Strong coupling asymptotics for Schrödinger operators with an interaction supported by an open arc in three dimensions

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Abstract

We consider Schrödinger operators with a strongly attractive singular interaction supported by a finite curve Γ of lenghth L in $\mathbb{R}^3$. We show that if Γ is C^4-smooth and has regular endpoints, the j-th eigenvalue of such an operator has the asymptotic expansion $\lambda_j(H_{\alpha,\Gamma}) = \xi_\alpha + \lambda_j(S) + \mathcal{O}(\mathrm{e}^{\pi\alpha})$ as the coupling parameter $\alpha \to \infty$, where $\xi_\alpha = -4\,\mathrm{e}^{2(-2\pi\alpha+\psi(1))}$ and $\lambda_j(S)$ is the j-th eigenvalue of the Schrödinger operator $S = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma^2(s)$ on $L^2(0,L)$ with Dirichlet condition at the interval endpoints in which γ is the curvature of Γ.

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1 Introduction

Schrödinger operators with singular interactions supported by zero-measure subsets of the configuration space attracted attention of mathematicians already several decades ago. One of the reasons was that their spectral analysis can be often done explicitly to a degree. The simplest situation the interaction support is a discrete set of points has been studied thoroughly, see the monograph [1]. Later singular interactions supported by manifolds of codimension one were analyzed [2, 3]. From 2001 one witnessed a new wave of interest to such operators with attractive interactions. It was motivated by two facts. On the one hand such operators appeared to be good models for a number of tiny

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structures studied in solid state physics, and on the other hand, an intriguing connection between spectral properties of such operators and the geometry of the interaction support was found in [7]. The most prominent manifestation of this connection is the existence of purely geometrically induced bound states [7, 8]; a review of the work done in this area can be found in the paper [6].

A question of a particular importance concerns the strong coupling behavior of the spectra of such operators. In this asymptotic regime the eigenfunctions are strongly localized around the interaction support and one expects an effective lower-dimensional dynamics to play role. The corresponding asymptotic expansion were demonstrated in several situations, for curves in $\mathbb{R}^2$ [12, 13] and $\mathbb{R}^3$ [8] as well as for surfaces in $\mathbb{R}^3$ [9]. In all those cases, the next to leading term was governed by a Schrödinger operator of the dimension of the interaction support with an effective, geometrically induced potential.

The technique used in all those papers was a combination of bracketing estimates with suitable coordinate transformations which allowed one to translate the geometry of the problem into coefficients of the comparison operator. It had a serious restriction as it required that the manifold supporting the interaction has no boundary, being either infinite or a closed curve or surface. Manifolds with a boundary have been also considered but only in situations when the latter is connected with a shrinking 'hole' in a surface [14] or a shrinking hiatus in a curve [8]. The methods used in those cases were perturbative and did not help to address the problem of strong coupling asymptotics for a fixed manifolds with a boundary.

A way to overcome the difficulties with the boundary was proposed in [11]. It used a bracketing estimate again, this time in the neighborhood of an extended curve, together with a suitable integral representation of the eigenfunctions. In this way two-dimensional Schrödinger operators with an interaction supported an open arc, i.e. a finite non-closed curve in $\mathbb{R}^2$, were treated in [11]. It was shown that next-to-leading term is again given by an auxiliary one-dimensional Schrödinger operator with the curvature-induced potential, this time with the Dirichlet conditions at the endpoints of the interval that parametrizes the curve. Our aim in the present paper is to analyze the analogous problem for Schrödinger operators with interaction support of codimension two being a finite non-closed curve in $\mathbb{R}^3$.

Such an extension is no way trivial, in particular, due to a different and more singular character of the interaction. To be specific, we consider a non-relativistic spinless particle exposed to a singular interaction supported by a finite curve $\Gamma \subset \mathbb{R}^3$ with 'free' ends. In the following section we shall describe how one can construct Hamiltonian of such a system, in brief it will be identified with a self-adjoint extension of $-\dot{\Delta} = -\Delta \upharpoonright_{C_0^\infty(\mathbb{R}^3 \setminus \Gamma)}$, where the latter denotes the restriction of the Laplacian $-\Delta : W^{2,2}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ to the set $C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$. The self-adjoint extensions are determined by means of boundary conditions imposed at Γ and classified by a parameter $\alpha \in \mathbb{R}$ which can be regarded coupling constant[1]. We denote those operators as $H_{\alpha,\Gamma}$.

We are going to find the *asymptotics of eigenvalues of* $H_{\alpha,\Gamma}$ *in the regime of strong coupling,* $\alpha \to -\infty$. As in the other cases mentioned above the expansion

[1]A caution is needed, though, due the particular character of singular interactions with support of codimension larger than one. In particular, it is better to avoid formal expressions of the type $-\Delta - \alpha\delta(x - \Gamma)$, because the limits $\alpha \to \pm\infty$ for $H_{\alpha,\Gamma}$ correspond to absence of the interaction and the strong attraction asymptotics, respectively.

starts with a divergent term. We are interested in the next one, expected to be a one-dimensional Schrödinger operator with the same symbol as in the case when Γ is a loop. When the eigenfunctions are strongly localized aroud Γ one may expect their rapid falloff not only transversally but also with the distance from the curve ends. This suggests that the effective dynamics should involve Dirichlet boundary conditions as in the case of codimension one. We are going to show that under mild regularity assumptions it is indeed the case: the jth eigenvalue of $H_{\alpha,\Gamma}$ admits the expansion

$$\lambda_j(H_{\alpha,\Gamma}) = \xi_\alpha + \lambda_j(S) + \mathcal{O}(\mathrm{e}^{\pi\alpha}) \quad \text{as} \quad \alpha \to -\infty\,,$$

where $\lambda_j(S)$ stands for the jth eigenvalue of

$$S = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma^2(s) \;:\; D(S) \to L^2(0,L)\,,$$

with $D(S) := W_0^{1,2}(0,L) \cap W^{2,2}(0,L)$, where L and γ are the length of Γ and its signed curvature, respectively, and ξ_α is given by (2.2) below.

The result will be stated properly together with the outline of the proof in Sec. 3, cf. Theorem 3.1. Before that we collect in the next section the needed preliminaries, Secs. 4.2 and 5 are devoted to completion of the proof.

2 Preliminaries

2.1 Strongly singular interactions

As we have mentioned the character of interactions with support of codimension two is different and more singular than in the case of codimension one. Let us first recall well known facts about point interactions in dimension two which illuminate how our curve-supported potential behaves in the transverse plane to Γ. Consider a single point interactions placed at $y \in \mathbb{R}^2$. The corresponding Hamiltonian is constructed as a self-adjoint extension of the symmetric operator $-\dot{\Delta} := -\Delta \restriction_{C_0^\infty(\mathbb{R}^2\setminus\{y\})}$, i.e. the restriction of $-\Delta : W^{2,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ to the set $C_0^\infty(\mathbb{R}^2 \setminus \{y\})$. Functions from the domain of the adjoint of $-\dot{\Delta}$ admit a logarithmic singularity at the point y, in its vicinity they can be written as $f(x) = -\Xi(f)\ln|x-y| + \Omega(f) + \mathcal{O}(|x-y|)$. Self-adjoint extensions are then characterized by a parameter $\alpha \in \mathbb{R}\cup\{\infty\}$ being characterized by the boundary condition

$$2\pi\alpha\Xi(f) = \Omega(f)\,, \tag{2.1}$$

which in the case $\alpha = \infty$ is a just shorthand for $\Xi(f) = 0$. With the exception of this case, each extension has a single negative eigenvalue equal to

$$\xi_\alpha = -4\,\mathrm{e}^{2(-2\pi\alpha+\psi(1))}\,, \tag{2.2}$$

where ψ is the digamma function. In the following we will use notation $f \in \mathrm{bc}(\alpha,\Gamma)$ for a a function $f \in L^2(\mathbb{R}^2)$ satisfying (2.1). We refer to [1, Chap. I.5] for these and other facts concerning two-dimensional point interactions.

2.2 Geometry of the potential support and its neighborhood

Geometry of Γ. Let Γ be a finite non-closed C^4 smooth curve in $\mathbb{R}^3$ of the length L. In addition, we suppose that Γ has no self-intersections. Without loss of generality we may assume that it is parameterized by its arc length, and we keep the notation $\Gamma : I \to \mathbb{R}^3,\ I := (0\,, L)$, for the corresponding function. Furthermore, we assume that the curve has *regular ends*, i.e. there exists $d_0 \geq 0$ such that for any $d \in [0\,, d_0]$ the curve Γ admits a regular extension Γ_d^{ex}. By this we mean that Γ_d^{ex} is the graph of a C^4 smooth function $\Gamma_d^{\mathrm{ex}} : I_d \to \mathbb{R}^3$ with $I_d := (-d\,, L+d)$ and the restriction of Γ_d^{ex} to I coincides with the original curve, in other words, $\Gamma_0^{\mathrm{ex}} = \Gamma$. Finally, we also assume that the extended curve Γ_d^{ex} admits the global Frenet frame, i.e. the triple of vectors $(t(s), b(s), n(s))$ for any $s \in I_d$.

Remark 2.1. The tangetial, binormal and normal vectors are, by assumption, C^2 functions mapping from I_d to $\mathbb{R}^3$. The assumption about global existence of the (unique) Frenet frame is satisfied provided $\frac{\mathrm{d}}{\mathrm{d}s}\Gamma_d^{\mathrm{ex}}(s) \neq 0$ for all $s \in I_d$. Let us emphasize, however, that we adopt this hypothesis for simplicity only. Our main result requires only piecewise existence of the Frenet frame from which a global coordinate transformation we need can be constructed rotating the coordinate frame on a fixed angle if necessary. A discussion how this can be done curves with straight segment can be found in [8].

The extended curve Γ_d^{ex} is, uniquely up to Euclidean transformations, determined by its curvature γ_d^{ex} and torsion τ_d^{ex}. The same quantities for the original curve Γ are respectively denoted as γ and τ

'Thin' neighborhoods of Γ_d^{ex}. Consider a disc of radius d parametrized by polar coordinates, $\mathcal{B}_d := \{r \in [0\,, d)\,, \varphi \in [0\,, 2\pi)\}$. Using it, we define the cylindrical set $\mathcal{D}_d^{\mathrm{ex}} := I_d \times \mathcal{B}_d$ and the map $\phi_d : \mathcal{D}_d^{\mathrm{ex}} \to \mathbb{R}^3$

$$\phi_d(s\,, r\,, \varphi) = \Gamma_d^{\mathrm{ex}}(s) - r[n(s)\cos(\varphi - \beta(s)) + b(s)\sin(\varphi - \beta(s))]\,,$$

where the function β will be specified latter. For d small enough the function ϕ_d is injective and its image determines a tubular neighborhood Ω_d of Γ_d^{ex}.

The geometry of Ω_d can be described in terms of the metric tensor written in the matrix form as

$$(g_{ij}) = \begin{pmatrix} h^2 + r^2\zeta^2 & 0 & r^2\zeta \\ 0 & 1 & 0 \\ r^2\zeta & 0 & r^2 \end{pmatrix}\,,$$

where $\zeta := \tau - \beta_{,s}$ and $h := 1 + r\gamma\cos(\varphi - \beta)$; we employ the shorthand $\beta_{,s}$ for the derivative of β with respect to the variable s. Choosing $\beta_{,s} = \tau$ we can achieve that the metric tensor becomes diagonal, $g_{ij} = \mathrm{diag}(h^2, 1, r^2)$. This means we choose what is usually called a *Tang frame*, a coordinate system which rotates with respect to the Frenet triple with the angular velocity equal to the curve torsion.

The volume element of Ω_d can be expressed in the coordinates $q \equiv (q_1, q_2, q_3) = (s, r, \varphi)$ as $\mathrm{d}\Omega_d = g^{1/2}\mathrm{d}q$ where $g := \det g_{ij}$. The following elementary inequality will be useful in the further discussion,

$$|h - 1| \leq d \max \gamma\,. \tag{2.3}$$

Shifted curves. Keeping in mind a latter purpose we define now a family of 'shifted' curves $\tilde{\Gamma}_d^{\rm ex}(\rho)$ located in the distance $\rho \in (0, d]$ from $\Gamma_d^{\rm ex}$. Using the Frenet frame we define $\tilde{\Gamma}_d^{\rm ex}(\rho)$ as graph of the function

$$\tilde{\Gamma}_d^{\rm ex} + \eta_n n + \eta_b b \,:\, (-d\,, L+d) \to \mathbb{R}^3\,, \quad \sqrt{|\eta_n|^2 + |\eta_b|^2} = \rho\,.$$

Following the above introduced convention we use the symbol $\tilde{\Gamma}(\rho) = \tilde{\Gamma}_0^{\rm ex}(\rho)$ for the curves shifted with respect to the original Γ. Although we do not mark it explicitly, one has to keep in mind that a shifted curve depends not only on the distance ρ but also on the angular variable encoded in the parameters η_n, η_b.

Let us also list some *notation* we are going to use:

• Let $\mathcal{A} \subset \mathbb{R}^3$ be an open set. We use the abbreviation $(\cdot, \cdot)_{\mathcal{A}}$ for the scalar product $(\cdot, \cdot)_{L^2(\mathcal{A}, {\rm d}x)}$. If $\mathcal{A} = \mathbb{R}^3$ we shortly write $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\mathbb{R}^3, {\rm d}x)}$.
• We denote by $\mathcal{D}_d^{\rm ex} = I_d \times \mathcal{B}_d$ the tubular neighborhood of the extended curve $\Gamma_d^{\rm ex}$, and similarly, $\mathcal{D}_d = I \times \mathcal{B}_d$ corresponds to the original curve Γ.
• Given a self-adjoint operator A, we denote by $\lambda_j(A)$ its jth eigenvalue.

2.3 Definition of Hamiltonian and the Birman-Schwinger principle

Boundary conditions. The definition of the singular Schrödinger operator presented below is a summary of the discussion provided in [8] which we include the make this article self-contained; we refer to the mentioned paper for more details. Suppose given a function $f \in W^{2,2}_{\rm loc}(\mathbb{R}^3 \setminus \Gamma)$, its restriction to $\tilde{\Gamma}(\rho)$ is well defined as a distribution from $D'(0, L)$ which we denote as $f \restriction_{\tilde{\Gamma}(\rho)}$. Furthermore, we assume that the following limits

$$\Xi(f)(s) := -\lim_{\rho \to 0} \frac{1}{\ln \rho} f \restriction_{\tilde{\Gamma}(\rho)} (s)\,, \tag{2.4}$$

$$\Omega(f)(s) := \lim_{\rho \to 0} \Big[f \restriction_{\tilde{\Gamma}(\rho)} (s) + \Xi(f)(s) \ln \rho \Big] \tag{2.5}$$

exist a.e. in $(0\,, L)$. We write $f \in {\rm bc}(\alpha, \Gamma)$ if a $f \in W^{2,2}_{\rm loc}(\mathbb{R}^3 \setminus \Gamma)$ satisfies

$$2\pi\alpha \Xi(f) = \Omega(f)\,. \tag{2.6}$$

Equation (2.6) plays the role of generalized boundary conditions, [4]. Then we define the set

$$D(H_{\alpha,\Gamma}) := \{ f \in W^{2,2}_{\rm loc}(\mathbb{R}^3 \setminus \Gamma) \cap L^2 \,:\, f \in {\rm bc}(\alpha, \Gamma) \}$$

and the operator $H_{\alpha,\Gamma} \,:\, D(H_{\alpha,\Gamma}) \to L^2$ which acts as

$$H_{\alpha,\Gamma} f(x) = -\Delta f(x)\,, \quad x \in \mathbb{R}^3 \setminus \Gamma\,.$$

This operator is self-adjoint, cf. [8, Thm. 2.3] and defines the Hamiltonian we are going to study.

Free resolvent kernel. We start with the resolvent of the 'free' Laplacian, $-\Delta$: $W^{2,2}(\mathbb{R}^3) \to L^2$. It is well known that $R(-\kappa^2) = (-\Delta + \kappa^2)^{-1}$ is for any $\kappa > 0$ an integral operator with the kernel

$$G(\kappa; x, y) = \frac{1}{4\pi} \frac{{\rm e}^{-\kappa|x-y|}}{|x-y|}\,. \tag{2.7}$$

In the following we also use the notation $G(\kappa;\rho) = \frac{1}{4\pi}\frac{\mathrm{e}^{-\kappa\rho}}{\rho}$ where $\rho > 0$. It is well known, see for example [2], that the operator $R(-\kappa^2)$ admits the embedding into $L^2(I)$. To be more precise, consider an $\omega \in L^2(I)$ and define

$$f = f_\kappa^\omega = G(\kappa)\omega * \delta_\Gamma = \frac{1}{4\pi}\int_I \frac{\mathrm{e}^{-\kappa|\cdot-\Gamma(s)|}}{|\cdot-\Gamma(s)|}\,\omega(s)\mathrm{d}s\,. \tag{2.8}$$

Then $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma) \cap L^2$ and the limit $\Omega(f)$ defines one-parameter family of operators $\mathbb{R}_+ \ni \kappa \mapsto Q_{-\kappa^2} : L^2(I) \to L^2(I)$ acting as

$$Q_{-\kappa^2}\omega = \Omega(f^\omega)\,, \quad \omega \in L^2(I)\,, \tag{2.9}$$

cf. [10] for more details.

The Birman-Schwinger principle. The stability of the essential spectrum,

$$\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = \sigma_{\mathrm{ess}}(-\Delta) = [0\,,\infty)\,,$$

is a consequence of the fact that the singular potential in our model is supported by a compact set. Using the results of [17] we can formulate conditions for the existence of discrete eigenvalues. Specifically, we have,

$$\lambda = -\kappa^2 \in \sigma_{\mathrm{d}}(H_{\alpha,\Gamma}) \;\Leftrightarrow\; \ker(Q_{-\kappa^2} - \alpha) \neq \emptyset$$

and the multiplicity of λ is equal to $\dim\ker(Q_{-\kappa^2} - \alpha)$. Moreover, the corresponding eigenspaces are spanned by the functions

$$f = G(\kappa)\omega * \delta_\Gamma\,, \quad \omega \in \ker(Q_{-\kappa^2} - \alpha)\,. \tag{2.10}$$

3 Main result and the proof scheme

Now we are in position to state the main result of this paper.

Theorem 3.1. *Let $H_{\alpha,\Gamma}$ be the singular Schrödinger operator defined by means of the boundary conditions* (2.6) *corresponding to a finite, non-closed C^4 smooth curve with regular ends which has the global Frenet frame.*
(i) The cardinality of the discrete spectrum admits the same asymptotics as in the case of the closed curved, i.e.

$$\sharp\sigma_{\mathrm{d}}(H_{\alpha,\Gamma}) = \frac{L}{\pi}\,\zeta_\alpha(1 + \mathcal{O}(\mathrm{e}^{\alpha\pi}))\,, \tag{3.11}$$

where

$$\zeta_\alpha = (-\xi_\alpha)^{1/2}\,.$$

(ii) Furthermore, the jth eigenvalue of $H_{\alpha,\Gamma}$ has the expansion

$$\lambda_j(H_{\alpha,\Gamma}) = \xi_\alpha + \lambda_j(S) + \mathcal{O}(\mathrm{e}^{\pi\alpha}) \quad \text{for} \quad \alpha \to -\infty\,, \tag{3.12}$$

where $\lambda_j(S)$ stands for the jth eigenvalue of the operator

$$S = -\frac{d^2}{ds^2} - \frac{1}{4}\gamma^2(s) \,:\, D(S) \to L^2(0,L) \tag{3.13}$$

with the domain $D(S) := W^{1,2}_0(0,L) \cap W^{2,2}(0,L)$.

Remark 3.2. One may also ask a question on a varying interaction. Of course, there is not a unique answer for a general case, however, admitting a varying coupling $\tilde{\alpha} = \alpha + \omega(s)$, where $\omega(s) \in C_0^2(\mathbb{R})$ instead of α, we may expect the asymtotics

$$\lambda_j(H_{\tilde{\alpha},\Gamma}) = \lambda_j(\tilde{S}) + \mathcal{O}(\mathrm{e}^{\pi\alpha}) \quad \text{for} \quad \alpha \to -\infty\,, \tag{3.14}$$

where

$$\tilde{S} = -\frac{d^2}{ds^2} - \frac{\gamma^2(s)}{4} - 4\mathrm{e}^{2(-2\pi\tilde{\alpha}(s)+\psi(1))}\,.$$

However, the model requires detailed analysis and further generalizations of the methods used in this paper.

3.1 The proof scheme

Dirichlet–Neumann bracketing. The asymptotics (3.12) can not be obtained directly from the Dirichlet–Neumann bracketing on tubular neighborhoods of the curve Γ in the way analogous to the loop case [6, 8], because in the lower bound the operator S would be replaced by the operator S^N acting as (3.13) but Neumann boundary conditions. Nevertheless, this technique is powerful enough to yield claim (i) of the theorem. More specifically, using the Dirichlet–Neumann bracketing and repeating the argument of [8] we get

$$\lambda_j(H_{\alpha,\Gamma}) = -\kappa_j(\alpha)^2 = \xi_\alpha + c_j + \mathcal{O}(\mathrm{e}^{\pi\alpha})\,, \tag{3.15}$$

where the numbers c_j satisfy the inequalities

$$\lambda_j(S^N) \le c_j \le \lambda_j(S)\,, \tag{3.16}$$

and $S^N : D(S^N) = \{f \in W^{2,2}(0,L) : f'(0) = f'(L) = 0\} \to L^2(0,L)$; recall that S^N has the same differential symbol as S. Note that the second inequality of (3.16) reproduces a right upper bound. In order to prove Theorem 3.1 we obviously have to replace the first inequality by a better lower bound. The remaining part of the paper is devoted to this problem.

A few ideas. Let us mention three concepts we are going to use in the proof of Theorem 3.1. The first is the observation that the properties of the discrete spectrum are reflected in the behavior of the eigenfunctions in the vicinity of the curve Γ. Specifically, let f_j stand for the jth eigenfunction of $H_{\alpha,\Gamma}$ corresponding to $\lambda_j(H_{\alpha,\Gamma})$. Then we have

$$\lambda_j(H_{\alpha,\Gamma}) = \frac{(H_{\alpha,\Gamma} f_j, f_j)}{\|f_j\|^2} = \frac{(-\Delta_{\alpha,\Gamma} f_j, f_j)_{\Omega_d}}{\|f_j\|^2_{\Omega_d}}\,,$$

where the second one of the equalities follows from the natural embedding $L^2(\mathbb{R}^3) \supset L^2(\Omega_d)$ in combination with the fact that f_j satisfies away from Γ the appropriate differential equation: the symbol $-\Delta_{\alpha,\Gamma}$ is understood not as a self-adjoint operator, rather as the differential expression, $-(\Delta_{\alpha,\Gamma} f)(x) = -(\Delta f)(x)$ for $x \ne \Gamma$ and $f \in W^{2,2}_{\mathrm{loc}}(\Omega_d)$.

The second idea is to employ a suitable 'straightening' transformation which allows us to translate the geometry of the problem into the coefficients of the operator. In particular, we obtain an effective potential expressed in terms of the curvature of Γ and its derivatives. To this aim we introduce two unitary transformations,

$$Uf = f \circ \phi_d \,:\, L^2(\Omega_d) \to L^2(\mathcal{D}_d^{\rm ex}, g^{1/2}\mathrm{d}q)$$

and the other one removing the weight factor in the inner product,

$$\hat{U}f = g^{1/4}f\,, \quad \hat{U} \,:\, L^2(\mathcal{D}_d^{\rm ex}, g^{1/2}\mathrm{d}q) \to L^2(\mathcal{D}_d^{\rm ex}, \mathrm{d}q)\,;$$

we combine them denoting

$$f^g := \hat{U}Uf\,. \tag{3.17}$$

Since f_j is by assumption the jth eigenfunction of $H_{\alpha,\Gamma}$, in view of (2.6) we have $g^{-1/4}f_j^g \in \mathrm{bc}(\alpha,\Gamma)$. After a straightforward calculation [8], we get

$$(-\Delta_{\alpha,\Gamma}f_j, f_j)_{\Omega_d} = \big((-\partial_s h^{-2}\partial_s + T_\alpha + V)f_j^g, f_j^g\big)_{\mathcal{D}_d^{\rm ex}}\,, \tag{3.18}$$

where T_α is defined by the differential expression

$$T_\alpha = -\partial_r^2 - r^{-2}\partial_\varphi^2 - \frac{1}{4}r^{-2} \tag{3.19}$$

and

$$V = -\frac{\gamma^2}{4h} + \frac{h_{,ss}}{2h^3} - \frac{5(h_{,s})^2}{4h^4}\,. \tag{3.20}$$

Note that the above described idea was used, for example, in the context of waveguides, cf. [5, 15].

Finally, the third concept is to use is an approximation $f_j \restriction_{\Omega_d}$ by functions vanishing on $\partial\Omega_d$. To explain why it is possible note that in view of $f_j \in \mathrm{bc}(\alpha,\Gamma)$ the eigenfunctions have a logarithmic singularity at Γ, however, away from the curve they decay rapidly: relations (2.7) and (2.10) show that $f_j(x) \sim \mathrm{e}^{-\kappa_j(\alpha)|x-\Gamma|}$ holds for $x \in \Omega_d \setminus \Gamma$, where $\kappa_j(\alpha) := \sqrt{-\lambda_j(\alpha)}$. It shows, in particular, that f_j 'accumulates' at the curve Γ as $\alpha \to -\infty$. This suggests that one might get a good estimate replacing $f_j \restriction_{\Omega_d}$ by suitable functions vanishing on $\partial\Omega_d$ and relate simultaneously the transverse size of Ω_d to the parameter α. To this aim we assume in the following that

$$d = d(\alpha) = \mathrm{e}^{\pi\alpha}\,. \tag{3.21}$$

Proof steps. We are going to use the described ideas in the following way:

- We construct a self-adjoint operator W in $L^2(\mathcal{D}_d^{\rm ex})$ acting as

$$W = -\partial_s h^{-2}\partial_s + T_\alpha + V \,:\, D(W) \to L^2(\mathcal{D}_d^{\rm ex}) \tag{3.22}$$

with the domain $D(W)$ consisting of the functions that satisfies Dirichlet boundary conditions on $\partial\mathcal{D}_d^{\rm ex}$ and $g^{-1/4}f \in \mathrm{bc}(\alpha,\Gamma)$. Our aim is to find a lower bound for eigenvalues of $H_{\alpha,\Gamma}$ in terms of W. Specifically, we are going to show that the following asymptotic inequality

$$\lambda_j(W) \le \lambda_j(H_{\alpha,\Gamma}) + \mathcal{O}(d^{-18}\mathrm{e}^{-C/d}) \tag{3.23}$$

holds.

- The next step is to recover a lower bounds for $\lambda_j(W)$. Using a variational argument we prove that

$$\lambda_j(W) \ge \xi_\alpha + \lambda_j(S) + \mathcal{O}(d)\,. \tag{3.24}$$

Combining it with (3.23) and (3.16) we obtain the claim of Theorem 3.1.

4 Approximating f_j by Dirichlet functions

For the sake of brevity we shall speak of the functions $f \in D(W)$ involved in the first step as of Dirichlet functions; we are sure that the reader would not confuse them with other objects bearing in mathematics the same name.

We keep the notation $\lambda_j(H_{\alpha,\Gamma}) = -\kappa_j(\alpha)^2$ for the eigenvalues of our original operator. To investigate the behavior of the corresponding eigenfunction f_j we employ the expression

$$f_j = G(\kappa_j(\alpha))\omega_j * \delta_\Gamma \,,$$

where

$$(Q_{-\kappa_j(\alpha)^2} - \alpha)\omega_j = 0 \,, \tag{4.25}$$

following from (2.10). For brevity again we shall write in the following shortly $f_j = G(\kappa_j(\alpha))\omega_j$; without loss of generality we may assume that ω_j is normalized function, i.e. $\|\omega_j\|_I = 1$. Combining (3.21) and (3.15) we get

$$\kappa_j(\alpha)d(\alpha) = Cd(\alpha)^{-1} + \mathcal{O}(d(\alpha)^3) \tag{4.26}$$

with the constant $C := 2\mathrm{e}^{2\psi(1)}$.

4.1 Approximate orthogonality of Dirichlet functions

Now we approximate the eigenfunctions f_j by suitable Dirichlet functions. We set $f_j^D = \eta f_j^g$, where $\eta \in C_0^\infty(\mathcal{D}_d^{\mathrm{ex}})$ is a positive function such that

$$\eta(x) = 1 \quad \text{for } \; x \in \mathcal{D}_{d/2}^{\mathrm{ex}}$$

and f^g is the 'straightened' function defined by (3.17).

Lemma 4.1. *Let d be given by (3.21), then the following asymptotics,*

$$(f_j^D, f_k^D)_{\mathcal{D}_d^{\mathrm{ex}}} = \|f_j^D\|^2_{\mathcal{D}_d^{\mathrm{ex}}}\delta_{jk} + R(d) \quad \textit{as} \;\; \alpha \to -\infty \,, \tag{4.27}$$

holds with the remainder term satisfying

$$|R(d)| = \mathcal{O}(d^{-2}\mathrm{e}^{-C/d}) \,.$$

Proof. We start from the self-evident statement that

$$(f_j, f_k) = \|f_j\|^2\delta_{jk} = (f_j, f_k)_{\Omega_d} + (f_j, f_k)_{\mathbb{R}^3\setminus\Omega_d} \,.$$

In view of the unitarity of the straightening transformation the first term on the right-hand side can be written as $(f_j, f_k)_{\Omega_d} = (f_j^g, f_k^g)_{\mathcal{D}_d^{\mathrm{ex}}}$ which implies

$$(f_j^g, f_k^g)_{\mathcal{D}_d^{\mathrm{ex}}} = \|f_j\|^2\delta_{jk} - (f_j, f_k)_{\mathbb{R}^3\setminus\Omega_d} \,. \tag{4.28}$$

The remaining part of the argument can be divided into two parts:

Step 1. Approximating f_k^g by means of f_k^D. Consider a point$x \in \Omega_d$ and denote $x_q := \phi_d(x)$. Combining the inequality $|x_q - \Gamma(s)| \geq r$ with (2.8) and (2.3) we obtain

$$
\begin{aligned}
&\big|(f_j^g, f_k^g)_{\mathcal{D}_d^{\mathrm{ex}}} - (f_j^D, f_k^D)_{\mathcal{D}_d^{\mathrm{ex}}}\big| \\
&\quad = \int_{\mathcal{D}_d^{\mathrm{ex}} \setminus \mathcal{D}_{d/2}^{\mathrm{ex}}} g^{1/2}(1-\eta^2) G(\kappa_j(\alpha))\omega_j \,\overline{G(\kappa_k(\alpha))\omega_k}\,\mathrm{d}q \\
&\quad \le C_1 \int_{d/2}^{d} \int_0^{2\pi} \int_{I_d} G(\kappa_j(\alpha))\omega_j \,\overline{G(\kappa_k(\alpha))\omega_k}\, r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}s \\
&\quad \le \frac{C_1}{2}\,(2d+L)\,\|\omega_j\|_{L^1(I)}\|\omega_k\|_{L^1(I)} \int_{d/2}^{d} \frac{\mathrm{e}^{-(\kappa_j(\alpha)+\kappa_k(\alpha))r}}{r}\,\mathrm{d}r \\
&\quad \le \frac{C_1}{2}\,L^2(2d+L)\,\mathrm{e}^{-C/d}
\end{aligned}
\tag{4.29}
$$

with some constant $C_1 > 0$. The last estimate comes from (4.26) and Schwartz inequality which gives $\|\omega_j\|_{L^1(I)} \le L\|\omega_j\|_I = L$ for any $j \in 1,\dots,N$; we have also used here $|I_d| = 2d + L$. Combining (4.28) and (4.29) we get

$$
(f_j^D, f_k^D)_{\mathcal{D}_d^{\mathrm{ex}}} = \|f_j^D\|^2_{\mathcal{D}_d^{\mathrm{ex}}}\delta_{jk} + \|f_j\|^2_{\mathbb{R}^3\setminus\Omega_d}\delta_{jk} - (f_j, f_k)_{\mathbb{R}^3\setminus\Omega_d} + R_1(d) \tag{4.30}
$$

with the remainder term satisfying

$$
|R_1(d)| = \mathcal{O}(\mathrm{e}^{-C/d})\,;
$$

it remains to estimate the parts of (4.30) referring to $L^2(\mathbb{R}^3 \setminus \Omega_d)$.

Step 2. Estimates of $\|f_j\|_{\mathbb{R}^3\setminus\Omega_d}$. Consider the ball $\mathcal{B} = B(\Gamma(L/2), L)$ of the radius L centered at $\Gamma(L/2)$, the midpoint of the curve. For d small enough we obviously have $\Omega_d \subset \mathcal{B}$, and consequently, we can decompose the norm $\|f_j\|_{\mathbb{R}^3\setminus\Omega_d}$ as

$$
\|f_j\|^2_{\mathbb{R}^3\setminus\Omega_d} = \|f_j\|^2_{\mathbb{R}^3\setminus\mathcal{B}} + \|f_j\|^2_{\mathcal{B}\setminus\Omega_d}\,. \tag{4.31}
$$

Let us introduce the spherical coordinates $(\hat r, \hat\theta, \hat\varphi)$, where $\hat r$ is the radius measuring the distance from the ball center at $\Gamma(L/2)$ and $\hat\theta, \hat\varphi$ are appropriate polar and azimuthal angles. Employing the inequality $|x - \Gamma(s)| \ge \hat r - L/2$ for $x \in \mathbb{R}^3 \setminus \mathcal{B}$ we get by a straightforward computation

$$
\begin{aligned}
\|f_j\|^2_{\mathbb{R}^3\setminus\mathcal{B}} &= \int_{\mathbb{R}^3\setminus\mathcal{B}} \left|\int_I \frac{\mathrm{e}^{-\kappa_j|x-\Gamma(s)|}}{4\pi|x-\Gamma(s)|}\,\omega_j(s)\mathrm{d}s\right|^2 \mathrm{d}x \\
&\le \|\omega_j\|^2_{L^1(I)} \int_0^{2\pi}\int_0^{\pi}\int_L^{\infty} \left(\frac{\hat r}{4\pi(\hat r - L/2)}\right)^2 \mathrm{e}^{-2\kappa_j(\hat r - L/2)}\mathrm{d}\hat r\mathrm{d}\hat\theta\mathrm{d}\hat\varphi \\
&\le \frac{L^2}{16\kappa_j}\mathrm{e}^{-L\kappa_j} = \mathcal{O}(d^2\,\mathrm{e}^{-CL/d^2})\,,
\end{aligned}
\tag{4.32}
$$

where we have used (4.26) and $\|\omega_j\|_{L^1(I)} \le L$. The second norm at the right-hand side of the decomposition (4.31) can estimated as

$$
\begin{aligned}
\|f_j\|^2_{\mathcal{B}\setminus\Omega_d} &= \int_{\mathcal{B}\setminus\Omega_d} \left|\int_I \frac{\mathrm{e}^{-\kappa_j|x-\Gamma(s)|}}{4\pi|x-\Gamma(s)|}\,\omega_j(s)\mathrm{d}s\right|^2 \mathrm{d}x \\
&\le \mathrm{vol}(\mathcal{B}\setminus\Omega_d)\,\frac{\mathrm{e}^{-2\kappa_j d}}{(4\pi d)^2}\,\|\omega_j\|^2_{L^1(I)} = \mathcal{O}(d^{-2}\mathrm{e}^{-C/d})\,.
\end{aligned}
\tag{4.33}
$$

Combining (4.32) and (4.33) we get

$$\|f_j\|^2_{\mathbb{R}^3\setminus\Omega_d} = \mathcal{O}(d^{-2}\mathrm{e}^{-C/d})\,,$$

which together with the result of the first step yields the sought claim. $\square$

4.2 Estimates for the operator W

We also have to find how the 'Dirichlet trimming' influences the operator W defined by (3.22). The idea of replacing the true 'straightened' eigenfunctions f_j^g by the Dirichlet approximants is based on the fact that the contribution coming from

$$\breve{\mathcal{D}}_d := \mathcal{D}_d^{\mathrm{ex}} \setminus \mathcal{D}_{d/2}^{\mathrm{ex}}$$

is asymptotically negligible. Note that, on the one hand, the operator W acts up to the unitary transformation $\hat{U}U$ as $H_{\alpha,\Gamma}$ on the functions supported by $\mathcal{D}_{d/2}^{\mathrm{ex}}$. On the other hand, the following two lemmata justify the just made claim by gauging the component coming from $\breve{\mathcal{D}}_d$.

Lemma 4.2. *The asymptotical relation*

$$|(Wf_j^D, f_k^D)|_{\breve{\mathcal{D}}_d} = \mathcal{O}(d^{-8}\mathrm{e}^{-C/d}) \tag{4.34}$$

holds for d defined by (3.21) and $\alpha \to -\infty$.

Proof. We start from an elementary Schwarz inequality estimate,

$$|(Wf_j^D, f_k^D)_{\breve{\mathcal{D}}_d}| \le \|Wf_j^D\|_{\breve{\mathcal{D}}_d}\|f_k^D\|_{\breve{\mathcal{D}}_d}\,.$$

Proceeding in analogy with Step 1 in the proof of Lemma 4.1, cf. (4.29), we get for the norm $\|f_k^D\|^2_{\breve{\mathcal{D}}_d}$ the bound

$$\|f_k^D\|^2_{\breve{\mathcal{D}}_d} = \|\eta f_k^g\|^2_{\breve{\mathcal{D}}_d} = \int_{\breve{\mathcal{D}}_d} g^{1/2}|\eta G(\kappa_j(\alpha))\omega_j|^2\mathrm{d}q = \mathcal{O}(\mathrm{e}^{-C/d})\,.$$

Next we estimate $\|Wf_k^D\|_{\breve{\mathcal{D}}_d}$. Applying (3.18) we obtain

$$\begin{aligned} Wf_k^D = W(\eta f_k^g) &= \eta\left(-\partial_s h^{-2}\partial_s + T_\alpha + V\right) f_k^g && (4.35)\\ &+ \left(-\partial_{q_1}\partial_{\tilde{q}_1}\eta - \sum_{i=2}^{3}\frac{d^2}{d\tilde{q}_i^2}\eta\right) f_k^g && (4.36)\\ &-(\partial_{q_1}\eta)(\partial_{\tilde{q}_1}f_k^g) - (\partial_{q_1}f_k^g)(\partial_{\tilde{q}_1}\eta) && (4.37)\\ &-2\sum_{i=2}^{3}\partial_{\tilde{q}_i}\eta\partial_{\tilde{q}_i}f_k^g\,, && (4.38)\end{aligned}$$

where we use the shorthands $\partial_{\tilde{q}_1} = h^{-2}\partial_s$, $\partial_{\tilde{q}_2} = \partial_r$, and $\partial_{\tilde{q}_3} = \frac{1}{r}\partial_\varphi$, and the involved differential expressions have been defined in (3.19) and (3.20). Since $(-\partial_s h^{-2}\partial_s + T_\alpha + V)f_k^g(q) = \lambda_k(H_{\alpha,\Gamma})f_k^g(q)$ holds for $q \in \mathcal{D}_d^{\mathrm{ex}}$ and $|\eta|$ is bounded by assumption, the norm of the right-hand-side expression of (4.35) can be estimated by means of $|\lambda_k(H_{\alpha,\Gamma})|^2\|f_k^g\|^2_{\mathcal{D}_d^{\mathrm{ex}}}$. Moreover, it is easy to see that the factor appearing in the longitudinal part of the operator satisfies $h^{-2} = 1+\mathcal{O}(d)$

and $\partial_s h^{-2} = \mathcal{O}(d)$ as $d \to 0$, which implies $|\partial_{q_1}\partial_{\tilde{q}_1}\eta| \leq \text{const}\, d^{-2}$. Using further inequality $|\partial^2_{\tilde{q}_i}\eta| \leq \text{const}\, d^{-2}$, $i = 2,3$, we can estimate the norm of the expression (4.36) by means of $d^{-4}\|f_k^g\|^2_{\mathcal{D}_d^{\text{ex}}}$.

Suppose that $x_q \in \Omega_d$. We put again $x_q = \phi_d(x)$ and denote $\rho(q;s') := |x_q - \Gamma(s')|$. Since $|\partial_{\tilde{q}_i}\rho| \leq \text{const}\, d^{-1}$ we have

$$\left|\partial_{\tilde{q}_i}\frac{\mathrm{e}^{-\kappa\rho}}{\rho}\right| \leq \text{const}\,\frac{\mathrm{e}^{-\kappa\rho}}{\rho^2}\left(\kappa + \frac{1}{\rho}\right).$$

Applying the above inequality to the expression (4.37) and combining this with the fact that the quantity $|\partial_{\tilde{q}_i}\eta|$ entering (4.38) is bounded by $\text{const}\, d^{-1}$ we get

$$\begin{aligned}\|W(\eta f_k^g)\|^2_{\tilde{\mathcal{D}}_d} &\leq C_3|\lambda_k(H_{\alpha,\Gamma})|^2\|f_k^g\|^2_{\mathcal{D}_d^{\text{ex}}} + C_3'\, d^{-4}\|f_k^g\|^2_{\mathcal{D}_d^{\text{ex}}}\\ &+ C_3'' d^{-2}\|\omega_k\|^2_{L^1(I)}\int_{d/2}^{d}\frac{\mathrm{e}^{-2\kappa r}}{r^4}\left(\kappa + \frac{1}{r}\right)^2 r\,\mathrm{d}r = \mathcal{O}(d^{-8}\mathrm{e}^{-C/d})\end{aligned} \tag{4.39}$$

with appropriate constants. It completes the proof. □

The aim of the next lemma is to find out a lower bound for $\|f_j^D\|_{\mathcal{D}_d^{\text{ex}}}$ which will give us a possibility to compare this norm with the small terms appearing in relations (4.27) and (4.34).

Lemma 4.3. *Let d be given by (3.21). Then there exists a $c > 0$ such that*

$$\|f_j^D\|^2_{\mathcal{D}_d^{\text{ex}}} \geq c\, d^8\,. \tag{4.40}$$

Proof. We inspect first the behavior of the eigenfunction f_j in Ω_d. Combining the boundary conditions (2.4) and (2.6) with (4.25) we get

$$f_j \restriction_{\Gamma(r)} = G(\kappa_j)\omega_j \restriction_{\Gamma(r)} = -\frac{1}{2\pi}\,\omega_j \ln r + \alpha\,\omega_j + o(r)\,,$$

where the error term on the right-hand side means a function from $L^2(I)$ the norm of which is $o(r)$ uniformly in α, cf. [10]. Consider the curve distances $r \in (0, d^4)$, then in view of (3.21) the inequality

$$-\ln r > -4\pi\alpha$$

holds for any α, in particular, for $\alpha \to -\infty$. This implies

$$\|f_j \restriction_{\Gamma(r)}\|^2_I \geq \left(\frac{\ln r}{4\pi}\right)^2\|\omega_j\|^2_I + o(\ln r) = \left(\frac{\ln r}{4\pi}\right)^2 + o(\ln r)\,, \tag{4.41}$$

where we have used the fact that the functions ω_j are normalized by assumption. Consequently, for d small enough we can estimate

$$\|f_j^D\|^2_{\mathcal{D}_d^{\text{ex}}} \geq \|f_j^g\|^2_{\mathcal{D}_{d/2}^{\text{ex}}} \geq 4\pi\int_0^{d^4}\|f_j \restriction_{\Gamma(r)}\|^2_I\, r\,\mathrm{d}r \geq c\, d^8 + o(d^8)\,,$$

where c is a positive constant. □

Combining the asymptotics (4.27) with the bound (4.40) we obtain

$$(f_j^D, f_k^D)_{\mathcal{D}_d^{\rm ex}} = \|f_j^D\|^2_{\mathcal{D}_d^{\rm ex}} \delta_{jk} + \|f_j^D\|_{\mathcal{D}_d^{\rm ex}} \|f_k^D\|_{\mathcal{D}_d^{\rm ex}} R_2(d)\,, \tag{4.42}$$

where

$$R_2(d) = \mathcal{O}(d^{-10} \mathrm{e}^{-C/d})\,;$$

on the other hand, a combination of (4.34) with (4.40) yields

$$|(W f_j^D, f_k^D)|_{\breve{\mathcal{D}}_d} = \|f_j^D\|_{\breve{\mathcal{D}}_d} \|f_k^D\|_{\breve{\mathcal{D}}_d} R_3(d)\,, \tag{4.43}$$

where

$$R_3(d) = \mathcal{O}(d^{-16} \mathrm{e}^{-C/d})\,.$$

5 Eigenvalues of W

In this section we are going to conclude the proof of Theorem 3.1 by demonstrating the inequalities (3.23) and (3.24).

5.1 A lower bound for $H_{\alpha,\Gamma}$ in the terms of W

Our first aim is to derive inequality (3.23) in a way partially inspired by [11].

Lemma 5.1. *Let d be given by (3.21), then for $\alpha \to -\infty$ we have*

$$\lambda_j(W) \le \lambda_j(H_{\alpha,\Gamma}) + \mathcal{O}(d^{-18} \mathrm{e}^{-\mathrm{C}/\mathrm{d}})\,. \tag{5.44}$$

Proof. Fix a number $k \in \mathbb{N}$. According to the minimax principle we have

$$\lambda_k(W) = \sup_{\mathcal{S}_k} \inf_{f \in \mathcal{S}_k^\perp} \frac{(Wf, f)_{\mathcal{D}_d^{\rm ex}}}{\|f\|^2_{\mathcal{D}_d^{\rm ex}}}\,, \quad f \in D(W)\,, \tag{5.45}$$

where S_k runs through $(k-1)$-dimensional subspaces of $L^2(\mathcal{D}_d^{\rm ex}) \cap D(W)$. It follows from Lemma 4.1 that the functions $\{f_j^D\}_{j=1}^N$ are linearly independent for all d small enough, and consequently, at least one of the function from each $\mathcal{S}_k^\perp$ admits the decomposition

$$h = \sum_{j=1}^k h_j f_j^D\,, \quad h_j \in \mathbb{C}\,,$$

which means that

$$\inf_{f \in \mathcal{S}_k^\perp} \frac{(Wf, f)_{\mathcal{D}_d^{\rm ex}}}{\|f\|^2_{\mathcal{D}_d^{\rm ex}}} \le \frac{(Wh, h)_{\mathcal{D}_d^{\rm ex}}}{\|h\|^2_{\mathcal{D}_d^{\rm ex}}}\,. \tag{5.46}$$

Using next the fact that $(W f_k^D)(q) = \lambda_k(H_{\alpha,\Gamma}) f_k^D(q)$ holds for any $q \in \mathcal{D}_{d/2}^{\rm ex}$ together with the asymptotic relations (4.43) and (4.42) we get

$$(Wh, h)_{\mathcal{D}_d^{\rm ex}} = \sum_{j=1}^k \lambda_j(H_{\alpha,\Gamma})\, |h_j|^2\, \|f_j^D\|^2_{D_{d/2}^{\rm ex}} + \sum_{i,j=1}^k h_i \overline{h_j}\, S_{ij}(d)\,, \tag{5.47}$$

where

$$S_{ij}(d) := \|f_i^D\|_{\mathcal{D}_d^{\rm ex}} \|f_j^D\|_{\mathcal{D}_d^{\rm ex}} \big(\lambda_j(H_{\alpha,\Gamma}) R_2(d/2) + R_3(d)\big)\,.$$

Consequently, using (4.26), (4.34), (3.11) and (3.15), we can estimate the last term of (5.47) as

$$\Big|\sum_{i,j=1}^{k} h_i\overline{h_j}S_{ij}(d)\Big| \le k\big(|\lambda_k(H_{\alpha,\Gamma})R_2(d/2)| + |R_3(d)|\big)\sum_{i=1}^{k}|h_i|^2\|f_i^D\|^2_{\mathcal{D}_d^{\rm ex}}$$

$$= R_4(d)\sum_{i=1}^{k}|h_i|^2\|f_i^D\|^2\,, \tag{5.48}$$

where

$$R_4(d) = \mathcal{O}(d^{-18}\mathrm{e}^{-C/d})\,.$$

This yields

$$(Wh,h)_{\mathcal{D}_d^{\rm ex}} = \sum_{j=1}^{k}\big(\lambda_j(H_{\alpha,\Gamma}) + \mathcal{O}(d^{-18}\mathrm{e}^{-C/d})\big)\,|h_j|^2\|f_j^D\|^2_{D_d^{\rm ex}}\,. \tag{5.49}$$

In the analogous way we can get an asymptotic expression for the norm,

$$\|h\|^2_{\mathcal{D}_d^{\rm ex}} = \sum_{j=1}^{k}\big(1 + \mathcal{O}(d^{-12}\mathrm{e}^{-C/d})\big)\,|h_j|^2\|f_j^D\|^2_{\mathcal{D}_d^{\rm ex}}\,. \tag{5.50}$$

Combining now the relations (5.48) and (5.50), taking into account (5.45) and (5.46), we arrive at the desired result. □

5.2 A lower bound for W

Finally, we are going to prove (3.24). It will be done in two steps.
An auxiliary lower bound. Our first aim is to show

$$\lambda_j(W) \ge \xi_\alpha + \lambda_j(S_d^{\rm ex}) + \mathcal{O}(d)\,, \tag{5.51}$$

where

$$S_d^{\rm ex} = -\frac{d^2}{ds^2} - \frac{1}{4}(\gamma_d^{\rm ex})^2\ : W_0^{2,2}(I_d) \to L^2(I_d)\,.$$

To prove this statement we recall that the operator W is defined as

$$W = -\partial_s h^{-2}\partial_s + T_\alpha + V\ :\ D(W) \to L^2(\mathcal{D}_d^{ex})\,,$$

where the functions $f \in D(W)$ satisfy Dirichlet boundary conditions on $\partial\mathcal{D}_d^{ex}$ and $g^{-1/4}f \in \mathrm{bc}(\alpha,\Gamma)$. In particular, those functions are continuous away from Γ. Given an $s \in [-d, L+d]$ we denote by $f_s \in L^2(\mathcal{B}_d)$ the 'cut' function, $f_s(r,\varphi) := f(s,r,\varphi)$ where $f \in D(W) \subset L^2(\mathcal{D}_d^{ex})$. Operator T_α can be decomposed into a direct integral, $T_\alpha = \int_{[-d,L+d]}^{\oplus} T_\alpha(s)\,\mathrm{d}s$, on $L^2(\mathcal{D}_d^{ex}) = \int_{[-d,L+d]}^{\oplus} L^2(\mathcal{B}_d)\,\mathrm{d}s$. In other words, for any $s \in [0,L]$ the operators $T_\alpha(s)$ act as

$$T_\alpha(s)f_s = \left(-\partial_r^2 - r^{-2}\partial_\varphi^2 - \frac{1}{4}r^{-2}\right)f_s\,, \tag{5.52}$$

where $g_s^{1/4}f_s \in \mathrm{bc}(\alpha, r=0)$ and f_s satisfies Dirichlet boundary conditions on $\partial\mathcal{B}_d$. Furthermore, for $s \in [-d,0) \cup (L, L+d]$ operators $T_\alpha(s)$ act as (5.52), however, functions from their domains are regular at the origin as the point interaction is absent at the extended parts of the curve. Of course, they still satisfy Dirichlet boundary conditions on $\partial\mathcal{B}_d$. For a fixed $s \in [-d, L+d]$ we denote by $\nu(s)$ the lowest eigenvalue of $T_\alpha(s)$. Using the results of [8, Lemma 3.6] we conclude that

$$\nu(s) = \xi_\alpha + \mathcal{O}(d^{-9/2}\mathrm{e}^{C/d}) \quad \text{for } s \in [0, L]\,,$$

and

$$\nu(s) > 0 \quad \text{for } s \in [-d, 0) \cup (L, L+d]\,.$$

Suppose that $\psi \in D(W)$ is normalized, $\|\psi\|_{\mathcal{D}_d^{ex}} = 1$. Using (3.20) together with the above inequalities we get

$$\begin{aligned}
(W\psi, \psi)_{\mathcal{D}_d^{ex}} &= \Big(\Big(-\partial_s h^{-2}\partial_s - \frac{1}{4}(\gamma^{\mathrm{ex}})^2\Big)\psi, \psi\Big)_{\mathcal{D}_d^{\mathrm{ex}}} + (T_\alpha\psi, \psi)_{\mathcal{D}_d^{\mathrm{ex}}} + \mathcal{O}(d) \\
&\geq \Big(\Big(-\partial_s h^{-2}\partial_s - \frac{1}{4}(\gamma^{ex})^2\Big)\psi, \psi\Big)_{\mathcal{D}_d^{ex}} + (\nu\psi, \psi)_{\mathcal{D}_d^{\mathrm{ex}}} + \mathcal{O}(d) \\
&\geq \Big(\Big(-\partial_s h^{-2}\partial_s - \frac{1}{4}(\gamma^{\mathrm{ex}})^2\Big)\psi, \psi\Big)_{\mathcal{D}_d^{\mathrm{ex}}} + \xi_\alpha + \mathcal{O}(d)\,. \qquad (5.53)
\end{aligned}$$

Using now the minimax principle in combination with the result of [12] we arrive at (5.51).

Estimates for eigenvalues of S_d^{ex}. The change-of-variable transformation

$$s \to \frac{L}{L+2d}(s+d)$$

turns S_d^{ex} into the operator acting in $L^2(I)$ as

$$\tilde{S}_d = -\left(\frac{L}{L+2d}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}(\tilde{\gamma}_d)^2\,,$$

where $\tilde{\gamma}_d(s) = \gamma_d^{\mathrm{ex}}\Big(\frac{L}{L+2d}(s+d)\Big)$. By construction we have

$$\lambda_j(S_d^{\mathrm{ex}}) = \lambda_j(\tilde{S}_d)\,. \qquad (5.54)$$

Moreover, since $|\tilde{\gamma}_d - \gamma| = \mathcal{O}(d)$ we get

$$\tilde{S}_d = -\big(1 + \mathcal{O}(d)\big)\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{\gamma^2}{4}\,,$$

which in view of (5.54) implies

$$\lambda_j(S_d^{\mathrm{ex}}) = \lambda_j(S) + \mathcal{O}(d)\,. \qquad (5.55)$$

Combining this relation with (5.51) we arrive at the sought lower bound (3.24).

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