mu_i^\bullet(g) - \mu_i^\bullet(f)\big) + \big(\nu_i^\bullet(g) - \nu_i^\bullet(f)\big)\\ &\leqslant \big(\mu_i^\bullet(g) - \mu_i^\bullet(f)\big) + \nu_i^\bullet(\mathrm{M}\varphi_{\mathrm{G}}).\end{aligned}$$

One has $\nu_i^\bullet(\mathrm{M}\varphi_{\mathrm{G}}) = \mu^\bullet(\mathrm{M}\varphi_{\mathrm{G}}) - \mu_i^\bullet(\mathrm{M}\varphi_{\mathrm{G}})$ and $\mu^\bullet(\mathrm{M}\varphi_{\mathrm{G}}) = \sup \mu_i^\bullet(\mathrm{M}\varphi_{\mathrm{G}})$ (No. 7, Prop. 6); the number $\nu_i^\bullet(\mathrm{M}\varphi_{\mathrm{G}})$ may therefore be made arbitrarily small by a suitable choice of i. Thus everything comes down to showing that one can find, for any number $c > 0$ and any index $i \in \mathrm{I}$, a lower semi-continuous function g between f and $\mathrm{M}\varphi_{\mathrm{G}}$, such that $\mu_i^\bullet(g) - \mu_i^\bullet(f) \leqslant c$. Now, let L be the compact support of the measure μ_i, and let λ be the measure $(\mu_i)_{\mathrm{L}}$; since μ_i is concentrated on L, one has $\mu_i^\bullet(h) = \mu_i^\bullet(h\varphi_{\mathrm{L}}) = \lambda^\bullet(h_{\mathrm{L}})$ for every function $h \in \mathscr{F}_+(\mathrm{T})$ (No. 1, Lemma 1 and No. 2, Prop. 2); therefore

$$\mu_i^\bullet(g) - \mu_i^\bullet(f) = \lambda^\bullet(g_{\mathrm{L}}) - \lambda^\bullet(f_{\mathrm{L}}).$$

[1] The cited appendix on Lindelöf spaces does not appear in GT. Lindelöf spaces are defined in GT in Ch. I, §9, Exer. 14. Souslin spaces (and Lusin spaces) are defined in TG for Hausdorff spaces (TG, IX, §6, No. 2, Def. 2 and No. 4, Def. 7); in GT they are required to be metrizable (GT, IX, §6, No. 2, Def. 2 and No. 4, Def. 6). (*Transl.*)

The two measures μ and μ' thus have the same essential upper integral, which implies their equality (§1, No. 2, Cor. of Prop. 2).

Remark. — Suppose that π is injective. Let θ be a complex measure such that π is θ-proper and $\pi(\theta) = 0$; then $\theta = 0$. Indeed, by separating θ into its real and imaginary parts, one can reduce to the case that θ is real. We then have $\pi(\theta^+) = \pi(\theta^-)$, therefore $\theta^+ = \theta^-$ (Prop. 8), and finally $\theta = 0$.

Here is an important case where condition *a*) of Prop. 8 is always satisfied.

PROPOSITION 9. — *Let* T *be a Souslin space* (TG, IX, §6, No. 2, Def. 2), X *a Hausdorff space,* π *a continuous mapping of* T *onto* X, *and* ν *a bounded measure on* X. *Then there exists a bounded measure* μ *on* T *such that* $\pi(\mu) = \nu$.

The hypotheses obviously imply that X is a Souslin space.

Let us consider the set function $c : \mathrm{A} \mapsto \nu^\bullet\big(\pi(\mathrm{A})\big)$ on $\mathfrak{P}(\mathrm{T})$. The relation $\mathrm{A} \subset \mathrm{B}$ implies $c(\mathrm{A}) \leqslant c(\mathrm{B})$; if (A_n) is an increasing sequence of subsets of T, and if $\mathrm{A} = \bigcup_{n \in \mathbf{N}} \mathrm{A}_n$, then $c(\mathrm{A}) = \sup_n c(\mathrm{A}_n)$ from the fact that $\nu^\bullet$ is an encumbrance. Finally, let $\mathrm{A} \subset \mathrm{T}$ and let ε be a number > 0; choose an open subset G of X containing $\pi(\mathrm{A})$, such that $\nu^\bullet(\mathrm{G}) \leqslant \nu^\bullet\big(\pi(\mathrm{A})\big) + \varepsilon$ (§1, No. 9, Prop. 13); the open subset $\mathrm{H} = \overset{-1}{\pi}(\mathrm{G})$ of T contains A, and $c(\mathrm{H}) \leqslant c(\mathrm{A}) + \varepsilon$. The function c is therefore a right-continuous capacity on T (TG, IX, §6, No. 10, Def. 9)[(1)] and the theorem on capacitability (*loc. cit.*, Th. 6) implies the equality $c(\mathrm{T}) = \sup_{\mathrm{K}} c(\mathrm{K})$, where K runs over the set of compact subsets of T. Prop. 8 then implies the existence of the desired measure μ.

5. Product of two measures

Let S and T be two topological spaces, equipped respectively with two (positive) premeasures λ and μ, and let X be the product space $\mathrm{S} \times \mathrm{T}$. Let K be a compact subset of X; let us denote by A and B the projections of K on S and T respectively, and set

$$\nu_{\mathrm{K}} = (\lambda_{\mathrm{A}} \otimes \mu_{\mathrm{B}})_{\mathrm{K}}\,. \tag{3}$$

We thus define a premeasure on X. For, let L be a compact subset of X containing K, and let C and D be its two projections; then $\mathrm{A} \subset \mathrm{C}$, $\mathrm{B} \subset \mathrm{D}$,

[(1)] A capacity f on T is said to be right-continuous if, for every compact set K in T, $f(\mathrm{K}) = \inf_{\mathrm{U}} f(\mathrm{U})$ as U runs over the open sets $\mathrm{U} \supset \mathrm{K}$. In GT, a "capacity" is defined by three axioms (GT, IX, §6, No. 9, Def. 8). In TG, a function satisfying only the first two is called a capacity, but a right-continuous capacity also satisfies the third (TG, *loc. cit.*, *Remarque*). (*Transl.*)

immediate that the mapping $t \mapsto \lambda_t$ satisfies the conditions *a*) and *b*) of the statement.

D) *Proof of c*):

Let f be a universally measurable function on X that is positive and bounded; we are going to show that the universally measurable function $h_f : t \mapsto \lambda_t^\bullet(f)$ on T is a density for the measure $\mu_f = p(f \cdot \nu)$ with respect to $\mu = p(\nu)$. Let K be a compact subset of T and let $\mathrm{A} = \overset{-1}{p}(\mathrm{K})$. For every $t \in \mathrm{T}$, the measure λ_t is carried by $\overset{-1}{p}(t)$; if t belongs to K then $\overset{-1}{p}(t) \subset \mathrm{A}$, whence $\lambda_t^\bullet(f\varphi_\mathrm{A}) = \lambda_t^\bullet(f)$; on the other hand, if t belongs to $\mathrm{T} - \mathrm{K}$ then $\overset{-1}{p}(t) \subset \mathrm{X} - \mathrm{A}$, whence $\lambda_t^\bullet(f\varphi_\mathrm{A}) = 0$. Applying the formula (12) to $f \cdot \varphi_\mathrm{A}$,[1] we obtain

$$\mu_f(\mathrm{K}) = \int_\mathrm{A}^\bullet f(x)\,d\nu(x) = \int_\mathrm{K}^\bullet d\mu(t) \int_\mathrm{X}^\bullet f(x)\,d\lambda_t(x) = \int_\mathrm{K}^\bullet h_f(t)\,d\mu(t)\,,$$

which establishes the relation $\mu_f = h_f \cdot \mu$.

Letting $f = 1$, one sees that the function $h_1 : t \mapsto \|\lambda_t\|$ is a density of the measure $\mu_1 = \mu$ with respect to μ, hence is equal to 1 locally μ-almost everywhere in T.

E) *Uniqueness*:

Let $t \mapsto \lambda_t^i$ (for $i = 1, 2$) be two mappings of T into $\mathscr{M}_+(\mathrm{X})$ satisfying the conditions *a*) and *b*) of the statement. As in C), choose a μ-crushing $(\mathrm{X}_n)_{n\in\mathbf{N}}$ of X such that p_{X_n} is continuous for every $n \in \mathbf{N}$, and set $\mathrm{N} = \mathrm{X} - \bigcup_{n\in\mathbf{N}} \mathrm{X}_n$. For every integer $n \in \mathbf{N}$, choose a countable set D_n of positive functions on X, zero outside X_n, whose restrictions to X_n form a dense set in the normed space $\mathscr{C}(\mathrm{X}_n)$ (apply Th. 1 of GT, X, §3, No. 3 to the metrizable compact space X_n). We set $\mathrm{D} = \bigcup_{n\in\mathbf{N}} \mathrm{D}_n$.

Let $f \in \mathrm{D}$; by D), the functions $t \mapsto (\lambda_t^1)^\bullet(f)$ and $t \mapsto (\lambda_t^2)^\bullet(f)$ are densities of the measure μ_f with respect to μ, and so there exists a locally μ-negligible set E_f in T such that $(\lambda_t^1)^\bullet(f) = (\lambda_t^2)^\bullet(f)$ for $t \in \mathrm{T} - \mathrm{E}_f$. Moreover, by (12), the set F_i of $t \in \mathrm{T}$ such that $(\lambda_t^i)^\bullet(\mathrm{N}) \neq 0$ is locally μ-negligible for $i = 1, 2$. Since D is countable, the set $\mathrm{G} = \big(\bigcup_{f\in\mathrm{D}} \mathrm{E}_f\big) \cup \mathrm{F}_1 \cup \mathrm{F}_2$ is locally μ-negligible; for $t \in \mathrm{T} - \mathrm{G}$, we have $(\lambda_t^1)^\bullet(\mathrm{N}) = (\lambda_t^2)^\bullet(\mathrm{N}) = 0$ and $(\lambda_t^1)_{\mathrm{X}_n} = (\lambda_t^2)_{\mathrm{X}_n}$, whence $\lambda_t^1 = \lambda_t^2$ by Prop. 9 of §1, No. 8.

Q.E.D.

Remarks. — 1) If X is a Souslin space, then every compact subspace of X is a Souslin space, hence is metrizable (TG, IX, Appendix I, Cor. 2 of Prop. 3),[2]

[1] If $f \cdot \varphi_\mathrm{A}$ is not universally measurable, cf. *Remark* 2) below. (*Transl.*)

[2] Cf. the footnote to *Remark* 1 of §1, No. 9. (*Transl.*)

Set $K' = \bigcup_{n\in\mathbf{N}} K_n$; K' is Borel in X, $K' \subset T \subset C$, $\lambda^\bullet(K') = \lambda^\bullet(C)$, therefore these three sets differ only by λ-negligible sets, and so T is λ-measurable. This completes the proof of *a*).

Let us pass to *b*). Suppose that X is a Radon space, and that T is universally measurable in X. Let I be a positive function on $\mathfrak{B}(T)$ that is countably additive and bounded; the function $A \mapsto I(A \cap T)$ on $\mathfrak{B}(X)$ is then positive, countably additive and bounded, therefore there exists a bounded measure ν on X such that $I(A\cap T) = \nu^\bullet(A)$ for all $A \in \mathfrak{B}(X)$. Now, T is ν-measurable; the preceding relation shows that $\nu^\bullet(K) = 0$ for every compact subset K of X that is disjoint from T, therefore ν is concentrated on T. Consequently, for every Borel set A of X, we have $I(A\cap T) = \nu^\bullet(A\cap T) = \mu^\bullet(A\cap T)$, where μ is the measure induced by ν on T. Finally, it follows that $I(B) = \mu^\bullet(B)$ for every set $B \in \mathfrak{B}(T)$ (GT, IX, §6, No. 3, *Remark* 2), and I is indeed inner regular.

COROLLARY. — *If* X *is a Radon space, then every Borel subset* T *of* X *is Radon.*

For, T is universally measurable in X.

PROPOSITION 3. — *Every Souslin space (in particular, every Polish or Lusin space) is strongly Radon.*

Let T be a Souslin space; since T is a Lindelöf space (TG, IX, Appendix I, Cor. of Prop. 1),(2) it suffices to show that T is Radon (Prop. 1). Let I be a function defined on $\mathfrak{B}(T)$, positive, countably additive and bounded. We extend I to $\mathfrak{P}(T)$ by setting, for every subset A of T,

$$I(A) = \inf_{\substack{B\in\mathfrak{B}(T)\\ B\supset A}} I(B).$$

Let us show that this extension is a *capacity* on T (TG, IX, §6, No. 10).(3) It is clear that the relation $A \subset A'$ implies $I(A) \leqslant I(A')$. Let (A_n) be an increasing sequence of subsets of T, and let $A = \bigcup_n A_n$. The set of Borel sets that contain A_n being stable for countable intersections, there exists for each n a Borel set B_n such that $A_n \subset B_n$ and $I(A_n) = I(B_n)$ (cf. the proof of Prop. 2). Set $C_n = \bigcap_{p\geqslant n} B_p$; C_n is Borel, and $A_n \subset C_n \subset B_n$, therefore $I(A_n) = I(C_n)$. On the other hand, the sequence (C_n) is increasing. Let $C = \bigcup_n C_n$: the relation $A \subset C$ implies that

$$I(A) \leqslant I(C) = \lim_n I(C_n) = \lim_n I(A_n),$$

(2) In GT, every Souslin space has a countable base for open sets (GT, IX, §6, No. 2, Prop. 4), hence is Lindelöf (GT, I, §9, Exer. 14); but see the footnote on p. IX.18. (*Transl.*)

(3) See footnote (1) on p. IX.31. (*Transl.*)

whence the equality $\mathrm{I}(\mathrm{A}) = \lim_n \mathrm{I}(\mathrm{A}_n)$ is immediate. Consequently, I is a capacity.

If (H_n) is a decreasing sequence of closed sets in T, obviously $\mathrm{I}\big(\bigcap_n \mathrm{H}_n\big) = \inf_n \mathrm{I}(\mathrm{H}_n)$. It follows that every Souslin subset F of T is capacitable for I (TG, IX, §6, No. 10, Prop. 15). In particular, every Borel set A of T is capacitable (*loc. cit.*, §6, No. 3, Prop. 10).[4] In other words,

$$\mathrm{I}(\mathrm{A}) = \sup_{\mathrm{K}} \mathrm{I}(\mathrm{K}),$$

where K runs over the set of compact sets contained in A; we have proved that I is inner regular.

Remark. — Let X be a Lusin space (in particular, any Polish space), and f a bijective continuous mapping of X onto a (Lusin) regular space Y. One knows (TG, IX, §6, No. 7, Prop. 14) that the mapping $\mathrm{B} \mapsto f^{-1}(\mathrm{B})$ is a bijection of the Borel tribe of Y onto the Borel tribe of X. The spaces X and Y are Lusin, hence strongly Radon (Prop. 3). It follows immediately that the mapping $\mu \mapsto f(\mu)$ is a bijection of the set of bounded measures on X onto the set of bounded measures on Y.

§4. INVERSE LIMITS OF MEASURES

Throughout this section, I *denotes a nonempty set, equipped with a preorder relation, denoted* $i \leqslant j$, *and directed for this relation. Recall* (GT, I, §4, No. 4) *that an inverse system of topological spaces indexed by* I *is a family* (T_i, p_{ij}) *where* T_i *is a topological space and* p_{ij} *is a continuous mapping of* T_j *into* T_i *for* $i \leqslant j$, *where* p_{ii} *is the identity mapping of* T_i, *and where* $p_{ik} = p_{ij} \circ p_{jk}$ *for* $i \leqslant j \leqslant k$. *Let* T *be a topological space and* $(p_i)_{i\in\mathrm{I}}$ *a family of continuous mappings* $p_i : \mathrm{T} \to \mathrm{T}_i$. *The family* $(p_i)_{i\in\mathrm{I}}$ *is said to be coherent if* $p_i = p_{ij} \circ p_j$ *for* $i \leqslant j$, *and it is said to be separating if for distinct* x, y *in* T *there exists an* $i \in \mathrm{I}$ *such that* $p_i(x) \neq p_i(y)$. *When* $\mathrm{T} = \varprojlim \mathrm{T}_i$ *and* p_i *is the canonical mapping of* T *into* T_i, *the family* $(p_i)_{i\in\mathrm{I}}$ *is coherent and separating.*

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[4] In GT, every Borel set in a Souslin space is a Souslin set (GT, IX, §6, No. 3, Prop. 11); but see the footnotes on pp. IX.18 and IX.31. (*Transl.*)

total masses of the measures in A are bounded by a number M ; it therefore suffices to verify that

$$\lim_{\lambda,\,\mathfrak{F}} \int_{\mathrm{X}} g\, d\big(i(\lambda)\big) = \int_{\mathrm{X}} g\, d\big(i(\mu)\big) \tag{5}$$

for functions $g \in \mathscr{C}^b(\mathrm{X})$ forming a total set in $\mathscr{C}^b(\mathrm{X})$. Now, this equality is satisfied when g has compact support in T, because of the vague convergence of $\mathfrak{F}$ to μ, and also when g is a constant function on X, from the fact that $\lim_{\lambda,\,\mathfrak{F}} \lambda(1) = \mu(1)$. Since the functions of the preceding two types form a total set in $\mathscr{C}^b(\mathrm{X})$ (Ch. III, §1, No. 2, Prop. 3), this completes the proof.

4. Application: topological properties of the space $\mathscr{M}^b_+(\mathrm{T})$

We first observe that if T is completely regular, then $\mathscr{M}^b(\mathrm{T})$ is a Hausdorff topological vector space, hence is completely regular. Consequently, $\mathscr{M}^b_+(\mathrm{T})$ is completely regular.

PROPOSITION 10. — *Let* T *be a Polish space*; *the space* $\mathscr{M}^b_+(\mathrm{T})$ *is then Polish for the tight topology.*

We begin by treating the case that T is Polish and *compact*. The set U of positive measures with mass $\leqslant 1$ is then compact (Ch. III, §1, No. 9, Cor. 2 of Prop. 15), and the topology induced on U by the tight topology (which here coincides with the vague topology) is also induced by the topology of pointwise convergence on a total subset of $\mathscr{C}(\mathrm{T})$ (*loc. cit.*, No. 10, Prop. 17). Now, there exists in $\mathscr{C}(\mathrm{T})$ a countable total set (GT, X, §3, No. 3, Th. 1); consequently, U is a metrizable compact space. The set V of positive measures of mass < 1 is open in U, hence is a Polish locally compact space. Now, the mapping $\mu \mapsto \dfrac{1}{1+\mu(1)}\,\mu$ of $\mathscr{M}^b_+(\mathrm{T})$ onto V is a homeomorphism, the mapping $\lambda \mapsto \dfrac{1}{1-\lambda(1)}\,\lambda$ being the inverse homeomorphism.

Let us pass to the case that T is Polish; we can suppose that T is the intersection of a decreasing sequence (G_n) of open sets in a metrizable compact space X (GT, IX, §6, No. 1, Cor. 1 of Th. 1); the space $\mathscr{M}^b_+(\mathrm{T})$ is then homeomorphic to the subspace W of $\mathscr{M}^b_+(\mathrm{X})$ consisting of the measures concentrated on T (No. 3, Prop. 8), and it will suffice to show that W is the intersection of a sequence of open sets in the Polish space $\mathscr{M}^b_+(\mathrm{X})$ (*loc. cit.*, Cor. 1 of Th. 1). Now, let W_n be the set of measures $\mu \in \mathscr{M}^b_+(\mathrm{X})$ concentrated on G_n ; the mapping $h_n : \mu \mapsto \mu^\bullet(\mathrm{X} - \mathrm{G}_n)$ on $\mathscr{M}^b_+(\mathrm{X})$ is upper semi-continuous (No. 3, Prop. 6), and the set A_k^n of measures $\mu \in \mathscr{M}_+^b(X)$ such that $h_n(\mu) < 1/k$ is therefore open for every $k \geqslant 1$ and every $n \in \mathbf{N}$. The proof is completed by observing that $W = \bigcap_n W_n = \bigcap_{n,k} A_k^n$.

Corollary 1. — *If* T *is a metrizable space of countable type, then* $\mathscr{M}_+^b(T)$ *is metrizable of countable type for the tight topology.*

For, let $\widehat{T}$ be the completion of T for a metric defining the topology of T; the space $\widehat{T}$ is Polish, and $\mathscr{M}_+^b(T)$ is homeomorphic to the subspace of the Polish space $\mathscr{M}_+^b(\widehat{T})$ consisting of the measures concentrated on T (No. 3, Prop. 8). But every subspace of a Polish space is metrizable of countable type (GT, IX, §2, No. 8).

Corollary 2. — *If* T *is a completely regular Souslin* (resp. *Lusin*) *space, then the space* $\mathscr{M}_+^b(T)$ *is Souslin* (resp. *Lusin*).[1]

For, consider a Polish space P and a continuous mapping f of P onto T (TG, IX, §6, No. 2, Def. 2). Let $\widetilde{f}$ be the continuous mapping $\mu \mapsto f(\mu)$ of $\mathscr{M}_+^b(P)$ into $\mathscr{M}_+^b(T)$; the space $\mathscr{M}_+^b(P)$ is Polish by Prop. 10, and $\widetilde{f}$ is surjective (§2, No. 4, Prop. 9); the space $\mathscr{M}_+^b(T)$ is therefore Souslin. Similarly, if T is Lusin, then f may be assumed to be injective (TG, *loc. cit.*, No. 4, Prop. 11); then $\widetilde{f}$ is injective (§2, No. 4, Prop. 8), and so $\mathscr{M}_+^b(T)$ is Lusin (TG, *loc. cit.*, No. 4, Prop. 11).

Let T be a completely regular Souslin space (recall that for this, it suffices that T be Souslin and *regular* (TG, IX, App. I, Cor. of Prop. 2)), and let H be a compact subset of $\mathscr{M}_+^b(T)$; then H is compact and Souslin, hence *metrizable*, for the tight topology (*loc. cit.*, App. I, Cor. 2 of Prop. 3).

5. Compactness criterion for tight convergence

Definition 2. — *Let* T *be a topological space, and let* H *be a subset of* $\mathscr{M}^b(T)$; *one says that* H *satisfies Prokhorov's condition if*

a) $\sup_{\mu \in H} |\mu|(1) < +\infty$;

b) *for every number* $\varepsilon > 0$, *there exists a compact subset* K_ε *of* T *such that*

$$|\mu|(T - K_\varepsilon) \leqslant \varepsilon \quad \text{for every measure } \mu \in H. \tag{6}$$

It can be shown that if T is completely regular, the set of conditions *a*) and *b*) is equivalent to the following condition: there exists a real function $f \geqslant 1$ on T, such that the set of points t of T satisfying $f(t) \leqslant c$ is compact for every $c \in \mathbf{R}_+$ (which in particular implies that f is lower semi-continuous), and

[1] Cf. the footnote on p. IX.18. (*Transl.*)

It can be shown that the inverse limit of the inverse system $\mathscr{Q}(\mathrm{E})$ is canonically isomorphic to the algebraic dual E'^* of E', equipped with the weak topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$.

DEFINITION 1. — *Let* E *be a locally convex space. One calls promeasure on* E *every inverse system*[1] *of measures* (§4, No. 2, Def. 1) *on the inverse system of finite-dimensional quotients of* E.

In other words, a promeasure μ on E is a family $(\mu_{\mathrm{V}})_{\mathrm{V}\in\mathscr{F}(\mathrm{E})}$, where μ_{V} is a bounded (positive) measure on the finite-dimensional space E/V, and where $\mu_{\mathrm{V}} = p_{\mathrm{VW}}(\mu_{\mathrm{W}})$ when $\mathrm{V} \supset \mathrm{W}$. All of the measures μ_{V} have the same total mass, which is called the *total mass* of the promeasure μ.

For a subspace V of E to belong to $\mathscr{F}(\mathrm{E})$, it is necessary and sufficient that there exist a finite number of elements $x'_1, \ldots, x'_n$ of E' such that V consists of the $x \in \mathrm{E}$ satisfying $\langle x, x'_i \rangle = 0$ for $1 \leqslant i \leqslant n$ (TVS, II, §6, No. 3, Cor. 2 of Th. 1 and No. 5, Cor. 2 of Prop. 7). Moreover, on a finite-dimensional vector space there exists one and only one Hausdorff topological vector space topology (TVS, I, §2, No. 3, Th. 2). Consequently, the concept of promeasure on E depends only on the dual E' of E.

Let λ be a bounded measure on E. For every $\mathrm{V} \in \mathscr{F}(\mathrm{E})$, let us denote by $\widetilde{\lambda}_{\mathrm{V}}$ the image of λ under the canonical mapping p_{V} of E onto E/V. One has $p_{\mathrm{V}} = p_{\mathrm{VW}} \circ p_{\mathrm{W}}$ for any two elements V and W of $\mathscr{F}(\mathrm{E})$ such that $\mathrm{V} \supset \mathrm{W}$; consequently, the family $\widetilde{\lambda} = (\widetilde{\lambda}_{\mathrm{V}})_{\mathrm{V}\in\mathscr{F}(\mathrm{E})}$ is a promeasure on E. We shall say that $\widetilde{\lambda}$ is the promeasure *associated* with the measure λ. One sees immediately that λ and $\widetilde{\lambda}$ have the same total mass.

PROPOSITION 1. — *Let* E *be a locally convex space. The mapping* $\lambda \mapsto \widetilde{\lambda}$ *is a bijection of the set of bounded measures on* E *onto the set of promeasures* $(\mu_{\mathrm{V}})_{\mathrm{V}\in\mathscr{F}(\mathrm{E})}$ *on* E *satisfying the following condition*:

For every $\varepsilon > 0$, *there exists a compact subset* K *of* E *such that* $\mu_{\mathrm{V}}\big(\mathrm{E}/\mathrm{V} - p_{\mathrm{V}}(\mathrm{K})\big) \leqslant \varepsilon$ *for all* $\mathrm{V} \in \mathscr{F}(\mathrm{E})$.

One knows that the intersection of the kernels of the continuous linear forms on E is equal to 0 (TVS, II, §4, No. 1, Cor. 1 of Prop. 2); consequently $\bigcap_{\mathrm{V}\in\mathscr{F}(\mathrm{E})} \mathrm{V} = \{0\}$ and the family $(p_{\mathrm{V}})_{\mathrm{V}\in\mathscr{F}(\mathrm{E})}$ is coherent and separating. The proposition then follows from Th. 1 of §4, No. 2.

In particular, the mapping $\lambda \mapsto \widetilde{\lambda}$ is injective. If μ is a promeasure on E, and if there exists a bounded measure λ on E such that $\mu = \widetilde{\lambda}$, we shall say, by an abuse of language, that μ is a measure. If E is finite-dimensional, every promeasure $\mu = (\mu_{\mathrm{V}})_{\mathrm{V}\in\mathscr{F}(\mathrm{E})}$ is a measure: for, $\{0\} \in \mathscr{F}(\mathrm{E})$, $\mathrm{E}/\{0\} = \mathrm{E}$ and $p_{\mathrm{V},\{0\}} = p_{\mathrm{V}}$, whence $\mu_{\mathrm{V}} = p_{\mathrm{V}}(\mu_{\{0\}})$ for all $\mathrm{V} \in \mathscr{F}(\mathrm{E})$; in other words, $\mu = \widetilde{\lambda}$ with $\lambda = \mu_{\{0\}}$.

(1) Also called a 'projective system'. (*Transl.*)

In particular, if $t_1, \ldots, t_n$ are elements of T, and $c_1, \ldots, c_n$ are real numbers, then

$$\mathrm{W}\Big(\sum_{j=1}^{n} c_j \varepsilon_{t_j}\Big) = \sum_{j,k=1}^{n} c_j c_k \inf(t_j, t_k)$$

and since W is positive, the function $(t, t') \mapsto \inf(t, t')$ is a kernel of positive type on T.

THEOREM 1 (Wiener). — *Let w be the image under $P : \mathscr{H} \to \mathscr{C}$ of the canonical Gaussian promeasure on the Hilbert space $\mathscr{H}$. Then w is a Gaussian* measure *on $\mathscr{C}$ with variance W.*

By construction, $\mathrm{W}(\mu) = \|{}^t P(\mu)\|_2^2$; Prop. 5 of No. 5 shows that w is a Gaussian promeasure with variance W. It remains to prove that w is a measure on $\mathscr{C}$.

A) *Construction of an auxiliary measured space*[2] (Ω, m):

For every integer $n \geqslant 0$, denote by D_n the set of numbers of the form $k/2^n$ with $k = 1, 2, 3, \ldots, 2^n$. Set $\mathrm{D} = \bigcup_{n \geqslant 0} \mathrm{D}_n$ (the set of dyadic numbers contained in T) and $\Omega = \mathbf{R}^{\mathrm{D}}$. For every $t \in \mathrm{D}$, denote by $\mathrm{X}(t)$ the linear form $f \mapsto f(t)$ on Ω.

For t, t' in D, set $\mathrm{M}(t, t') = \inf(t, t')$; we have seen that M is a kernel of positive type on D. Since the set D is countable, one can define the Gaussian *measure* m on Ω with covariance M (No. 6, *Example* 2).

Lemma 3. — *For any t, t' in* D,

$$\int_\Omega \Big| \mathrm{X}\Big(\frac{t+t'}{2}\Big) - \frac{\mathrm{X}(t) + \mathrm{X}(t')}{2} \Big|^3 dm = \frac{1}{(8\pi)^{1/2}} |t - t'|^{3/2}. \tag{38}$$

Note that $\dfrac{t+t'}{2}$ belongs to D. One knows (No. 6, *Example* 2) that the family $\big(\mathrm{X}(t)\big)_{t \in \mathrm{D}}$ is a basis of the topological dual Ω' of Ω; therefore there exists a symmetric bilinear form $\widehat{\mathrm{M}}$ on $\Omega' \times \Omega'$ characterized by $\widehat{\mathrm{M}}\big(\mathrm{X}(t), \mathrm{X}(t')\big) = \inf(t, t')$. By construction, the variance of the Gaussian measure m on Ω is the quadratic form $\xi \mapsto \widehat{\mathrm{M}}(\xi, \xi)$ on Ω'. Set, in particular,

$$\xi = \mathrm{X}\Big(\frac{t+t'}{2}\Big) - \frac{\mathrm{X}(t) + \mathrm{X}(t')}{2}; \tag{39}$$

an easy calculation yields

$$\widehat{\mathrm{M}}(\xi, \xi) = \frac{|t - t'|}{4}. \tag{40}$$

[2] The term *espace mesuré* was used in the first edition of Ch. III (§2, No. 2, p. 52) for a space (locally compact, there) equipped with a measure. (*Transl.*)

C) *Construction of a Gaussian measure on* $\mathscr{C}$:

Let w' be the bounded measure on $\mathscr{C}$ that is the image of m under the m-measurable mapping $u : \Omega \to \mathscr{C}$. We are going to show that w' is a Gaussian measure on $\mathscr{C}$, with variance W , whence $w = w'$. Denote by $\mathscr{D}$ the linear subspace of $\mathscr{M}^1$ generated by the measures ε_t for t running over D .

Lemma 5. — *For every measure* $\mu \in \mathscr{D}$,

$$\int_{\mathscr{C}} e^{i\langle f,\mu\rangle}\, dw'(f) = e^{-\mathrm{W}(\mu)/2} . \tag{48}$$

Set $\mu = c_1\varepsilon_{t_1} + c_2\varepsilon_{t_2} + \cdots + c_n\varepsilon_{t_n}$ with $t_1, \ldots, t_n$ in D and $c_1, \ldots, c_n$ in $\mathbf{R}$. For every $g \in \Omega_0$, the function $u(g)$ coincides with g on D ; therefore

$$\langle u(g), \mu\rangle = \sum_{j=1}^{n} c_j g(t_j) \qquad (g \in \Omega_0) . \tag{49}$$

Also,

$$\mathrm{W}(\mu) = \sum_{j,k=1}^{n} c_j c_k \, \inf(t_j, t_k) , \tag{50}$$

and, since m is the Gaussian measure on Ω with covariance M , and $\Omega - \Omega_0$ is m-negligible, we have

$$\int_{\Omega_0} e^{i\sum_{j=1}^{n} c_j g(t_j)}\, dm(g) = \exp\Big(-\frac{1}{2}\sum_{j,k=1}^{n} c_j c_k \, \inf(t_j, t_k)\Big). \tag{51}$$

Now, $\Omega - \Omega_0$ is m-negligible and $w' = u(m)$; it follows that

$$\int_{\mathscr{C}} e^{i\langle f,\mu\rangle}\, dw'(f) = \int_{\Omega_0} e^{i\langle u(g),\mu\rangle}\, dm(g) . \tag{52}$$

The formula (48) follows immediately from the formulas (49) to (52).

Lemma 6. — *Let* $\mu \in \mathscr{M}^1$. *There exists a sequence of measures* $\mu_n \in \mathscr{D}$ *such that* $\mu(f) = \lim_{n\to\infty} \mu_n(f)$ *for all* $f \in \mathscr{C}$ *and* $\mathrm{W}(\mu) = \lim_{n\to\infty} \mathrm{W}(\mu_n)$.

Let $\mathrm{I} = [0, 1]$. The space $\mathscr{M}^1$ of bounded measures on $\mathrm{T} = \,]0, 1]$ will be identified with the subspace of $\mathscr{M}(\mathrm{I})$ formed by the measures that place no weight at 0 . (3) We equip $\mathscr{M}(\mathrm{I})$ with the vague topology. The mapping

(3) That is, the measures on I that are concentrated on $\mathrm{T} = \mathrm{I} - \{0\}$. (*Transl.*)

$t \mapsto \varepsilon_t$ of I into $\mathscr{M}(\mathrm{I})$ is continuous (Ch. III, §1, No. 9, Prop. 13); since D is dense in I, the closure $\overline{\mathscr{D}}$ of $\mathscr{D}$ contains all of the point measures. Let A be the set of measures $\nu \in \mathscr{D}$ such that $\|\nu\| \leqslant \|\mu\|$; the measure μ is in the closure of A (Ch. III, §2, No. 4, Cor. 1 of Th. 1). The set A is relatively compact in $\mathscr{M}(\mathrm{I})$ (Ch. III, §1, No. 9, Prop. 15) and the compact subsets of $\mathscr{M}(\mathrm{I})$ are metrizable (TVS, III, §3, No. 4, Cor. 2 of Prop. 6,[(4)] and GT, X, §3, No. 3, Th. 1). Therefore there exists a sequence of measures $\mu_n \in \mathrm{A}$ converging to μ in $\mathscr{M}(\mathrm{I})$. Since $\mathscr{C}$ is identified with the subspace of continuous functions on I zero at the origin, we have $\mu(f) = \lim_{n\to\infty} \mu_n(f)$ for all $f \in \mathscr{C}$. Moreover, since $\mathscr{C}(\mathrm{I}) \otimes \mathscr{C}(\mathrm{I})$ is dense in the normed space $\mathscr{C}(\mathrm{I}\times\mathrm{I})$ (Ch. III, §4, No. 1, Lemma 1), the relations $\lim_{n\to\infty} \mu_n = \mu$ and $\|\mu_n\| \leqslant \|\mu\|$ imply that $\lim_{n\to\infty} (\mu_n \otimes \mu_n) = \mu \otimes \mu$ (Ch. III, §1, No. 10, Prop. 17); since the measures μ_n and μ place no weight at 0, we have

$$\mathrm{W}(\mu_n) = \int_{\mathrm{I}}\int_{\mathrm{I}} \inf(t,t')\, d\mu_n(t)\, d\mu_n(t'),$$
$$\mathrm{W}(\mu) = \int_{\mathrm{I}}\int_{\mathrm{I}} \inf(t,t')\, d\mu(t)\, d\mu(t'),$$

whence $\lim_{n\to\infty} \mathrm{W}(\mu_n) = \mathrm{W}(\mu)$.

It remains to prove that the Fourier transform of w' is equal to $e^{-\mathrm{W}/2}$. Let $\mu \in \mathscr{M}^1$; choose measures $\mu_n \in \mathscr{D}$ as in Lemma 6. The measure w' is bounded, and $|e^{i\langle f,\mu_n\rangle}| = 1$ for all n; Lemma 5 and Lebesgue's convergence theorem (Ch. IV, §4, No. 3, Th. 2) then imply

$$\int_{\mathscr{C}} e^{i\langle f,\mu\rangle}\, dw'(f) = \lim_{n\to\infty} \int_{\mathscr{C}} e^{i\langle f,\mu_n\rangle}\, dw'(f) = \lim_{n\to\infty} e^{-\mathrm{W}(\mu_n)/2} = e^{-\mathrm{W}(\mu)/2}.$$

Q.E.D.

The measure w on $\mathscr{C}$ whose Fourier transform is equal to $e^{-\mathrm{W}/2}$ is called the *Wiener measure on* $\mathscr{C}$.

Remark. — For every semi-open interval $\mathrm{J} =]a,b]$ contained in T, let us set $l(\mathrm{J}) = b-a$ (the length of J) and denote by $\mathrm{A_J}$ the linear form $f \mapsto f(b) - f(a)$ on $\mathscr{C}$. It can be shown that the Wiener measure is characterized by the following property:

Let $\mathrm{J}_1, \dots, \mathrm{J}_n$ *be semi-open intervals contained in* T *and pairwise disjoint. The image of the measure* w *under the linear mapping* $f \mapsto (\mathrm{A}_{\mathrm{J}_1}(f), \dots, \mathrm{A}_{\mathrm{J}_n}(f))$ *of* $\mathscr{C}$ *into* $\mathbf{R}^n$ *is equal to* $\gamma_{a_1} \otimes \cdots \otimes \gamma_{a_n}$ *with* $a_i = l(\mathrm{J}_i)^{1/2}$ *for* $1 \leqslant i \leqslant n$.

(4) In the cited Cor. 2, read 'second' (axiom of countability) instead of 'first'. (*Transl.*)

PROPOSITION 9. — *If* F *is barreled, then the Fourier transform of every bounded measure on* F' *is a uniformly continuous function on* F.

Let μ be a bounded measure on F' and Φ its Fourier transform. Let $\varepsilon > 0$. There exists a compact subset K of F' such that $\mu(F' - K) \leqslant \varepsilon$. Now, K is compact for the weak topology $\sigma(F', F)$, hence is equicontinuous because F is barreled (TVS, III, §4, No. 1, *Scholium*). Therefore there exists a symmetric neighborhood U of 0 in F whose polar U° contains K. Let x be in εU; then

$$\Phi(0) - \mathscr{R}\Phi(x) = \int_{F'} (1 - \cos\langle x, x'\rangle)\, d\mu(x').$$

Now, $0 \leqslant 1 - \cos\langle x, x'\rangle \leqslant 2$ for every $x' \in F' - K$, and

$$1 - \cos\langle x, x'\rangle \leqslant \tfrac{1}{2}\langle x, x'\rangle^2 \leqslant \frac{\varepsilon^2}{2}$$

for $x' \in K \subset U^\circ$; it follows that

$$0 \leqslant \Phi(0) - \mathscr{R}\Phi(x) \leqslant 2\,\mu(F' - K) + \frac{\varepsilon^2}{2}\,\mu(K) \leqslant 2\varepsilon + \frac{\varepsilon^2}{2}\,\mu(F').$$

The second member of this inequality tends to 0 with ε; thus $\mathscr{R}\Phi$ is continuous at 0 and the proposition follows from the Cor. of Prop. 8.

9. Minlos's lemma

Let T be a finite-dimensional vector space and μ a bounded measure on T'; we shall identify T with the dual of T', so that the Fourier transform Φ of μ is a function on T. We assume given two positive quadratic forms h and q on T and a number $\varepsilon > 0$. For every real number $r > 0$, we denote by C_r the set of $x' \in T'$ such that $\langle x, x'\rangle^2 \leqslant r^2 h(x)$ for all $x \in T$.

PROPOSITION 10. — *Under the hypothesis* $\Phi(0) - \mathscr{R}\Phi \leqslant \varepsilon + q$, *we have*

$$\mu(T' - C_r) \leqslant 3\big(\varepsilon + r^{-2}\,\mathrm{Tr}\,(q/h)\big) \tag{55}$$

for every $r > 0$.

One writes $\mathrm{Tr}\,(q/h)$ for the trace of q with respect to h (cf. Annex, No. 1). The formula (55) is trivial when $\mathrm{Tr}\,(q/h)$ is infinite. We assume henceforth that $\mathrm{Tr}\,(q/h)$ is finite, hence that $h(x) = 0$ implies $q(x) = 0$ for $x \in T$.

Set $r = \big(12\,\Phi(0)\,\mathrm{Tr}(\mathrm{Q}/\mathrm{H})\,\varepsilon^{-1}\big)^{1/2}$ and denote by K the set of $x' \in \mathrm{F}'_{\mathscr{T}}$ such that $\langle x, x'\rangle^2 \leqslant r^2\mathrm{H}(x)$ for all $x \in \mathrm{F}$. Since $\mathrm{H}^{1/2}$ is a continuous seminorm on F, the set K is equicontinuous and closed in $\mathrm{F}'_{\mathscr{T}}$; it is therefore compact in $\mathrm{F}'_{\mathscr{T}}$ by Ascoli's theorem (GT, X, §2, No. 5, Cor. 1 of Th. 2).

Let V be a closed linear subspace of $\mathrm{F}'_{\mathscr{T}}$ with finite codimension; then, V is the orthogonal of a finite-dimensional linear subspace T of F. Let μ_{V} be the measure on T' that is the image of the promeasure μ on $\mathrm{F}'_{\mathscr{T}}$ under the mapping p_{V} that is the transpose of the canonical injection of T into F; its Fourier transform is the restriction of Φ to T. Finally, by the Hahn–Banach theorem (TVS, II, §3, No. 2, Cor. 1 of Th. 1), $p_{\mathrm{V}}(\mathrm{K})$ is equal to the set C_r of $x' \in \mathrm{T}'$ such that $\langle x, x'\rangle^2 \leqslant r^2\mathrm{H}(x)$ for all $x \in \mathrm{T}$. By the inequality (63), one can apply Prop. 10 of No. 9 to the measure μ_{V} on T', on taking for q the restriction of $2\Phi(0)\mathrm{Q}$ to T and for h that of H. Then $\mathrm{Tr}(q/h) \leqslant 2\,\Phi(0)\,\mathrm{Tr}\mathrm{Q}/\mathrm{H})$, whence

$$\mu_{\mathrm{V}}(\mathrm{T}' - \mathrm{C}_r) \leqslant 3\Big(\frac{\varepsilon}{6} + 2\,\Phi(0)\,\mathrm{Tr}(\mathrm{Q}/\mathrm{H})\,r^{-2}\Big) = \varepsilon\,.$$

Since p_{V} defines, by passage to the quotient, an isomorphism of $\mathrm{F}'_{\mathscr{T}}/\mathrm{V}$ onto T', Prop. 1 of No. 1 then shows that μ is a measure on $\mathrm{F}'_{\mathscr{T}}$.

Q.E.D.

COROLLARY. — *Let* F *be a barreled nuclear space*, $\mathscr{T}$ *a locally convex topology on* F' *intermediate to* $\mathscr{T}_s$ *and* $\mathscr{T}_c$, μ *a promeasure on* $\mathrm{F}'_{\mathscr{T}}$, *and* Φ *the Fourier transform of* μ. *For* μ *to be a measure, it is necessary and sufficient that* Φ *be continuous on* F.

Necessity follows from Prop. 9 of No. 8 and sufficiency from Th. 2.

Remark. — Let F be a barreled space and $\mathscr{T}$ a locally convex topology on F' intermediate to $\mathscr{T}_s$ and $\mathscr{T}_c$. Every subset of F' compact for $\mathscr{T}$ is compact for the coarser topology $\mathscr{T}_s$. Conversely, let K be a subset of F' compact for $\mathscr{T}_s$. Since F is barreled, K is equicontinuous (TVS, III, §4, No. 1, *Scholium*); but by Ascoli's theorem, every equicontinuous subset of F' is relatively compact for $\mathscr{T}_c$ and *a fortiori* for $\mathscr{T}$, therefore K is contained in a subset of F' compact for $\mathscr{T}$. It is not difficult to infer from this that the identity mapping of $\mathrm{F}'_{\mathscr{T}}$ onto $\mathrm{F}'_{\mathscr{T}_s}$ defines a bijection between the sets of measures on these two spaces.

11. Measures on a Hilbert space

Let E be a real Hilbert space, in which the scalar product is denoted $(x|y)$. There exists an isomorphism j of E onto its dual E', characterized by the formula $\langle x, j(y)\rangle = (x|y)$ for x, y in E (TVS, V, §1, No. 7, Th. 3). We will identify E and E' by means of j. The Fourier transform of a promeasure μ on E is therefore a function $\mathscr{F}\mu$ on E; when μ is a