On the mathematical error of Aspect/Bell, and its resolution

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Abstract

Long claimed violations of Bell's inequality by the predictions of quantum mechanics are found to arise from the neglect of symmetric functional relations among the products of paired photon polarization observations. Bruno de Finetti's fundamental theorem of probability is used to derive a 4-dimensional coherent space of expectations that quantum theory actually supports and which respect the inequality everywhere. Aspect's empirical estimations which have been understood to support the violations are shown to depend on the same oversight. Their corrected versions are stable with simulation data, yielding estimates well within Bell's recognized bounds. The error is fundamental, impinging on all subsequent experimental results from more sophisticated experimental procedures.

1 Introduction

The apparent violations of standard probability inequalities by stochastic predictions of quantum mechanics have had fundamental consequences for the understanding of QM theory. First characterised in the works of John Bell (1964, 1966), the violations are widely considered to undermine Einstein's view that "hidden variables" must underlie scientific uncertainty which is codified in quantum probabilities, and to identify that such probabilities follow laws that are different from those pertinent to the mundane world of classical scale. Reformulation of the violation structure by Clauser, Horne, Shimony and Holt (1969) made the challenging results more easily amenable to experimental confrontation and empirical assessment. The seminal research of Alain Aspect (1982) which seemed to validate the violations in optical experiments have led to increasingly more complex experimental designs for investigating related matters.

Bell had begun his analytical investigations with the expectation of confirming the hidden variables interpretation of quantum theory proposed in the controversial article of Einstein, Podalsky, and Rosen (1935). Although he was surprised by the conundrum his results raised, he trusted their mathematical derivation and he sought to explain them conceptually throughout the years of his subsequent research. Nonetheless, he remained somewhat leery of it all, continuing (Bell, 1987) to voice a suspicion that the implied boundary between relevant laws of physical behaviour at quantum and classical scales would eventually be eliminated.

This present article provides grounds for dismissing the considered boundary as unwarranted. However, it attends exclusively to the mathematical error in the formulation of Aspect/Bell and its resolution, leaving interpretative discussion of related matters to other venues. The entire presentation relies merely on results of computational linear algebra accessible to university level understanding. In brief, we identify a mathematical error in the derivation of the considered violations as it is ably presented in the instructive review of Aspect (2002), and we display a resolution that relies on the constructive probabilistic methods of Bruno de Finetti (1974, 1975). The error derives from a neglect of symmetric functional relations among the polarization products of four conceivable optical experiments on a single pair of photons.

The experimental setup is reviewed in Section 2, followed by a presentation of Aspect's account of the situation in Section 3. The neglected functional relations among polarization products are exposed in Section 4 using a matrix that clarifies the mathematical structure of all empirical considerations relevant to the experiments under consideration. The heart of the resolution appears in Section 5, with several subsections. These firstly review the fundamental theorem of probability as specified by Bruno de Finetti, and then apply it to the particulars of the Aspect/Bell problem. The theorem provides the result that QM probabilities do not support a violation of Bell's inequality at all. Rather they support a 4-D configuration of probability distributions whose every member honours all relevant inequalities of probability theory. We further present an intriguing display of this polytope geometrically as it appears in three dimensions via slices along its four axes. Section 6 reassesses the empirical work of Aspect, and displays how its content would be transformed by the recognition of the symmetric functional relations.

2 An Experimental Setup for Bell's Inequality

We shall follow the context of Aspect's treatment of an optical variant of the EPR experiment. The original discussions were couched in terms of observations of spins of paired electrons. Although specific algebraic details differ for the two types of experimental situation, the conclusions we reach here would be identical.

An experiment is conducted on a pair of photons traveling in opposite directions along an axis, $\mathbf{z}$, from a common source. The direction of photon A toward detector A on the left is opposite to the direction its paired photon B travels toward detector B on the right: $\mathbf{z}_A = -\mathbf{z}_B$. At the end of their respective journeys, the paired photons pass through polarizers that are angled in directions $\mathbf{a}^*$ and $\mathbf{b}^*$ with respect to these incident photons, respectively. The subsequent behaviour of the photons, detected either as reflected or absorbed by their polarizers, is registered by the statistical indicators $A(\mathbf{a}^*)$ and $B(\mathbf{b}^*)$. These are valued as $+1$ if detection occurs parallel to the polarizer and as -1 if perpendicular.

There are two possible settings for the polarization directions $\mathbf{a}^*$ and $\mathbf{b}^*$ at the opposite ends of the emission line: $\mathbf{a}^*$ may be either of direction $\mathbf{a}$ or direction $\mathbf{a}'$, and $\mathbf{b}^*$ may be either of $\mathbf{b}$ or $\mathbf{b}'$. Experimental choices of the two directions yields a specific *relative angle* between them at A and B in any given experiment. Using Aspect's notation that parentheses around a pair of directions denotes the relative angle between them, the experimental detection angle settings $(\mathbf{a}^*, \mathbf{z}_A)$ and $(\mathbf{b}^*, \mathbf{z}_B)$ imply the *relative* angle between polarizers as $(\mathbf{a}^*, \mathbf{b}^*)$. Bell's inequality is relevant to this context in which the two photon polarization directions can be paired at any one of four distinct relative angles, denoted by the parenthetic pairs $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}', \mathbf{b})$, or $(\mathbf{a}', \mathbf{b}')$.

A feature crucial to the supposed violation of the inequality is that it pertains to experimental results conducted with a *single photon pair*. Although four distinct relative polarization angles can be considered for the experiment, only one of them will actually be prepared for empirical analysis. Nonetheless, as any of these experimental setups is possible, the prescriptions of quantum mechanics can be recognized as relevant to any of them. When *considering* all four possible experimental arrangements, the presumption of "locality" of experimental activity holds that the observation of indicator A in an $(\mathbf{a}, \mathbf{b})$ setup must be identical to its observation in an experimental $(\mathbf{a}, \mathbf{b}')$ setup. For the determination of polarization behaviour of the photon at A is understood not to depend on whether the polarization direction for the paired photon observed at detector B is designed to be $\mathbf{b}$ or $\mathbf{b}'$. After all, activity at B occurs some distance away from the determination at A, and the photons themselves are each traveling half of this distance in opposite directions at the speed of light. Thus, the statistical recording of A in an experimental design $(\mathbf{a}, \mathbf{b})$ is denoted merely by $A(\mathbf{a})$, and that at B by $B(\mathbf{b})$.

3 Examining Aspect's Account

Aspect's (2002) presentation begins with the quantity denoted by $s(\lambda)$ in his equation (21):

$$
\begin{aligned}
s(\lambda) \;&\equiv\; A(\lambda,\mathbf{a})\,B(\lambda,\mathbf{b}) \;-\; A(\lambda,\mathbf{a})\,B(\lambda,\mathbf{b}') \;+\; A(\lambda,\mathbf{a}')\,B(\lambda,\mathbf{b}) \;+\; A(\lambda,\mathbf{a}')\,B(\lambda,\mathbf{b}')\,, \;\; \text{for } \lambda \in \Lambda\,, \\
&=\; A(\lambda,\mathbf{a})\,[B(\lambda,\mathbf{b}) - B(\lambda,\mathbf{b}')] \;+\; A(\lambda,\mathbf{a}')\,[B(\lambda,\mathbf{b}) + B(\lambda,\mathbf{b}')] \qquad (1) \\
&=\; B(\lambda,\mathbf{b})\,[A(\lambda,\mathbf{a}) + A(\lambda,\mathbf{a}')] \;-\; B(\lambda,\mathbf{b}')\,[A(\lambda,\mathbf{a}) - A(\lambda,\mathbf{a}')]\;\;.
\end{aligned}
$$

Components of the vector λ are meant to be numerical indicators of proposed "hidden variables" that may be involved in the determination of the photons' behaviour, and Λ is the space of their possible values. Issues regarding hidden variables are irrelevant to the limited focus of this review, for they are eliminated at this point by assessing the expectation of $s(\lambda)$ with respect to a density $\rho(\lambda)$ which is rotationally invariant. This yields the central equation,

$$
E[s(\lambda)] \;=\; E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b})] - E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b}')] + E[A(\lambda,\mathbf{a}')B(\lambda,\mathbf{b})] + E[A(\lambda,\mathbf{a}')B(\lambda,\mathbf{b}')]. \quad (2)
$$

Aspect uses the simpler notation $E(\mathbf{a},\mathbf{b})$ for the term designated in (2) as $E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b})]$, which we shall now reduce to $E[A(\mathbf{a})B(\mathbf{b})]$. However his designation can be confusing, because the notation $(\mathbf{a},\mathbf{b})$ has already been used to define the relative angle between the detector directions $\mathbf{a}$ and $\mathbf{b}$. Also worthy of remark is that a well-known equation identifies the correlation (product moment) $E[A(\mathbf{a})B(\mathbf{b})]$ as equivalent to $2\,P[A(\mathbf{a}) = B(\mathbf{b})] - 1$ in this context. For this reason we refer to QM-motivated assertions equivalently as "expectations" or as "probabilities". For even deeper reasons, de Finetti refers to both such assessments as "previsions".

If the first defining line of equation (1) were meant to pertain to actual experimental results on four different photon pairs, the value of $s(\lambda)$ might equal any integer from -4 to $+4$. Each summand polarization product might equal $+1$ or -1. In this case the factorization lines that follows do not hold, for the value of $A(\mathbf{a})$ observed for one photon pair along with $B(\mathbf{b})$ need not equal the same value as that observed for another photon pair along with $B(\mathbf{b}')$. However, the second and third lines assure us that when pertinent to four distinct "thought experiments" on the *same* pair of photons, the value of $s(\lambda)$ can equal only -2 or $+2$. One of the bracketed terms in the second or third lines must equal zero, while the other equals ± 2. Thus, this limitation.

We now address the QM-motivated probability assessments for the exact problem assessed by Aspect and Bell, as they pose it. It is presumed that all the polarization observations used to define the quantity $s(\lambda)$ in equation (1) involve the *same pair* of photons. It is recognized that the quantity $s(\lambda)$ is actually unobservable, even though each of its summands *is* observable. For only one of the four possible relative angle settings $(\mathbf{a}^*,\mathbf{b}^*)$ can be implemented in an experiment with a specific pair of photons. Nonetheless, the Aspect/Bell account proposes the quantity $s(\lambda)$ to be of interest as defined in (1). Its expectation specified in (2) is considered to be at least estimable according to procedures introduced in the work of Aspect.

The theory of quantum mechanics motivates four related probabilistic assertions regarding polarization observations at any relative detector angle setting used in such an experiment on a pair of photons. For any relative angle pairing $(\mathbf{a}^*,\mathbf{b}^*)$,

$$
\begin{aligned}
P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = +1)] \;&=\; P[(A(\mathbf{a}^*) = -1)(B(\mathbf{b}^*) = -1)] \;=\; \tfrac{1}{2}\,cos^2(\mathbf{a}^*,\mathbf{b}^*)\,, \quad \text{and} \\
P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = -1)] \;&=\; P[(A(\mathbf{a}^*) = -1)(B(\mathbf{b}^*) = +1)] \;=\; \tfrac{1}{2}\,sin^2(\mathbf{a}^*,\mathbf{b}^*)\,. \qquad (3)
\end{aligned}
$$

Together, these imply $E[A(\mathbf{a}^*)B(\mathbf{b}^*)] \;=\; cos\,2(\mathbf{a}^*,\mathbf{b}^*)$ on account of the cosine double angle formula which arises in this derivation, and the marginal probabilities $P[A(\mathbf{a}^*) = +1] = P[B(\mathbf{b}^*) = +1] \;=\; 1/2$. For efficiency in what follows, we shall denote the four probabilities appearing in equations (3) by P_{++}, P_{--}, P_{+-}, and P_{-+} when the pertinent angle setting is evident.

Aspect/Bell presume that the specification equations (3) can be applied to all four of the angle settings involved in the quantity $s(\lambda)$ defined for measurements on a single pair of photons. As is currently and has long been understood, the most striking violation of Bell's inequality is said to occur at the four relative angle settings $(\mathbf{a},\mathbf{b}) = -\pi/8$, $(\mathbf{a},\mathbf{b}') = -3\pi/8$, $(\mathbf{a}',\mathbf{b}) = \pi/8$, and $(\mathbf{a}',\mathbf{b}') = -\pi/8$. Evaluating the contents of equations (3) at the angles $(\mathbf{a}^*,\mathbf{b}^*) = \pi/8$ *or* $-\pi/8$ yields identical results: $P_{++} = P_{--} = 0.4268$ while $P_{+-} = P_{-+} = 0.0732$. These imply as well that $E[A(\mathbf{a}^*)B(\mathbf{b}^*)] = 1/\sqrt{2} = .7071$. For the angle $(\mathbf{a}^*,\mathbf{b}^*) = -3\pi/8$, these probabilities reverse: $P_{++} = P_{--} = 0.0732$ and $P_{+-} = P_{-+} = 0.4268$, implying that $E[A(\mathbf{a})B(\mathbf{b}')] = -1/\sqrt{2} = -.7071$. These computations yield the quandary of Aspect/Bell, for they seem to imply that $E[s(\lambda)] = 2\sqrt{2}$ according to equation (2). Such an expectation would exceed the bounds $[-2,+2]$ required by the CHSH form of Bell's inequality. The standard conclusion from this has been that quantum mechanical probabilities have special properties that are different from those of mundane probabilities pertinent to classical scales of magnitude.

4 A neglected functional dependence

In specifying the QM motivated expectation as they do, however, Aspect/Bell fail to recognize a functional dependence among the values of the four proposed polarization products $A(\mathbf{a})B(\mathbf{b})$, $A(\mathbf{a})B(\mathbf{b}')$, $A(\mathbf{a}')B(\mathbf{b})$, and $A(\mathbf{a}')B(\mathbf{b}')$, which drastically affects their considerations. This neglected functional relationship might have been expected. As Aspect noted following his equation (23), the four relative angle possibilities themselves are linearly related: $(\mathbf{a},\mathbf{b}') = (\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{a}') + (\mathbf{a}',\mathbf{b}')$ at the considered angles of drastic violation. In fact, this same linear dependency would hold among any four angles determined by dual detection angle choices at A and B. Of course the values of the paired polarization indicators $A(\mathbf{a}^*)$ and $B(\mathbf{b}^*)$ vary from experiment to experiment at any of these relative angle settings. They are unknown prior to observation in any particular setting, and are amenable to probability assessment. Nonetheless, they are subject to identifiable restrictions. Perhaps surprisingly, the achieved values of any three *products* of the paired polarization indicators imply a unique value for the fourth product. We now engage to substantiate this claim.

4.1 The realm matrix of experimental quantities

Consider the "realm matrix" of all quantities relevant to the observations in these proposed experiments on a pair of photons under investigation. This matrix is displayed below in blocks that will now be discussed in turn. The sixteen columns of four-dimensional vectors in the first block exhaustively list all the speculative vector observations that could possibly arise among the four experimental polarization quantities. Every other component quantity in the columns displayed in subsequent blocks of the realm matrix is computed via some function of these possibilities. For example, the second block of components identifies the four designated *products* of the paired polarization indicators that yield the value of $s(\lambda)$ defined in equation (1). The first row of this block, identifying the product $A(\mathbf{a})B(\mathbf{b})$, is the componentwise product of the first two rows of the first block. The second row of this block, identifying the product $A(\mathbf{a})B(\mathbf{b}')$, is the componentwise product of the first and fourth rows of the first block, and so on.

The first item to notice about this realm matrix is that, whereas the sixteen columns of the first block of polarization observations are distinct, the second block contains only *eight* distinct column vectors. Columns 9 through 16 in block two of the realm matrix reproduce columns 1 through 8 in reverse order. Moreover, examining the first *three* rows of this second block more closely, it can be recognized that the first eight columns of these rows exclusively exhaust the simultaneous measurement possibilities for the three product quantities they identify. These are the eight vectors of the cartesian product $\{+1,-1\}^3$, which are repeated in columns nine through sixteen. Together, what these two observations mean is that the fourth product quantity in this second block of vector components is derivable as a function of the first three. What is more, any one of the product quantities identified in block two is determined by the same computational function of the other three! This can be seen by examining the columns of the *fourth* block of the matrix, which we shall do shortly.

$$
\mathbf{R}\begin{pmatrix}
A(\mathbf{a}) \\
B(\mathbf{b}) \\
A(\mathbf{a}') \\
B(\mathbf{b}') \\
\ast\ast\ast\ast \\
A(\mathbf{a})B(\mathbf{b}) \\
A(\mathbf{a})B(\mathbf{b}') \\
A(\mathbf{a}')B(\mathbf{b}) \\
A(\mathbf{a}')B(\mathbf{b}') \\
\ast\ast\ast\ast \\
\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}') \\
\ast\ast\ast\ast \\
\Sigma_{/(\mathbf{a},\mathbf{b})} \\
\Sigma_{/(\mathbf{a},\mathbf{b}')} \\
\Sigma_{/(\mathbf{a}',\mathbf{b})} \\
\Sigma_{/(\mathbf{a}',\mathbf{b}')} \\
\ast\ast\ast\ast \\
s(\lambda) \\
s_{\mathcal{A}/\mathcal{B}}(\mathbf{a}',\mathbf{b}') \\
1 \\
\ast\ast\ast\ast \\
(A(\mathbf{a})=+1)(B(\mathbf{b})=+1) \\
(A(\mathbf{a})=-1)(B(\mathbf{b})=-1) \\
(A(\mathbf{a})=+1)(B(\mathbf{b})=-1) \\
\ast\ast\ast\ast \\
(A(\mathbf{a})=+1)(B(\mathbf{b}')=+1) \\
(A(\mathbf{a})=-1)(B(\mathbf{b}')=-1) \\
(A(\mathbf{a})=+1)(B(\mathbf{b}')=-1) \\
\ast\ast\ast\ast \\
(A(\mathbf{a}')=+1)(B(\mathbf{b})=+1) \\
(A(\mathbf{a}')=-1)(B(\mathbf{b})=-1) \\
(A(\mathbf{a}')=+1)(B(\mathbf{b})=-1) \\
\ast\ast\ast\ast \\
(A(\mathbf{a}')=+1)(B(\mathbf{b}')=+1) \\
(A(\mathbf{a}')=-1)(B(\mathbf{b}')=-1) \\
(A(\mathbf{a}')=+1)(B(\mathbf{b}')=-1)
\end{pmatrix}
=
\begin{pmatrix}
1&1&1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1 \\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1 \\
1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1 \\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1 \\
&&&&&&&&&&&&&&& \\
1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1&1&1&1&1 \\
1&-1&1&-1&-1&1&-1&1&1&-1&1&-1&-1&1&-1&1 \\
1&1&-1&-1&1&1&-1&-1&-1&-1&1&1&-1&-1&1&1 \\
1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1 \\
&&&&&&&&&&&&&&& \\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1 \\
&&&&&&&&&&&&&&& \\
3&-1&-1&-1&1&1&-3&1&1&-3&1&1&-1&-1&-1&3 \\
3&1&-1&1&1&-1&-3&-1&-1&-3&-1&1&1&-1&1&3 \\
3&-1&1&1&-1&-1&-3&1&1&-3&-1&-1&1&1&-1&3 \\
3&1&1&-1&-1&1&-3&-1&-1&-3&1&-1&-1&1&1&3 \\
&&&&&&&&&&&&&&& \\
2&2&-2&2&2&-2&-2&-2&-2&-2&-2&2&2&-2&2&2 \\
2&4&0&2&2&0&0&-2&-4&-2&-2&0&0&-2&2&0 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
&&&&&&&&&&&&&&& \\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\
&&&&&&&&&&&&&&& \\
1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\
0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\
0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0 \\
&&&&&&&&&&&&&&& \\
1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\
0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0 \\
&&&&&&&&&&&&&&& \\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0
\end{pmatrix}
$$

The third block of the realm matrix contains only a single row, corresponding to a quantity we designate as $\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')$. This quantity takes values only of ± 1, but it is logically independent of the product quantities appearing in the first three rows of block two. This is the quantity that Aspect/Bell think they are assessing when they freely specify the quantum expectations for all four angle settings as they do, defying Bell's inequality. We denote its name with calligraphic type to distinguish it from the actual polarization product $A(\mathbf{a}')B(\mathbf{b}')$ whose functional relation to the other three products we are now identifying. Peculiar, it is not an "Alice and Bob" observation quantity, but rather an "Aspect/Bell" imagined quantity. It is logically independent of the first three "Alice and Bob" products. Whatever values these products may be, the value of $\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')$ may equal $+1$ in the appropriate row among the first eight columns, or it may equal -1 in the corresponding column among the second eight. However, it does not represent the polarization product $A(\mathbf{a}')B(\mathbf{b}')$ in the experiment to which it is meant to pertain.

4.2 Specifying the functional form via block four

Quantities in the fourth block of the realm matrix are designated with the names $\Sigma_{/(\mathbf{a},\mathbf{b})}$, $\Sigma_{/(\mathbf{a},\mathbf{b}')}$, $\Sigma_{/(\mathbf{a}',\mathbf{b})}$, and $\Sigma_{/(\mathbf{a}',\mathbf{b}')}$. These quantities are defined by sums of column elements in those rows of *the second block* that are *not* marked in the notational subscript. For examples,

$$\begin{aligned}\Sigma_{/(\mathbf{a},\mathbf{b})} &\equiv A(\mathbf{a})B(\mathbf{b}') + A(\mathbf{a}')B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}'), \quad \text{and} \\ \Sigma_{/(\mathbf{a},\mathbf{b}')} &\equiv A(\mathbf{a})B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}') \ .\end{aligned}$$

The quantities $\Sigma_{/(\mathbf{a}',\mathbf{b})}$, and $\Sigma_{/(\mathbf{a}',\mathbf{b}')}$ are defined similarly.

Next to notice is that the fourth row of the *second* matrix block, corresponding to $A(\mathbf{a}')B(\mathbf{b}')$, has an entry of 1 if and only if the fourth row of the *fourth* block, corresponding to $\Sigma_{/(\mathbf{a}',\mathbf{b}')}$, has an entry of -1 or 3 in the same column. When that entry is 1 or -3, the corresponding entry of the second block is -1. What this recognition does is to identify the functional relation of the fourth polarization product to the first three polarization products, viz.,

$$\begin{aligned}A(\mathbf{a}')B(\mathbf{b}') &= \mathbf{G}[A(\mathbf{a})B(\mathbf{b}),\, A(\mathbf{a}')B(\mathbf{b}),\, A(\mathbf{a})B(\mathbf{b}')] \\ &\equiv \Big(\Sigma_{/(\mathbf{a}',\mathbf{b}')} = -1 \ or \ 3\Big) \ - \ \Big(\Sigma_{/(\mathbf{a}',\mathbf{b}')} = 1 \ or \ -3\Big) \ . \end{aligned} \qquad (4)$$

Here and throughout this note I am using notation in which parentheses surrounding a mathematical statement that might be true and might be false signifies the number 1 when the interior statement is true, and signifies 0 when it is false.

Some eyeball work is required to recognize functional relationship (4) by examining the final row of block two and block four together. It may take even more concentration to recognize that this very same functional rule identifies each of the other three polarization products as a function of the other three as well! The four product quantities $A(\cdot)B(\cdot)$ are related by four symmetric functional relationships, each of them being calculable via the same functional rule applied to the other three! This surprising recognition identifies the source of the Aspect/Bell error in assessing the QM-motivated expectation for $s(\lambda)$ in the way they do.

It is surely true that $E[s(\lambda)]$ equals a linear combination of four expectations of polarization products, as specified in equation (2). Moreover, if the definition of $s(\lambda)$ in equation (1) were understood to represent the combination of observed products from experiments on four distinct pairs of photons, then the possible values of $s(\lambda)$ would span the integers from -4 through $+4$; the expectation of each product $E[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ would equal $-1/\sqrt{2}$ or $+1/\sqrt{2}$ as appropriate to the angle $(\mathbf{a}^*, \mathbf{b}^*)$; and $E[s(\lambda)]$ would equal $2\sqrt{2}$ as proposed by Aspect/Bell. This involves no violation of any probabilistic inequality at all, and there is no suggestion of mysterious activity of quantum mechanics.

However, when it is proposed that the paired polarization experiments at all four considered angles pertain to the same photon pair, then each of the products is restricted to equal the specified function value of the other three that we identified explicitly for $A(\mathbf{a}')B(\mathbf{b}')$ in equation (4) as $\mathbf{G}[A(\mathbf{a})B(\mathbf{b}),\, A(\mathbf{a}')B(\mathbf{b}),\, A(\mathbf{a})B(\mathbf{b}')]$ via the function $\Sigma_{/(\mathbf{a}',\mathbf{b}')}$. In this context, Aspect's expected quantity would be representable equivalently as any one of the following:

$$\begin{aligned}E[s(\lambda)] &= E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b})] + E[\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b})]] \\ &= E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b}')] + E[\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b}')]] \\ &= E[A(\mathbf{a})B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b}')] - E[\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}')]] \\ &= -E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b}')] + E[\mathbf{G}[A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}')]].\end{aligned}$$

The symmetries imposed on this problem would yield an identical result in each case, which would surely *not* yield $2\sqrt{2}$ at all. What might it yield?

The functional relation we have exposed in (4) is *not* linear. If it were, then the specification of an expectation for its arguments would imply the expectation value for the function value. As it is not, the specification of expectation values for the arguments only imply bounds on any cohering expectation value for the fourth. These numerical bounds can be computed via Bruno de Finetti's fundamental theorem of probability, or prevision as applies more generally. We shall review the content of de Finetti's theorem shortly, and then examine its relevance to assessing the expectation of $s(\lambda)$ motivated by considerations of quantum mechanics. We need first to air some further brief remarks about the remaining blocks of the realm matrix.

4.3 Remaining blocks of quantities and their realm components

The first row of block five of the realm matrix merely identifies the values of $s(\lambda)$ associated with the polarization observation possibilities enumerated in the columns of block one. Each component of this row is computed from the corresponding column of block two according to the defining equation (1). The second row of this block pertains to a quantity denoted as $s_{\mathcal{A}/\mathcal{B}}(\mathbf{a}', \mathbf{b}')$. Its value is defined similarly to equation (1), but its final summand is specified as the Aspect/Bell quantity $\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')$ rather than the actual polarization product quantity $A(\mathbf{a}')B(\mathbf{b}')$ that appears in this equation defining $s(\lambda)$. Again peculiar, its realm can be seen to include the elements $\{-4, -2, 0, 2, 4\}$ whereas the realm of $s(\lambda)$ includes only $\{-2, 2\}$.

The third row of block five is merely an accounting device, denoting that the "sure" quantity, 1, is equal to 1 no matter what the observed results of the four imagined optic experiments of Aspect/Bell might be. Its relevance will become apparent when the need arises to apply de Finetti's fundamental theorem to quantum assertions. The remaining blocks of quantities and their realm elements are also pertinent to that application. They will be discussed in the context of assertions that arise from considerations of quantum mechanics.

5 The relevance of the fundamental theorem of probability

The fundamental theorem of probability (FTP) specifies that when probabilities or expectations for any N quantities whatsoever are assessed with the vector of values $\mathbf{p}_N$, then bounds on a cohering expectation for any further $(N+1)^{\text{st}}$ quantity can be computed via a linear programming routine. The size of the bounding interval depends on the logical relations entailed in the definitions of the considered quantities. The theorem was first *named* in de Finetti (1974, Chapter 3.10), though it is as old as his famous lectures at the Institute Henri Poincaré in 1935. It extends naturally to specify bounds on expectations for general quantities ("previsions" in de Finetti's nomenclature) as presented in the article of Lad, Dickey, and Rahman (1990). It is discussed pedagogically in Lad (1996, Ch 2.10, pp 99-113). Following is what the theorem says.

Suppose the realm matrix $\mathbf{R}(\mathbf{X}_{N+1})$ for the vector of quantities X_1 through X_{N+1} has K columns. These columns exhaust all possibilities for prospective quantity observations under consideration. Define the vector $\mathbf{r}_{N+1}$ as the final *row* of this realm matrix corresponding to the possibilities for the quantity X_{N+1} as the last component of the observation vector, and the matrix $\mathbf{R}_{N,K}$ as the N initial rows of the realm matrix corresponding to the concomitant possibilities for the first N components of $\mathbf{X}_{N+1}$. The design of the linear programming routine is to find the column vectors $\mathbf{q}_K$ for which the linear combination $\mathbf{r}_{N+1}\,\mathbf{q}_K$ achieves minimum and maximum values subject to the N linear restrictions that $\mathbf{R}_{N,K}\,\mathbf{q}_K$ equals $\mathbf{p}_N$, along with the restrictions that the components of $\mathbf{q}_K$ are non-negative and that they sum to 1. These latter restrictions ensure that as long as $E(X_{N+1})$ is within the extremes determined by the theorem, the expectation of the full vector $E(\mathbf{X}_{N+1})$ would then lie within the convex hull of the columns of its realm matrix. This is the general condition of coherency. If there is no feasible solution to these problems, then the assertion of the N expectations that have been presumed is incoherent.

5.1 The incoherence of a naive application of *all* QM assertions

Straightforward applications of the fundamental theorem of prevision identify the extent of QM-motivated coherent probability assertions and expectations relevant to $s(\lambda)$. To begin, examine the final four blocks of three events each in the realm matrix we have displayed. Within each block are event quantities whose QM probabilities would be specified by equations (3) if they were applied to a photon pair at any *one* of the four experimental detection angles. Asserting the QM probabilities we have identified at the end of Section 3 for any specific experimental setting block would amount to three linear restrictions on the vector $\mathbf{q}_{16}$ involved in the statement of de Finetti's theorem. That components of $\mathbf{q}_{16}$ are non-negative and sum to 1 would complete the linear restrictions that quantum theory requires for this experiment. Specifying the fourth probability motivated by quantum theory in (3) would be redundant in light of the universal constraint on the components of $\mathbf{q}_{16}$ that they sum to 1.

For example, applying QM probability specifications of equation (3) to the relative angle $(\mathbf{a}, \mathbf{b}) = -\pi/8$ would instigate three linear relations among the components of the probability vector $\mathbf{q}_{16}$. These can be identified by viewing the rows of block six of the realm matrix, viz.,

$$q_1 + q_2 + q_3 + q_4 \;=\; q_{13} + q_{14} + q_{15} + q_{16} \;=\; \tfrac{1}{2}cos^2(-\pi/8) \;=\; .4268\,, \quad \text{and}$$
$$q_5 + q_6 + q_7 + q_8 \;=\; \tfrac{1}{2}sin^2(-\pi/8) \;=\; .0732.$$

A fourth linear restriction we might have written would be redundant in the face of the general restriction that the sum of all components of $\mathbf{q}_{16}$ must equal 1.

Computational evaluations show that these specified assertions are coherent. The feasible set of vectors $\mathbf{q}_{16}$ is non-empty, supporting QM probabilities for any experimentation angle considered. The same would be true of assertions regarding experimental results achieved at any of the other three possible experimental angles. These would be represented by similar algebraic restrictions identified in blocks seven through nine of the realm matrix. To repeat: there is nothing incoherent about QM-motivated probabilities assessed for any actual optic experiment we are considering. Moreover, *concomitantly augmenting* the assertions represented in block six by further specifications of QM probabilities for the same pair of photons relevant to any second detection angle would be coherent as well. (Along with Aspect/Bell we are ignoring the fact that it is impossible to conduct *both* such experiments, only either one of them.) Even augmenting these restrictions further with assertions appropriate to a third experimental angle would be coherent. However, to ignore the functional relation among the polarization products, and to assert the QM probabilities for *all four* detection angles concurrently is found to be incoherent, as we now detail.

Suppose we assert QM-motivated probabilities for all twelve events appearing in the final four blocks of the displayed realm matrix. With the four relative detector angles set at the notorious arrangement $-\pi/8$, $-3\pi/8$, $\pi/8$, and $-\pi/8$, the QM probabilities for the three events listed in each of the sixth, eighth, and ninth (final) blocks of the realm matrix would be .4268, .4268, .0732. These derive from the well known QM formulas involving $\frac{1}{2}\,cos^2(\cdot,\cdot\,)$ and $\frac{1}{2}\,sin^2(\cdot,\cdot\,)$ as described at the end of Section 3. For the *seventh* block of events corresponding to the angle $-3\pi/8$, the ordered probabilities would be .0732, .0732, .4268. However, the result of applying a linear programming routine to these *twelve* assertions concurrently is to conclude that there is no feasible solution to this proposal! Thus, the full array of such assertions applied *to the results of a single photon pair* are incoherent. We might have expected this in light of the recognition that the value of $s(\lambda)$ can only equal 2 or -2 in any column of the realm matrix. For these incoherent assertions would imply the Aspect/Bell assessment that $E[s(\lambda)] \;=\; 2\sqrt{2}$.

5.2 What *do* coherent assertions of QM probabilities specify?

While the concurrent assertion of QM-motivated probabilities for the same photon pair at *all four* relative angle settings is incoherent, asserting QM probabilities for any one, two or three of the paired detector settings *is* coherent. We are left to identify what *is* the subspace of probability vectors for all four settings that are coherently motivated by quantum theory, and what is the range of expectations for the quantity $s(\lambda)$ they imply. We shall do this by presuming the assertion of only the various *three* blocks of QM probabilities according to the sin^2 and cos^2 specifications, aware that the fourth polarization product quantity is functionally determined by the three products so considered. This would amount to nine linear restrictions for the linear programming setup rather than twelve. We can then assess the bounds on cohering probabilities for the fourth block of probability components using the LP formulation prescribed by the fundamental theorem of probability.

To these nine specific trigonometric numerical restrictions for the three designated blocks, however, we do need to append two further QM-motivated restrictions which are still pertinent to the events whose cohering probabilities we seek. Whichever angle setting is being considered, we still need require that the probability for observing two parallel polarizations in either direction are equal: $P_{++} = P_{--}$; and that the probabilities of observing perpendicular polarizations are equal to one another too: $P_{+-} = P_{-+}$. These requirements are enforced computationally by appending two more rows to the realm matrix for the "difference" quantities at the appropriate angle $(\mathbf{a}^*, \mathbf{b}^*)$,

$$(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = +1) \; - \; (A(\mathbf{a}^*) = -1)(B(\mathbf{b}^*) = -1) \qquad \text{and}$$
$$(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = -1) \; - \; (A(\mathbf{a}^*) = -1)(B(\mathbf{b}^*) = +1) \qquad ,$$

and restricting the two ensuing linear combinations of the vector $\mathbf{q}_{16}$ to equal 0. This would make 11 linear restrictions for the problem, along with a twelfth for the summation constraint on $\mathbf{q}_{16}$. Notice that the probability restrictions $P_{++} = P_{--}$ and $P_{+-} = P_{-+}$ together imply the equality $P[A(\mathbf{a}) = +1] = P[B(\mathbf{b}) = 1] = 1/2$. We will be aware of this prescription, but we need not introduce these marginal probabilities explicitly as another restriction. Understanding the format of such a linear programming setup, we can now report an array of interesting results in the next subsections, commenting upon them in conclusion.

5.2.1 Presuming QM probabilities regarding experimental detection angle settings $(\mathbf{a}, \mathbf{b}) = -\pi/8$, $(\mathbf{a}', \mathbf{b}) = \pi/8$, and $(\mathbf{a}', \mathbf{b}') = -\pi/8$

If the identical assertion values $[P_{++}, P_{--}, P_{+-}] = [.4268, .4268, .0732]$ are applied to the angle settings $(\mathbf{a}, \mathbf{b}) = -\pi/8$, $(\mathbf{a}', \mathbf{b}) = \pi/8$, and $(\mathbf{a}', \mathbf{b}') = -\pi/8$, then for the setting $(\mathbf{a}, \mathbf{b}') = -3\pi/8$ the coherent bounds on probabilities for identical polarizations, $P[(A(\mathbf{a}) = +1)(B(\mathbf{b}') = +1)] = P[(A(\mathbf{a}) = -1)(B(\mathbf{b}') = -1)]$, are found to be $[.2803, .5000]$, and the bounds on probabilities for mixed polarization observations $P[(A(\mathbf{a}) = +1)(B(\mathbf{b}') = -1)]$ are $[0, .2197]$. Actually, these results provide *joint* bounds for these probabilities $P(A(\mathbf{a}) = +1)(B(\mathbf{b}') = +1)$ and $P(A(\mathbf{a}) = +1)(B(\mathbf{b}') = -1)$ as pairs, because quantum theory assures that the marginal probability for observing a photon polarization parallel to either indicator A or B alone equals 1/2. This is to say that the probability pair (P_{++}, P_{+-}) can range continuously from (.2803, .2197) through to (.5, 0) as P_{++} increases while P_{+-} decreases. These results are entailed in the determination that bounds on cohering assertions of $E[s(\lambda)]$ are $[1.1213, 2.0]$. At the bounding solutions giving these extremes, the expected polarization products $E[A(\mathbf{a})B(\mathbf{b}')]$ that get subtracted in the computation of $E[s(\lambda)]$ are 1.0 and .1213, respectively.

This range of coherent expectations takes into account the functional relationships among the polarization products that we have identified in Sections 4.1 and 4.2. When Aspect/Bell do not take these functional relations into account in their assessment, they presume the non-cohering value of $-.7071$ for $E[A(\mathbf{a})B(\mathbf{b}')]$. This is what promotes their bound-breaking assessment value for $E_{QM}(s)$ as $2\sqrt{2}$.

We shall now assess three alternative QM-motivated assertion triples appropriate to the experimental detection angle settings $(\mathbf{a},\mathbf{b}) = -\pi/8$, $(\mathbf{a},\mathbf{b}') = -3\pi/8$, and $(\mathbf{a}',\mathbf{b}) = \pi/8$, and investigate their coherent implications for assessing the polarization product at the setting $(\mathbf{a}',\mathbf{b}') = \pi/8$.

5.2.2 Presuming QM probabilities regarding experimental detection angle settings $(\mathbf{a},\mathbf{b}) = -\pi/8$, $(\mathbf{a},\mathbf{b}') = -3\pi/8$, and $(\mathbf{a}',\mathbf{b}) = \pi/8$

The assertions we presume now include the polarization product probabilities P_{++}, P_{--} and P_{+-} at the angle $(\mathbf{a}',\mathbf{b}) = -3\pi/8$, along with those appropriate to the angles at $(\mathbf{a},\mathbf{b}) = -\pi/8$ and $(\mathbf{a}',\mathbf{b}) = \pi/8$. Again we use linear programming routines to specify the bounds on the cohering assertion of $E[s(\lambda)]$ and for probabilities they imply for products relevant to the angle $(\mathbf{a}',\mathbf{b}') = \pi/8$. The result is that the values of $P_{++}(\mathbf{a}',\mathbf{b}') = P_{--}(\mathbf{a},\mathbf{b}')$ are bounded within $[0, .2197]$, while the values of the mixed polarization probabilities $P_{+-}(\mathbf{a}',\mathbf{b}') = P_{-+}(\mathbf{a},\mathbf{b}')$ range accordingly downward from .5 to .2803. This should not be surprising, in that these bounds are precisely the reverse of those impinging on the values of P_{++}, P_{--} and P_{+-} at the angle $(\mathbf{a},\mathbf{b}') = -3\pi/8$ that we studied in subsection 5.2.1. Corresponding to this range of assertions for these probabilities is the range of cohering values for $E[s(\lambda)]$ which is bounded once again between 1.1213 and 2.0.

5.2.3 The complete space of coherent QM-motivated assertion boundaries

On account of the symmetries we have already observed in the structure of this problem, one would correctly expect that the two other QM motivated settings in which the three sets of asserted probabilities include one relevant to the angle $-3\pi/8$ and the other two relevant to the angles $\pm\,\pi/8$ would give similar results to those just reported. Since they do, and as we need now discuss the global implications of all four of these groups of bounded probabilities, the results of all eight linear programs are reported in Table 1. Each row of the Table presents in column two a minimum or a maximum value of a linear programming problem in which the objective function is the value of $E[s(\lambda)]$, and the constraints are provided by applying the quantum theoretic assertions of equation (3) to the three relative detector angles that are *not* identified in the row heading at the left. The last three columns present the associated extreme values of P_{++}, P_{+-} and $E[A(\mathbf{a}^*)\,B(\mathbf{b}^*)]$ pertaining to the angle that *is* identified in the row heading. Values of P_{++} and P_{+-} associated with the other angles not identified there are the appropriate QM-motivated probabilities that constrained the LP problem.

Table 1: Bounding values of coherent QM expectations for Aspect/Bell $s(\lambda)$

LP problem	$E[s(\lambda)]$	$P_{++}(\mathbf{a}^*,\mathbf{b}^*)$	$P_{+-}(\mathbf{a}^*,\mathbf{b}^*)$	$E[A(\mathbf{a}^*)\,B(\mathbf{b}^*)]$
min $E[s(\lambda)](\mathbf{a},\mathbf{b}')$	1.1213	.5	0	1.0
max $E[s(\lambda)](\mathbf{a},\mathbf{b}')$	2.0	.2803	.2197	.1213
min $E[s(\lambda)](\mathbf{a}',\mathbf{b}')$	1.1213	0	.5	-1.0
max $E[s(\lambda)](\mathbf{a}',\mathbf{b}')$	2.0	.2197	.2803	$-.1213$
min $E[s(\lambda)](\mathbf{a},\mathbf{b})$	1.1213	0	.5	-1.0
max $E[s(\lambda)](\mathbf{a},\mathbf{b})$	2.0	.2197	.2803	$-.1213$
min $E[s(\lambda)](\mathbf{a}',\mathbf{b})$	1.1213	0	.5	-1.0
max $E[s(\lambda)](\mathbf{a}',\mathbf{b})$	2.0	.2197	.2803	$-.1213$

Recognizing the coherency of all four of these LP solution pairs pertaining to expectation assessments for a fourth polarization product warrants remembering that linear combinations of coherent expectations are coherent as well. What we are doing when tracing through these related linear programming computations is identifying faces of a multidimensional configuration of coherent probability vectors defined over all possibilities of polarization products. Focus on the fact that each of the linear programming problems we have reported involves a specification of twelve linear constraints on the sixteen-dimensional vector $\mathbf{q}_{16}$. Nine of these arise from the three QM probabilities applied to three of the four angles; two arise from the equalities $P{+}{+} = P{-}{-}$ and $P_{+-} = P_{-+}$ at the objective function angle; and one arises from the summation constraint. Actually, linear dependencies among the constraints of the several LP problems entail that the total number of linearly independent constraints is only eight.

Geometrically, this means that the space of feasible solutions to the several linear programming programs has eight dimensions. The solution to each of the minimum and maximum programming problems identifies a vertex vector of the simplex of feasible solutions, identified by the way it supports an extreme value of a different linear objective function. The eight 16-dimensional extreme vectors that solve the linear programming problems of Table 1 are the columns of the matrix $\mathbf{q}_{16}\mathbf{VertexMat}$, which has a column rank of eight:

$$\mathbf{q}_{16}\mathbf{VertexMat} = \begin{pmatrix}
.3902 & .2803 & 0 & .0732 & 0 & .0732 & 0 & .0732 \\
0 & .0732 & .3902 & .2803 & 0 & .0732 & 0 & .0732 \\
.0366 & 0 & .0366 & 0 & 0 & 0 & .0366 & 0 \\
0 & .0732 & 0 & .0732 & 0 & .0732 & .3902 & .2803 \\
0 & .0732 & 0 & .0732 & .3902 & .2803 & 0 & .0732 \\
.0366 & 0 & .0366 & 0 & .0366 & 0 & 0 & 0 \\
0 & 0 & .0366 & 0 & .0366 & 0 & .0366 & 0 \\
.0366 & 0 & 0 & 0 & .0366 & 0 & .0366 & 0 \\
.0366 & 0 & 0 & 0 & .0366 & 0 & .0366 & 0 \\
0 & 0 & .0366 & 0 & .0366 & 0 & .0366 & 0 \\
.0366 & 0 & .0366 & 0 & .0366 & 0 & 0 & 0 \\
0 & .0732 & 0 & .0732 & .3902 & .2803 & 0 & .0732 \\
0 & .0732 & 0 & .0732 & 0 & .0732 & .3902 & .2803 \\
.0366 & 0 & .0366 & 0 & 0 & 0 & .0366 & 0 \\
0 & .0732 & .3902 & .2803 & 0 & .0732 & 0 & .0732 \\
.3902 & .2803 & 0 & .0732 & 0 & .0732 & 0 & .0732
\end{pmatrix}$$

These eight linearly independent vectors specify vertices of a 4-dimensional polytope of coherent QM-motivated probability vectors, as each of them is constrained to honour twelve linear restrictions (different but intermingled). The 4-dimensional space could be described in many different ways using different coordinate systems. However, the simplest way to state the result and to visualize it is to report the column vectors of $P_{++}(\mathbf{a}^*\mathbf{b}^*)$ values at the four relative detector angle settings projected from the eight column vectors of $\mathbf{q}_{16}\mathbf{VertexMat}$. Computationally, these projected vectors are produced by premultiplying $\mathbf{q}_{16}\mathbf{VertexMat}$ with a 4×16 matrix constituted by the first rows of blocks six through nine of our realm matrix.

Table 2 displays these projected vertex vectors, followed by the values of $E[s(\lambda)]$ they support. Notice how the second row of the Table associated with the angle $(\mathbf{a}, \mathbf{b}')$ is complementary to the other three rows which are permutations of one another. The complementarity is determined by the fact that .5 times the unit vector minus row two of this matrix yields still another permutation of any of the other three rows. The pleasing symmetry of the column matrix should be evident without further comment. Recognize that the specification of $P_{++}(\mathbf{a}^*, \mathbf{b}^*)$ in any row actually identifies the full distribution of possible polarization outcome pairs for that angle setting. For P_{--} must equal this P_{++}, and then the values of $P_{+-} = P_{-+}$ are determined by the fact that the sum of all four of these probabilities must equal 1.

Table 2: Vertex vectors of the coherent QM probability polytope

$P_{++}(\mathbf{a}, \mathbf{b})$	0.4268	0.4268	0.4268	0.4268	0.0000	0.2197	0.4268	0.4268
$P_{++}(\mathbf{a}, \mathbf{b}')$	0.5000	0.2803	0.0732	0.0732	0.0732	0.0732	0.0732	0.0732
$P_{++}(\mathbf{a}', \mathbf{b})$	0.4268	0.4268	0.4268	0.4268	0.4268	0.4268	0.0000	0.2197
$P_{++}(\mathbf{a}', \mathbf{b}')$	0.4268	0.4268	0.0000	0.2197	0.4268	0.4268	0.4268	0.4268
$E[s(\lambda)]$	1.1213	2.0000	1.1213	2.0000	1.1213	2.0000	1.1213	2.0000

The way to think about the content of quantum theory relevant to the Aspect/Bell problem is to think of it as supporting a convex four-dimensional subspace of probabilities and expectations rather than a precise point in sixteen-dimensional space. The space surely does *not* contain the point asserted by Aspect/Bell as the expectation motivated by quantum mechanics. Readers unfamiliar with this way of thinking would be interested to know that there has emerged in recent years a sizable contingent of research statisticians who devote attention to a wide range of problems requiring this type of solution. Early influential expositions can be found in Dempster (1967), Shafer (1976), and Walley (1991). By now a Society for Imprecise Probability has been established featuring biennial international conferences since 1999. Conference proceedings can be traced on the internet. Although a variety of evidential theories are expounded by participants, the mathematics underlying such structures of coherent "prevision polytopes" is exhausted by de Finetti's fundamental theorem of prevision.

The convex hull of the 4-D column vectors shown in Table 2 can be visualized through a sequence of 3-D intersections it affords with slices perpendicular to any one of its axes. Figure 1 displays such a sequence, by slices perpendicular to the $P_{++}(\mathbf{a}', \mathbf{b}')]$ axis at values increasing from 0 to .4268. When $P_{++}(\mathbf{a}', \mathbf{b}') = 0$, the intersection of the slice identifies only a single vertex point $(.0732, .4268, .4268)$ which appears in the subplot $(1, 1)$. See also column three of the matrix in Table 2. As the slice level increases to a small positive value in subplot $(2, 1)$, the intersection appears as a tetrahedron. The size of the intersecting tetrahedron increases when $P_{++}(\mathbf{a}', \mathbf{b}')$ increases to level .2197 in subplot $(3, 2)$. The tetrahedrons continue to increase in size as the level of the $P_{++}(\mathbf{a}', \mathbf{b}')$ increases still further, but a corner of their intersections begins to be cut off more and more severely, as displayed in subplots $(1, 2)$ through $(3, 2)$.

The symmetry of the configuration implies that slices along the other axes would create intersection sequences that look identical. This Figure was produced by my colleague Rachael Tappenden. Discussion and display of the remaining four dimensions of the coherent QM-motivated polytope will be presented elsewhere. A movie sequence of this progressing intersection is available.

6 Reassessing Aspect's empirical results

We have yet to deal with the awkward claim of Aspect and the several subsequent research groups who have developed more extensive experimental techniques, that experimental violations of Bell's inequality have been observed. What are we to make of Aspect's estimations?

Aspect (2002, page 15) estimates the expectation $E[s(\lambda)]$, as defined in equation (2), from experimental data by means of a classical method of moments estimator applied to each of its four expectation components. Of course repeated experiments using the same pair of photons is impossible, so an experimental sequence of observations on distinct photon pairs was generated at the various angle settings in order to estimate the expected polarization product at each angle. Using the notations $N_{\pm\pm}(\mathbf{a}, \mathbf{b})$ for the coincidence counts of +1 and −1 observations of A polarization and B polarization among N experimental runs at the relative polarization angle

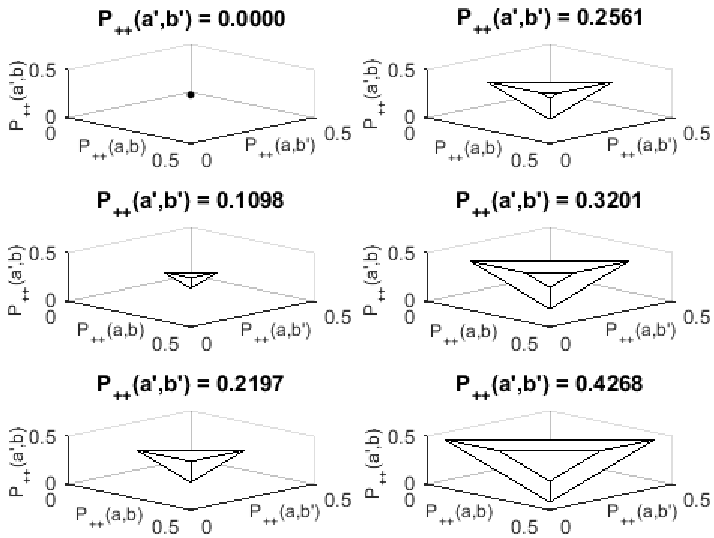

Figure 1: Sequential intersections of the 4-D convex hull of vectors $[P_{++}(\mathbf{a},\mathbf{b}), P_{++}(\mathbf{a},\mathbf{b}'), P_{++}(\mathbf{a}',\mathbf{b}), P_{++}(\mathbf{a}',\mathbf{b}')]$ with slices perpendicular to the $P_{++}(\mathbf{a}',\mathbf{b}')$ axis, at levels increasing from 0 to .4268. View the sequence column by column as $P_{++}(\mathbf{a}',\mathbf{b}')$ increases.

$(\mathbf{a},\mathbf{b})$, he specifies the estimator for this angle's component of $E[s(\lambda)]$ (*Aspect equation* 18) as

$$\hat{E}[A(\mathbf{a})B(\mathbf{b})] \;=\; \frac{[N_{++}(\mathbf{a},\mathbf{b}) - N_{+-}(\mathbf{a},\mathbf{b}) - N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})]}{[N_{++}(\mathbf{a},\mathbf{b}) + N_{+-}(\mathbf{a},\mathbf{b}) + N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})]} \;, \tag{5}$$

with a similar specification for the components of $E[s(\lambda)]$ deriving from the relative angles $(\mathbf{a}',\mathbf{b})$, $(\mathbf{a},\mathbf{b}')$, and $(\mathbf{a}',\mathbf{b}')$. He then inserts these independent estimates into equation (2) to yield his touted estimate $\hat{E}[s(\lambda)]$ near to $2\sqrt{2}$.

We can now recognize that Aspect's estimation procedure allows complete liberty for all four polarization product expectations $E[A(\mathbf{a}^*)B(\mathbf{b}^*)]$, using experimental incidence values of $N_{\pm\pm}(\mathbf{a}^*,\mathbf{b}^*)$ from many experimental runs *with distinct photon pairs*. Each of his experimental observations may be whatever value it happens to be at its experimental angle setting, identifying whatever value of polarization product that it does. However, if Aspect's estimation is meant to apply to the ontological understanding of $s(\lambda)$ within which he and Bell couch their theoretical claims, he would have to adjust this methodology. One may pick experimental runs using three different photon pairs at any three angles one wishes, to simulate the behaviours $A(\mathbf{a}^*)B(\mathbf{b}^*)$ for the corresponding polarization products of a single pair of photons. However, to be consistent with the Aspect/Bell problem as posed, one then would need to compute the implied value of the polarization product observation for the fourth angle according to the functional form that we have identified in equation (4).

Statistical estimation values reported in Aspect (2002) have no relevance to the estimation of $E[s(\lambda)]$ as it is understood to pertain to four spin products on a single pair of photons. It is perfectly reasonable to find estimation values near to $2\sqrt{2}$ as Aspect has. For they could pertain to an estimate of $E[s(\lambda)]$, but only with $s(\lambda)$ defined as a combination of polarization products on *four different pairs of photons*. In such a context, $E[s(\lambda)]$ is not bound by the Bell bounds of $[-2, +2]$, but rather by the interval $[-4, +4]$ which is unchallenged in this context.

6.1 Exposition by simulation

Because Aspect's experimental observation data is not available in full, the implications of correcting his estimation procedure need to be displayed here via simulation. To begin, four columns of one million (10^6) pseudo random numbers, uniform on $[0, 1]$, were generated with a MATLAB routine. These were then transformed into simulated observations of photon polorization experiments at the same four angles we have been studying. These transformations were performed using the QM probabilities based on calculations of $\frac{1}{2}\, cos^2(\mathbf{a}^*, \mathbf{b}^*)$ and $\frac{1}{2}\, sin^2(\mathbf{a}^*, \mathbf{b}^*)$ as described in equations (3) of Section 3. Each resulting simulated polarization pair was then multiplied together to yield a polarization product. In this way, the four columns correspond to simulated observations of polarization products from one million experiments at each of the four angles, ordered in the manner we have been following in the linear programming analysis: $(\mathbf{a}', \mathbf{b}')$, $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}', \mathbf{b})$.

Aspect's estimation equation (5) was applied to each of these columns, yielding an "Aspect estimate" of the expected polarization product associated with the relative detector angle appropriate to that column. These appear in the first row of Table 3. These four estimates were then inserted into equation (2) appropriately to yield an Aspect estimate $\hat{E}[s(\lambda)] = 2.827738$, appearing in the second row of the Table under *each* of these columns. As noted, it is quite near to $2\sqrt{2} \approx 2.828427$, as was Aspect's reported empirical estimate, proposed as an evidential violation of Bell's inequality. As we now know, the problem is that when the product observations are supposed to apply to *the same* photon pair, the value of any polarization product at any angle is meant to be related to the product at the other three angles via the functional equation we generated as our equation (3).

Table 3: Corrections to Aspect's estimate of $E[s(\lambda)]$

	$(\mathbf{a}', \mathbf{b}')$	$(\mathbf{a}, \mathbf{b}')$	$(\mathbf{a}', \mathbf{b})$	$(\mathbf{a}', \mathbf{b}')$
$\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$	0.707232	−0.706186	0.706840	0.707480
Aspect $\hat{E}[s]$	2.827738	2.827738	2.827738	2.827738
Functional $\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$	−0.353078	0.354348	−0.354766	−0.353934
Corrected $\hat{E}[s]$	1.767180	1.767204	1.765740	1.766964

The third row of Table 3 is generated then by first applying this function to the simulated observations of the three angles that *differ* from the angle ID that heads the column. Aspect's estimation equation (5) was then applied to this resulting column. This procedure generates the "Functional" estimates $\hat{E}[A(\mathbf{a}^*)\, B(\mathbf{b}^*)]$ that appear in row three of the Table. Finally, this "correct" estimate of that expected product is inserted into equation (2) along with the expectations that appear in *the other columns of row one* of the Table, to yield the results shown in row four. The elements of this row display corrected estimates of $E[s(\lambda)]$ as they should be calculated with the simulated Aspect data. Averaging these four estimates over the four ways of generating a column of polarization products from the other three columns of simulated products would yield a "Corrected estimate" of $E[s(\lambda)]$ as 1.766772 , well within the Bell bounds of $[-2, +2]$.

Results on the order of this peculiar number are stable in repeated runs of this simulation as described. Since the theoretical analysis reported in this article yields only an interval of cohering possibilities for $E[s(\lambda)]$, this simulation leaves us with a tantalizing problem of how to account for this stable result, which is quite near to $[3/\sqrt{2} - 1/(2\sqrt{2})] \approx 1.767766952966369$. Most likely, this specific result is a construct of the independence feature embedded in the simulation results across angle settings. Such a feature would be highly suspect in nature, given what we know now about quantum entanglement itself in a single experiment. A number deriving from Aspect's experimental data could be calculated too, with appropriate adjustment to his estimation method. However, there can be no real empirical evidence on the issue since it is impossible to activate the setup of the four imagined simultaneous experiments on a single pair of photons. Thus, the physicists' long reliance on the fabled gedankenexperiment.

6.2 A comment on empirical work and statistical estimation

While Aspect's conception of statistical estimates appropriate to the photon detection problem is understandable, and corrections can be made to improve its relevance to the Aspect/Bell problem, developments of statistical theory and practice during the past fifty years have surely generated superior methods for evaluating the physical theory of quantum behavior. Without detailing such now, it is apparent that calls for open access to raw data (Khrennikov, 2015) from several well-known research programs that publish summary results need to be heeded.

7 Concluding comments

I have described the mathematical structure of the Aspect/Bell problem from the viewpoint of a proponent of subjective probability, and thus the impinging probabilities have been referred to as "assertions". This viewpoint is in keeping with Einstein's interpretation of quantum mechanics, known by his famous adage of not believing that god rolls dice. Readers more comfortable with the standard realist interpretation of quantum mechanics may consider the probabilities as ontic properties of the photons themselves, without disturbing the mathematical issues we have engaged. Anyone who professes uncertain knowledge about the possible values of a quantum optical experiment may assert whatever probabilities are deemed appropriate for the sixteen possible observation vectors displayed in block one, whether based on the theory of quantum mechanics or not. (Of course the probabilities so asserted need to be assessed on the basis of observations according to some scoring rule, an issue we have not addressed here.) Similarly, realist proponents of quantum theory may hypothesize whatever probability values they think it prescribes. However, since the sixteen events of possible vector observations listed in block one of our realm matrix are both exclusive and exhaustive, the sum of these probabilities must equal 1 for anyone who makes coherent assertions. This understanding is what resolves the conundrum posed by apparent violations of Bell's inequality.

Virtually all discussion of quantum probabilities since the original work of Bell supports the conclusion that probabilities pertinent to quantum behaviour can violate the seemingly innocuous inequality that he identified. The mathematical error that has been discovered and reported in this article substantiates the end of an era of accepting this conclusion. The results aired here will have ramifications for many published estimations based on more sophisticated experimentation as well. There are further consequences for a host of theoretical issues that have been studied and discussed in the context of a mistaken understanding. These include related notions of hidden variables, entangled particles, and information transfer. Discussion of these topics do require philosophical attention to a variety of conceptual constructs in which they are imbedded.

However, the analysis of Aspect/Bell presented here has nothing to do with philosophical distinctions. It has identified a mathematical error in their work that must be recognized no matter what might be the philosophical positions of interested parties. Probabilistic forecasts motivated by quantum mechanics do not violate any laws of probability theory. Full stop.

Discussions of related issues proceeding henceforth will need to begin with this new recognition. Interestingly, this resolution was suspected in some way by Bell himself, though not the analytical detail. This was clearly evident in his musings on the hidden variables question in Bell (1971) which he himself had reprinted in a collection of his publications, Bell (1987).

A final reference relevant to the analysis of this review is the article of Romano Scozzafava (2000) on the role of probability in statistical physics. In the context of the constructive mathematics of Bruno de Finetti's operational subjective statistical method, he discusses several issues that clarify fundamental matters.

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