ON THE SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON FREDHOLM OPERATORS RELATED TO THE ANOMALOUS DIFFUSION

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Abstract: We study solvability of some linear nonhomogeneous elliptic problems and prove that under reasonable technical conditions the convergence in $L^2(\mathbb{R}^d)$ of their right sides implies the existence and the convergence in $H^{2s}(\mathbb{R}^d)$ of the solutions. The equations involve the second order non Fredholm differential operators raised to certain fractional powers s and we use the methods of spectral and scattering theory for Schrödinger type operators developed in our preceding work [26].

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1. Introduction

Consider the equation

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and $V(x)$ is a function decaying to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.1) contains the origin. As a consequence, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article is devoted to the studies of some properties of the operators of this kind raised to a fractional power. We recall that elliptic problems with non-Fredholm operators were treated extensively in recent years (see [18], [21], [19], [23], [20], [22], [24], [25], also [5]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). In the particular case when $a = 0$ the operator A satisfies the Fredholm property in some properly chosen weighted spaces (see [1], [2], [3], [4], [5]). However, the case with $a \neq 0$ is significantly different and the method developed in these articles cannot be applied.

One of the important questions concerning problems with non-Fredholm operators is their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n,\ n \in \mathbb{N}.$$

Because the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [17]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of the non Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In this case, solution in the sense of sequences may not imply the existence of the usual solution. In the present article we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. Solvability in the sense of sequences for the sums of non Fredholm Schrödinger type operators was studied in [27]. In the work we deal with the situation when a second order differential operator without Fredholm property is raised to a certain fractional power. The resulting operator will be defined via the spectral calculus.

Let us consider the equation

$$(-\Delta)^s u - au = f(x),\ x \in \mathbb{R}^d,\ d \in \mathbb{N}, \tag{1.2}$$

where $s \in (0, 1)$, $a \geq 0$ is a constant and the right side is square integrable. The operator $(-\Delta)^s$ is actively used, for instance in the studies of the anomalous diffusion problems (see e.g. [28] and the references therein). Anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. Asymptotic behavior at infinity of the probability density function determines the value of the power of the Laplacian (see [14]). The problem analogous to (1.2) but with the standard Laplacian in the context of the solvability in the sense of sequences was studied in [26]. The case when the power of the negative Laplace operator $s = \dfrac{1}{2}$ was treated recently in [30]. Evidently, for the operator $(-\Delta)^s - a : H^{2s}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the essential spectrum fills the semi-axis $[-a,\ \infty)$ such that its inverse from $L^2(\mathbb{R}^d)$ to $H^{2s}(\mathbb{R}^d)$ is not bounded.

Let us write down the corresponding sequence of equations with $n \in \mathbb{N}$ as

$$(-\Delta)^s u_n - a u_n = f_n(x),\ x \in \mathbb{R}^d,\ d \in \mathbb{N}, \tag{1.3}$$

where the right sides converge to the right side of (1.2) in $L^2(\mathbb{R}^d)$ as $n \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.4}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x)$ is bounded, then clearly the integral in the right side of (1.4) makes sense, like for instance in the case of functions involved in the orthogonality relations of Theorems 1 and 2 below. Let us use the space $H^{2s}(\mathbb{R}^d)$ equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^s u\|^2_{L^2(\mathbb{R}^d)}. \tag{1.5}$$

Throughout the article, the sphere of radius $r > 0$ in $\mathbb{R}^d$ centered at the origin will be designated by S^d_r. When $r = 1$, such unit sphere will be denoted by S^d and $|S^d|$ will stand for its Lebesgue measure. The unit ball in $\mathbb{R}^d$ centered at the origin will be designated by B^d and $|B^d|$ will denote its Lebesgue measure. Let us first formulate the solvability conditions for problem (1.2).

Theorem 1. *Let* $f(x) \in L^2(\mathbb{R}^d),\ d \in \mathbb{N}$ *and* $s \in (0,1)$.

a) Let $a = 0,\ d = 1$. *If* $s \in \left(0, \frac{1}{4}\right)$ *and in addition* $f(x) \in L^1(\mathbb{R})$, *then equation (1.2) has a unique solution* $u(x) \in H^{2s}(\mathbb{R})$.

Suppose that $s \in \left[\frac{1}{4}, \frac{3}{4}\right)$ *and in addition* $xf(x) \in L^1(\mathbb{R})$. *Then problem (1.2) admits a unique solution* $u(x) \in H^{2s}(\mathbb{R})$ *if and only if the equality*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.6}$$

holds.

Suppose that $s \in \left[\frac{3}{4}, 1\right)$ *and in addition* $x^2 f(x) \in L^1(\mathbb{R})$. *Then equation (1.2) has a unique solution* $u(x) \in H^{2s}(\mathbb{R})$ *if and only if orthogonality conditions (1.6) along with*

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{1.7}$$

hold.

b) Let $a = 0,\ d = 2$. *Then when* $s \in \left(0, \frac{1}{2}\right)$ *and additionally* $f(x) \in L^1(\mathbb{R}^2)$, *equation (1.2) admits a unique solution* $u(x) \in H^{2s}(\mathbb{R}^2)$.

Suppose that $s \in \left[\frac{1}{2}, 1\right)$ *and additionally* $xf(x) \in L^1(\mathbb{R}^2)$. *Then equation (1.2) has a unique solution* $u(x) \in H^{2s}(\mathbb{R}^2)$ *if and only if*

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{1.8}$$

holds.

c) Let $a=0,\ d=3$. *If* $s\in\left(0,\frac{3}{4}\right)$ *and in addition* $f(x)\in L^1(\mathbb{R}^3)$, *then problem (1.2) has a unique solution* $u(x)\in H^{2s}(\mathbb{R}^3)$.

Suppose that $s\in\left[\frac{3}{4},1\right)$ *and in addition* $xf(x)\in L^1(\mathbb{R}^3)$. *Then equation (1.2) admits a unique solution* $u(x)\in H^{2s}(\mathbb{R}^3)$ *if and only if*

$$(f(x),1)_{L^2(\mathbb{R}^3)}=0 \tag{1.9}$$

holds.

d) If $a=0,\ d\geq 4$ *with* $s\in(0,1)$ *and additionally* $f(x)\in L^1(\mathbb{R}^d)$, *then problem (1.2) possesses a unique solution* $u(x)\in H^{2s}(\mathbb{R}^d)$.

e) Suppose that $a>0,\ d=1$ *with* $s\in(0,1)$ *and in addition* $xf(x)\in L^1(\mathbb{R})$. *Then equation (1.2) admits a unique solution* $u(x)\in H^{2s}(\mathbb{R})$ *if and only if*

$$\left(f(x),\frac{e^{\pm ia^{\frac{1}{2s}}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})}=0 \tag{1.10}$$

holds.

f) Suppose that $a>0,\ d\geq 2$ *with* $s\in(0,1)$ *and additionally* $xf(x)\in L^1(\mathbb{R}^d)$. *Then problem (1.2) has a unique solution* $u(x)\in H^{2s}(\mathbb{R}^d)$ *if and only if*

$$\left(f(x),\frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)}=0,\quad p\in S^d_{a^{\frac{1}{2s}}} \tag{1.11}$$

holds.

Then we turn our attention to the issue of the solvability in the sense of sequences for our problem.

Theorem 2. *Let* $n\in\mathbb{N}$ *and* $f_n(x)\in L^2(\mathbb{R}^d),\ d\in\mathbb{N}$, *such that* $f_n(x)\to f(x)$ *in* $L^2(\mathbb{R}^d)$ *as* $n\to\infty$.

a) Let $a=0,\ d=1$. *If* $s\in\left(0,\frac{1}{4}\right)$ *and additionally* $f_n(x)\in L^1(\mathbb{R}),\ n\in\mathbb{N}$, *such that* $f_n(x)\to f(x)$ *in* $L^1(\mathbb{R})$ *as* $n\to\infty$, *then equations (1.2) and (1.3) admit unique solutions* $u(x)\in H^{2s}(\mathbb{R})$ *and* $u_n(x)\in H^{2s}(\mathbb{R})$ *respectively, such that* $u_n(x)\to u(x)$ *in* $H^{2s}(\mathbb{R})$ *as* $n\to\infty$.

Suppose that $s\in\left[\frac{1}{4},\frac{3}{4}\right)$. *Let in addition* $xf_n(x)\in L^1(\mathbb{R}),\ n\in\mathbb{N}$, *such that* $xf_n(x)\to xf(x)$ *in* $L^1(\mathbb{R})$ *as* $n\to\infty$ *and the orthogonality conditions*

$$(f_n(x),1)_{L^2(\mathbb{R})}=0 \tag{1.12}$$

hold for all $n\in\mathbb{N}$. *Then equations (1.2) and (1.3) admit unique solutions* $u(x)\in H^{2s}(\mathbb{R})$ *and* $u_n(x)\in H^{2s}(\mathbb{R})$ *respectively, such that* $u_n(x)\to u(x)$ *in* $H^{2s}(\mathbb{R})$ *as* $n\to\infty$.

Suppose that $s \in \left[\frac{3}{4}, 1\right)$. *Let in addition* $x^2 f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, *such that* $x^2 f_n(x) \to x^2 f(x)$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$ *and the orthogonality conditions*

$$(f_n(x), 1)_{L^2(\mathbb{R})} = 0, \quad (f_n(x), x)_{L^2(\mathbb{R})} = 0 \tag{1.13}$$

hold for all $n \in \mathbb{N}$. *Then equations (1.2) and (1.3) have unique solutions* $u(x) \in H^{2s}(\mathbb{R})$ *and* $u_n(x) \in H^{2s}(\mathbb{R})$ *respectively, such that* $u_n(x) \to u(x)$ *in* $H^{2s}(\mathbb{R})$ *as* $n \to \infty$.

b) Let $a = 0,\ d = 2$. *If* $s \in \left(0, \frac{1}{2}\right)$ *and additionally* $f_n(x) \in L^1(\mathbb{R}^2),\ n \in \mathbb{N}$, *such that* $f_n(x) \to f(x)$ *in* $L^1(\mathbb{R}^2)$ *as* $n \to \infty$, *then equations (1.2) and (1.3) possess unique solutions* $u(x) \in H^{2s}(\mathbb{R}^2)$ *and* $u_n(x) \in H^{2s}(\mathbb{R}^2)$ *respectively, such that* $u_n(x) \to u(x)$ *in* $H^{2s}(\mathbb{R}^2)$ *as* $n \to \infty$.

Suppose that $s \in \left[\frac{1}{2}, 1\right)$. *Let in addition* $x f_n(x) \in L^1(\mathbb{R}^2),\ n \in \mathbb{N}$, *such that* $x f_n(x) \to x f(x)$ *in* $L^1(\mathbb{R}^2)$ *as* $n \to \infty$ *and the orthogonality relations*

$$(f_n(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{1.14}$$

hold for all $n \in \mathbb{N}$. *Then equations (1.2) and (1.3) admit unique solutions* $u(x) \in H^{2s}(\mathbb{R}^2)$ *and* $u_n(x) \in H^{2s}(\mathbb{R}^2)$ *respectively, such that* $u_n(x) \to u(x)$ *in* $H^{2s}(\mathbb{R}^2)$ *as* $n \to \infty$.

c) Let $a = 0,\ d = 3$. *Suppose that* $s \in \left(0, \frac{3}{4}\right)$ *and additionally* $f_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, *such that* $f_n(x) \to f(x)$ *in* $L^1(\mathbb{R}^3)$ *as* $n \to \infty$. *Then problems (1.2) and (1.3) possess unique solutions* $u(x) \in H^{2s}(\mathbb{R}^3)$ *and* $u_n(x) \in H^{2s}(\mathbb{R}^3)$ *respectively, such that* $u_n(x) \to u(x)$ *in* $H^{2s}(\mathbb{R}^3)$ *as* $n \to \infty$.

Suppose that $s \in \left[\frac{3}{4}, 1\right)$. *Let in addition* $x f_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, *such that* $x f_n(x) \to x f(x)$ *in* $L^1(\mathbb{R}^3)$ *as* $n \to \infty$ *and*

$$(f_n(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{1.15}$$

holds for all $n \in \mathbb{N}$. *Then equations (1.2) and (1.3) have unique solutions* $u(x) \in H^{2s}(\mathbb{R}^3)$ *and* $u_n(x) \in H^{2s}(\mathbb{R}^3)$ *respectively, such that* $u_n(x) \to u(x)$ *in* $H^{2s}(\mathbb{R}^3)$ *as* $n \to \infty$.

d) Let $a = 0,\ d \geq 4$ *with* $s \in (0, 1)$ *and additionally* $f_n(x) \in L^1(\mathbb{R}^d),\ n \in \mathbb{N}$, *such that* $f_n(x) \to f(x)$ *in* $L^1(\mathbb{R}^d)$ *as* $n \to \infty$. *Then problems (1.2) and (1.3) possess unique solutions* $u(x) \in H^{2s}(\mathbb{R}^d)$ *and* $u_n(x) \in H^{2s}(\mathbb{R}^d)$ *respectively, such that* $u_n(x) \to u(x)$ *in* $H^{2s}(\mathbb{R}^d)$ *as* $n \to \infty$.

e) Let $a > 0,\ d = 1$ *with* $s \in (0, 1)$ *and in addition* $x f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, *such that* $x f_n(x) \to x f(x)$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$. *Let*

$$\left(f_n(x), \frac{e^{\pm i a^{\frac{1}{2s}} x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \tag{1.16}$$

hold for all $n \in \mathbb{N}$. Then equations (1.2) and (1.3) admit unique solutions $u(x) \in H^{2s}(\mathbb{R})$ and $u_n(x) \in H^{2s}(\mathbb{R})$ respectively, such that $u_n(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $n \to \infty$.

f) Let $a > 0,\ d \geq 2$ with $s \in (0,1)$ and additionally $xf_n(x) \in L^1(\mathbb{R}^d),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Let

$$\left(f_n(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{a^{\frac{1}{2s}}} \tag{1.17}$$

hold for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) have unique solutions $u(x) \in H^{2s}(\mathbb{R}^d)$ and $u_n(x) \in H^{2s}(\mathbb{R}^d)$ respectively, such that $u_n(x) \to u(x)$ in $H^{2s}(\mathbb{R}^d)$ as $n \to \infty$.

Let us note that when $a = 0$ each of the cases $a)-d)$ above contains the situation when orthogonality conditions are not required.

We use the hat symbol to denote the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x)e^{-ipx}dx,\ p \in \mathbb{R}^d,\ d \in \mathbb{N}, \tag{1.18}$$

such that

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}}\|f(x)\|_{L^1(\mathbb{R}^d)}. \tag{1.19}$$

In the second part of the article we consider the equation

$$(-\Delta + V(x))^s u - au = f(x), \quad x \in \mathbb{R}^3, \quad a \geq 0, \quad s \in (0,1), \tag{1.20}$$

with the square integrable right side. The corresponding sequence of equations for $n \in \mathbb{N}$ will be

$$(-\Delta + V(x))^s u_n - au_n = f_n(x), \quad x \in \mathbb{R}^3, \quad a \geq 0, \tag{1.21}$$

with $s \in (0,1)$ and the right sides converging to the right side of (1.20) in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Note that the situation when the power $s = \dfrac{1}{2}$ was studied in the recent work [30]. Let us make the following technical assumptions on the scalar potential involved in the problems above. Note that the conditions on $V(x)$, which is shallow and short-range will be analogous to those stated in Assumption 1.1 of [21] (see also [19], [23]). The essential spectrum of such a Schrödinger operator $-\Delta + V(x)$ fills the nonnegative semi-axis (see e.g. [11]).

Assumption 3. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\delta}}$$

with some $\delta > 0$ *and* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *a.e. such that*

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad and \quad \sqrt{c_{HLS}} \|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi. \tag{1.22}$$

Here and further down C will stand for a finite positive constant and c_{HLS} given on p.98 of [13] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} dx dy \Big| \le c_{HLS} \|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

By virtue of Lemma 2.3 of [21], under Assumption 3 above on the potential function, the operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta$ via the wave operators (see [12], [16])

$$\Omega^\pm := s - \lim_{t \to \mp\infty} e^{it(-\Delta+V)} e^{it\Delta},$$

where the limit is understood in the strong L^2 sense (see e.g. [15] p.34, [6] p.90). Hence $(-\Delta + V(x))^s$ on $L^2(\mathbb{R}^3)$ defined via the spectral calculus has only the essential spectrum

$$\sigma_{ess}((-\Delta + V(x))^s - a) = [-a, \infty)$$

and no nontrivial $L^2(\mathbb{R}^3)$ eigenfunctions. By means of the spectral theorem, its functions of the continuous spectrum satisfy

$$(-\Delta + V(x))^s \varphi_k(x) = |k|^{2s} \varphi_k(x), \quad k \in \mathbb{R}^3, \tag{1.23}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [15] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V\varphi_k)(y) dy \tag{1.24}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3. \tag{1.25}$$

Particularly, when the vector $k = 0$, we have $\varphi_0(x)$. Let us denote the generalized Fourier transform with respect to these functions using the tilde symbol as

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \; k \in \mathbb{R}^3. \tag{1.26}$$

(1.26) is a unitary transform on $L^2(\mathbb{R}^3)$. The integral operator involved in (1.24) is being denoted as

$$(Q\varphi)(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V\varphi)(y) dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Under Assumption 3, via Lemma 2.1 of [21] the operator norm $\|Q\|_\infty$ is bounded above by the quantity $I(V)$, which is the left side of the first inequality in (1.22), such that $I(V) < 1$. Corollary 2.2 of [21] under our assumptions gives us the bound

$$|\tilde{f}(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f(x)\|_{L^1(\mathbb{R}^3)}. \tag{1.27}$$

We have the following result concerning the solvability of equation (1.20).

Theorem 4. *Let Assumption 3 hold and $f(x) \in L^2(\mathbb{R}^3)$.*

a) Let $a = 0,\ s \in \left(0, \frac{3}{4}\right)$ and additionally $f(x) \in L^1(\mathbb{R}^3)$. Then equation (1.20) possesses a unique solution $u(x) \in L^2(\mathbb{R}^3)$.

Let $a = 0,\ s \in \left[\frac{3}{4}, 1\right)$ and in addition $xf(x) \in L^1(\mathbb{R}^3)$. Then problem (1.20) admits a unique solution $u(x) \in L^2(\mathbb{R}^3)$ if and only if

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.28}$$

holds.

b) Let $a > 0,\ s \in (0, 1)$ and in addition $xf(x) \in L^1(\mathbb{R}^3)$. Then equation (1.20) has a unique solution $u(x) \in L^2(\mathbb{R}^3)$ if and only if

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{a^{\frac{1}{2s}}} \tag{1.29}$$

holds.

Our final main statement is devoted to the solvability in the sense of sequences of problem (1.20).

Theorem 5. *Let Assumption 3 hold, $n \in \mathbb{N}$ and $f_n(x) \in L^2(\mathbb{R}^3)$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.*

a) Let $a = 0$. If $s \in \left(0, \frac{3}{4}\right)$ and additionally $f_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$, then equations (1.20) and (1.21) possess unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.

Suppose that $s \in \left[\frac{3}{4}, 1\right)$. Let in addition $xf_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ and

$$(f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.30}$$

holds for all $n \in \mathbb{N}$. Then equations (1.20) and (1.21) admit unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.

b) Suppose that $a > 0,\ s \in (0,1)$. Let in addition $xf_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ and

$$(f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{a^{\frac{1}{2s}}} \tag{1.31}$$

holds for all $n \in \mathbb{N}$. Then problems (1.20) and (1.21) have unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.

Let us note that (1.28) and (1.29) are the orthogonality relations to the functions of the continuous spectrum of our Schrödinger operator, as distinct from the Limiting Absorption Principle in which one needs to orthogonalize to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [9]).

2. Solvability in the sense of sequences in the free Laplacian case

Proof of Theorem 1. Let us note that the case a) of the theorem was stated in Lemma 4.1 of [29] and the case c) in Lemma 5 of [28].

Clearly, if $u(x) \in L^2(\mathbb{R}^d)$ is a solution of (1.2) with a square integrable right side, it belongs to $H^{2s}(\mathbb{R}^d)$ as well. Indeed, in this case from (1.2) we easily deduce $(-\Delta)^s u(x) \in L^2(\mathbb{R}^d)$, such that via norm definition (1.5) we have $u(x) \in H^{2s}(\mathbb{R}^d)$.

To prove the uniqueness of solutions for our equation, let us suppose that (1.2) has two square integrable solutions $u_1(x)$ and $u_2(x)$. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^d)$ as well. Obviously, it is a solution of the equation

$$(-\Delta)^s w = aw.$$

Since the operator $(-\Delta)^s$ has no nontrivial square integrable eigenfunctions in the whole space, we have $w(x) = 0$ a.e. in $\mathbb{R}^d$.

We apply the standard Fourier transform (1.18) to both sides of problem (1.2) with $a = 0$. This gives us

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s}}\chi_{\{|p|>1\}}. \tag{2.32}$$

Here and further down χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}^d$. Clearly, the second term in the right side of (2.32) can be bounded from above in the absolute value by $|\widehat{f}(p)| \in L^2(\mathbb{R}^d)$ due to the one of our assumptions.

First we consider the case b) of the theorem when the dimension of the problem $d = 2$. Let us estimate the first term in the right side of (2.32) from above in the absolute value using (1.19) by $\dfrac{\|f(x)\|_{L^1(\mathbb{R}^2)}}{2\pi|p|^{2s}}\chi_{\{|p|\le 1\}}$. It can be easily verified that such expression is square integrable when $s \in \left(0, \frac{1}{2}\right)$.

To treat the case when $s \in \left(\frac{1}{2}, 1\right)$, we use the formula

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds.$$

Here and throughout the article σ will denote the angle variables on the sphere. This enables us to express the first term in the right side of (2.32) as

$$\frac{\widehat{f}(0)}{|p|^{2s}} \chi_{\{|p|\le 1\}} + \frac{\int_0^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds}{|p|^{2s}} \chi_{\{|p|\le 1\}}. \tag{2.33}$$

Note that by means of the definition of the Fourier transform (1.18), we easily derive for the space of an arbitrary dimension

$$\left| \frac{\partial \widehat{f}(p)}{\partial |p|} \right| \le \frac{\|xf(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}}, \quad d \in \mathbb{N}. \tag{2.34}$$

Therefore, the second term in (2.33) can be bounded from above in the absolute value by

$$\frac{\|xf(x)\|_{L^1(\mathbb{R}^2)}}{2\pi} |p|^{1-2s} \chi_{\{|p|\le 1\}} \in L^2(\mathbb{R}^2).$$

It can be easily verified that the first term in (2.33) is square integrable if and only if $\widehat{f}(0)$ vanishes, which is equivalent to orthogonality relation (1.8).

Then we turn our attention to the case d) of the theorem. Let us estimate the first term in the right side of (2.32) from above in the absolute value via (1.19) by $\frac{\|f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}} |p|^{2s}} \chi_{\{|p|\le 1\}}$, $d \ge 4$. It can be easily checked that this expression is square integrable for $s \in (0,1)$.

Let us apply the standard Fourier transform (1.18) to both sides of equation (1.2) when $a > 0$. This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - a}.$$

First of all we consider the case e) of the theorem, namely when the dimension of the problem $d = 1$. For $s \in (0,1)$ we define the following sets on the real line

$$I_\delta^+ := [a^{\frac{1}{2s}} - \delta, a^{\frac{1}{2s}} + \delta], \quad I_\delta^- := [-a^{\frac{1}{2s}} - \delta, -a^{\frac{1}{2s}} + \delta], \quad 0 < \delta < a^{\frac{1}{2s}}, \tag{2.35}$$

such that

$$I_\delta := I_\delta^+ \cup I_\delta^-, \quad \mathbb{R} = I_\delta \cup I_\delta^c.$$

Here and further down $A^c \subseteq \mathbb{R}^d$ stands for the complement of the set $A \subseteq \mathbb{R}^d$. This allows us to express $\widehat{u}(p)$ as the sum

$$\frac{\widehat{f}(p)}{|p|^{2s} - a} \chi_{I_\delta^c} + \frac{\widehat{f}(p)}{p^{2s} - a} \chi_{I_\delta^+} + \frac{\widehat{f}(p)}{(-p)^{2s} - a} \chi_{I_\delta^-}. \tag{2.36}$$

Evidently, we have $I_\delta^c = I_\delta^{c+} \cup I_\delta^{c-}$, where

$$I_\delta^{c+} := I_\delta^c \cap \mathbb{R}^+, \quad I_\delta^{c-} := I_\delta^c \cap \mathbb{R}^-. \tag{2.37}$$

Here $\mathbb{R}^+$ and $\mathbb{R}^-$ are the nonnegative and the negative semi-axes of the real line respectively. Clearly,

$$\left| \frac{\widehat{f}(p)}{p^{2s}-a} \chi_{I_\delta^{c+}} \right| \leq C|\widehat{f}(p)| \in L^2(\mathbb{R})$$

due to the one of our assumptions. Analogously,

$$\left| \frac{\widehat{f}(p)}{(-p)^{2s}-a} \chi_{I_\delta^{c-}} \right| \leq C|\widehat{f}(p)| \in L^2(\mathbb{R}).$$

We express

$$\widehat{f}(p) = \widehat{f}(a^{\frac{1}{2s}}) + \int_{a^{\frac{1}{2s}}}^p \frac{d\widehat{f}(s)}{ds} ds.$$

(2.34) easily gives us the upper bound

$$\left| \frac{\int_{a^{\frac{1}{2s}}}^p \frac{d\widehat{f}(s)}{ds} ds}{p^{2s}-a} \chi_{I_\delta^+} \right| \leq \frac{1}{\sqrt{2\pi}} \|xf(x)\|_{L^1(\mathbb{R})} \left| \frac{p - a^{\frac{1}{2s}}}{p^{2s}-a} \right| \chi_{I_\delta^+} \leq$$

$$\leq C\|xf(x)\|_{L^1(\mathbb{R})} \chi_{I_\delta^+} \in L^2(\mathbb{R}).$$

Apparently,

$$\frac{\widehat{f}(a^{\frac{1}{2s}})}{p^{2s}-a} \chi_{I_\delta^+} \in L^2(\mathbb{R})$$

if and only if $\widehat{f}(a^{\frac{1}{2s}})$ vanishes, which is equivalent to the orthogonality condition

$$\left(f(x), \frac{e^{ia^{\frac{1}{2s}}x}}{\sqrt{2\pi}} \right)_{L^2(\mathbb{R})} = 0, \quad s \in (0,1).$$

To study the singularity of the problem on the negative semi-axis, we apply the formula

$$\widehat{f}(p) = \widehat{f}(-a^{\frac{1}{2s}}) + \int_{-a^{\frac{1}{2s}}}^p \frac{d\widehat{f}(s)}{ds} ds.$$

Via (2.34) we have the upper bound

$$\left| \frac{\int_{-a^{\frac{1}{2s}}}^p \frac{d\widehat{f}(s)}{ds} ds}{(-p)^{2s}-a} \chi_{I_\delta^-} \right| \leq \frac{1}{\sqrt{2\pi}} \|xf(x)\|_{L^1(\mathbb{R})} \left| \frac{p + a^{\frac{1}{2s}}}{(-p)^{2s}-a} \right| \chi_{I_\delta^-} \leq$$

$$\le C\|xf(x)\|_{L^1(\mathbb{R})}\chi_{I_\delta^-} \in L^2(\mathbb{R}).$$

Evidently,

$$\frac{\widehat{f}(-a^{\frac{1}{2s}})}{(-p)^{2s}-a}\chi_{I_\delta^-} \in L^2(\mathbb{R})$$

if and only if $\widehat{f}(-a^{\frac{1}{2s}}) = 0$, which is equivalent to the orthogonality relation

$$\left(f(x), \frac{e^{-ia^{\frac{1}{2s}}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0, \quad s \in (0,1).$$

We complete the proof of the theorem with establishing the part f). When the dimension $d \ge 2$, we define the set

$$A_\delta := \{p \in \mathbb{R}^d \mid a^{\frac{1}{2s}} - \delta \le |p| \le a^{\frac{1}{2s}} + \delta\}, \quad 0 < \delta < a^{\frac{1}{2s}} \tag{2.38}$$

and express

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}-a}\chi_{A_\delta} + \frac{\widehat{f}(p)}{|p|^{2s}-a}\chi_{A_\delta^c}. \tag{2.39}$$

Obviously, we have the estimate from above

$$\left|\frac{\widehat{f}(p)}{|p|^{2s}-a}\chi_{A_\delta^c}\right| \le C|\widehat{f}(p)| \in L^2(\mathbb{R}^d)$$

via the one of our assumptions. To treat the first term in the right side of (2.39), we will use the representation formula

$$\widehat{f}(p) = \widehat{f}(a^{\frac{1}{2s}}, \sigma) + \int_{a^{\frac{1}{2s}}}^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds.$$

Inequality (2.34) enables us to estimate

$$\left|\frac{\int_{a^{\frac{1}{2s}}}^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds}{|p|^{2s}-a}\chi_{A_\delta}\right| \le \frac{\|xf(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}}\left|\frac{|p| - a^{\frac{1}{2s}}}{|p|^{2s}-a}\right|\chi_{A_\delta} \le$$

$$\le C\|xf(x)\|_{L^1(\mathbb{R}^d)}\chi_{A_\delta} \in L^2(\mathbb{R}^d).$$

It can be easily verified that the remaining term

$$\frac{\widehat{f}(a^{\frac{1}{2s}}, \sigma)}{|p|^{2s}-a}\chi_{A_\delta} \in L^2(\mathbb{R}^d)$$

if and only if $\widehat{f}(a^{\frac{1}{2s}}, \sigma)$ vanishes, which is equivalent to orthogonality relation (1.11) for the dimensions $d \ge 2$. ■

Then we proceed to establishing the solvability in the sense of sequences for our equation in the no potential case.

Proof of Theorem 2. Suppose $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique solutions of equations (1.2) and (1.3) in $H^{2s}(\mathbb{R}^d),\ d \in \mathbb{N}$ with $a \geq 0$ respectively, $s \in (0,1)$ and it is known that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Then $u_n(x) \to u(x)$ in $H^{2s}(\mathbb{R}^d)$ as $n \to \infty$ as well. Indeed,

$$(-\Delta)^s(u_n(x) - u(x)) = a(u_n(x) - u(x)) + f_n(x) - f(x),$$

which clearly gives us

$$\|(-\Delta)^s(u_n(x)-u(x))\|_{L^2(\mathbb{R}^d)} \leq a\|u_n(x)-u(x)\|_{L^2(\mathbb{R}^d)}+\|f_n(x)-f(x)\|_{L^2(\mathbb{R}^d)} \to 0$$

as $n \to \infty$ via our assumptions. Norm definition (1.5) yields $u_n(x) \to u(x)$ in $H^{2s}(\mathbb{R}^d)$ as $n \to \infty$.

If $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique solutions of equations (1.2) and (1.3) in $H^{2s}(\mathbb{R}^d),\ d \in \mathbb{N}$ respectively with $a = 0$ as in the cases a)-d) of the theorem, by applying the standard Fourier transform (1.18) we easily derive

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s}}\chi_{\{|p|\leq 1\}} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s}}\chi_{\{|p|>1\}}. \tag{2.40}$$

Evidently, the second term in the right side of equality (2.40) can be estimated from above in the absolute value in the space of any dimension by $|\widehat{f}_n(p) - \widehat{f}(p)|$, such that

$$\left\|\frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s}}\chi_{\{|p|>1\}}\right\|_{L^2(\mathbb{R}^d)} \leq \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

due to the one of our assumptions.

First we treat the case a) of the theorem when the dimension $d = 1$. Then, when $s \in \left(0, \frac{1}{4}\right)$ via the part a) of Theorem 1, equation (1.2) and each of equations (1.3) admit unique solutions $u(x) \in H^{2s}(\mathbb{R})$ and $u_n(x) \in H^{2s}(\mathbb{R}), \quad n \in \mathbb{N}$ respectively. Clearly, the first term in the right side of equality (2.40) can be bounded from above in the absolute value via (1.19) by $\dfrac{1}{\sqrt{2\pi}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R})}\dfrac{\chi_{\{|p|\leq 1\}}}{|p|^{2s}}$, such that its $L^2(\mathbb{R})$ norm can be estimated from above by

$$\frac{1}{\sqrt{\pi}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R})}\frac{1}{\sqrt{1-4s}} \to 0, \quad n \to \infty$$

due to the one of our assumptions and with $s \in \left(0, \frac{1}{4}\right)$. This shows that in this case $u_n(x) \to u(x)$ in $L^2(\mathbb{R})$ as $n \to \infty$.

Then we turn our attention to the case of $s \in \left[\frac{1}{4}, \frac{3}{4}\right)$. Note that by means of the parts a) and b) of Lemma 6 below, under our assumptions we have $f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$. Then, via (1.12) we obtain

$$|(f(x), 1)_{L^2(\mathbb{R})}| = |(f(x) - f_n(x), 1)_{L^2(\mathbb{R})}| \le \|f_n(x) - f(x)\|_{L^1(\mathbb{R})} \to 0$$

as $n \to \infty$. Thus,

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.41}$$

holds. By means of the part a) of Theorem 1, when $s \in \left[\frac{1}{4}, \frac{3}{4}\right)$, equations (1.2) and (1.3) admit unique solutions $u(x), u_n(x) \in H^{2s}(\mathbb{R}),\ n \in \mathbb{N}$ respectively. Orthogonality relations (2.41) and (1.12) yield

$$\widehat{f}(0) = 0, \quad \widehat{f}_n(0) = 0, \quad n \in \mathbb{N}$$

in this case. This allows us to use the expressions

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds} ds, \quad \widehat{f}_n(p) = \int_0^p \frac{d\widehat{f}_n(s)}{ds} ds, \quad n \in \mathbb{N},$$

which enables us to write the first term in the right side of equality (2.40) as

$$\frac{\int_0^p \left(\frac{d\widehat{f}_n(s)}{ds} - \frac{d\widehat{f}(s)}{ds} \right) ds}{|p|^{2s}} \chi_{\{|p| \le 1\}}. \tag{2.42}$$

Using inequality (2.34), we easily estimate

$$\left| \frac{d\widehat{f}_n(p)}{dp} - \frac{d\widehat{f}(p)}{dp} \right| \le \frac{1}{\sqrt{2\pi}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})}, \tag{2.43}$$

such that expression (2.42) can be bounded from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} |p|^{1-2s} \chi_{\{|p| \le 1\}}.$$

Hence, we arrive at

$$\left\| \frac{\int_0^p \left(\frac{d\widehat{f}_n(s)}{ds} - \frac{d\widehat{f}(s)}{ds} \right) ds}{|p|^{2s}} \chi_{\{|p| \le 1\}} \right\|_{L^2(\mathbb{R})} \le \frac{1}{\sqrt{\pi}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \to 0$$

as $n \to \infty$ due to the one of our assumptions. This implies that

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}), \quad n \to \infty$$

when the dimension $d = 1$ and $a = 0$ with $s \in \left[\frac{1}{4}, \frac{3}{4}\right)$.

Then we proceed to the proof of the theorem when the power of the negative Laplacian $s \in \left[\frac{3}{4}, 1\right)$ in dimension $d = 1$ with $a = 0$. By means of the parts c) and d) of Lemma 6 below under our assumptions we have $xf_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$. Then via the parts a) and b) of Lemma 6 we have $f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$. Orthogonality condition (2.41) here can be easily obtained via the limiting argument as above. By means of the second orthogonality relation in (1.13), we derive

$$|(f(x), x)_{L^2(\mathbb{R})}| = |(f(x) - f_n(x), x)_{L^2(\mathbb{R})}| \leq \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \to 0$$

as $n \to \infty$. Hence

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{2.44}$$

holds. By virtue of the part a) of Theorem 1, when $s \in \left[\frac{3}{4}, 1\right)$, equations (1.2) and (1.3) possess unique solutions $u(x), u_n(x) \in H^{2s}(\mathbb{R}),\ n \in \mathbb{N}$ respectively. Via the definition of the standard Fourier transform (1.18), orthogonality relations (2.41), (1.13) and (2.44) give us for $n \in \mathbb{N}$

$$\widehat{f}(0) = 0, \quad \widehat{f}_n(0) = 0, \quad \frac{d\widehat{f}}{dp}(0) = 0, \quad \frac{d\widehat{f}_n}{dp}(0) = 0,$$

such that

$$\widehat{f}(p) = \int_0^p \left(\int_0^s \frac{d^2\widehat{f}(q)}{dq^2} dq\right) ds, \quad \widehat{f}_n(p) = \int_0^p \left(\int_0^s \frac{d^2\widehat{f}_n(q)}{dq^2} dq\right) ds, \quad n \in \mathbb{N}.$$

By means of definition (1.18), we easily estimate

$$\left|\frac{d^2\widehat{f}_n(p)}{dp^2} - \frac{d^2\widehat{f}(p)}{dp^2}\right| \leq \frac{1}{\sqrt{2\pi}} \|x^2 f_n(x) - x^2 f(x)\|_{L^1(\mathbb{R})}.$$

This yields the inequality

$$|\widehat{f}_n(p) - \widehat{f}(p)| \leq \frac{1}{\sqrt{2\pi}} \|x^2 f_n(x) - x^2 f(x)\|_{L^1(\mathbb{R})} \frac{p^2}{2},$$

which allows us to obtain the upper bound on the absolute value of the first term in the right side of identity (2.40) by

$$\frac{1}{2\sqrt{2\pi}} \|x^2 f_n(x) - x^2 f(x)\|_{L^1(\mathbb{R})} |p|^{2-2s} \chi_{\{|p| \leq 1\}}.$$

Therefore,

$$\left| \frac{\widehat{f_n}(p) - \widehat{f}(p)}{|p|^{2s}} \chi_{\{|p| \le 1\}} \right|_{L^2(\mathbb{R})} \le \frac{1}{2\sqrt{\pi(5-4s)}} \|x^2 f_n(x) - x^2 f(x)\|_{L^1(\mathbb{R})} \to 0$$

when $n \to \infty$ as assumed. Thus

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}), \quad n \to \infty$$

when the dimension $d = 1$ and $a = 0$ with $s \in \left[\frac{3}{4}, 1\right)$.

In the case of the dimension $d = 2$ and $a = 0$, let us first treat the situation when $s \in \left(0, \frac{1}{2}\right)$. Due to the part b) of Theorem 1, problem (1.2) and each of problems (1.3) have unique solutions $u(x) \in H^{2s}(\mathbb{R})$ and $u_n(x) \in H^{2s}(\mathbb{R}), \quad n \in \mathbb{N}$ respectively. Obviously, the first term in the right side of (2.40) can be estimated from above in the absolute value via (1.19) by $\frac{1}{2\pi}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^2)} \frac{\chi_{\{|p|\le 1\}}}{|p|^{2s}}$, such that its $L^2(\mathbb{R}^2)$ norm can be bounded from above by

$$\frac{1}{2\sqrt{\pi(1-2s)}} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad n \to \infty$$

by means the one of our assumptions and with $s \in \left(0, \frac{1}{2}\right)$.

For the higher values of the power of the two dimensional negative Laplacian $s \in \left(\frac{1}{2}, 1\right)$, the orthogonality relation

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{2.45}$$

can be derived via the easy limiting argument, analogously to (2.41). By virtue of the part b) of Theorem 1, problems (1.2) and (1.3) possess unique solutions $u(x) \in H^{2s}(\mathbb{R}^2)$ and $u_n(x) \in H^{2s}(\mathbb{R}^2),\ n \in \mathbb{N}$ respectively. Orthogonality relations (2.45) and (1.12) imply

$$\widehat{f}(0) = 0, \quad \widehat{f_n}(0) = 0, \quad n \in \mathbb{N}$$

when the dimension $d = 2$ and $a = 0$ with $s \in \left(\frac{1}{2}, 1\right)$. This enables us to express

$$\widehat{f}(p) = \int_0^{|p|} \frac{\partial \widehat{f}(s,\sigma)}{\partial s} ds, \quad \widehat{f_n}(p) = \int_0^{|p|} \frac{\partial \widehat{f_n}(s,\sigma)}{\partial s} ds, \quad n \in \mathbb{N} \tag{2.46}$$

and to write the first term in the right side of identity (2.40) as

$$\frac{\int_0^{|p|} \left(\frac{\partial \widehat{f_n}(s,\sigma)}{\partial s} - \frac{\partial \widehat{f}(s,\sigma)}{\partial s} \right) ds}{|p|^{2s}} \chi_{\{|p|\le 1\}}. \tag{2.47}$$

Inequality (2.34) gives us

$$\left| \frac{\partial \widehat{f_n}(p)}{\partial |p|} - \frac{\partial \widehat{f}(p)}{\partial |p|} \right| \le \frac{1}{2\pi} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^2)}. \tag{2.48}$$

Thus, expression (2.47) can be bounded from above in the absolute value by

$$\frac{1}{2\pi} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^2)} |p|^{1-2s} \chi_{\{|p| \le 1\}}.$$

Hence

$$\left\| \frac{\int_0^{|p|} \left(\frac{\partial \widehat{f_n}(s,\sigma)}{\partial s} - \frac{\partial \widehat{f}(s,\sigma)}{\partial s} \right) ds}{|p|^{2s}} \chi_{\{|p| \le 1\}} \right\|_{L^2(\mathbb{R}^2)} \le \frac{\|x f_n(x) - x f(x)\|_{L^1(\mathbb{R}^2)}}{2\sqrt{2\pi(1-s)}} \to 0$$

as $n \to \infty$ via the one of our assumptions. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^2), \quad n \to \infty$$

when the dimension $d = 2$ and $a = 0$ with $s \in \left(\frac{1}{2}, 1\right)$.

Let us proceed to the proof of the part c) of the theorem, when the dimension $d = 3$ and $a = 0$ with $s \in \left(0, \frac{3}{4}\right)$. In such case, by virtue of the part c) of Theorem 1, problems (1.2) and (1.3) admit unique solutions $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ respectively, belonging to $H^{2s}(\mathbb{R}^3)$. Using (1.19), we obtain the upper bound on the first term in the right side of (2.40) in the absolute value by

$$\frac{\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}} |p|^{2s}} \chi_{\{|p| \le 1\}},$$

such that its $L^2(\mathbb{R}^3)$ norm can be estimated from above by

$$\frac{1}{\pi\sqrt{2(3-4s)}} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

via the one of our assumptions. Thus,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^3), \quad n \to \infty$$

in the case of the dimension $d = 3$ and $a = 0$ with $s \in \left(0, \frac{3}{4}\right)$.

For the higher values of the power of the three dimensional negative Laplacian $s \in \left[\frac{3}{4}, 1\right)$, the orthogonality condition

$$(f(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{2.49}$$

can be obtained via the trivial limiting argument, similarly to (2.41). By means of the part c) of Theorem 1, equations (1.2) and (1.3) have unique solutions $u(x) \in H^{2s}(\mathbb{R}^3)$ and $u_n(x) \in H^{2s}(\mathbb{R}^3)$, $n \in \mathbb{N}$ respectively. Orthogonality conditions (2.49) and (1.15) yield

$$\widehat{f}(0) = 0, \quad \widehat{f_n}(0) = 0, \quad n \in \mathbb{N}$$

when the dimension $d = 3$ and $a = 0$ with $s \in \left[\frac{3}{4}, 1\right)$. This allows us to obtain here the expressions analogous to (2.46). Let us use the three dimensional analog of inequality (2.48) to derive the upper bound on the first term in the right side of (2.40) in the absolute value by

$$\frac{\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}}|p|^{1-2s}\chi_{\{|p|\le 1\}},$$

such that its $L^2(\mathbb{R}^3)$ norm can be estimated from above by

$$\frac{1}{\pi\sqrt{2(5-4s)}}\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

due to the one of our assumptions. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^3), \quad n \to \infty$$

in the case of the dimension $d = 3$ and $a = 0$ with $s \in \left[\frac{3}{4}, 1\right)$.

Then we turn our attention to the case d) of the theorem. By virtue of the part d) of Theorem 1 equations (1.2) and (1.3) admit a unique solutions $u(x) \in H^{2s}(\mathbb{R}^3)$ and $u_n(x) \in H^{2s}(\mathbb{R}^3)$, $n \in \mathbb{N}$ respectively. Using inequality (1.19), we estimate the first term in the right side of (2.40) in the absolute value by

$$\frac{\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}|p|^{2s}}\chi_{\{|p|\le 1\}}, \quad d \ge 4,$$

such that its $L^2(\mathbb{R}^d)$ norm can be bounded from above by

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\sqrt{\frac{|S^d|}{d-4s}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

by virtue of the one of our assumptions. Hence,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^d), \quad d \ge 4, \quad n \to \infty$$

when $a = 0$ and $s \in (0, 1)$.

If $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique solutions of equations (1.2) and (1.3) in $H^{2s}(\mathbb{R}^d),\ d \in \mathbb{N}$ respectively with $a > 0$ as in the cases e) and f) of the theorem, by applying the standard Fourier transform (1.18) we easily obtain

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - a}, \quad \widehat{u}_n(p) = \frac{\widehat{f}_n(p)}{|p|^{2s} - a}, \quad n \in \mathbb{N}. \tag{2.50}$$

First of all, we consider the case e) of the theorem, when the dimension $d = 1$ and $a > 0$. Thus, due to the result of the part e) of Theorem 1, equation (1.3) has a unique solution $u_n(x) \in H^{2s}(\mathbb{R}),\ n \in \mathbb{N}$. Clearly, $f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$ via the parts a) and b) of Lemma 6 below. By means of the limiting argument, analogously to the proof of (2.41) we obtain the orthogonality relations

$$\left(f(x), \frac{e^{\pm i a^{\frac{1}{2s}} x}}{\sqrt{2\pi}} \right)_{L^2(\mathbb{R})} = 0, \quad s \in (0, 1). \tag{2.51}$$

Then by virtue of the result of the part e) of Theorem 1, problem (1.2) admits a unique solution $u(x) \in H^{2s}(\mathbb{R})$. Using (2.50), we express $\widehat{u}_n(p) - \widehat{u}(p)$ as

$$\frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^{2s} - a} \chi_{I_\delta^+} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^{2s} - a} \chi_{I_\delta^{c+}} +$$
$$+ \frac{\widehat{f}_n(p) - \widehat{f}(p)}{(-p)^{2s} - a} \chi_{I_\delta^-} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{(-p)^{2s} - a} \chi_{I_\delta^{c-}}, \tag{2.52}$$

with $I_\delta^+,\ I_\delta^-$ are given by (2.35) and $I_\delta^{c+},\ I_\delta^{c-}$ are defined in (2.37). Evidently, the second term in (2.52) can be estimated from above in the absolute value by $C|\widehat{f}_n(p) - \widehat{f}(p)|$, such that

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{p^{2s} - a} \chi_{I_\delta^{c+}} \right\|_{L^2(\mathbb{R})} \le C \|f_n(x) - f(x)\|_{L^2(\mathbb{R})} \to 0, \quad n \to \infty$$

as assumed. Analogously, the last term in (2.52) can be bounded from above in the absolute value by $C|\widehat{f}_n(p) - \widehat{f}(p)|$. Thus

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{(-p)^{2s} - a} \chi_{I_\delta^{c-}} \right\|_{L^2(\mathbb{R})} \le C \|f_n(x) - f(x)\|_{L^2(\mathbb{R})} \to 0, \quad n \to \infty$$

due to the one of our assumptions. Orthogonality relations (2.51) and (1.16) give us

$$\widehat{f}(a^{\frac{1}{2s}}) = 0, \quad \widehat{f}_n(a^{\frac{1}{2s}}) = 0, \quad n \in \mathbb{N},$$

such that

$$\widehat{f}(p) = \int_{a^{\frac{1}{2s}}}^{p} \frac{d\widehat{f}(s)}{ds} ds, \quad \widehat{f}_n(p) = \int_{a^{\frac{1}{2s}}}^{p} \frac{d\widehat{f}_n(s)}{ds} ds, \quad n \in \mathbb{N},$$

which enables us to express the first term in (2.52) as

$$\frac{\int_{a^{\frac{1}{2s}}}^{p} \left[\frac{d\widehat{f}_n(s)}{ds} - \frac{d\widehat{f}(s)}{ds}\right] ds}{p^{2s} - a} \chi_{I_\delta^+}. \tag{2.53}$$

By means of (2.43), we obtain the upper bound on (2.53) in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \left|\frac{p - a^{\frac{1}{2s}}}{p^{2s} - a}\right| \chi_{I_\delta^+} \leq C\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \chi_{I_\delta^+}.$$

Thus, the $L^2(\mathbb{R})$ norm of (2.53) can be estimated from above by

$$C\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \to 0, \quad n \to \infty$$

due to the one of our assumptions. Orthogonality conditions (2.51) and (1.16) yield

$$\widehat{f}(-a^{\frac{1}{2s}}) = 0, \quad \widehat{f}_n(-a^{\frac{1}{2s}}) = 0, \quad n \in \mathbb{N}$$

with $s \in (0, 1)$. Therefore, at the negative singularity

$$\widehat{f}(p) = \int_{-a^{\frac{1}{2s}}}^{p} \frac{d\widehat{f}(s)}{ds} ds, \quad \widehat{f}_n(p) = \int_{-a^{\frac{1}{2s}}}^{p} \frac{d\widehat{f}_n(s)}{ds} ds, \quad n \in \mathbb{N}.$$

This gives us the upper bound on the third term in (2.52) in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \left|\frac{p + a^{\frac{1}{2s}}}{(-p)^{2s} - a}\right| \chi_{I_\delta^-} \leq C\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \chi_{I_\delta^-}.$$

Thus, its $L^2(\mathbb{R})$ norm can be estimated from above by

$$C\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \to 0, \quad n \to \infty$$

as assumed. This shows that in dimension $d = 1$, when $a > 0$ and $s \in (0, 1)$ we have

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}), \quad n \to \infty.$$

We conclude the proof of the theorem with treating the case f) when the dimension $d \geq 2$ and $a > 0$ with $s \in (0, 1)$. Then under our assumptions, by virtue of the part f) of Theorem 1, problem (1.3) has a unique solution $u_n(x) \in H^{2s}(\mathbb{R}^d), \ n \in \mathbb{N}$. A trivial limiting argument analogous to the proof of (2.41) gives us

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{a^{\frac{1}{2s}}}. \tag{2.54}$$

Then by means of the part f) of Theorem 1, problem (1.2) admits a unique solution $u(x) \in H^{2s}(\mathbb{R}^d)$. Using (2.50), we easily arrive at

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s} - a}\chi_{A_\delta} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s} - a}\chi_{A_\delta^c}, \tag{2.55}$$

with the set A_δ defined in (2.38). Evidently, the second term in the right side of (2.55) can be estimated from above in the absolute value by $C|\widehat{f}_n(p) - \widehat{f}(p)|$. Thus,

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s} - a}\chi_{A_\delta^c} \right\|_{L^2(\mathbb{R}^d)} \leq C\|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0$$

as $n \to \infty$ via the one of our assumptions. Orthogonality relations (2.54) and (1.17) imply that

$$\widehat{f}(a^{\frac{1}{2s}}, \sigma) = 0, \quad \widehat{f}_n(a^{\frac{1}{2s}}, \sigma) = 0, \quad n \in \mathbb{N},$$

such that

$$\widehat{f}(p) = \int_{a^{\frac{1}{2s}}}^{|p|} \frac{\partial \widehat{f}(s, \sigma)}{\partial s} ds, \quad \widehat{f}_n(p) = \int_{a^{\frac{1}{2s}}}^{|p|} \frac{\partial \widehat{f}_n(s, \sigma)}{\partial s} ds, \quad n \in \mathbb{N}.$$

By means of the definition of the Fourier transform (1.18), analogously to inequalities (2.43) and (2.48) in lower dimensions, we easily obtain

$$\left| \frac{\partial \widehat{f}_n(p)}{\partial |p|} - \frac{\partial \widehat{f}(p)}{\partial |p|} \right| \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^d)}.$$

We derive the upper bound in the absolute value on the first term in the right side of (2.55) by

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^d)} \left| \frac{|p| - a^{\frac{1}{2s}}}{|p|^{2s} - a} \right| \chi_{A_\delta} \leq C\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^d)}\chi_{A_\delta}.$$

This implies that

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s} - a}\chi_{A_\delta} \right\|_{L^2(\mathbb{R}^d)} \leq C\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

as assumed. Therefore, in dimensions $d \geq 2$, when $a > 0$ and $s \in (0, 1)$, we have

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^d)$$

as $n \to \infty$. ■

3. Solvability in the sense of sequences with a scalar potential

Proof of Theorem 4. Note that the case a) of the theorem is the result of Lemma 7 of [28]. Then we proceed to proving the case of $a > 0$.

To prove the uniqueness of solutions of our equation, let us suppose that there exist both $u_1(x)$ and $u_2(x)$ which are square integrable in $\mathbb{R}^3$ and solve (1.20). Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ is a solution of the problem

$$(-\Delta + V(x))^s w = aw, \quad s \in (0, 1).$$

The fact that the operator $(-\Delta + V(x))^s$ has no nontrivial $L^2(\mathbb{R}^3)$ eigenfunctions as discussed above yields that $w(x)$ vanishes a.e. in $\mathbb{R}^3$.

Let us apply the generalized Fourier transform (1.26) with the functions of the continuous spectrum of the Schrödinger operator to both sides of problem (1.20), which yields

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^{2s} - a}, \quad s \in (0, 1).$$

We introduce the spherical layer in the space of three dimensions as

$$B_\delta := \{k \in \mathbb{R}^3 \mid a^{\frac{1}{2s}} - \delta \leq |k| \leq a^{\frac{1}{2s}} + \delta\}, \; 0 < \delta < a^{\frac{1}{2s}}. \tag{3.56}$$

This allows us to express

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^{2s} - a} \chi_{B_\delta} + \frac{\tilde{f}(k)}{|k|^{2s} - a} \chi_{B_\delta^c}. \tag{3.57}$$

The second term in the right side of (3.57) can be trivially bounded from above in the absolute value by

$$C|\tilde{f}(k)| \in L^2(\mathbb{R}^3),$$

because $f(x)$ is square integrable as assumed. We express

$$\tilde{f}(k) = \tilde{f}(a^{\frac{1}{2s}}, \sigma) + \int_{a^{\frac{1}{2s}}}^{|k|} \frac{\partial \tilde{f}(q, \sigma)}{\partial q} dq.$$

Therefore, the first term in the right side of (3.57) can be written as

$$\frac{\tilde{f}(a^{\frac{1}{2s}}, \sigma)}{|k|^{2s} - a} \chi_{B_\delta} + \frac{\int_{a^{\frac{1}{2s}}}^{|k|} \frac{\partial \tilde{f}(q,\sigma)}{\partial q} dq}{|k|^{2s} - a} \chi_{B_\delta}. \tag{3.58}$$

The second term in sum (3.58) can be easily bounded above in the absolute value by

$$\|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)} \left| \frac{|k| - a^{\frac{1}{2s}}}{|k|^{2s} - a} \right| \chi_{B_\delta} \leq C \|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)} \chi_{B_\delta} \in L^2(\mathbb{R}^3).$$

Note that under the stated assumptions $\nabla_k \tilde{f}(k) \in L^\infty(\mathbb{R}^3)$ due to Lemma 2.4 of [21]. Apparently, the first term in (3.58) is square integrable if and only if $\tilde{f}(a^{\frac{1}{2s}}, \sigma)$ vanishes, which yields orthogonality relation (1.29). ∎

Then we proceed to the establishing of our last main statement dealing with the solvability in the sense of sequences.

Proof of Theorem 5. In the case a) when $s \in \left(0, \frac{3}{4}\right)$ problems (1.20) and (1.21) have unique solutions $u(x),\ u_n(x) \in L^2(\mathbb{R}^3),\ n \in \mathbb{N}$ respectively due to the part a) of Theorem 4 above. Let us apply the generalized Fourier transform (1.26) to both sides of equations (1.20) and (1.21). We obtain

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^{2s}}, \quad \tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{|k|^{2s}}, \quad n \in \mathbb{N}.$$

Therefore

$$\tilde{u}_n(k) - \tilde{u}(k) = \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s}} \chi_{\{|k| \le 1\}} + \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s}} \chi_{\{|k| > 1\}}. \tag{3.59}$$

Obviously, the second term in the right side of (3.59) can be easily estimated from above in the absolute value by $|\tilde{f}_n(k) - \tilde{f}(k)|$. Hence

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s}} \chi_{\{|k|>1\}} \right\|_{L^2(\mathbb{R}^3)} \le \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

via the one of our assumptions. Using (1.27) we obtain the upper bound for the first term in the right side of (3.59) in the absolute value by

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \frac{\chi_{\{|k| \le 1\}}}{|k|^{2s}}.$$

Apparently, this yields

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s}} \chi_{\{|k| \le 1\}} \right\|_{L^2(\mathbb{R}^3)} \le$$

$$\le \frac{1}{\sqrt{2(3-4s)\pi}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

due to the one of our assumptions. Therefore, $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$ in the case when the parameter $a = 0$ and $s \in \left(0, \frac{3}{4}\right)$.

Then we turn our attention to the situation when $a = 0$ and $s \in \left[\frac{3}{4}, 1\right)$. By means of orthogonality relation (1.30) along with the Corollary 2.2 of [21] and the part b) of Lemma 6 below we obtain

$$|(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}| = |(f(x) - f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}| \le$$

$$\le \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Therefore,

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{3.60}$$

holds. Hence equations (1.20) and (1.21) admit unique solutions $u(x),\ u_n(x) \in L^2(\mathbb{R}^3),\ n \in \mathbb{N}$ respectively via the part a) of Theorem 4. As discussed above, it is sufficient to consider the first term in the right side of (3.59). Orthogonality relations (3.60) and (1.30) yield

$$\tilde{f}(0) = 0, \quad \tilde{f}_n(0) = 0, \quad n \in \mathbb{N},$$

such that

$$\tilde{f}(k) = \int_0^{|k|} \frac{\partial \tilde{f}(s, \sigma)}{\partial s} ds, \quad \tilde{f}_n(k) = \int_0^{|k|} \frac{\partial \tilde{f}_n(s, \sigma)}{\partial s} ds, \quad n \in \mathbb{N}.$$

This enables us to estimate the first term in the right side of (3.59) from above in the absolute value by $\|\nabla_k[\tilde{f}_n(k) - \tilde{f}(k)]\|_{L^\infty(\mathbb{R}^3)} |k|^{1-2s} \chi_{\{|k| \le 1\}}$. Therefore

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s}} \chi_{\{|k| \le 1\}} \right\|_{L^2(\mathbb{R}^3)} \le C \|\nabla_k[\tilde{f}_n(k) - \tilde{f}(k)]\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

due to Lemma 3.4 of [26] under the given assumptions. This shows that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$ when $a = 0$ and $s \in \left[\frac{3}{4}, 1\right)$.

We complete the proof of the theorem by establishing the result of the part b). By virtue of the limiting argument similar to the proof of relation we have (3.60)

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \quad k \in S^3_{a^{\frac{1}{2s}}}, \quad s \in (0, 1). \tag{3.61}$$

holds. Thus by means of the result the part b) of Theorem 4, equations (1.20) and (1.21) possesses unique solutions $u(x), u_n(x) \in L^2(\mathbb{R}^3)$. We apply the generalized Fourier transform (1.26) to both sides of problems (1.20) and (1.21). Hence, we obtain

$$\tilde{u}_n(k) - \tilde{u}(k) = \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s} - a} \chi_{B_\delta} + \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s} - a} \chi_{B_\delta^c} \tag{3.62}$$

with B_δ defined in (3.56). Obviosly, the second term in the right side of (3.62) can be estimated from above in the absolute value by $C|\tilde{f}_n(k) - \tilde{f}(k)|$, such that

$$\left\|\frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s} - a}\chi_{B_\delta^c}\right\|_{L^2(\mathbb{R}^3)} \leq C\|f_n(x) - f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

due to the one of our assumptions. By means of orthogonality conditions (3.61) and (1.31), we have

$$\tilde{f}(a^{\frac{1}{2s}}, \sigma) = 0, \quad \tilde{f}_n(a^{\frac{1}{2s}}, \sigma) = 0, \quad n \in \mathbb{N}.$$

This gives us the representations

$$\tilde{f}(k) = \int_{a^{\frac{1}{2s}}}^{|k|} \frac{\partial \tilde{f}(s,\sigma)}{\partial s} ds, \quad \tilde{f}_n(k) = \int_{a^{\frac{1}{2s}}}^{|k|} \frac{\partial \tilde{f}_n(s,\sigma)}{\partial s} ds, \quad n \in \mathbb{N},$$

such that the first term in the right side of (3.62) can be expressed as

$$\frac{\int_{a^{\frac{1}{2s}}}^{|k|} \left[\frac{\partial \tilde{f}_n(s,\sigma)}{\partial s} - \frac{\partial \tilde{f}(s,\sigma)}{\partial s}\right] ds}{|k|^{2s} - a} \chi_{B_\delta}. \tag{3.63}$$

Clearly, (3.63) can be trivially estimated from above in the absolute value by

$$\|\nabla_k[\tilde{f}_n(k) - \tilde{f}(k)]\|_{L^\infty(\mathbb{R}^3)} \left|\frac{|k| - a^{\frac{1}{2s}}}{|k|^{2s} - a}\right| \chi_{B_\delta} \leq C\|\nabla_k[\tilde{f}_n(k) - \tilde{f}(k)]\|_{L^\infty(\mathbb{R}^3)}\chi_{B_\delta}.$$

Therefore, the $L^2(\mathbb{R}^3)$ norm of (3.63) can be bounded from above by

$$C\|\nabla_k[\tilde{f}_n(k) - \tilde{f}(k)]\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

by virtue of Lemma 3.4 of [26] under the given assumptions. This yields that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$ when $a > 0$ with $s \in (0, 1)$. ∎

4. Auxiliary results

The following technical lemma is useful for proving the solvability in the sense of sequences in our theorems. Note that its parts a) and b) were established in Lemma 6 of [30].

Lemma 6. *a) Let $f(x) \in L^2(\mathbb{R}^d), \ d \in \mathbb{N}$ and $xf(x) \in L^1(\mathbb{R}^d)$. Then $f(x) \in L^1(\mathbb{R}^d)$.*

b) Let $n \in \mathbb{N}, \ f_n(x) \in L^2(\mathbb{R}^d), \ d \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Let $xf_n(x) \in L^1(\mathbb{R}^d)$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Then $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$.

c) Let $f(x) \in L^2(\mathbb{R}^d), \ d \in \mathbb{N}$ and $x^2 f(x) \in L^1(\mathbb{R}^d)$. Then $xf(x) \in L^1(\mathbb{R}^d)$.

d) Let $n \in \mathbb{N}$, $f_n(x) \in L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Let $x^2 f_n(x) \in L^1(\mathbb{R}^d)$, such that $x^2 f_n(x) \to x^2 f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Then $x f_n(x) \to x f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$.

Proof. To prove the part c) of the lemma, we express the norm $\|xf(x)\|_{L^1(\mathbb{R}^d)}$ as

$$\int_{|x|\le 1} |x||f(x)|dx + \int_{|x|>1} |x||f(x)|dx \le \int_{|x|\le 1} |f(x)|dx + \int_{|x|>1} |x|^2|f(x)|dx.$$

This sum can be easily bounded from above via the Schwarz inequality by

$$\|f(x)\|_{L^2(\mathbb{R}^d)}\sqrt{|B^d|} + \|x^2 f(x)\|_{L^1(\mathbb{R}^d)} < \infty$$

as assumed. Let us complete the proof of the lemma with establishing its part d). Clearly, the norm $\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^d)}$ can be written as

$$\int_{|x|\le 1} |x||f_n(x) - f(x)|dx + \int_{|x|>1} |x||f_n(x) - f(x)|dx \le$$

$$\le \int_{|x|\le 1} |f_n(x) - f(x)|dx + \int_{|x|>1} |x|^2|f_n(x) - f(x)|dx.$$

By means of the Schwarz inequality this sum can be trivially estimated from above by

$$\|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)}\sqrt{|B^d|} + \|x^2 f_n(x) - x^2 f(x)\|_{L^1(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

due to our assumptions. ■

References

[1] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl. (9), **76** (1997), no. 1, 55–81.

[2] C. Amrouche, F. Bonzom, Mixed exterior Laplace's problem, J. Math. Anal. Appl., **338** (2008), no. 1, 124–140.

[3] P. Bolley, T.L. Pham, *Propriétés d'indice en théorie höldérienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl. (9), **72** (1993), no 1, 105–119.

[4] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Höldérienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 315–334.

[5] N. Benkirane, *Propriétés d'indice en théorie höldérienne pour des opérateurs elliptiques dans R^n*, CRAS, **307**, Série I (1988), no. 11, 577–580.

[6] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer-Verlag, Berlin (1987), 319 pp.

[7] A. Ducrot, M. Marion, V. Volpert, *Systémes de réaction-diffusion sans propriété de Fredholm*, CRAS, **340** (2005), no. 9, 659–664.

[8] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192.

[9] M. Goldberg, W. Schlag, *A limiting absorption principle for the three-dimensional Schrödinger equation with L^p potentials*, Int. Math. Res. Not., (2004), no. 75, 4049–4071.

[10] P.D. Hislop, I.M. Sigal, Introduction to spectral theory. With applications to Schrödinger operators, Springer-Verlag, New-York (1996), 337 pp.

[11] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting, *Applied Analysis*, In preparation.

[12] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., **162** (1965/1966), 258–279.

[13] E.H. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence, RI (1997), 278 pp.

[14] R. Metzler, J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., **339**, (2000), 1–77.

[15] M. Reed, B. Simon, *Methods of modern mathematical physics. III. Scattering theory*, Academic Press (1979), 463 pp.

[16] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math., **155** (2004), no. 3, 451–513.

[17] V. Volpert. Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains. Birkhäuser (2011), 639 pp.

[18] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski, *Solvability conditions for elliptic problems with non-Fredholm operators*, Appl.Math., **29** (2002), no. 2, 219–238.

[19] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators*, Int. J. Pure Appl. Math., **60** (2010), no. 2, 169–191.

[20] V. Vougalter, V. Volpert. *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl. Math., **2** (2010), no. 1, 75–83.

[21] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271

[22] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation*, Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[23] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), no. 1, 365–373.

[24] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), no. 2, 146–154.

[25] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), no. 4, 473–496.

[26] V. Volpert, V. Vougalter. *Solvability in the sense of sequences to some non-Fredholm operators*, Electron. J. Differential Equations, **2013**, No. 160, 16 pp.

[27] V. Vougalter, V. Volpert. *On the solvability in the sense of sequences for some non-Fredholm operators,* Dyn. Partial Differ. Equ., **11** (2014), no. 2, 109–124.

[28] V. Vougalter, V. Volpert. *Existence of stationary solutions for some integro-differential equations with anomalous diffusion,* J. Pseudo-Differ. Oper. Appl., **6** (2015), no.4, 487–501.

[29] V. Vougalter, V. Volpert. *On the solvability of some systems of integro-differential equations with anomalous diffusion. Pseudo- differential operators: groups, geometry and applications*, Trends Math., *Birkhäuser / Springer, Cham,* (2017), 161–179.

[30] V. Vougalter, V. Volpert. *Solvability in the sense of sequences for some non Fredholm operators related to the superdiffusion,* to appear in J. Pseudo-Differ. Oper. Appl.