A first integral to the partially averaged Newtonian potential of the three–body problem[*]

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Abstract

We consider the partial average **i.e.**, the Lagrange average with respect to *just one* of the two mean anomalies, of the Newtonian part of the perturbing function in the three–body problem Hamiltonian. We prove that such a partial average exhibits a non–trivial first integral. We next show how this integral is responsible of three known occurrences in the averaged Newtonian potential: Harrington property, Herman resonance and certain strange symmetries in the planetary torsion.

Contents

1 Results

This paper deals with the discovery of a first integral for some partial average of the Newtonian part of perturbing function of the planetary three–body problem (Theorem 1.1). We then relate such a first integral to three facts that have been observed in the field literature: a property noticed by Harrington in the 60s, a linear relation (called Herman resonance) between the first order Birkhoff invariants and certain symmetries in the second–order Birkhoff invariants to the Birkhoff normal form around the co–planar, co–circular equilibrium.

We consider the three–body problem Hamiltonian, reduced by translations accordingly to the heliocentric method. We label as "1" the inner and "2" the outer planet. Referring to Appendix A (and references therein) for notations and a basic background, we begin with providing some definition. The reader is invited to compare such definitions to the more conventional ones of *double Kepler map* and *doubly averaged Newtonian potential* (Definition A.1).

Definition 1.1 (maps)

⋆ Let $d = 2, 3$ the dimension of configuration space. We denote as $\mathcal{O}$, and call *outer Kepler maps*, the class of canonical changes of coordinates

$$\mathcal{C}: \qquad (\Lambda_2, \ell_2, u, v) \in \mathcal{A} \times \mathbb{T} \times V \to (y_{\mathcal{C}}^{(1)}, y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(1)}, x_{\mathcal{C}}^{(2)}) \in (\mathbb{R}^d)^4$$

where V is a open and connected set of $\mathbb{R}^{4d-2}$, such that there exist positive numbers $\mathfrak{m}_2$, $\mathfrak{M}_2$ and possibly two others $\mathfrak{m}_1$, $\mathfrak{M}_1$ and one of the actions, $u_1 := \Lambda_1$, such that the equality

$$\frac{|y_{\mathcal{C}}^{(i)}|^2}{2\mathfrak{m}_i} - \frac{\mathfrak{m}_i \mathfrak{M}_i}{|x_{\mathcal{C}}^{(i)}|} = -\frac{\mathfrak{m}_i^3 \mathfrak{M}_i^2}{2\Lambda_i^2} =: h_{\mathrm{Kep}}^{(i)}(\Lambda_i) \tag{1}$$

holds *at least for $i = 2$.*

⋆ For a given $\mathcal{C} \in \mathcal{O}$ we denote, for short

$$\mathrm{r}_1 := |x_{\mathcal{C}}^{(1)}|, \quad a_2 := \frac{1}{\mathfrak{M}_2} \frac{\Lambda_2^2}{\mathfrak{m}_2^2}, \quad \varepsilon := \frac{\mathrm{r}_1}{a_2}, \quad \mathrm{G}_2 := |\mathrm{C}_{\mathcal{C}}^{(2)}|, \quad \mathrm{e}_2 := \sqrt{1 - \frac{\mathrm{G}_2^2}{\Lambda_2^2}}, \quad \iota := \widehat{\mathrm{C}_{\mathcal{C}}^{(1)} \mathrm{C}_{\mathcal{C}}^{(2)}}$$

where $\widehat{uv} \in [0, \pi)$ denotes the convex angle of two non vanishing vectors $u, v \in \mathbb{R}^d$, $\mathrm{C}_{\mathcal{C}}^{(i)} := x_{\mathcal{C}}^{(i)} \times y_{\mathcal{C}}^{(i)}$.
We also denote as $\mathrm{P}^{(2)}$ the perihelion of the Keplerian orbit $\ell_2 \to (y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(2)})$.

⋆ We denote as $\mathcal{O}_{\mathrm{Harr}} \subset \mathcal{O}$ and call *(outer) Harrington Kepler maps* the class of outer Kepler map $\mathcal{C} = (\Lambda_2, \ell_2, \mathrm{g}_2, u, v_{\mathrm{g}_2})$ including, among the v–coordinates, $v = (\mathrm{g}_2, v_{\mathrm{g}_2})$, the angle $\mathrm{g}_2 := \alpha_{\mathrm{C}^{(2)}}(\nu_2, \mathrm{C}^{(2)} \times \mathrm{P}^{(2)})$, for some $0 \neq \nu_2 \perp \mathrm{C}^{(2)}$, where, for a given ordered triple $(u, v, w) \in \mathbb{R}_*^3 \times \mathbb{R}_*^3 \times \mathbb{R}_*^3$ with $u, v \perp w$, we denote as $\alpha_w(u, v)$ the oriented angle (u, v), as seen from w.

⋆ We denote as $\mathcal{O}_{\mathrm{Herm}} \subset \mathcal{O}$ and call *(outer) Herman Kepler maps* the class of outer Kepler map $\mathcal{C} = (\Lambda_2, \ell_2, \delta_1, u, v_{\delta_1})$ including, among the v–coordinates, $v = (\delta_1, v_{\delta_1})$, the angle $\delta_1 = \varphi_1 := \alpha_{\mathrm{C}^{(1)}}(\nu_1, \mathrm{C}^{(1)} \times x^{(1)})$, or, when $\mathcal{C}$ is a double Kepler map, $\delta_1 = \mathrm{g}_1 := \alpha_{\mathrm{C}^{(1)}}(\nu_1, \mathrm{C}^{(1)} \times P^{(1)})$, for some $0 \neq \nu_1 \perp \mathrm{C}^{(1)}$.

Finer classes $\mathcal{O}^\star \subset \mathcal{O}$, $\mathcal{O}_{\mathrm{Herm}}^\star \subset \mathcal{O}_{\mathrm{Herm}}$, $\mathcal{O}_{\mathrm{Harr}}^\star \subset \mathcal{O}_{\mathrm{Harr}}$ will be defined along the way (compare Definitions 3.1÷4.2). The following objects will be studied in the paper.

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Definition 1.2 (averages)

⋆ The function, defined for $\mathcal{C} \in \mathcal{O}$, as

$$h_1(\Lambda_2, u, v) := \langle f_{\mathcal{C}} \rangle_{\ell_2}(\Lambda_2, u, v) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\ell_2}{|x_{\mathcal{C}}^{(1)}(\Lambda_2, \ell_2, u, v) - x_{\mathcal{C}}^{(2)}(\Lambda_2, \ell_2, u, v)|}$$

will be referred to as *outer averaged Newtonian potential*, or, simply, *outer average*;

⋆ The function, defined for $\mathcal{C} \in \mathcal{O}$, as

$$h_2(\Lambda_2, u, v) := \langle g_{\mathcal{C}} \rangle_{\ell_2}(\Lambda_2, u, v) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\ell_2}{|w_{\mathcal{C}}^{(1)}(\Lambda_2, \ell_2, u, v) - x_{\mathcal{C}}^{(2)}(\Lambda_2, \ell_2, u, v)|}$$

where

$$w^{(1)} = |x^{(1)}|\widehat{C}^{(1)} \qquad \widehat{C}^{(1)} := \frac{C^{(1)}}{|C^{(1)}|}$$

will be referred to as the *dual averaged Newtonian potential*;

⋆ The function, defined for $\mathcal{C} \in \mathcal{O}_{\mathrm{Herm}}$, as

$$h_3(\Lambda_2, u, v_{\delta_1}) := \langle f_{\mathcal{C}} \rangle_{\delta_1, \ell_2}(\Lambda_2, u, v_{\delta_1}) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\delta_1 d\ell_2}{|x_{\mathcal{C}}^{(1)}(\Lambda_2, \ell_2, u, v) - x_{\mathcal{C}}^{(2)}(\Lambda_2, \ell_2, u, v)|}$$

where v_{δ_1} is v deprived of δ_1, will be referred to as *mixed doubly averaged Newtonian potential*.

Similarly, one might define, in the *inner* case, classes $\mathcal{I}$, $\mathcal{I}_{\mathrm{Harr}}$, $\mathcal{I}_{\mathrm{Herm}}$, and, for $\mathcal{C} \in \mathcal{I}$, $\mathcal{C} \in \mathcal{I}_{\mathrm{Herm}}$, respectively, suitable functions $\langle f_{\mathcal{C}} \rangle_{\ell_1}$, $\langle g_{\mathcal{C}} \rangle_{\ell_1}$ and $\langle f_{\mathcal{C}} \rangle_{\ell_1, \delta_2}$. Here we focus on the outer case, leaving the reader the work of extending results for the inner one. Beware that such an extension is possible in a neat way for all the results quoted below, *except for* Theorem 1.2.

The first result of the paper concerns the outer average $h_1 = \langle f_{\mathcal{C}} \rangle_{\ell_1}$. Firstly, we observe that this function is integrable by quadratures. Indeed, for $d = 3$, as a six–degrees of freedom system, it possess, besides itself, the following five, independent, commuting integrals:

$I_1 :=$ the Euclidean length $G := |C|$ of the total angular momentum C and

$I_2 :=$ its third component $Z := C_3$;

$I_3 :=$ the semi–major axis action $\Lambda_2 := \mathfrak{m}_2 \sqrt{\mathfrak{M}_2 a_2}$;

$I_4 :=$ the Euclidean length $r_1 := |x^{(1)}|$ of $x^{(1)}$;

$I_5 :=$ the projection $\Theta := C^{(2)} \cdot \frac{x^{(1)}}{r_1}$ of the angular momentum $C^{(2)}$ along the direction $x^{(1)}$.

(for $d = 2$, just exclude Z and Θ).

We incidentally mention two facts concerning this item. The former is that *for small values of the ratio ε, $\langle f_{\mathcal{C}} \rangle_{\ell_2}$ is Liouville–Arnold integrable* [2]. Indeed, in Section 3.1, we shall exhibit an outer Kepler map $\mathcal{K}$ such that $\langle f_{\mathcal{K}} \rangle_{\ell_2}$ has (up to re–scalings) the classical close–to be integrable structure:

$$\langle f_{\mathcal{K}} \rangle_{\ell_2} \sim f_0(A) + \varepsilon f_1(A, \varphi) + \cdots .$$

where $A \in \mathbb{R}^3$ are "actions", $\varphi \in \mathbb{T}$ is a one–dimensional "angle" (see Equation (12)).

The latter concerns the full Hamiltonian of the three–body problem. As well known, this is a four degrees of freedom system, therefore partial average $\langle H_{\mathcal{C}} \rangle_{\ell_2}$ of the whole Hamiltonian is a three–degrees–of freedom system. In view of the integrability of $\langle f_{\mathcal{C}} \rangle_{\ell_2}$, it turns out that the integrability of $\langle H_{\mathcal{C}} \rangle_{\ell_2}$ is broken just by the kinetic energy of the inner body (see Section 3.1 for more details). This may have some importance in the study of Arnold diffusion for the three–body problem (compare [15]).

Remark 1.1 It might seem against intuition, but, while the partially averaged Newtonian potential $h_1 = \langle f_{\mathcal{C}} \rangle_{\ell_2}$ is integrable, the doubly averaged one (38) *is not.* One suddenly sees that the count of first integrals for such a double average stops to four (the two components Z and G of the total angular momentum, and the two semi–major axes a_1, a_2). A definitive negative answer to the question has been once and forever provided by a recent work by J. Féjoz and M. Guardia [15], where the authors prove the *splitting of separatices* this function, which, as well known, is a proof of non–integrability.

Definition 1.3 Let h be a n–degrees of freedom Hamiltonian, integrable by quadratures. We say that a first integral I to h belongs to $\sigma(h)$ if, letting h, I_1, $\cdots$, I_{n-1} a set of independent, commuting first integrals, I commutes with I_1, $\cdots$, I_{n-1}, is functionally independent of them, and $I \neq h$.

One reasonably expects that, when a situation like this occurs, a functional dependence between h, I_1, $\cdots$, I_{n-1} and I should exist.

We shall prove that the situation of Definition 1.3 actually occurs for the function h_1 above.

Theorem 1.1 *The function $\mathcal{G}_{\mathcal{C}}$ defined via*

$$\frac{\mathcal{G}_{\mathcal{C}}^2}{\Lambda_2^2} := \frac{G_2^2}{\Lambda_2^2} - e_2 \frac{x_{\mathcal{C}}^{(1)} \cdot P^{(2)}}{a_2}$$

belongs to $\sigma(\langle f_{\mathcal{C}} \rangle_{\ell_2})$, for any $\mathcal{C} \in \mathcal{O}$.

Theorem 1.1 has two strong consequences on h_1. The former of them is specific of h_1, but holds for a larger class of $\mathcal{G}$–commuting functions. It is as follows.

Theorem 1.2 *There exists a class $\mathcal{O}_{\mathrm{Harr}}^\star \subset \mathcal{O}_{\mathrm{Harr}}$, including the maps mentioned after Definition A.1 and a class of functions $\mathcal{F}$, including h_1, such that, if $\{h, \mathcal{G}\} = 0$ and $h \in \mathcal{F}$, then $h_{\mathcal{C}}$ affords an expansion Taylor–Fourier expansion*

$$h_{\mathcal{C}} = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} \varepsilon^n h_{\mathcal{C},nm} e^{im g_2}$$

with

$$h_{\mathcal{C},nm} \equiv 0 \qquad \text{for} \qquad |m| \geq \max\{1, \; n\} \; .$$

Applying the result above to h_1, in light of what we know a priori about the expansion of this function (see Equation (40)), the theorem merely says that, for $n \geq 1$, $h_{1\mathcal{C},nm} \equiv 0$ when $m = \pm n$. Observe that, however, Theorem 1.2 deals with the outer average, not double. Therefore, even in the case $n = 2$, it is more general than Harrington property.

The second consequence of Theorem 1.1 is that there actually exists, up to a zero–measure set, a functional dependence between h_1 and $\mathcal{G}$. In the next, we shall write such a dependence explicitly. To this end, we introduce the following notation.

Definition 1.4

$\star$ For a given Hamiltonian $h(y, Y)$ defined on a phase space $\mathcal{M}$, we denote as $\mathcal{L}_{\mathrm{E}}(h)$ the E–energy level to h, namely, the set $\left\{ (y, Y) \in \mathcal{M} : \; h(y, Y) = \mathrm{E} \right\}$.

We shall prove that

Proposition 1.1 *There exists a suitable* $\mathcal{K} \in \mathcal{O}_{\mathrm{Harr}}$ *such that, for any fixed value of* r_1, a_2 *and* Θ, *there exists an open set* $\mathcal{D}(\mathrm{r}_1, a_2, \Theta) \subset \mathbb{R}$ *such that, for all* $\mathcal{G} \in \mathcal{D}(\mathrm{r}_1, a_2, \Theta)$ *there exists a unique* $\mathrm{E} = \mathrm{E}(\mathrm{r}_1, a_2, \Theta, \mathcal{G})$ *such that*

$$\mathcal{L}_{\mathcal{G}}(\mathcal{G}_{\mathcal{K}}) = \mathcal{L}_{\mathrm{E}}(\langle f_{\mathcal{K}} \rangle_{\ell_{\mathcal{K}2}}) \ . \tag{2}$$

The function

$$(\mathrm{r}_1, a_2, \Theta, \mathcal{G}) \to \mathrm{E}(\mathrm{r}_1, a_2, \Theta, \mathcal{G})$$

has the analytical expression

$$\mathrm{E} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1 - \mathcal{E}\cos\zeta)d\zeta}{\sqrt{\mathrm{r}_1^2 + a_2^2 - 2a_2(\mathrm{r}_1 \mathcal{I}\sin\zeta + a_2\mathcal{E}\cos\zeta) + a_2^2\mathcal{E}^2\cos^2\zeta}} \ , \tag{3}$$

where

$$\mathcal{E} = \mathcal{E}(\Lambda_2, \mathcal{G}) = \frac{\sqrt{\Lambda_2^2 - \mathcal{G}^2}}{\Lambda_2} \qquad \mathcal{I} = \mathcal{I}(\Lambda_2, \mathcal{G}, \Theta) = \frac{\sqrt{\mathcal{G}^2 - \Theta^2}}{\Lambda_2} \ .$$

Let us comment a bit Proposition 1.1.

⋆ The equality (2) is based on a strong peculiarity of the mentioned map $\mathcal{K}$. Namely, the functions $\langle f_{\mathcal{K}} \rangle_{\ell_2}$ and $\mathcal{G}_{\mathcal{K}}$ besides being *both* integrable, exhibit, using such map, just *one degree of freedom*: compare Equation (13) below.

⋆ In view of the formulae in (3), of the definitions of $\mathcal{E}$, $\mathcal{I}$ below, and of the mentioned expression of $\mathcal{G}_{\mathcal{K}}$, it would be nice to know if $\mathcal{G}$ in an action. Namely, if, on the compact level sets for $\mathcal{G}_{\mathcal{K}}$, its naturally associated coordinate takes values in $\mathbb{T}$. If so, in view of the comment below, the formula (3) might be regarded as a Birkhoff–normalization of h_1.

⋆ The function E in (3), regarded as a function of r_1, a_2, $\mathcal{E}$, $\mathcal{I}$, is even separately in all of its arguments. It is worth noticing that $\mathcal{E}$ and $\mathcal{I}$ are a sort of "eccentricity–inclination" coordinates, where by "eccentricity" we mean the one of the Keplerian orbit $\ell_2 \to (y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(2)})$, and, by "inclination" the one of this orbit with respect to $x_{\mathcal{C}}^{(1)}$. Indeed, we shall see in Section 3.2 that $\mathcal{E}$ vanishes when e_2 does and, similarly, $\mathcal{I}$ vanishes when $\mathrm{C}_{\mathcal{C}}^{(2)} \parallel x_{\mathcal{C}}^{(1)}$. By the claimed parity, E affords a Taylor expansion in $(\mathcal{E}, \mathcal{I})$ including just even powers with respect to each. In such an expansion, a "degeneracy" appears: see Corollary 1.1.

⋆ It might give some information the behavior inverse function

$$\mathcal{G} = \mathcal{G}(\mathrm{r}_1, a_2, \Theta, \mathrm{E}) \ .$$

Such a function will cease of exist as soon as the non–degeneracy condition required by the Implicit Function Theorem fails, namely, when

$$\partial_{\mathcal{G}} \mathrm{E}(\mathrm{r}_1, a_2, \Theta, \mathcal{G}) = 0 \ . \tag{4}$$

Looking at the Taylor expansion of $\mathrm{E}(\mathrm{r}_1, a_2, \Theta, \mathcal{G})$ in (even) powers of ε, which is given by

$$\mathrm{E}(\mathrm{r}_1, a_2, \Theta, \mathcal{G}) = \frac{1}{a_2}\left(1 - \frac{\varepsilon^2}{4}\frac{\Lambda_2^3(3\Theta^2 - \mathcal{G}^2)}{\mathcal{G}^5} + \mathrm{O}(\varepsilon^4; \mathrm{r}_1, a_2, \Theta, \mathcal{G})\right)$$

we find that, as soon as ε is sufficiently small, Equation (4) has two solutions, along the two *separatrices*

$$\mathcal{S}_{\pm}: \qquad \sqrt{3}\mathcal{G} \pm \sqrt{5}\Theta + \mathrm{O}(\varepsilon^2; \mathrm{r}_1, a_2, \mathcal{G}) = 0 \ .$$

Along such separatrices, the commuting functions h_1 and $\mathcal{G}$ become *independent*. Therefore, h_1 is *an example of Hamiltonian system possessing two invariant manifolds of co–dimension* 1, *different*

from the energy–level manifold. This example naturally does not contradict the non–existence Poincaré–Fermi Theorem [29, 16, 5], due to the integrability of h_1 (see [11] for more).

The following identity - expressing the conservation of the perihelion of such Keplerian orbit for the "central" averaged Newtonian potential -

$$\mathrm{E}(0, a_2, \Theta, \mathcal{G}) \equiv \frac{1}{a_2} \qquad \forall\, \Theta,\ \mathcal{G}$$

is precisely the reason (see Section 4.1 for more details) of the "$\rho : \sigma$–resonance" in the second–order term, claimed by the following

Corollary 1.1 *There exist ρ, $\sigma \in \mathbb{Q}$ and suitable homogeneous functions of degree $-\frac{1}{2}$*

$$(\alpha, \beta) \to b_i(\alpha, \beta)\ , \quad i = 0,\ 1\ ,$$

such that

$$\begin{aligned}
\mathrm{E}(\mathrm{r}_1, a_2, \Theta, \mathcal{G}) &= b_0(\mathrm{r}_1^2, a_2^2) + \rho b_1(\mathrm{r}_1^2, a_2^2)\mathcal{E}(\Lambda_2, \mathcal{G})^2 + \sigma b_1(\mathrm{r}_1^2, a_2^2)\mathcal{I}(\Lambda_2, \mathcal{G}, \Theta)^2 \\
&+ \mathrm{O}\Big((\mathcal{E}^2, \mathcal{I}^2)^2\Big)\ .
\end{aligned}$$

We shall see that the result here is strictly related to Herman resonance and the symmetries in the planetary torsion. To explain how such facts are related, something more is needed.

Two "duality" relations occur, linking the functions h_1, h_2 and h_3 above altogether. Note that, as well as h_1, also the functions h_2 and h_3 are integrable by quadratures and, for small ε, in the sense of Liouville–Arnold.

The former relation links h_1 and h_2, and is as follows.

Proposition 1.2 *Let $\mathcal{K}$ as in Proposition 1.1, and $\mathcal{C} \in \mathcal{O}^\star$. There exists $\Psi_{\mathcal{C}} : \mathcal{C} \to \mathcal{K}$ such that*

$$\langle f_{\mathcal{K}} \rangle_{\ell_{\mathcal{K}2}} \circ \Psi_{\mathcal{C}} = \langle g_{\mathcal{C}} \rangle_{\ell_2}\ .$$

The latter relation links h_2 and h_3. To quote it, we need the following

Definition 1.5

$\star$ We call ε–*projection* any operator Π that acts on the power series

$$g_\varepsilon = \sum_n a_n \varepsilon^n$$

as

$$\Pi[g_\varepsilon] = \sum_n \delta_n a_n \varepsilon^n$$

where $\{\delta_n\}_{n \in \mathbb{N}}$ is a rational sequence.

Then we have

Theorem 1.3 *There exists a suitable ε–projection Π, such that, for all $\mathcal{C} \in \mathcal{O}_{\mathrm{Herm}}$,*

$$\langle f_{\mathcal{C}} \rangle_{\delta_1, \ell_2} = \Pi[\langle g_{\mathcal{C}} \rangle_{\ell_2}]\ .$$

The nature of this theorem is (up to our understanding) purely computational. It is intimately based on a technical and somewhat surprising "self–similarity" result involving the classical Legendre polynomials (Lemma 5.1 below).

Working out Corollary 1.1, Proposition 1.2 and Theorem 1.3 (but never using direct computation), in Section 4.4 we shall prove the following

Corollary 1.2 *Let ρ, σ be as in Corollary [1.1]. There exist suitable functions β_0, β_1 such that*

$$\langle f_{\mathcal{C}} \rangle_{\delta_1, \ell_2} = \frac{1}{a_2}\left[\beta_0(r_1^2/a_2^2) + \rho\beta_1(r_1^2/a_2^2)e_2^2 + \sigma\beta_1(r_1^2/a_2^2)\frac{G_2^2}{\Lambda_2^2}\sin^2\iota + O_4(e_2, \iota)\right]. \qquad (5)$$

for all $\mathcal{C} \in \mathcal{O}^\star_{\text{Herm}}$.

A byproduct of Corollary [1.2] s the following (a–priori not obvious) fact:

Corollary 1.3 *For any $\mathcal{C} \in \mathcal{O}^\star_{\text{Herm}} \cap \mathcal{O}^\star_{\text{Harr}}$, $\langle f_{\mathcal{C}} \rangle_{\delta_1, \ell_2}$ does not depend of g_2, up to $O_4(e_2, \iota)$.*

It is not difficult to see (we omit the details) that Corollary [1.2] and its specular version for $\mathcal{C} \in \mathcal{I}^\star_{\text{Herm}}$ are responsible of Herman resonance and the symmetries (44) in the planetary torsion.

2 The first integral

Here we prove Theorem 1.1.

We equivalently prove that $\langle f_\mathcal{C} \rangle_{\ell_2}$ commutes with the function

$$\widehat{\mathcal{G}}^2_\mathcal{C} := \frac{\Lambda^2_2}{\mathfrak{m}_2} \mathcal{G}^2_\mathcal{C} = \frac{1}{\mathfrak{m}_2} \left(G^2_2 - \mathfrak{M}_2 \mathfrak{m}^2_2 e_2 x^{(1)}_\mathcal{C} \cdot P^{(2)} \right) . \tag{6}$$

To this end, we introduce the "auxiliary" Hamiltonian

$$H_{aux} = h^{(2)}_{2P} - \mu m_1 m_2 f_{2P,Newt} .$$

with

$$h^{(2)}_{2P} := \frac{|y^{(2)}|^2}{2\mathfrak{m}_2} - \frac{\mathfrak{m}_2 \mathfrak{M}_2}{|x^{(2)}|} , \qquad f_{2P,Newt} := \frac{1}{|x^{(1)} - x^{(2)}|}$$

where $\mathfrak{m}_2$, $\mathfrak{M}_2$, m_1, m_2 are arbitrary masses, and μ is a positive a–dimensional parameter. Note that H_{aux} is obtained from H_{2P} in (36) truncating the term $h^{(1)}_{2P}$ and the indirect part $-\mu m_1 m_2 f_{2P,indir}$ of the perturbing function f_{2P}. We preliminarily rescale of H_{aux}, letting

$$\begin{aligned}
\widehat{H}_{aux}(\widehat{y}^{(2)}, \widehat{x}^{(2)}, \widehat{x}^{(1)}) &= \mathfrak{m}^{-1}_2 H_{aux}(\mathfrak{m}_2 \widehat{y}^{(2)}, \widehat{x}^{(2)}, \widehat{x}^{(1)}) \\
&= \frac{|\widehat{y}^{(2)}|^2}{2} - \frac{\mathfrak{M}_2}{|\widehat{x}^{(2)}|} - \frac{\mu m_1 m_2}{\mathfrak{m}_2} \frac{1}{|\widehat{x}^{(1)} - \widehat{x}^{(2)}|} .
\end{aligned} \tag{7}$$

Letting further

$$\widehat{y}^{(1)} = \frac{1}{2}(y_0 - y) \qquad \widehat{y}^{(2)} = y \qquad \widehat{x}^{(1)} = 2x_0 \qquad \widehat{x}^{(2)} = x_0 + x \tag{8}$$

we approach the two–centre Hamiltonian H_{2C} in (45), with masses

$$m_- = \frac{\mu m_1 m_2}{\mathfrak{m}_2} \qquad m_+ = \mathfrak{M}_2 .$$

But H_{2C} admits the integral $\mathcal{N}$ in (47), and hence, applying the inverse transformations of (8) and (7) we find that H_{aux} has the integral

$$\begin{aligned}
\widehat{\mathcal{N}}_{aux} &= \frac{1}{\mathfrak{m}_2} |(x^{(2)} - \frac{x^{(1)}}{2}) \times y^{(2)}|^2 + \frac{1}{4\mathfrak{m}_2}(x^{(1)} \cdot y^{(2)})^2 \\
&+ x^{(1)} \cdot (x^{(2)} - \frac{x^{(1)}}{2})\left(\frac{\mathfrak{M}_2 \mathfrak{m}_2}{|x^{(2)}|} - \frac{\mu m_1 m_2}{|x^{(1)} - x^{(2)}|}\right)
\end{aligned}$$

We rewrite this integral as

$$\widehat{\mathcal{N}}_{aux} = \mathcal{G}^2_{aux} + \mu \mathcal{H}_{aux} + \frac{|x^{(1)}|^2}{2} H_{aux} \tag{9}$$

where

$$\mathcal{G}^2_{aux} := \frac{1}{\mathfrak{m}_2}\left(|C^{(2)}|^2 - x^{(1)} \cdot L^{(2)}\right) \qquad \mathcal{H}_{aux} := m_1 m_2 \frac{(x^{(1)} - x^{(2)}) \cdot x^{(1)}}{|x^{(1)} - x^{(2)}|} \tag{10}$$

with

$$C^{(2)} := x^{(2)} \times y^{(2)} \qquad L^{(2)} = y^{(2)} \times C^{(2)} - \mathfrak{M}_2 \mathfrak{m}^2_2 \frac{x^{(2)}}{|x^{(2)}|} .$$

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Since the last term in (9) is itself an integral for $\mathrm{H}_{\mathrm{aux}}$, we can neglect it and conclude that the function

$$\mathcal{N}_{\mathrm{aux}} := \mathcal{G}^2_{\mathrm{aux}} + \mu \mathcal{H}_{\mathrm{aux}}$$

is an integral to $\mathrm{H}_{\mathrm{aux}}$.

We recognize that $\mathrm{C}^{(2)}$, $\mathrm{L}^{(2)}$ are, respectively, the angular momentum and the Lenz vector associated to $\mathrm{h}^{(2)}_{\mathrm{2P}}$. Since such quantities, as well as $x^{(1)}$, commute with $\mathrm{h}^{(2)}_{\mathrm{2P}}$, we have that $\mathcal{G}^2_{\mathrm{aux}}$ commutes with $\mathrm{h}^{(2)}_{\mathrm{2P}}$. Recalling that $\mathrm{L}^{(2)}$ is related to the eccentricity e_2 of the ellipse and its perihelion $\mathrm{P}^{(2)}$ via the classical relation

$$\mathrm{L}^{(2)} = \mathfrak{M}_2 \mathfrak{m}_2^2 \mathrm{e}_2 \mathrm{P}^{(2)} \ ,$$

we easily recognize that $\mathcal{G}^2_{\mathrm{aux}}$, written in terms of the $\mathcal{C}$–coordinates, is just the function $\widehat{\mathcal{G}}^2_{\mathcal{C}}$ in (6).

Note also that $\mathcal{H}_{\mathrm{aux}}$ commutes with $f_{\mathrm{2P,Newt}}$.

We then have, by these observations,

$$
\begin{aligned}
0 &= \left\{ \mathrm{H}_{\mathrm{aux}},\, \mathcal{N}_{\mathrm{aux}} \right\} \\
&= \left\{ \mathrm{h}^{(2)}_{\mathrm{2P}} - \mu m_1 m_2 f_{\mathrm{2P,Newt}},\, \widehat{\mathcal{G}}^2_{\mathrm{aux}} + \mu \mathcal{H}_{\mathrm{aux}} \right\} \\
&= \left\{ \mathrm{h}^{(2)}_{\mathrm{2P}},\, \widehat{\mathcal{G}}^2_{\mathrm{aux}} \right\} - \mu m_1 m_2 \left\{ f_{\mathrm{2P,Newt}},\, \widehat{\mathcal{G}}^2_{\mathrm{aux}} \right\} + \mu \left\{ \mathrm{h}^{(2)}_{\mathrm{2P}},\, \mathcal{H}_{\mathrm{aux}} \right\} - \mu^2 m_1 m_2 \left\{ f_{\mathrm{2P,Newt}},\, \mathcal{H}_{\mathrm{aux}} \right\} \\
&= -\mu m_1 m_2 \left\{ f_{\mathrm{2P,Newt}},\, \mathcal{G}^2_{\mathrm{aux}} \right\} + \mu \left\{ \mathrm{h}^{(2)}_{\mathrm{2P}},\, \mathcal{H}_{\mathrm{aux}} \right\} \qquad (11)
\end{aligned}
$$

We now turn to the coordinates $\mathcal{C} = (\Lambda_2, \ell_2, u, v)$ and aim to rewrite (11) in terms of them. As already observed,

$$\mathcal{G}^2_{\mathrm{aux}} \circ \mathcal{C} = \widehat{\mathcal{G}}^2_{\mathcal{C}} \ ,$$

and, moreover

$$f_{\mathrm{2P,Newt}} \circ \mathcal{C} = f_{\mathcal{C},\mathrm{Newt}} = \frac{1}{|x_{\mathcal{C}}^{(1)} - x_{\mathcal{C}}^{(2)}|} \ , \qquad \mathrm{h}^{(2)}_{\mathrm{2P}} \circ \mathcal{C} = \mathrm{h}^{(2)}_{\mathrm{Kep}} = -\frac{\mathfrak{m}_2^3 \mathfrak{M}_2^2}{2\Lambda_2^2} \ .$$

We define

$$\mathcal{H}_{\mathcal{C}} := \mathcal{H}_{\mathrm{aux}} \circ \mathcal{C} \ .$$

Since $\widehat{\mathcal{G}}^2_{\mathcal{C}}$ does not depend of ℓ_2, we have

$$\{ f_{\mathcal{C},\mathrm{Newt}}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \} = \{ f_{\mathcal{C},\mathrm{Newt}}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \}_{\Lambda_2, \ell_2} + \{ f_{\mathcal{C},\mathrm{Newt}}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \}_{u,v} = -\partial_{\ell_2} f_{\mathcal{C},\mathrm{Newt}} \partial_{\Lambda_2} \widehat{\mathcal{G}}^2_{\mathcal{C}} + \{ f_{\mathcal{C},\mathrm{Newt}}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \}_{u,v} \ .$$

Also,

$$\left\{ \mathrm{h}^{(2)}_{\mathrm{Kep}},\, \mathcal{H}_{\mathcal{C}} \right\} = \left\{ -\frac{\mathfrak{m}_2^3 \mathfrak{M}_2^2}{2\Lambda_2^2},\, \mathcal{H}_{\mathcal{C}} \right\} = \frac{\mathfrak{m}_2^3 \mathfrak{M}_2^2}{\Lambda_2^3} \partial_{\ell_2} \mathcal{H}_{\mathcal{C}} \ .$$

Therefore, the equality (11) becomes

$$-\mu m_1 m_2 \left(-\partial_{\ell_2} f_{\mathcal{C},\mathrm{Newt}} \partial_{\Lambda_2} \widehat{\mathcal{G}}^2_{\mathcal{C}} + \{ f_{\mathcal{C},\mathrm{Newt}}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \}_{u,v} \right) + \mu \frac{\mathfrak{m}_2^3 \mathfrak{M}_2^2}{\Lambda_2^3} \partial_{\ell_2} \mathcal{H}_{\mathcal{C}} = 0 \ .$$

We now take the ℓ_2–average of this equality. Since the first and the third term do not survive such average, we find the thesis:

$$0 = \left\langle \left\{ f_{\mathcal{C},\mathrm{Newt}}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \right\}_{u,v} \right\rangle_{\ell_2} = \left\{ \langle f_{\mathcal{C},\mathrm{Newt}} \rangle_{\ell_2}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \right\}_{u,v} = \left\{ \langle f_{\mathcal{C}} \rangle_{\ell_2}, \widehat{\mathcal{G}}^2_{\mathcal{C}} \right\}_{u,v} \ . \qquad \blacksquare$$

3 Harrington property

3.1 The $\mathcal{K}$–map

To specify the class $\mathcal{O}_{\mathrm{Harr}}^{\star}$, we need to define a certain outer Kepler map for the three–body problem, that we denote as

$$\mathcal{K}: \quad (\Lambda_2, \ell_{\mathcal{K}2}, u_{\mathcal{K}}, v_{\mathcal{K}}) := \big(\Lambda_2, \mathrm{l}_2, \mathrm{Z}, \mathrm{G}, \mathrm{R}_1, \Gamma_2, \Theta, \mathrm{z}, \gamma_2, \mathrm{g}, r_1, \vartheta\big) \to (y_{\mathcal{K}}^{(1)}, y_{\mathcal{K}}^{(2)}, x_{\mathcal{K}}^{(1)}, x_{\mathcal{K}}^{(2)}) \ .$$

Let

$$\mathrm{C}^{(1)} := x^{(1)} \times y^{(1)} \qquad \mathrm{C}^{(2)} := x^{(2)} \times y^{(2)} \qquad \mathrm{C} := \mathrm{C}^{(1)} + \mathrm{C}^{(2)} \ .$$

Define the "nodes"

$$\nu_0 := k^{(3)} \times \mathrm{C} \ , \quad \nu_1 := \mathrm{C} \times x^{(1)} \ , \quad \nu_2 := x^{(1)} \times \mathrm{C}^{(2)} \ .$$

Next, assuming that the instantaneous conic $\mathfrak{E}$ generated by the two–body Hamiltonian (1) with $i = 2$ has non–vanishing eccentricity e_2, let $\mathrm{P}^{(2)}$, with $|\mathrm{P}^{(2)}| = 1$ be the direction of its perihelion, a_2 its semi–major axis. Finally, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a vector w, let $\alpha_w(u, v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the "right hand rule"). Then define $\mathcal{K}^{-1}$ via the relations

$$\mathcal{K}^{-1}: \quad \begin{cases} \mathrm{Z} := \mathrm{C} \cdot k^{(3)} \\[2mm] \mathrm{G} := |\mathrm{C}| \\[2mm] \mathrm{R}_1 := \dfrac{y^{(1)} \cdot x^{(1)}}{|x^{(1)}|} \\ \Lambda_2 = \mathfrak{m}_2 \sqrt{\mathfrak{M}_2 a_2} \\[2mm] \Gamma_2 := |\mathrm{C}^{(2)}| \\[2mm] \Theta := \dfrac{\mathrm{C}^{(2)} \cdot x^{(1)}}{|x^{(1)}|} \end{cases} \qquad \begin{cases} \mathrm{z} := \alpha_{k^{(3)}}(k^{(1)}, \nu_0) \\[2mm] \mathrm{g} := \alpha_{\mathrm{C}}(\nu_0, \nu_1) \\[2mm] r_1 := |x^{(1)}| \\[2mm] \mathrm{l}_2 := \text{mean anomaly of } x^{(2)} \text{ on } \mathfrak{E} \\[2mm] \gamma_2 := \alpha_{\mathrm{C}^{(2)}}(\nu_2, \nu_3) \\[2mm] \vartheta := \alpha_{x^{(1)}}(\nu_1, \nu_2) \end{cases}$$

Some properties of $\mathcal{K}$ are collected below.

$\star$ The map $\mathcal{K}$ is canonical, since it can be obtained as the composition of the map of two canonical maps: the canonical map (see Appendix B.1 for more)

$$\mathcal{P}: \quad \big(\mathrm{Z}, \mathrm{G}, \mathrm{R}_1, \mathrm{R}_2, \Gamma_2, \Theta, \zeta, \mathrm{g}, r_1, r_2, \varphi_2, \vartheta\big) \to (y_{\mathcal{P}}^{(1)}, y_{\mathcal{P}}^{(2)}, x_{\mathcal{P}}^{(1)}, x_{\mathcal{P}}^{(2)})$$

defined via

$$\mathcal{P}^{-1}: \quad \begin{cases} \mathrm{Z} := \mathrm{C} \cdot k^{(3)} \\[2mm] \mathrm{G} := |\mathrm{C}| \\[2mm] \mathrm{R}_1 := \dfrac{y^{(1)} \cdot x^{(1)}}{|x^{(1)}|} \\ \mathrm{R}_2 := \dfrac{y^{(2)} \cdot x^{(2)}}{|x^{(2)}|} \\ \Gamma_2 = |\mathrm{C}^{(2)}| \\ \Theta := \dfrac{\mathrm{C}^{(2)} \cdot x^{(1)}}{|x^{(1)}|} \end{cases} \qquad \begin{cases} \mathrm{z} := \alpha_{k^{(3)}}(k^{(1)}, \nu_0) \\[2mm] \mathrm{g} := \alpha_{\mathrm{C}}(\nu_0, \mathrm{C} \times x^{(1)}) \\[2mm] r_1 := |x^{(1)}| \\[2mm] r_2 = |x^{(2)}| \\[2mm] \varphi_2 = \alpha_{\mathrm{C}^{(2)}}(\nu_2, \mathrm{C}^{(2)} \times x^{(2)}) \\[2mm] \vartheta := \alpha_{x^{(1)}}(\mathrm{C} \times x^{(1)}, \nu_2) \end{cases}$$

(note that the coordinates $\big(Z, G, \Theta, R_1, z, g, \vartheta, r_1\big)$ are as in the definition of $\mathcal{K}$) and the classical canonical map

$$\mathrm{K}_{\mathrm{ep}}: \qquad (\mathrm{R}_2, \Gamma_2, r_2, \varphi_2) \to (\Lambda_2, \Gamma_2, \mathrm{l}_2, \gamma_2)$$

which integrates the two–body Hamiltonian, i.e., such that

$$\left(\frac{\mathrm{R}_2^2}{2\mathfrak{m}_2} + \frac{\Gamma_2^2}{2\mathfrak{m}_2 r_2^2} - \frac{\mathfrak{m}_2 \mathfrak{M}_2}{r_2} \right) \circ \mathrm{K}_{\mathrm{ep}} = -\frac{\mathfrak{M}_2^2 \mathfrak{m}_2^3}{2\Lambda_2^2} \; .$$

$\star$ $\mathcal{K}$ is a outer Kepler map, since in fact it satisfies (1) with $i = 2$. Moreover, $\mathcal{K} \in \mathcal{O}_{\mathrm{Harr}}$, according to Definition 1.1.

$\star$ The coordinates $\mathcal{K}$, as well as $\mathcal{P}$, provide a reduction of the angular momentum which is *regular* for planar motions (the planar case, i.e., $\mathrm{C}^{(1)} \parallel \mathrm{C}^{(2)} \parallel \mathrm{C}$, corresponds to $\Theta = 0$ and $\vartheta = \pi$). This should be compared to the *Jacobi reduction of the nodes*, which is singular for planar motions.

$\star$ Using the coordinates $\mathcal{K}$, one has the following expansion for $\langle f_{\mathcal{K}} \rangle_{\ell_2}$, which, exhibiting a leading part completely independent of angles, shows its Liouville–Arnold integrability:

$$\begin{aligned}
\langle f_{\mathcal{K}} \rangle_{\ell_2} &= \frac{1}{a_2}\Bigg[1 + \varepsilon^2 \Big(-\frac{1}{4} \frac{\Lambda_2^3 (3\Theta^2 - \Gamma_2^2)}{\Gamma_2^5} - \frac{3}{8}\varepsilon \sqrt{1 - \frac{\Gamma_2^2}{\Lambda_2^2}} \sqrt{1 - \frac{\Theta^2}{\Gamma_2^2}} \frac{\Lambda_2^5}{\Gamma_2^5} \Big(1 - 5\frac{\Theta^2}{\Gamma_2^2} \Big) \cos\gamma_2 \\
&\quad + \frac{\varepsilon^2}{128} \frac{\Lambda_2^7}{\Gamma_2^7} \Big((5 - 3\frac{\Gamma_2^2}{\Lambda_2^2})(35\frac{\Theta^4}{\Gamma_2^4} - 30\frac{\Theta^2}{\Gamma_2^2} + 3) \\
&\quad + 10(1 - \frac{\Gamma_2^2}{\Lambda_2^2})(1 - \frac{\Theta}{\Gamma_2})(1 - 7\frac{\Theta}{\Gamma_2}) \cos 2\gamma_2 + \cdots
\end{aligned} \tag{12}$$

$\star$ The *three–body problem Hamiltonian* (36), in terms of the $\mathcal{K}$–coordinates, has the expression

$$\mathrm{H}_{\mathcal{K}} = \frac{\mathrm{R}_1^2}{2\mathfrak{m}_1} + \frac{|\mathrm{C}_{\mathcal{K}}^{(1)}|^2}{2\mathfrak{m}_1 r_1^2} - \frac{\mathfrak{m}_1 \mathfrak{M}_1}{r_1} - \frac{\mathfrak{m}_2^3 \mathfrak{M}_2^2}{2\Lambda_2^2} - \mu m_1 m_2 f_{\mathcal{K}}$$

where

$$|\mathrm{C}_{\mathcal{K}}^{(1)}|^2 = \mathrm{G}^2 + \Gamma_2^2 - 2\Theta^2 + 2\sqrt{\mathrm{G}^2 - \Theta^2} \sqrt{\Gamma_2^2 - \Theta^2} \cos\vartheta \; .$$

Switching to the ℓ_2–average, one has

$$\langle \mathrm{H}_{\mathcal{K}} \rangle_{\ell_2} = \frac{\mathrm{R}_1^2}{2\mathfrak{m}_1} + \frac{|\mathrm{C}_{\mathcal{K}}^{(1)}|^2}{2\mathfrak{m}_1 r_1^2} - \frac{\mathfrak{m}_1 \mathfrak{M}_1}{r_1} - \frac{\mathfrak{m}_2^3 \mathfrak{M}_2^2}{2\Lambda_2^2} - \mu m_1 m_2 \langle f_{\mathcal{K}} \rangle_{\ell_2}$$

As expected, $\langle \mathrm{H}_{\mathcal{K}} \rangle_{\ell_2}$ is a three–degrees of freedom system. As mentioned in the introduction, because of the integrability of $\langle f_{\mathcal{K}} \rangle_{\ell_2}$, the part of $\langle \mathrm{H}_{\mathcal{K}} \rangle_{\ell_2}$ which results truncating away the kinetic energy term of the inner body

$$\mathrm{T}_1 := \frac{\mathrm{R}_1^2}{2\mathfrak{m}_1} + \frac{|\mathrm{C}_{\mathcal{K}}^{(1)}|^2}{2\mathfrak{m}_1 r_1^2}$$

keeps to be integrable.

3.2 Proof of Theorem 1.2

In terms of the outer Kepler map $\mathcal{C} = \mathcal{K}$, the functions $\langle f_{\mathcal{K}} \rangle_{\ell_2}$ and $\frac{\mathcal{G}_{\mathcal{K}}^2}{\Lambda_2^2}$ depend just on the coordinates $(r_1, \Lambda_2, \Theta, \Gamma_2, \gamma_2)$. Their expressions are indeed

$$\langle f_{\mathcal{K}} \rangle_{\ell_2}(r_1, \Lambda_2, \Theta, \Gamma_2, \gamma_2) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\mathrm{l}_2}{\sqrt{r_1^2 + 2r_1 a_2 \rho_2 \sqrt{1 - \frac{\Theta^2}{\Gamma_2^2}} \cos(\gamma_2 + \nu_2) + a_2^2 \rho_2^2}}$$

$$\frac{\mathcal{G}_{\mathcal{K}}(r_1, \Lambda_2, \Theta, \Gamma_2, \gamma_2)^2}{\Lambda_2^2} = \frac{\Gamma_2^2}{\Lambda_2^2} + \frac{r_1}{a_2} \sqrt{1 - \frac{\Gamma_2^2}{\Lambda_2^2}} \sqrt{1 - \frac{\Theta^2}{\Gamma_2^2}} \cos\gamma_2 \; . \tag{13}$$

where, classically,

$$\rho_2 \cos\nu_2 = \cos\zeta_2 - \sqrt{1 - \frac{\Gamma_2^2}{\Lambda_2^2}} \qquad \rho_2 \sin\nu_2 = \frac{\Gamma_2}{\Lambda_2}\sin\zeta_2 \qquad \rho_2 = 1 - \sqrt{1 - \frac{\Gamma_2^2}{\Lambda_2^2}}\cos\zeta_2$$

$$dl_2 = \rho_2 d\zeta_2\ , \quad a_2 = \frac{1}{\mathfrak{M}_2}\frac{\Lambda_2^2}{\mathfrak{m}_2^2}\ , \quad a_2\rho_2 = |x_{\mathcal{K}}^{(2)}|\ . \tag{14}$$

with ζ_2, ν_2 the eccentric-, true anomaly of $x_{\mathcal{K}}^{(2)}$, respectively.

The formulae (13) give us the chance of making two observations.

The former concerns $\mathcal{G}_{\mathcal{K}}$, and is that

$$\mathcal{G}_{\mathcal{K}}\Big|_{\Gamma_2=\Lambda_2} = \Lambda_2 \qquad \mathcal{G}_{\mathcal{K}}\Big|_{\Gamma_2=\Theta} = \Theta\ .$$

This shows, as mentioned in the introduction, that the quantities $\mathcal{E}$, $\mathcal{I}$ in Proposition 1.1 vanish simultaneously to the eccentricity and the inclination of the orbit $\ell_2 \to (y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(2)})$.

The latter concerns instead $\langle f_{\mathcal{K}}\rangle_{\ell_2}$: such a function is even in γ_2, therefore, it affords a Taylor–Fourier expansion with respect to ε and γ_2 "including just cosines":

$$\langle f_{\mathcal{K}}\rangle_{\ell_2} = \frac{1}{a_2}\sum_{n=0}^{\infty}\varepsilon^n \sum_{m\in\mathbb{Z}} f_{nm}(\Lambda_2, \Gamma_2, \Theta)\cos m\gamma_2\ .$$

Lemma 3.1 *The functions*

$$h_\varepsilon(\Gamma, \gamma) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varepsilon^n h_{nm}(\Gamma)\cos m\gamma \qquad g_\varepsilon(\Gamma, \gamma) = a(\Gamma) + \varepsilon b(\Gamma)\cos\gamma$$

commute with respect to $\left\{\,\cdot\,,\,\cdot\,\right\}_{\Gamma,\gamma}$ if and only if the following equalities are satisfied

$$mh_{nm}\partial_\Gamma a + \frac{1}{2}\Big((m-1)h_{n-1,m-1} + (m+1)h_{n-1,m+1}\Big)\partial_\Gamma b$$

$$= \frac{1}{2}\Big(\partial_\Gamma h_{n-1,m-1} - \partial_\Gamma h_{n-1,m+1} + \partial_\Gamma h_{n-1,0}\delta_{m,1}\Big)b \qquad \forall\ n = 0,\ 1,\ \cdots,\ m = 1,\ 2,\ \cdots$$

where δ_{ij} is the Kronecker symbol, and $h_{-1,m} := 0$.

The proof of this elementary lemma is omitted, as well as the consequence below.

Corollary 3.1 *If h_ε commutes with g_ε and $\partial_\Gamma a \neq 0$, then $h_{nm} = 0$ for all $m \geq \max\{1,\ n\}$.*

With the following definitions of $\mathcal{O}_{\mathrm{Harr}}^\star$ and $\mathcal{F}$, the proof of Theorem 1.2 becomes trivial.

Definition 3.1

$\star$ We denote as $\mathcal{O}_{\mathrm{Harr}}^\star$ the subset of $\mathcal{O}_{\mathrm{Harr}}$ including maps $\mathcal{C} = (\Lambda_2, \ell_2, \mathsf{g}_2, u, v_{\mathsf{g}_2})$ related to $\mathcal{K}$ via (canonical) transformations of the form

$$\begin{cases} \Lambda_2 = \Lambda_2 \\ \ell_{\mathcal{K}2} = \ell_2 + \psi(\Lambda_2, \mathsf{g}_2, u, v_{\mathsf{g}_2}) \\ \gamma_2 = \mathsf{g}_2 + \varphi(\Lambda_2, u, v_{\mathsf{g}_2}) \\ u_{\mathcal{K}} = u(\Lambda_2, u, v_{\mathsf{g}_2}) \\ v_{\mathcal{K}\gamma_2} = v_{\mathsf{g}_2}(\Lambda_2, u, v_{\mathsf{g}_2}) \end{cases}$$

$\star$ We denote as $\mathcal{F}$ the class of functions such that $h_{\mathcal{K}}$ is a regular function of Θ, r_1, Λ_2, Γ_2, γ_2, Z, G, ζ, g, defined on a neighborhood of $\{\varepsilon = 0\ , \gamma_2 \in \mathbb{T}\}$, even with respect to γ_2.

4 Herman resonance and planetary torsion

4.1 Proof of Proposition 1.1

Definition 4.1 For a given Hamiltonian $h(y, Y)$ defined on a phase space $\mathcal{M}$,

⋆ we denote as $\mathrm{fix}(h)$ the set of phase points $(y, Y) \in \mathcal{M}$ which are rest points to h;

⋆ for fixed (r_1, a_2, Θ), we denote as $\mathcal{D} = \mathcal{D}(r_1, a_2, \Theta)$ the set of $\mathcal{G}_\star \in \mathbb{R}$ such that

$$\mathcal{L}_{\mathcal{G}_\star}(\mathcal{G}_\mathcal{K}) \cap \Big(\mathrm{fix}(\mathcal{G}_\mathcal{K}) \cup \mathrm{fix}(\langle f_\mathcal{K} \rangle_{\ell_2}) \Big) = \emptyset \ .$$

The sets $\mathcal{D}(r_1, a_2, \Theta)$ are open and non–empty, at least for ε small. This follows from the formula for $\mathcal{G}_\mathcal{K}$ in (13) and the ε–expansion of $\langle f_\mathcal{K} \rangle_{\ell_2}$ in (12).

Lemma 4.1 *Let $g(y, Y)$, $h(y, Y)$ two one–degrees of freedom commuting Hamiltonians on their common domain $D \subset \mathbb{R}^2$. Then for any $(y_0, Y_0) \in D \setminus \{\mathrm{fix}(h) \cup \mathrm{fix}(g)\}$, the g–level curve in D through (y_0, Y_0) and the h–level curve through (y_0, Y_0) in D coincide.*

Proof Obvious. ∎

Proof of Proposition 1.1 The former assertion, namely, Equation (2), is an immediate consequence of Lemma 4.1 and of the definition of $\mathcal{D}$. To prove the latter, let us denote as $h(r_1, a_2, \Theta, \Gamma_2, \gamma_2)$ the function $\langle f_\mathcal{K} \rangle_{\ell_2}$ in (13) and as $\Gamma_2(r_1, \Lambda_2, \Theta, \mathcal{G}, \gamma_2)$ the inverse function of

$$\mathcal{G}_\mathcal{K}(r_1, \Lambda_2, \Theta, \cdot, \gamma_2) = \mathcal{G}$$

with respect to Γ_2. This is certainly is well defined for small r_1. By (2), the function

$$h\Big(r_1, \Lambda_2, \Theta, \Gamma_2(r_1, \Lambda_2, \Theta, \mathcal{G}, \gamma_2), \gamma_2\Big)$$

is to be independent of γ_2. Hence, the following identity holds, for all $\gamma \in \mathbb{T}$

$$h\Big((r_1, \Lambda_2, \Theta, \Gamma_2(r_1, \Lambda_2, \Theta, \mathcal{G}, \gamma_2), \gamma_2)\Big) = h\Big((r_1, \Lambda_2, \Theta, \Gamma_2(r_1, \Lambda_2, \Theta, \mathcal{G}, \tfrac{\pi}{2}), \tfrac{\pi}{2})\Big) =: \mathrm{E}(r_1, a_2, \Theta, \mathcal{G}) \ .$$

But for $\gamma_2 = \frac{\pi}{2}$, we have $\Gamma_2 = \mathcal{G}$, and, using the formulae in (14),

$$\rho_2\Big|_{\Gamma_2 = \mathcal{G}} = 1 - \mathcal{E} \cos \zeta_2$$

$$dl_2\Big|_{\Gamma_2 = \mathcal{G}} = (1 - \mathcal{E} \cos \zeta_2) d\zeta_2$$

$$\rho_2 \sqrt{1 - \frac{\Theta^2}{\Gamma_2^2}} \cos(\gamma_2 + \nu_2)\Big|_{(\Gamma_2, \gamma_2) = (\mathcal{G}, \frac{\pi}{2})} = -\rho_2 \sqrt{1 - \frac{\Theta^2}{\Gamma_2^2}} \sin \nu_2 \Big|_{\Gamma_2 = \mathcal{G}} = -\frac{\sqrt{\mathcal{G}^2 - \Theta^2}}{\Lambda_2} \sin \zeta_2 \ .$$

Then the formula in (3) follows. ∎

Lemma 4.2 *Let $p \in \mathbb{Q}$. Then the function*

$$g_p := \frac{1}{2\pi} \int_\mathbb{T} \frac{(1 - \mathcal{E} \cos \zeta) d\zeta}{\Big(r_1^2 + a_2^2 - 2a_2(r_1 \mathcal{I} \sin \zeta + a_2 \mathcal{E} \cos \zeta) + a_2^2 \mathcal{E}^2 \cos^2 \zeta \Big)^p}$$

13

affords the expansion

$$g_p = \sum_{h,k} g_{hk}(r_1^2, a_2^2)\mathcal{E}^{2h}\mathcal{I}^{2k} \tag{15}$$

where

$$g_{hk}(\alpha, \beta) = \frac{\sum_{j=0}^{2h+2k} \rho_{hkj}\alpha^j \beta^{2h+2k-j}}{(\alpha + \beta)^{2h+2k+\frac{1}{2}}} \tag{16}$$

are homogeneous rational functions of degree $-\frac{1}{2}$ in the arguments (α, β) with rational coefficients ρ_{hkj} verifying

$$\rho_{hk,2h+2k} \equiv 0 \qquad for \qquad (h, k) \neq 0 . \tag{17}$$

In the special case $p = \frac{1}{2}$, one also has

$$\rho_{hk0} \equiv 0 \qquad for \qquad (h, k) \neq (0, 0) . \tag{18}$$

Proof Note firstly that g_p is even separately in r_1, a_2, $\mathcal{E}$ and $\mathcal{I}$. Indeed, when $a_2 \to -a_2$ or $\mathcal{E} \to -\mathcal{E}$ ($r_1 \to -r_1$ or $\mathcal{I} \to -\mathcal{I}$, respectively), it is sufficient to change the integration variable $\zeta \to \pi - \zeta$ ($\zeta \to -\zeta$, respectively) to have the result unvaried. Therefore, the expansion (15) follows, with

$$g_{hk}(r_1^2, a_2^2) = \frac{\partial_{\mathcal{E}}^{2h}\partial_{\mathcal{I}}^{2k} g\big|_{(\mathcal{E},\mathcal{I})=(0,0)}}{(2h)!(2k)!} . \tag{19}$$

On the other hand, the formula (19) implies that the functions $g_{hk}(r_1^2, a_2^2)$, must have the form

$$g_{h,k}(r_1^2, a_2^2) = \frac{p_{hk}(r_1^2, a_2^2)}{(r_1^2 + a_2^2)^{2h+2k+\frac{1}{2}}}$$

where

$$p_{hk}(\alpha, \beta) = \sum_{j=0}^{2h+2k} \rho_{hkj}\alpha^j \beta^{2h+2k-j}$$

are homogeneous polynomials in (α, β) of degree $2h + 2k$. The coefficients ρ_j in this formula have to be rational because this immediately follows from the Taylor formula (19), as soon as $p \in \mathbb{Q}$. Since

$$g_p\big|_{a_2=0} \equiv \frac{1}{r_1}$$

(17) follows.

In the special case $p = \frac{1}{2}$, one also has the identity

$$g_{\frac{1}{2}}\big|_{r_1=0} \equiv \frac{1}{a_2} . \tag{20}$$

Therefore, (18) follows. ∎

Remark 4.1 The identity (18), coming from (20), and immediate from the definitions, is not surprising, since it is nothing else than conservation of the perihelion of the Keplerian orbit $\ell_2 \to (y_\mathcal{C}^{(2)}, x_\mathcal{C}^{(2)})$ for the restriction

$$\langle f_\mathcal{C}\rangle_{\ell_2}\big|_{r_1=0} = \frac{1}{2\pi} \int_\mathbb{T} \frac{dl_2}{|x_\mathcal{C}^{(2)}|} = \frac{1}{a_2} .$$

Now we are ready to prove Corollary 1.1 as a direct consequence of Lemma 4.2.

Proof of Corollary 1.1 Let us use Lemma 4.2 with $p = \frac{1}{2}$. The identities (17) and (18) imply that, in the special case $h + k = 1$, in the summand in (16) just the term with $j = 1$ survives. Therefore, the claimed function $b_1(r_1^2, a_2^2)$ is

$$b_1(r_1^2, a_2^2) = \frac{r_1^2 a_2^2}{(r_1^2 + a_2^2)^{\frac{5}{2}}} \quad . \quad \blacksquare$$

4.2 Proof of Proposition 1.2

Definition 4.2

$\star$ We denote as $\mathcal{O}^\star$ the subset of the maps $\mathcal{C} = (\Lambda_2, \ell_2, u, v) \in \mathcal{O}$ such that, if l_2 denotes the mean anomaly of $x_{\mathcal{C}}^{(2)}$ on the orbit $\ell_2 \to (y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(2)})$, the difference $\ell_2 - \mathrm{l}_2$ does not depend of ℓ_2.

$\star$ We denote as $\mathcal{O}_{\mathrm{Herm}}^\star := \mathcal{O}^\star \cap \mathcal{O}_{\mathrm{Herm}}$.

For a given $\mathcal{C} \in \mathcal{O}^\star$, we let, for short

$$\mathrm{G}_i := |\mathrm{C}_{\mathcal{C}}^{(i)}| \ , \qquad \mathrm{C}_{\mathcal{C}} := \mathrm{C}_{\mathcal{C}}^{(1)} + \mathrm{C}_{\mathcal{C}}^{(2)} \ , \qquad \mathrm{G} := |\mathrm{C}_{\mathcal{C}}| \ , \qquad \iota_2 := \widehat{\mathrm{C}_{\mathcal{C}} \mathrm{C}_{\mathcal{C}}^{(2)}} \ , \qquad \iota := \widehat{\mathrm{C}_{\mathcal{C}}^{(1)} \mathrm{C}_{\mathcal{C}}^{(2)}}$$

$$\mathrm{g}_2 := \alpha_{\mathrm{C}_{\mathcal{C}}^{(2)}}\left(\mathrm{C}_{\mathcal{C}}^{(1)} \times \mathrm{C}_{\mathcal{C}}^{(2)}, \ \mathrm{C}_{\mathcal{C}}^{(2)} \times \mathrm{P}^{(2)}\right) \ , \qquad \mathrm{r}_1 := |x_{\mathcal{C}}^{(1)}| \ , \qquad \rho_2 = \frac{|x_{\mathcal{C}}^{(2)}|}{a_2} = 1 - \sqrt{1 - \frac{\mathrm{G}_2^2}{\Lambda_2^2}} \cos \zeta_2$$

$$(21)$$

where $\widehat{uv} \in [0, \pi)$ denotes the convex angle of two non vanishing vectors $u, v \in \mathbb{R}^d$. We also denote as ν_2, ζ_2, the true, eccentric anomaly of $x_{\mathcal{C}}^{(2)}$ on the Keplerian orbit $\ell_2 \to (y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(2)})$.
The following proposition refines Proposition 1.2.

Proposition 4.1 *Proposition 1.2 holds true with $\mathcal{O}^\star$ as above, and $\Psi_{\mathcal{C}}$ defined via*

$$\Psi_{\mathcal{C}} : \begin{cases} r_1 = \mathrm{r}_1 \\[2mm] \Gamma_1 = \mathrm{G}_1 \\[2mm] \Theta = \mathrm{G}_2 \cos \iota \\[2mm] \gamma_2 = \mathrm{g}_2 \end{cases} \quad .$$

Letting, moreover,

$$\frac{\mathfrak{G}^2}{\Lambda_2^2} = \frac{\mathrm{G}_2^2}{\Lambda_2^2} + \frac{\mathrm{r}_1}{a_2} \mathrm{e}_2 \sin \iota \cos \mathrm{g}_2$$

$$\mathcal{E}_{\mathcal{C}} := \frac{\sqrt{\Lambda_2^2 - \mathfrak{G}^2}}{\Lambda_2} \quad \mathcal{I}_{\mathcal{C}} := \frac{\sqrt{\mathfrak{G}^2 - \mathrm{G}_2^2 \cos^2 \iota}}{\Lambda_2} \quad (22)$$

then the following identity holds

$$\langle g_{\mathcal{C}} \rangle_{\ell_2} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(1 - \mathcal{E}_{\mathcal{C}} \cos \zeta) d\zeta}{\sqrt{\mathrm{r}_1^2 + a_2^2 - 2a_2(\mathrm{r}_1 \mathcal{I}_{\mathcal{C}} \sin \zeta + a_2 \mathcal{E}_{\mathcal{C}} \cos \zeta) + a_2^2 \mathcal{E}_{\mathcal{C}}^2 \cos^2 \zeta}} \quad (23)$$

Proof Observe that
$$|r_1 w_{\mathcal{C}}^{(1)} - x_{\mathcal{C}}^{(2)}|^2 = r_1^2 - 2r_1 a_2 w_{\mathcal{C}}^{(1)} \cdot x_{\mathcal{C}}^{(2)} + a_2^2 \rho_2^2 .$$

Since $C_{\mathcal{C}}^{(2)} \cdot x_{\mathcal{C}}^{(2)} \equiv 0$ and, from the analysis of the triangle with sides Γ_1, Γ_2, G there results

$$\frac{G}{\sin \iota} = \frac{\Gamma_1}{\sin \iota_2}$$

one finds

$$w_{\mathcal{C}}^{(1)} \cdot x_{\mathcal{C}}^{(2)} = \frac{C_{\mathcal{C}}^{(1)} \cdot x_{\mathcal{C}}^{(2)}}{\Gamma_1} = \frac{C_{\mathcal{C}} \cdot x_{\mathcal{C}}^{(2)}}{\Gamma_1} = -\frac{G}{\Gamma_1} \sin \iota_2 \cos(g_2 + \nu_2) = -\sin \iota \cos(g_2 + \nu_2) .$$

Therefore,

$$\langle g_{\mathcal{C}} \rangle_{\ell_2} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\ell_2}{|r_1 w_{\mathcal{C}}^{(1)} - x_{\mathcal{C}}^{(2)}|} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\ell_2}{\sqrt{r_1^2 + 2r_1 a_2 \rho_2 \sin \iota \cos(g_2 + \nu_2) + a_2^2 \rho_2^2}} .$$

Recalling the expression of $\langle f_{\mathcal{C}} \rangle_{\ell_2}$ in (13), this proves Proposition 1.2, with $\Psi_{\mathcal{C}}$ as claimed. The identity (23) is a consequence of this and of the thesis (3) in Proposition 1.1. ∎

4.3 Proof of Theorem 1.3

Definition 4.3

⋆ Given a power series in the parameter ε

$$g_\varepsilon := \sum_{n=0}^{\infty} a_n \varepsilon^n$$

we denote as $\Pi_\varepsilon g_\varepsilon$ the even power series

$$\Pi_\varepsilon g_\varepsilon := \sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)!!}{(2m)!!} a_{2m} \varepsilon^{2m}$$

where $(-1)!! := 1$.

Remark 4.2 Observe that

⋆ $\Pi_\varepsilon a g_\varepsilon = a \Pi_\varepsilon g_\varepsilon$

⋆ $\Pi_\varepsilon g_{a\varepsilon} = \Pi_{a\varepsilon} g_{a\varepsilon} \qquad \forall a \in \mathbb{R}$

Let us define, for a given $\mathcal{C} \in \mathcal{O}$, the quantities

$$\widehat{x}_{\mathcal{C}}^{(2)} := \frac{x_{\mathcal{C}}^{(2)}}{a_2} \qquad \widehat{C}_{\mathcal{C}}^{(1)} := \frac{C_{\mathcal{C}}^{(1)}}{|C_{\mathcal{C}}^{(1)}|} , \qquad g_{\varepsilon,\mathcal{C}} := \frac{1}{|\varepsilon \widehat{C}_{\mathcal{C}}^{(1)} - \widehat{x}_{\mathcal{C}}^{(2)}|}$$

Proposition 4.2 Let $\mathcal{C} = (\Lambda_2, \ell_2, u, v) \in \mathcal{O}_{\mathrm{Herm}}$. Then the the following identity holds

$$\langle f_{\mathcal{C}} \rangle_{\delta_1 \ell_2} = \frac{1}{a_2} \Pi_\varepsilon \langle g_{\varepsilon,\mathcal{C}} \rangle_{\ell_2} \Big|_{\varepsilon = \frac{r_1}{a_2}} .$$

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The proof of Proposition 4.2 is based on the following technical result, which is proved in Section 5.

Lemma 4.3 *Let $r_1 > 0$, $\varphi_1 \in \mathbb{T}$, $N^{(1)}$, $x^{(2)} \in \mathbb{R}^3$, with $|N^{(1)}| = 1$. Define $\nu := x^{(2)} \times N^{(1)}$. Let $x^{(1)}(r_1, \varphi_1, N^{(1)}, x^{(2)})$ be such that $x^{(1)} \perp N^{(1)}$, $|x^{(1)}| = r_1$ and $\alpha_{N^{(1)}}(\nu, N^{(1)} \times x^{(1)}) = \varphi_1$. Then, the following identity holds*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\varphi_1}{|x^{(1)}(r_1, \varphi_1, N^{(1)}, x^{(2)}) - x^{(2)}|} = \frac{1}{r_2} \Pi_\varepsilon \frac{1}{|\varepsilon N^{(1)} - \widetilde{x}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{r_2}}$$

with $r_2 := |x^{(2)}|$, $\widetilde{x}^{(2)} := \frac{x^{(2)}}{r_2}$.

Proof of Proposition 4.2 Since any γ_1–double Kepler map is a φ_1–outer Kepler map, with $\varphi_1 = \gamma_1 + \nu_1$, where ν_1 is the true anomaly, and $\langle f_\mathcal{C} \rangle_{\gamma_1 \ell_2} = \langle f_\mathcal{C} \rangle_{\varphi_1 \ell_2}$, we prove the assertion inn the case that $\mathcal{C}$ is a φ_1–outer Kepler map.

We use Proposition 4.2 with $N^{(1)} := \widehat{C}_\mathcal{C}^{(1)}$. Then we have

$$\langle f_\mathcal{C} \rangle_{\varphi_1} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\varphi_1}{|x_\mathcal{C}^{(1)} - x_\mathcal{C}^{(2)}|} = \frac{1}{r_2} \Pi_\varepsilon \frac{1}{|\varepsilon \widehat{C}_\mathcal{C}^{(1)} - \widetilde{x}_\mathcal{C}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{r_2}}$$

where $\widetilde{x}^{(2)} = \frac{x^{(2)}}{r_2}$. Writing $r_2 = a_2 \rho_2$ and $\widehat{x}^{(2)} := \rho_2 \widetilde{x}^{(2)}$ and using Remark 4.2, we write the previous equality as

$$
\begin{aligned}
\langle f_\mathcal{C} \rangle_{\varphi_1} &= \frac{1}{a_2} \frac{1}{\rho_2} \Pi_\varepsilon \frac{1}{|\varepsilon \widehat{C}_\mathcal{C}^{(1)} - \widetilde{x}_\mathcal{C}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{r_2}} \\
&= \frac{1}{a_2} \Pi_\varepsilon \frac{1}{\rho_2} \frac{1}{|\varepsilon \widehat{C}_\mathcal{C}^{(1)} - \widetilde{x}_\mathcal{C}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{r_2}} \\
&= \frac{1}{a_2} \Pi_\varepsilon \frac{1}{|\varepsilon \rho_2 \widehat{C}_\mathcal{C}^{(1)} - \rho_2 \widetilde{x}_\mathcal{C}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{r_2}} \\
&= \frac{1}{a_2} \Pi_{\varepsilon \rho_2} \frac{1}{|\varepsilon \rho_2 \widehat{C}_\mathcal{C}^{(1)} - \widehat{x}_\mathcal{C}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{r_2}} \\
&= \frac{1}{a_2} \Pi_{\widehat{\varepsilon}} \frac{1}{|\widehat{\varepsilon} \widehat{C}_\mathcal{C}^{(1)} - \widehat{x}_\mathcal{C}^{(2)}|}\Big|_{\widehat{\varepsilon} = \frac{r_1}{a_2}}
\end{aligned}
$$

Therefore, writing ε instead of $\widehat{\varepsilon}$, after averaging with respect to ℓ_2, we have the thesis:

$$
\begin{aligned}
\langle f_\mathcal{C} \rangle_{\varphi_1 \ell_2} &= \frac{1}{a_2} \left\langle \Pi_\varepsilon \frac{1}{|\varepsilon \widehat{C}_\mathcal{C}^{(1)} - \widehat{x}_\mathcal{C}^{(2)}|}\Big|_{\varepsilon = \frac{r_1}{a_2}} \right\rangle_{\ell_2} = \frac{1}{a_2} \Pi_\varepsilon \left\langle \frac{1}{|\varepsilon \widehat{C}_\mathcal{C}^{(1)} - \widehat{x}_\mathcal{C}^{(2)}|} \right\rangle_{\ell_2}\Big|_{\varepsilon = \frac{r_1}{a_2}} \\
&= \frac{1}{a_2} \Pi_\varepsilon \langle g_{\varepsilon, \mathcal{C}} \rangle_{\ell_2}\Big|_{\varepsilon = \frac{r_1}{a_2}} . \quad \blacksquare
\end{aligned}
$$

4.4 Proof of Corollary 1.2

Recall the notations in (21).

Proof Proposition 4.1 and Corollary 1.1 imply that, for $\mathcal{C} \in \mathcal{O}^\star$,

$$\langle g_\mathcal{C} \rangle_{\ell_2} = \frac{1}{a_2}\Big[b_0(\varepsilon^2, 1) + \rho b_1(\varepsilon^2, 1) \mathcal{E}_\mathcal{C}^2 + \sigma b_1(\varepsilon^2, 1) \mathcal{I}_\mathcal{C}^2 + O\big((\mathcal{E}_\mathcal{C}^2, \mathcal{I}_\mathcal{C}^2)^2\big) \Big]$$

with $\varepsilon = \frac{r_1}{a_2}$. Therefore, if $\mathcal{C} \in \mathcal{O}_{\text{Herm}}^\star$, by Proposition 4.2, we have

$$\langle f_{\overline{\mathcal{C}}} \rangle_{\delta_1 \ell_2} = \frac{1}{a_2} \Pi_\varepsilon \Big[b_0(\varepsilon^2, 1) + \rho b_1(\varepsilon^2, 1) \mathcal{E}_\mathcal{C}^2 + \sigma b_1(\varepsilon^2, 1) \mathcal{I}_\mathcal{C}^2 + O\big((\mathcal{E}_\mathcal{C}^2, \mathcal{I}_\mathcal{C}^2)^2\big) \Big] ,$$

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where Π_ε is as in Definition 4.3. Replacing $\mathcal{E}_\mathcal{C}, \mathcal{I}_\mathcal{C}$ as in (22), and next of $\mathfrak{G}$ in the same equation,

$$
\begin{aligned}
\langle f_{\overline{\mathcal{C}}} \rangle_{\delta_1 \ell_2} &= \frac{1}{a_2} \Pi_\varepsilon \left[b_0(\varepsilon^2, 1) + \rho b_1(\varepsilon^2, 1) \frac{\Lambda_2^2 - \mathfrak{G}^2}{\Lambda_2^2} + \sigma b_1(\varepsilon^2, 1) \frac{\mathfrak{G}^2 - \mathrm{G}_2^2 \cos^2 \iota}{\Lambda_2^2} + \mathrm{O}\left((\mathcal{E}_\mathcal{C}^2, \mathcal{I}_\mathcal{C}^2)^2 \right) \right. \\
&= \frac{1}{a_2} \Pi_\varepsilon \left[b_0(\varepsilon^2, 1) + \rho b_1(\varepsilon^2, 1) \frac{\Lambda_2^2 - \mathrm{G}_2^2}{\Lambda_2^2} + \sigma b_1(\varepsilon^2, 1) \frac{\mathrm{G}_2^2 \sin^2 \iota}{\Lambda_2^2} \right. \\
&\quad + \left. \mathcal{O}_{dd,\varepsilon} \right] + \mathrm{O}_4\left(\mathrm{e}_2, \iota \right)
\end{aligned}
$$

where $\mathcal{O}_{dd,\varepsilon}$ stands for a power series in ε containing just odd powers. Here we have used

$$
\mathcal{E}_\mathcal{C}^2 \; , \; \mathcal{I}_\mathcal{C}^2 = \mathrm{O}_2(\mathrm{e}_2, \iota)
$$

which are immediate to be checked. Since $\mathcal{O}_{dd,\varepsilon}$ is annihilated by Π_ε, we have the thesis, with

$$
\beta_0 := \Pi_\varepsilon b_0(\varepsilon^2, 1) \Big|_{\varepsilon = \frac{\mathrm{r}_1}{a_2}} \qquad \beta_1 := \Pi_\varepsilon b_1(\varepsilon^2, 1) \Big|_{\varepsilon = \frac{\mathrm{r}_1}{a_2}} \cdot \quad \blacksquare
$$

5   Proof of Lemma 4.3

The proof of Lemma 4.3 is based on a technical and somewhat surprising property of the classical Legendre polynomials, which is Lemma 5.1 below.

We recall that the Legendre polynomials $P_n(t)$ are defined via the ε–expansion

$$\frac{1}{\sqrt{1 - 2\varepsilon t + \varepsilon^2}} = \sum_{n=0}^{\infty} P_n(t)\varepsilon^n \ .$$

Many notices on such classical polynomials may be found, e.g., in [17, Appendix B].

We shall prove that

Lemma 5.1 *Let $x \in \mathbb{R}$, $|x| \leq 1$, P_n the nth Legendre polynomial. Then,*

$$\frac{1}{2\pi} \int_{\mathbb{T}} P_n(\sqrt{1 - t^2} \cos\theta) d\theta = \delta_n P_n(t) \tag{24}$$

where

$$\delta_n = \begin{cases} (-1)^m \dfrac{(2m - 1)!!}{(2m)!!} & \text{if} \quad n = 2m \quad \text{is even} \\[2em] 0 & \text{if} \quad n \quad \text{is odd} \end{cases}$$

Lemma 5.2 *The even Legendre polynomials $P_{2m}(t)$ verify, for any $h = 0, \cdots, m$,*

$$\begin{aligned} D_\tau^h P_{2m}(0) &= (-1)^{m-h} \frac{h!}{(2h)!} \frac{(2m - 2h - 1)!!}{(2m - 2h)!!} \\ D_\tau^h P_{2m}(1) &= \frac{1}{2^h} \frac{(2m + 2h - 1)!!}{(2m - 1)!!} \frac{(2m)!!}{(2h)!!(2m - 2h)!!} \end{aligned} \tag{25}$$

where $\tau := t^2$. In particular, the following relation holds

$$(-1)^h \frac{(2h - 1)!!}{(2h)!!} D_\tau^h P_{2m}(0) = (-1)^m \frac{(2m - 1)!!}{(2m)!!} D_\tau^h P_{2m}(1) \ .$$

Proof We first prove the former formula in (25). Let $n \in \mathbb{N}$, $k = 0, \cdots, n$ with $n - k$ even. We have

$$D_t^k \frac{1}{\sqrt{\varepsilon^2 - 2t\varepsilon + 1}}\Big|_{t=0} = (2k - 1)!! \frac{\varepsilon^k}{(1 + \varepsilon^2)^{\frac{2k+1}{2}}} \ .$$

Therefore, denoting as Π_n the projection over the monomial ε^n,

$$\begin{aligned} D_t^k P_n(0) &= D_t^k \Big(\Pi_n \frac{1}{\sqrt{\varepsilon^2 - 2t\varepsilon + 1}} \Big)\Big|_{t=0} = \Pi_n \Big(D_t^k \frac{1}{\sqrt{\varepsilon^2 - 2t\varepsilon + 1}}\Big|_{t=0} \Big) = (2k - 1)!! \Pi_{n-k} \frac{1}{(1 + \varepsilon^2)^{\frac{2k+1}{2}}} \\ &= \frac{(2k - 1)!!}{((n - k)/2)!} D_\eta^{(n-k)/2} \frac{1}{(1 + \eta)^{\frac{2k+1}{2}}}\Big|_{\eta=0} = (-1)^{(n-k)/2} \frac{(k + n - 1)!!}{2^{(n-k)/2}((n - k)/2)!} \\ &= (-1)^{(n-k)/2} \frac{(k + n - 1)!!}{(n - k)!!} \ . \end{aligned} \tag{26}$$

Then the desired formula follows, taking $n = 2m$, $k = 2h$ and noticing that

$$D_\tau^h P_{2m}(0) = \frac{h!}{(2h)!} D_t^{2h} P_{2m}(0) \ .$$

The proof of the latter formula in (25) is a bit more complicate. We propose an algebraic one. First of all, we change variable

$$t = \sqrt{\tau} = \sqrt{1 - 2z} .$$

Since

$$D_\tau^h = \frac{(-1)^h}{2^h} D_z^h$$

we are definitely reduced to prove the following identity

$$
\begin{aligned}
D_z^h P_{2m}(\sqrt{1 - 2z})\Big|_{z=0} &= D_{2m,2h} := \frac{(-1)^h}{(2h)!}(2m - 2h + 2)(2m - 2h + 4)\cdots(2m) \\
&\times (2m + 1)(2m + 3)\cdots(2m + 2h - 1) .
\end{aligned} \tag{27}
$$

To this end, we let

$$g(\varepsilon, z) := \frac{1}{\sqrt{\varepsilon^2 - 2\varepsilon\sqrt{1 - 2z} + 1}} ,$$

so that (analogously to (26)) we may identify

$$D_z^h P_{2m}(\sqrt{1 - 2z})\Big|_{z=0} = \Pi_{2m} D_z^h g(\varepsilon, z)\Big|_{z=0} . \tag{28}$$

We introduce the auxiliary functions

$$g_{\alpha,\beta}(\varepsilon, z) = \frac{1}{\left(\varepsilon^2 - 2\varepsilon\sqrt{1 - 2z} + 1\right)^{\alpha/2}} \frac{1}{\left(1 - 2z\right)^{\beta/2}} \qquad \alpha,\ \beta \in \mathbb{R}$$

so that $g_{1,0} = g$. Observe that the linear space generated by such functions is closed under the derivative operation, since in fact

$$D_z g_{\alpha,\beta}(\varepsilon, z) = -\varepsilon\alpha g_{\alpha+2,\beta+1}(\varepsilon, z) + \beta g_{\alpha,\beta+2}(\varepsilon, z) .$$

More in general, by iteration, one finds

$$D_z^h g_{\alpha,\beta}(\varepsilon, z) = \sum_{j=0}^{h} c_j^{(h)} \varepsilon^j g_{\alpha+2j,\beta+2h-j}(\varepsilon, z) \tag{29}$$

where, from the identity

$$D_z^{h+1} g_{\alpha,\beta}(\varepsilon, z) = D_z\left(D_z^h g_{\alpha,\beta}\right)(\varepsilon, z)$$

one easily sees that the coefficients $c_j^{(h)}$, with $j = 0, \cdots, h$ satisfy the following recursion

$$
\begin{cases}
c_0^{(0)} = 1 \\[2mm]
c_j^{(h+1)} = -c_{j-1}^{(h)}(\alpha + 2j - 2) + (\beta + 2h - j)c_j^{(h)} \\[2mm]
h = 0,\ 1,\ \cdots ; \qquad j = 0,\ 1,\ \cdots h + 1 \\[2mm]
c_{-1}^{(h)} := 0 , \qquad c_{h+1}^{(h)} := 0
\end{cases}
$$

Let $\bar{c}_j^{(h)}$'s be the numbers defined by

$$
\begin{cases}
\bar{c}_0^{(0)} = 1 \\[2mm]
\bar{c}_j^{(h+1)} = -\bar{c}_{j-1}^{(h)}(2j - 1) + (2h - j)\bar{c}_j^{(h)} \\[2mm]
h = 0,\ 1,\ \cdots ; \qquad j = 0,\ 1,\ \cdots h + 1 \\[2mm]
\bar{c}_{-1}^{(h)} := 0 , \qquad \bar{c}_{h+1}^{(h)} := 0
\end{cases} \tag{30}
$$

corresponding to the case

$$\alpha = 1 \ , \qquad \beta = 0 \ .$$

Then specializing the formula (29) to this case, we find

$$
\begin{aligned}
D_z^h g(\varepsilon, z)\Big|_{z=0} &= D_z^h g_{1,0}(\varepsilon, z)\Big|_{z=0} \\
&= \sum_{j=0}^{h} \bar{c}_j^{(h)} \varepsilon^j g_{1+2j, 2h-j}(\varepsilon, z)\Big|_{z=0} = \sum_{j=0}^{h} \bar{c}_j^{(h)} \frac{\varepsilon^j}{(1-\varepsilon)^{1+2j}}
\end{aligned}
$$

Therefore, applying (28), we find the desired derivatives

$$
D_z^h P_{2m}(\sqrt{1-2z})\Big|_{z=0} = \sum_{j=0}^{h} C_{2m,j} \bar{c}_j^{(h)} \tag{31}
$$

with

$$
C_{2m,j} := \frac{(2m-j+1)(2m-j+2)\cdots(2m+j)}{(2j)!}
$$

In order to check (27), let $\mathcal{P}_{2h}(\mu)$, $\mathcal{Q}_{2j}(\mu)$ the *polynomials* in the *real* variable μ defined as the extensions of $D_{2m,2h}$, $C_{2m,2h}$ on the reals, `i.e.`, such that

$$
\mathcal{P}_{2h}(2m) = D_{2m,2h} \ , \qquad \mathcal{Q}_{2j}(2m) = C_{2m,j} \tag{32}
$$

and let

$$
\mathcal{D}_{2h}(\mu) := \sum_{j=0}^{h} \bar{c}_j^{(h)} \mathcal{Q}_{2j}(\mu)
$$

the analogous polynomial extending the right hand side of (31). We shall prove that

$$
\mathcal{D}_{2h}(\mu) = \mathcal{P}_{2h}(\mu) \qquad \forall \, \mu \in \mathbb{R} \ , \quad h = 0, 1, \cdots \ ,
$$

which clearly implies (27). Note that $\mathcal{D}_{2h}(\mu)$, $\mathcal{P}_{2h}(\mu)$ have degree $2h$; $\mathcal{P}_{2h}(\mu)$ vanishes at the odd integers $-(2h-1)$, $-(2h-3)$, $\cdots$, -1, and the even integers 0, 2, $\cdots$, $2h-2$, while the $\mathcal{Q}_{2j}(\mu)$'s have degree $2j$ and vanish at the integers $-j$, $-j+1$, $\cdots$, $j-1$. The last formula in (32) provides a decomposition of $\mathcal{D}_{2h}(\mu)$ on the basis of the $\mathcal{Q}_{2j}$'s. We then do the same for $\mathcal{P}_{2h}$, `i.e.`, we decompose

$$
\mathcal{P}_{2h} = \sum_{j=0}^{h} \hat{c}_j^{(h)} \mathcal{Q}_{2j} \ .
$$

We now need to show that

$$
\hat{c}_j^{(h)} = \bar{c}_j^{(h)} \qquad \forall \, h = 0, \ 1, \ \cdots \ ; \qquad j = 0, \ 1, \cdots, h \ . \tag{33}
$$

From the relations

$$
\mathcal{P}_{2h+2}(\mu) = -\frac{(\mu - 2h)(\mu + 2h + 1)}{2h+2} \mathcal{P}_{2h}(\mu)
$$

and

$$
-(\mu - 2h)(\mu + 2h + 1) = (2h - j)(2h + j + 1) - (\mu - j)(\mu + j + 1)
$$

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the following recursion rule among the coefficients immediately follows

$$
\begin{cases}
\hat{c}_0^{(0)} = 1 \\[2mm]
\hat{c}_j^{(h+1)} = -\dfrac{j(2j-1)}{h+1}\hat{c}_{j-1}^{(h)} + \dfrac{4h^2 - j^2 + 2h - j}{2h+2}\hat{c}_j^{(h)} \\[2mm]
h = 0,\ 1,\ \cdots\ ; \qquad j = 0,\ 1,\ \cdots h+1 \\[2mm]
\hat{c}_{-1}^{(h)} := 0\ , \qquad \overline{c}_{h+1}^{(h)} := 0
\end{cases}
\tag{34}
$$

Let

$$
\delta_j^{(h)} := \hat{c}_j^{(h)} - \overline{c}_j^{(h)}\ .
$$

The formulae in (30) and (34) imply

$$
\begin{cases}
\delta_0^{(0)} = 0 \\[2mm]
\delta_j^{(h+1)} = -\dfrac{(2j+1)(j-h-1)}{h+1}\delta_{j-1}^{(h)} + \dfrac{(2h-j)(j-1)}{2(h+1)}\delta_j^{(h)} \\[2mm]
h = 0,\ 1,\ \cdots\ ; \qquad j = 0,\ 1,\ \cdots h+1 \\[2mm]
\delta_{-1}^{(h)} := 0\ , \qquad \delta_{h+1}^{(h)} := 0
\end{cases}
$$

Those relations immediately enforce, by induction, $\delta_j^{(h)} \equiv 0$ for all h, j, and hence (33). ∎

Proof of Lemma 5.1 Let $\mathcal{Q}_{2m}(t)$ denote the left hand side of (24). Observe that the $\mathcal{Q}_{2m}(t)$'s are polynomials of degree m in $\tau := t^2$, as it follows from its definition and the fact that

$$
\frac{1}{2\pi} \int_0^\pi (\cos\theta)^{2h+1} d\theta = 0 \qquad \forall\, h \in \mathbb{N}\ .
$$

Since also the he even Legendre polynomials P_{2m}'s are polynomial of degree m in τ, we only need to show, e.g., that

$$
D_\tau^h \mathcal{Q}_{2m}\big|_{\tau=1} = (-1)^m \frac{(2m-1)!!}{(2m)!!} D_\tau^h P_{2m}\big|_{\tau=1} \qquad \forall\, h = 0,\ \cdots,\ m\ .
$$

The definition of Q_{2m} implies that, for $h = 1,\ \cdots,\ m$

$$
D_\tau^h Q_{2m}(1) = (-1)^h \overline{(\cos\theta)^{2h}} D_\tau^h P_{2m}(0) = (-1)^h \frac{(2h-1)!!}{(2h)!!} D_\tau^h P_{2m}(0) \qquad h = 0,\ \cdots,\ m
$$

where

$$
\overline{(\cos\theta)^{2h}} := \frac{1}{2\pi} \int_0^\pi (\cos\theta)^{2h} d\theta = \frac{(2h-1)!!}{(2h)!!}\ .
$$

Using Lemma 5.2, we find

$$
D_\tau^h Q_{2m}(1) = (-1)^m \frac{(2m-1)!!}{(2m)!!} D_\tau^h P_{2m}(1) \qquad h = 0,\ \cdots,\ m
$$

and hence the thesis follows. ∎

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Now we are ready for the

Proof of Lemma 4.3 Let us decompose

$$x^{(2)} = (x^{(2)} \cdot \mathrm{N}^{(1)})\mathrm{N}^{(1)} + x_{\perp}^{(2)}$$

where $x_{\perp}^{(2)} := x^{(2)} - (x^{(2)} \cdot \mathrm{N}^{(1)})\mathrm{N}^{(1)}$ is orthogonal to $\mathrm{N}^{(1)}$. Since $x^{(1)}$ is orthogonal to $\mathrm{N}^{(1)}$ and $|x_{\perp}^{(2)}| = \sqrt{|x^{(2)}|^2 - (x^{(2)} \cdot \mathrm{N}^{(1)})^2|} = r_2\sqrt{1 - (\widehat{x}^{(2)} \cdot \mathrm{N}^{(1)})^2}$, we have

$$x^{(1)} \cdot x^{(2)} = x^{(1)} \cdot x_{\perp}^{(2)} = |x^{(1)}||x_{\perp}^{(2)}|\cos\psi = r_1 r_2 \sqrt{1 - (\widehat{x}^{(2)} \cdot \mathrm{N}^{(1)})^2}\cos\psi$$

where ψ is the convex angle formed by $x^{(1)}$ and $x_{\perp}^{(2)}$. But ψ is related to φ_1 via

$$\psi = |\pi - \varphi_1|$$

therefore, $\cos\psi = -\cos\varphi_1$. This readily implies

$$|x^{(1)}(r_1, \varphi_1, \mathrm{N}^{(1)}, x^{(2)}) - x^{(2)}| = \sqrt{r_1^2 + 2r_1 r_2 \sqrt{1 - (\mathrm{N}^{(1)} \cdot \widehat{x}^{(2)})^2}\cos\varphi_1 + r_2^2}$$

We now use this in the expansion of the inverse distance

$$\frac{1}{\mathrm{D}(r_1, \varphi_1, \mathrm{N}^{(1)}, x^{(2)})} = \frac{1}{\sqrt{r_1^2 + 2r_1 r_2 \sqrt{1 - (\mathrm{N}^{(1)} \cdot \widehat{x}^{(2)})^2}\cos\varphi_1 + r_2^2}}$$

in terms of Legendre polynomials

$$\frac{1}{\mathrm{D}(r_1, \varphi_1, \mathrm{N}^{(1)}, x^{(2)})} = \frac{1}{r_2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{r_1}{r_2}\right)^n P_n\left(\sqrt{1 - \frac{(x^{(2)} \cdot \mathrm{N}^{(1)})^2}{r_2^2}}\cos\varphi_1\right).$$

To conclude, we only need to use Lemma 5.1, so that

$$\frac{1}{2\pi}\int_{\mathbb{T}} P_n\left(\sqrt{1 - \frac{(x^{(2)} \cdot \mathrm{N}^{(1)})^2}{r_2^2}}\cos\varphi_1\right)d\varphi_1 = \delta_n P_n\left(\frac{x^{(2)} \cdot \mathrm{N}^{(1)}}{r_2}\right),$$

which is a rewrite of the thesis. ∎

A   Notations, background, etc.

Generalities   Let $d = 2, 3$ the dimension of configuration space. The three–body problem is the $3d$ *degrees of freedom* dynamical system of three gravitational particles.

If $\overline{m}_0$, $\overline{m}_1$, $\overline{m}_2$ are the masses of the particles and $(\overline{p}^{(i)}, \overline{q}^{(i)}) \in \mathbb{R}^d \times \mathbb{R}^d$ the impulse–position canonical coordinates in the configuration space $\mathbb{R}^d$, the problem admits the Hamiltonian

$$\mathrm{H}_{3\mathrm{B}}(p, q) = \sum_{i=0}^{2} \frac{|\overline{p}^{(i)}|^2}{2\overline{m}_i} - \sum_{0 \leq i < j \leq 2} \frac{\overline{m}_i \overline{m}_j}{|\overline{q}^{(i)} - \overline{q}^{(j)}|} \ ,$$

where $p = (p^{(0)}, p^{(1)}, p^{(2)})$, $q = (q^{(0)}, q^{(1)}, q^{(2)})$. The relative phase space has dimension $6d$, or, equivalently, the system has $3d$ degrees of freedom, and is given by

$$\overline{\mathcal{M}}^{6d} = (\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) \times \left((\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) \setminus \overline{\mathrm{D}} \right)$$

with

$$\overline{\mathrm{D}} := \left\{ (\overline{q}^{(0)}, \overline{q}^{(1)}, \overline{q}^{(2)}) : \qquad \overline{q}^{(i)} = \overline{q}^{(j)} \quad \text{for some } i < j \right\} \ .$$

The Hamiltonian $\mathrm{H}_{3\mathrm{B}}$ possess two main symmetries: by translations:

$$(\overline{p}^{(i)}, \overline{q}^{(i)}) \to (\overline{p}^{(i)}, \overline{q}^{(i)} + a) \qquad a \in \mathbb{R}^d \ , \qquad i = 0, \ 1, \ 2$$

and by rotations

$$(\overline{p}^{(i)}, \overline{q}^{(i)}) \to (\mathcal{R}\overline{p}^{(i)}, \mathcal{R}\overline{q}^{(i)}) \qquad \mathcal{R} \in \mathrm{SO}(3) \qquad i = 0, \ 1, \ 2 \ .$$

Reducing such symmetries has the benefit of lowering the number of degrees of freedom, and hence considering simpler equations of motion.

The reduction of translations leads to a system with $2d$ degrees of freedom. Two substantially equivalent procedures for obtaining it are known in the literature: the reduction via *Jacobi coordinates* and the *heliocentric reduction*. We shall dwell with the latter. This consists into switching to the Hamiltonian, defined, on the $4d$–dimensional phase space

$$\overline{\mathcal{M}}^{4d} = (\mathbb{R}^d \times \mathbb{R}^d) \times \left((\mathbb{R}^d \times \mathbb{R}^d) \setminus \overline{\Delta} \right)$$

with

$$\overline{\Delta} := \left\{ (\overline{x}^{(1)}, \overline{x}^{(2)}) : \qquad \overline{x}^{(1)} = 0 \quad \text{or} \quad \overline{x}^{(2)} = 0 \quad \text{or} \quad \overline{x}^{(1)} = x^{(2)} \right\} \ .$$

as

$$\overline{\mathrm{H}}_{2\mathrm{P}}(\overline{y}, \overline{x}) \quad = \quad \overline{\mathrm{h}}_{\mathrm{Kep}}(\overline{y}, \overline{x}) - \overline{m}_1 \overline{m}_2 \overline{f}_{2\mathrm{P}}(\overline{y}, \overline{x})$$

where, if

$$\overline{\mathrm{m}}_i = \frac{\overline{m}_0 \overline{m}_i}{\overline{m}_i + \overline{m}_0} \qquad \overline{\mathfrak{M}}_i = \overline{m}_i + \overline{m}_0 \qquad i = 1, \ 2$$

are the so–called *reduced masses* and $\overline{y} = (\overline{y}^{(1)}, \overline{y}^{(2)})$, $\overline{x} = (\overline{x}^{(1)}, \overline{x}^{(2)})$, then

$$\overline{\mathrm{h}}_{2\mathrm{P}}(\overline{y}, \overline{x}) = \sum_{i=1}^{2} \overline{\mathrm{h}}_{2\mathrm{P}}^{(i)}(\overline{y}^{(i)}, \overline{x}^{(i)}) \ , \qquad \overline{f}_{2\mathrm{P}}(\overline{y}, \overline{x}) = \overline{f}_{2\mathrm{P},\mathrm{Newt}}(\overline{y}, \overline{x}) + \overline{f}_{2\mathrm{P},\mathrm{indir}}(\overline{y}, \overline{x}) \tag{35}$$

where

$$\overline{\mathrm{h}}_{2\mathrm{P}}^{(i)}(\overline{y}^{(i)}, \overline{x}^{(i)}) := \frac{|\overline{y}_{\mathcal{C}}^{(i)}|^2}{2\overline{\mathrm{m}}_i} - \frac{\overline{\mathfrak{M}}_i \overline{\mathrm{m}}_i}{|\overline{x}_{\mathcal{C}}^{(i)}|} \ , \qquad i = 1, \ 2 \ ,$$

and the two parts the perturbing function f_{2P} splits in, called *Newtonian–* (or *direct–*) and the *indirect part*, respectively, are

$$\overline{f}_{2P,\mathrm{Newt}}(\overline{y}, \overline{x}) := \frac{1}{|\overline{x}^{(1)} - \overline{x}^{(2)}|} \ , \qquad \overline{f}_{2P,\mathrm{indir}}(\overline{y}, \overline{x}) := -\frac{\overline{y}^{(1)} \cdot \overline{y}^{(2)}}{\overline{m}_0 \overline{m}_1 \overline{m}_2} \ .$$

The planetary problem   The special case when one of the masses is much greater than the others simulates a sub–system (**e.g.**, the system Sun–Jupiter–Saturn) of our Solar System, and hence the problem is also called *planetary*. It is customary, for the planetary problem, to "rescale" the masses via a small a–dimensional parameter μ, letting

$$\overline{m}_0 = m_0 \ , \qquad \overline{m}_i = \mu m_i \qquad i = 1,\ 2 \qquad \mu \ll 1$$

where m_0, m_1, m_2 have the same strength, and to switch to the Hamiltonian

$$\mathrm{H}_{2P}(y, x) := \frac{1}{\mu} \overline{\mathrm{H}}_{2P}(\mu y, x) \Big|_{\overline{m}_0 = m_0,\ \overline{m}_1 = \mu m_1,\ \overline{m}_2 = \mu m_2} \ .$$

The new Hamiltonian H_{2P}, which governs the motion of the "rescaled" coordinates $(y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)})$, splits as

$$\mathrm{H}_{2P} = \mathrm{h}_{2P} - \mu m_1 m_2 f_{2P} \tag{36}$$

where, if

$$\mathfrak{m}_i = \frac{m_0 m_i}{m_0 + \mu m_i} \qquad \mathfrak{M}_i = m_0 + \mu m_i \qquad i = 1,\ 2 \ ,$$

$\mathrm{h}_{\mathrm{Kep}}$, f_{2P}, often referred to as *Keplerian part*, and[1] *perturbing function*, and defined as $\overline{\mathrm{h}}_{2P}$, $\overline{f}_{2P}$ in (35), without the "bars". The Keplerian part governs the fast unperturbed motions of the planets as if they interacted gravitationally with a fixed center at the origin of the $x^{(i)}$– frames, while the perturbing term $-\mu m_1 m_2 f_{2P}$, of smaller strength, governs slow displacements of the unperturbed motions due to the planet–planet interaction.

Assuming, once forever, that the planet labeled as "1" is the "inner", and the other, labeled as "2", the "outer", the larger phase space for this problem is

$$\mathcal{M}^{4d} = (\mathbb{R}^d \times \mathbb{R}^d) \times U$$

with

$$U = \left\{ (x^{(1)}, x^{(2)}) : \qquad 1 < |x^{(1)}| < |x^{(2)}| \right\} \ .$$

However, in view of looking for stable motions for the planetary problem, it is customary to further restrict the domain U introducing the "negative unperturbed energies constraint", namely

$$\mathrm{h}_{2P}^{(i)}(y^{(i)}, x^{(i)}) < 0 \ .$$

As soon as this condition is verified, it is possible, and quite natural, to look at canonical systems of coordinates

$$\mathcal{C}: \qquad (\Lambda_1, \Lambda_2, \ell_1, \ell_2, p, q) \in \mathcal{A} \times \mathbb{T}^2 \times V \to (y_{\mathcal{C}}^{(1)}, y_{\mathcal{C}}^{(2)}, x_{\mathcal{C}}^{(1)}, x_{\mathcal{C}}^{(2)}) \in (\mathbb{R}^d)^4 \tag{37}$$

with $\mathcal{A} \subset \mathbb{R}_+^2$, $V \subset \mathbb{R}^{4d-4}$ open and connected, where each of the unperturbed Hamiltonians, is set in the well known "action–angle" form (1).

---

[1] We choose a different normalization for the perturbing function (**i.e.**, $-\mu m_1, m_2 f_{2P}$ at the place of the common μf_{2P}) then done in most of existing literature, better suited to our purposes.

Then three–body Hamiltonian (36) becomes

$$\mathrm{H}_{\mathcal{C}}(\Lambda_1, \Lambda_2, \ell_1, \ell_2, p, q) = \mathrm{h}_{\mathrm{Kep}}(\Lambda_1, \Lambda_2) - \mu m_1 m_2 f_{\mathcal{C}}(\Lambda_1, \Lambda_2, \ell_1, \ell_2, p, q)$$

where

$$\mathrm{h}_{\mathrm{Kep}}(\Lambda_1, \Lambda_2) := \sum_{i=1}^{2} \mathrm{h}_{\mathrm{Kep}}^{(i)}(\Lambda_i) , \quad f_{\mathcal{C}} := f_{2\mathrm{P}} \circ \mathcal{C} = f_{\mathcal{C},\mathrm{Newt}} + f_{\mathcal{C},\mathrm{indir}} .$$

The idea of considering generalized maps verifying (37)–(1) goes back to Nekhorossev [24], who called them *Kepler maps*. A closely related object is[2]

$$\langle f_{\mathcal{C}} \rangle_{\ell_1, \ell_2}(\Lambda_1, \Lambda_2, p, q) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\ell_1 d\ell_2}{|x_{\mathcal{C}}^{(1)} - x_{\mathcal{C}}^{(2)}|} . \tag{38}$$

Definition A.1

$\star$ We refer to maps verifying (1) for $i = 1, 2$ as *double Kepler maps*, and to the function (38), where $\mathcal{C}$ is a double Kepler map, as *doubly averaged Newtonian potential*.

More generally, one might consider, when the number of planets is more than two, n–*Kepler maps* and n–*averaged Newtonian potential* $\langle f_{\mathcal{C}} \rangle_{\ell}$. Examples of n–Kepler maps are: the classical Delaunay–Poincaré coordinates [14], Deprit coordinates [10] as modified in [25, 8], the RPS coordinates [25, 8], the Perihelia reduction [28]. For notices on these latter, probably less known, sets, see the review [27] and references therein.

Three facts about the doubly averaged Newtonian potential The doubly averaged Newtonian potential (38) exhibits, with one or another choice of $\mathcal{C}$, some features that here we recall.

I: Harrington's property A historical example of map in the spirit of Definition A.1 is the well known reduction of the nodes for the three–body problem, introduced by Jacobi and Radau in the XIX Century [20, 30]. This is not properly a double Kepler map as in Definition A.1 (since it is not a change of coordinates), but is, rather, a family of immersions

$$\mathcal{J}_{\mathrm{G}} : (\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, \ell_1, \ell_2, \mathfrak{g}_1, \mathfrak{g}_2) \in \mathcal{A} \times \mathbb{T}^4 \to (y, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \qquad \mathcal{A} \subset \mathbb{R}^4$$

parametrized by the constant value $\mathrm{G} := |\mathrm{C}|$ of the length of the total angular momentum, such that (1) is satisfied and the Hamilton equations are preserved.

We refer, for notices on this map, to the vast existing literature: see, e.g., [3, 31], or also [27] for a unitary point of view, in the framework of the paper.

We just mention that, in the 1980s, the family $\{\mathcal{J}_{\mathrm{G}}\}_{\mathrm{G} \in \mathbb{R}_+}$ has been lifted (and also extended to any number n of planets) to a canonical full–dimensional map defined on a positive measure set of the phase space, thanks to the work of F. Boigey and A. Deprit [6, 10], and, in [25], such an extended map (for some reason forgotten since then) has been rediscovered just in the form of a n–*Kepler map*.

Letting

$$f_{\mathcal{J}}(\Lambda_1, \Lambda_2, \ell_1, \ell_2, \Gamma_1, \Gamma_2, \mathfrak{g}_1, \mathfrak{g}_2; \mathrm{G}) := \big(f_{2\mathrm{P}} \circ \mathcal{J}_{\mathrm{G}} \big)(\Lambda_1, \Lambda_2, \ell_1, \ell_2, \Gamma_1, \Gamma_2, \mathfrak{g}_1, \mathfrak{g}_2) ,$$

we now consider the doubly averaged Newtonian potential in $\mathcal{J}$–coordinates:

$$\langle f_{\mathcal{J}} \rangle_{\ell_1, \ell_2} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\ell_1 d\ell_2}{|x_{\mathcal{J}}^{(1)} - x_{\mathcal{J}}^{(2)}|} .$$

---

[2]It is well known that the indirect part of the perturbing function (36) does not contribute to such a double average, which so consists just of the double average of the Newtonian part.

We expand this function in powers of the semi–major axes ratio $\alpha := \dfrac{a_1}{a_2}$

$$\langle f_{\mathcal{J}} \rangle_{\ell_1, \ell_2} = \frac{1}{a_2} \sum_{n=0}^{\infty} \alpha^n f_n(\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, \mathfrak{g}_1, \mathfrak{g}_2; G) . \tag{39}$$

The following fact is, in the paper, referred to as "Harrington property":

The second–order term f_2 does not depend on the anomaly of the perihelion $\mathfrak{g}_2$. Hence, depending on just one angle only, $\mathfrak{g}_1$, it is integrable (Harrington, 1969, [18]).

It should be recalled here that Harrington property has been observed to persist for all the maps mentioned in just after Definition A.1.

Harrington's property has been successfully used, in order to find bifurcations for the secular systems [18, 22], or existence of KAM tori with suitable properties [12, 32, 28], or instability for the so–called secular three–body problem [15].

The non–triviality of Harrington's property relies on the fact that (as it is not difficult to see) an expansion like (39) for *any* Newton–like doubly averaged potential

$$h_p = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\ell_1 d\ell_2}{|x_{\mathcal{J}}^{(1)} - x_{\mathcal{J}}^{(2)}|^p}$$

is expected to hold with coefficients of the form

$$f_n(\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, \mathfrak{g}_1, \mathfrak{g}_2; G) = \sum_{\substack{|m| \leq n \\ m - n \in 2\mathbb{Z}}} f_{nm}(\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, \mathfrak{g}_1; G) e^{im\mathfrak{g}_2} . \tag{40}$$

Namely, the (say) $\mathfrak{g}_2$–Fourier expansion of f_n is expected to include Fourier modes f_{nm} with m *having the same parity as n and $|m| \leq n$.* Therefore, Harrington property might also be stated as follows: even though the Fourier expansion with respect to $\mathfrak{g}_2$ of f_2 in the double average (39) is expected to include Fourier modes with $m = 0$ and $m = \pm 2$, as a matter of fact, in the case of the (double average of the) Newtonian potential, it includes just the mode with $m = 0$.

II: Herman resonance In this paragraph and the next one we deal with the more general planetary problem with n planets, where n is any number greater of equal than two, even though at the end we shall restrict to $n = 2$. For this latter case, the chosen normalizations are a bit different[3] from the ones of the previous sections. We are confident that this abuse will not generate confusion.

Let

$$\mathrm{H}_{n\mathrm{P}}(y, x) = \mathrm{h}_{n\mathrm{P}}(y, x) + \mu f_{n\mathrm{P}}(y, x) \tag{41}$$

be the planetary Hamiltonian[4] reduced by translations via the Heliocentric coordinates.

Let $\mathcal{P}_{oinc} = (\Lambda, \lambda, \mathrm{z})$, where

$$\Lambda \in \mathbb{R}^n , \ \lambda \in \mathbb{T}^n , \ \mathrm{z} = (\eta, \xi, \mathrm{p}, \mathrm{q}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$

---

[3]Compare (36) to (41).

[4]For a generic number n of planets, the expression of $\mathrm{h}_{n\mathrm{P}}$, $f_{n\mathrm{P}}$ are

$$\mathrm{h}_{n\mathrm{P}}(y, x) = \sum_{i=1}^n \mathrm{h}_{2\mathrm{P}}^{(i)}(y^{(i)}, x^{(i)}) , \quad f_{n\mathrm{P}} = f_{n\mathrm{P},\mathrm{newt}} + f_{n\mathrm{P},\mathrm{indir}}$$

with

$$f_{n\mathrm{P},\mathrm{Newt}} = - \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|x^{(i)} - x^{(j)}|} , \qquad f_{n\mathrm{P},\mathrm{indir}} = \sum_{1 \leq i < j \leq n} \frac{y^{(i)} \cdot y^{(j)}}{m_0} .$$

with $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$, `etc.`, denote the Poincaré coordinates[5] for a n–planet system, and let

$$\mathrm{H}_{\mathcal{P}_{oinc}} = \mathrm{h}_{\mathrm{Kep}}(\Lambda) + \mu f_{\mathcal{P}_{oinc}}(\Lambda, \lambda, \mathrm{z})$$

the Hamiltonian (41) written in Poincaré coordinates.

Let us consider, in particular, the n–averaged Newtonian potential

$$\langle f_{\mathcal{P}_{oinc}} \rangle_\lambda (\Lambda, \mathrm{z}) = \frac{1}{(2\pi)^2} \sum_{1 \le i < j \le n} m_i m_j \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|x^{(i)}_{\mathcal{P}_{oinc}} - x^{(j)}_{\mathcal{P}_{oinc}}|}$$

and let us look at the Taylor expansion of $\langle f_{\mathcal{P}_{oinc}} \rangle_\lambda (\Lambda, \mathrm{z})$ around $\mathrm{z} = 0 \in \mathbb{R}^{4n}$.

Certain symmetries of $\langle f_{\mathcal{P}_{oinc}} \rangle_\lambda$, known as *D'Alembert rules*, ensure that such an expansion includes only even terms in z, and begins as

$$\langle f_{\mathcal{P}_{oinc}} \rangle_\lambda (\Lambda, \mathrm{z}) = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot (\eta^2 + \xi^2) + \mathcal{Q}_v(\Lambda) \cdot (\mathrm{p}^2 + \mathrm{q}^2) + \mathrm{O}(4) \qquad (42)$$

where $\mathcal{Q}_h$, $\mathcal{Q}_v$ are suitable $n \times n$ matrices, acting separately on the η, ξ, p, q [3, 19, 13]. This clearly shows that the $\mathrm{z} = 0$ point is an elliptic equilibrium point to $\langle f_{\mathcal{P}_{oinc}} \rangle_\lambda$ for all Λ, and hence calls, following an idea by V. Arnold [3], for a Birkhoff normal form. Such normal form, developed at a sufficiently high order, would allow (and in fact allowed; see below) to prove, constructively, the existence of a positive measure set of quasi–periodic motions for the planetary problem. We refer to the vast literature [3, 21, 13, 25, 8] (see also [14, 26, 27, 9] for reviews.) for more notices.

It is a fact that the matrices $\mathcal{Q}_h$, $\mathcal{Q}_v$ above verify, identically, the two following identities:

$$\det \mathcal{Q}_v \equiv 0 , \qquad \mathrm{tr}(\mathcal{Q}_h + \mathcal{Q}_v) \equiv 0 .$$

They are often called, altogether, *secular degeneracies*, or *secular resonances*.

This latter name, *secular resonances*, is related to the fact that, in terms of the respective eigenvalues $\sigma_1, \cdots, \sigma_n$ of $\mathcal{Q}_h$ $\zeta_1, \cdots, \zeta_n$ of $\mathcal{Q}_v$, they may be written as

$$\zeta_n \equiv 0 , \qquad \sum_{i=1}^{n} \sigma_i + \sum_{i=1}^{n-1} \zeta_i \equiv 0 . \qquad (43)$$

The former of such identities, also called *Laplace resonance*, has been firstly noticed by V. Arnold [3]. He related it to the existence of three integrals of motions to $\langle f_{\mathcal{P}_{oinc}} \rangle_{\lambda_1 \lambda_2}$, namely the the three components C_1, C_2 and C_3 of the total angular momentum C. More precisely, Arnold mentioned that just the fact of having *two* non–commuting integrals, say C_1 and C_2, is the cause of such a degeneracy.

The latter, commonly referred to as *Herman resonance*, and surprisingly non mentioned in [3], has been remarked by M. Herman [19]. It has been investigated, from a computational point of view, by K. Abdullah and A. Albouy [1].

III: Strange symmetries in the planetary torsion The secular resonances (43) represented, for about fifty years, an obstacle to construction of the Birkhoff normal form around the co–circular, co–planar equilibrium, for a generic number n of planets. This has been at the end found in [25, 8], via the production of a new set of canonical coordinates, named RPS coordinates. These coordinates are much similar in their structure, to Poincaré coordinates, *but*, as a strong characterization of them, include a couple (say, (p_n, q_n)) of integrals among the conjugated couple of coordinates, which so turn to be both cyclic.

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[5]Poincaré coordinates have been introduced by H. Poincaré at the end of the XIX Century [29], with the precise intention of regularizing the classical Delaunay coordinates in the neighborhood of almost circular, almost planar motions. A comprehensive definition of such coordinates may be found, `e.g.`, in [3, 14, 7, 28], (but also elsewhere).

The new Hamiltonian
$$H_{RPS} = h_{Kep}(\Lambda) + \mu f_{RPS}(\Lambda, \lambda, z)$$

where
$$\Lambda \in \mathbb{R}^n \ , \ \lambda \in \mathbb{T}^n \ , \ z = (\eta, \xi, p, q) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$$

has just one degree of freedom less than the old one, and a pleasant common feature with it.

In fact, the new n–averaged Newtonian potential $\langle f_{RPS} \rangle_\lambda$ retains *D'Alembert rules*, and hence, in particular affords an expansion completely analogue to (42), just with (the same $n \times n$ matrix $\mathcal{Q}_h$) a new $(n-1) \times (n-1)$ matrix $\overline{\mathcal{Q}}_v$. Moreover, D'Alembert rules allow, despite of Herman resonance (which is still there), for a $2n-1$ degrees of freedom Birkhoff normal form which turns to satisfy any KAM non–degeneracy.

What we aim to discuss here is the aspect of the second–order Birkhoff invariants matrix ("torsion") τ.

Let us consider the case $n = 2$. In this situation, τ is a 3×3 matrix given, at a first meaningful approximation, by

$$\tau = m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & \frac{3}{\Lambda_1^2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} & \frac{9}{4\Lambda_1\Lambda_2} \\ \frac{3}{\Lambda_1^2} & \frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix} + \text{h.o.t.}$$

Complete expressions of the entries τ_{ij} of τ are also available (see [8, Equations (8.1) and (8.6)]). The following identities among the entries of τ

$$\tau_{12} = -\tau_{23} + \text{h.o.t.} \qquad \text{and} \qquad \tau_{11} = -\tau_{33} + \text{h.o.t.} \tag{44}$$

are effectively remarkable.

B The two–centre problem

The two–center problem is the dynamical system of a gravitational particle that interacts with two fixed stars. Its Hamiltonian is usually written as

$$H_{2c} = \frac{|y|^2}{2} - \frac{m_+}{|x + x_0|} - \frac{m_-}{|x - x_0|} \ , \tag{45}$$

where $(y, x) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{\pm x_0\}$ are the impulse–position coordinates of the moving particle, $m = 1$ its mass, $m_\mp$ the masses of the stars, $\pm x_0$ their respective positions, and, finally $|\cdot|$ denotes the usual Euclidean norm.

The two–center problem is notoriously integrable, for having (when $x_0 \in \mathbb{R}^d$ is regarded as a constant vector) d degrees of freedom, and d independent, commuting integrals of motion.

$d - 1$ of such integrals are easy to be found: they are, with $C := x \times y$,

$$E := H_{2c} \ (d = 2, \ 3) \qquad \text{and} \qquad \Theta := \frac{x_0 \cdot C}{|x_0|} \ (d = 3) \ . \tag{46}$$

Such integrals are not precisely related to the features of the Newtonian potential. Indeed, one can consider generalized versions of the problem, like

$$H_{2c,p} = \frac{|y|^2}{2} - \frac{m_+}{|x + x_0|^p} - \frac{m_-}{|x - x_0|^p} \qquad \alpha \in \mathbb{R}_+$$

and have the energy $E_p = H_{2c,p}$ and Θ preserved again.

On the contrary, the third integral of H_{2c} has not an extension to $p \neq 1$. This has the expression

$$\mathcal{N} = |x \times y|^2 + (x_0 \cdot y)^2 + 2x \cdot x_0 \Big(\frac{m_+}{|x + x_0|} - \frac{m_-}{|x - x_0|} \Big) \ . \tag{47}$$

Observe that, when the two stars merge, `e.g.`, $x_0 = 0$, $\mathcal{N}$ reduces to $|C|^2$.

It is not easy to find, in the literature, a Hamiltonian discussion of the problem including a derivation of the integral $\mathcal{N}$ as above (for a Hamiltonian treatise see, `e.g.`, [4]). Therefore, for sake of completeness, we include, in the next section, a revisitation of the integration of H_{2c}, suited to our purposes.

B.1 Integration of H_{2c} (revisited)

We focus on the $d = 3$–case. For $d = 2$ it is sufficient to disregard the quantities Z Θ, z, ϑ below.

We regard H_{2c} as a Hamiltonian in the canonical coordinates (y_0, x_0), (y, x), with $y_0 \in \mathbb{R}^3$ conjugated to x_0, and cyclic. The associated canonical form being the standard one

$$\omega = dy_0 \wedge dx_0 + dy \wedge dx \ .$$

We shall perform the integration of H_{2c}, starting with a system of canonical coordinates, that we call $\mathcal{P}$–coordinates, that have been introduced in [26]. In this section, we recall their definition.

Let $(k^{(1)}, k^{(2)}, k^{(3)})$ be a prefixed orthonormal frame in $\mathbb{R}^3$.

Denote as

$$C_0 := x_0 \times y_0 \qquad C := x \times y \qquad C_{tot} := C_0 + C$$

For $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a vector w, let $\alpha_w(u, v)$ denote the positively oriented angle (mod 2π) between u and v, as seen from w according to the "right hand rule". Define the "nodes"

$$\nu_0 := k^{(3)} \times C_{tot} \ , \quad \nu_1 := C_{tot} \times x_0 \ , \quad \nu_2 := x_0 \times C \ .$$

Let $\mathcal{M}^{12}_\star$ denote the subset of $(\mathbb{R}^3)^4$ where C_{tot}, C, x_0, x, ν_0, ν_1 and ν_2 simultaneously do not vanish. On $\mathcal{M}^{12}_\star$ define a canonical map

$$\mathcal{P}^{-1}: \quad (y_0, x_0, y, x) \in \mathcal{M}^{12}_\star \to (Z, G, z, g, R_0, r_0, \Theta, \vartheta, R, \Phi, r, \varphi)$$

via the following formulae[6]

$$\mathcal{P}^{-1}: \quad \begin{cases} Z := C_{tot} \cdot k^{(3)} \\[2mm] G := |C_{tot}| \\[2mm] \Theta := \dfrac{C \cdot x_0}{|x_0|} \\[2mm] R_0 := \dfrac{y_0 \cdot x_0}{|x_0|} \\[2mm] R := \dfrac{y \cdot x}{|x|} \\[2mm] \Phi := |C| \end{cases} \qquad \begin{cases} z := \alpha_{k^{(3)}}(k^{(1)}, \nu_0) \\[2mm] g := \alpha_{C_{tot}}(\nu_0, \nu_1) \\[2mm] \vartheta := \alpha_{x_0}(\nu_1, \nu_2) \\[2mm] r_0 := |x_0| \\[2mm] r := |x| \\[2mm] \varphi := \alpha_C(\nu_2, x) \end{cases} \tag{48}$$

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[6]Even though $\mathcal{P}$ has been already presented in Section 3.1, due to a difference of notations here, we prefer, for sake of clarity, to rewrite it here once again. The reader will forgive some abusive overlap of notation (`e.g.`, C, C_{tot} here correspond to C_2, C of Section 3.1, `etc.`).

Note that the quantity Θ in (46) is a canonical action of $\mathcal{P}$ and that the coordinates (48) provide a reduction of the angular momentum which is *regular* for planar motions (the planar case, `i.e.`, $C_0 \parallel C \parallel C_{tot}$, corresponds to $\Theta = 0$ and $\vartheta = \pi$).

We denote as

$$\mathcal{R}_1(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}, \qquad \mathcal{R}_3(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition B.1 ([26]) *The map $\mathcal{P}^{-1}$ in (48) is invertible on $\mathcal{M}_\star^{12}$ The inverse map has the following analytical expression:*

$$\mathcal{P}: \begin{cases} x_0 = \mathcal{R}_3(z)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\begin{pmatrix} 0 \\ 0 \\ r_0 \end{pmatrix} \\[2em] y_0 := \dfrac{R_0}{r_0}x_0 + \dfrac{1}{r_0^2}C_0 \times x_0 \\[2em] x = \mathcal{R}_3(z)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} \\[2em] y = \mathcal{R}_3(z)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\begin{pmatrix} R\cos\varphi - \frac{\Phi}{r}\sin\varphi \\ R\sin\varphi + \frac{\Phi}{r}\cos\varphi \\ 0 \end{pmatrix} \end{cases}$$

where, if i, i_1, $i_2 \in (0,\pi)$ are defined by

$$\cos i = \frac{Z}{G}, \quad \cos i_1 = \frac{\Theta}{G}, \quad \cos i_2 = \frac{\Theta}{\Phi}$$

and C_{tot}, C by

$$C_{tot} := \mathcal{R}_3(z)\mathcal{R}_1(i)\begin{pmatrix} 0 \\ 0 \\ G \end{pmatrix} \qquad C := \mathcal{R}_3(z)\mathcal{R}_1(i)\mathcal{R}_3(g)\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2)\begin{pmatrix} 0 \\ 0 \\ \Phi \end{pmatrix}$$

then

$$C_0 := C_{tot} - C.$$

Proposition B.2 ([26]) *$\mathcal{P}$ preserves the standard Liouville 1–form $\lambda = \sum_{i=1}^{6} P_i dQ_i$.*

In terms of the coordinates $\mathcal{P}$, the scalar product $x_0 \cdot x$ takes the form

$$x_0 \cdot x = -r_0 r \sqrt{1 - \frac{\Theta^2}{\Phi^2}}\cos\varphi$$

and so H_{2c} in (45) becomes

$$H_{2c} = \frac{R^2}{2} + \frac{\Phi^2}{2r^2} - \frac{m_+}{r_+} - \frac{m_-}{r_-} \tag{49}$$

where

$$r_\pm^2 := r_0^2 \mp 2r_0 r \sqrt{1 - \frac{\Theta^2}{\Phi^2}}\cos\varphi + r^2.$$

H_{2c} has now two degrees of freedom, related to the motion of the two conjugated couples (R, r) and (Φ, φ).

At this point, one classically replaces the coordinates (R, r) and (Φ, φ) with new canonical coordinates (p_λ, λ), (p_μ, μ) where

$$\lambda = \frac{r_+ + r_-}{2r_0} \qquad \mu = \frac{r_+ - r_-}{2r_0} \ .$$

The expressions of the conjugated momenta p_λ, p_μ are found taking the inverse of these ones

$$r_+ = r_0(\lambda + \mu) \qquad r_- = r_0(\lambda - \mu) \tag{50}$$

and than squaring and summing, or subtracting. This gives

$$r = r_0\sqrt{\lambda^2 + \mu^2 - 1} \qquad \varphi = \cos^{-1}\left(-\frac{\lambda\mu}{\sqrt{\lambda^2 + \mu^2 - 1}\sqrt{1 - \frac{\Theta^2}{\Phi^2}}}\right) \tag{51}$$

Then one considers the generating function

$$\begin{aligned}
S(\Phi, \Theta, R_0, R, \lambda, \mu, \hat{r}_0, \hat{\vartheta}) \ = \ & \Theta\hat{\vartheta} + R_0\hat{r}_0 + R\hat{r}_0\sqrt{\lambda^2 + \mu^2 - 1} \\
& + \ \int^\Phi d\Phi \cos^{-1}\left(-\frac{\lambda\mu}{\sqrt{\lambda^2 + \mu^2 - 1}\sqrt{1 - \frac{\Theta^2}{\Phi^2}}}\right)
\end{aligned}$$

This leaves the coordinates

$$\Theta = \hat{\Theta} \qquad r_0 = \hat{r}_0$$

unvaried (therefore, we shall not change their names); shifts in an inessential way (since they do not appear into H_{2c}) the coordinates ϑ, R_0. Taking the derivatives with respect to λ, μ, one finds

$$\begin{cases}
p_\lambda = \dfrac{r_0\lambda R}{\sqrt{\lambda^2 + \mu^2 - 1}} - \dfrac{\mu\sqrt{(1 - \mu^2)(\lambda^2 - 1)\Phi^2 - (\lambda^2 + \mu^2 - 1)\Theta^2}}{(\lambda^2 + \mu^2 - 1)(\lambda^2 - 1)} \\[4mm]
p_\mu = \dfrac{r_0\mu R}{\sqrt{\lambda^2 + \mu^2 - 1}} + \dfrac{\lambda\sqrt{(1 - \mu^2)(\lambda^2 - 1)\Phi^2 - (\lambda^2 + \mu^2 - 1)\Theta^2}}{(\lambda^2 + \mu^2 - 1)(1 - \mu^2)}
\end{cases}$$

whence, taking the inverse with respect to R, Φ

$$\begin{cases}
R = \dfrac{\lambda(\lambda^2 - 1)p_\lambda + \mu(1 - \mu^2)p_\mu}{r_0(\lambda^2 - \mu^2)\sqrt{\lambda^2 + \mu^2 - 1}} \\[4mm]
\Phi^2 = \dfrac{(\lambda p_\mu - \mu p_\lambda)^2(\lambda^2 - 1)(1 - \mu^2)}{(\lambda^2 - \mu^2)} + \dfrac{\lambda^2 + \mu^2 - 1}{(1 - \mu^2)(\lambda^2 - 1)}\Theta^2
\end{cases}$$

Replacing these expressions and the one for r_+, r_-, r in (50), (51) into the Hamiltonian H_{2c} in (49), one finds the classical expression in Liouville coordinates

$$H_{2c} = \frac{p_\lambda^2(\lambda^2 - 1)}{2r_0^2(\lambda^2 - \mu^2)} + \frac{p_\mu^2(1 - \mu^2)}{2r_0^2(\lambda^2 - \mu^2)} + \frac{\Theta^2}{2r_0^2(\lambda^2 - \mu^2)}\left(\frac{1}{1 - \mu^2} + \frac{1}{\lambda^2 - 1}\right) - \frac{(m_+ + m_-)\lambda - (m_+ - m_-)\mu}{r_0^2(\lambda^2 - \mu^2)} \ .$$

Then one sees that equation

$$H_{2c} - E = 0$$

splits as

$$\mathcal{F}^{(\mu)}(p_\mu, \mu, \Theta, E, r_0) - \mathcal{F}^{(\lambda)}(p_\lambda, \lambda, \Theta, E, r_0) = 0 \tag{52}$$

where

$$\mathcal{F}^{(\mu)} = p_\mu^2(1-\mu^2) + \frac{\Theta^2}{1-\mu^2} + 2(m_+ - m_-)\mu + 2r_0^2\mu^2\mathrm{E}$$

$$\mathcal{F}^{(\lambda)} = -p_\lambda^2(\lambda^2-1) - \frac{\Theta^2}{\lambda^2-1} + 2(m_+ + m_-)\lambda + 2r_0^2\lambda^2\mathrm{E} \ .$$

Equation (52) implies then that $\mathcal{F}^{(\mu)}(p_\mu, \mu, \Theta, \mathrm{E}, r_0) = \mathcal{N}^{(\mu)}(\Theta, \mathrm{E}, r_0)$ is actually independent of (p_μ, μ); $\mathcal{F}^{(\lambda)}(p_\mu, \mu, \Theta, \mathrm{E}, r_0) = \mathcal{N}^{(\lambda)}(\Theta, \mathrm{E}, r_0)$ is actually independent of (p_λ, λ), and, a fortiori, since the partial derivatives of $\mathcal{N}^{(\mu)}$, $\mathcal{N}^{(\lambda)}$ depend explicitly on μ, λ, there must exists a $\mathcal{N} \in \mathbb{R}$ such that

$$\mathcal{F}^{(\mu)} = \mathcal{F}^{(\lambda)} = \mathcal{N}$$

where

$$\begin{aligned}
\mathcal{N} &= \frac{1}{2}(\mathcal{F}^{(\mu)} + \mathcal{F}^{(\lambda)}) \\
&= \frac{p_\mu^2}{2}(1-\mu^2) - \frac{p_\lambda^2}{2}(\lambda^2-1) + \frac{\Theta^2}{2}\Big(\frac{1}{1-\mu^2} - \frac{1}{\lambda^2-1}\Big) + m_+(\lambda+\mu) + m_-(\lambda-\mu) \\
&\quad + 2r_0^2(\lambda^2+\mu^2) \ .
\end{aligned}$$

After some elementary computations, one finds the expression of $\mathcal{N}$ in terms of the coordinates $\mathcal{P}$ is

$$\mathcal{N} = \Phi^2 + r_0^2(1 - \frac{\Theta^2}{\Phi^2})(-\mathrm{R}\cos\varphi + \frac{\Phi}{r}\sin\varphi)^2 - 2rr_0\cos\varphi\sqrt{1 - \frac{\Theta^2}{\Phi^2}}\Big(\frac{m_+}{r_+} - \frac{m_-}{r_-}\Big) \ .$$

While, in terms of the coordinates (y_0, x_0), (y, x), $\mathcal{N}$ has the expression in (47).

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