Multiscale Gevrey asymptotics in boundary layer expansions for some initial value problem with merging turning points

A. Lastra, S. Malek
University of Alcalá, Departamento de Física y Matemáticas,
Ap. de Correos 20, E-28871 Alcalá de Henares (Madrid), Spain,
University of Lille 1, Laboratoire Paul Painlevé,
59655 Villeneuve d'Ascq cedex, France,
alberto.lastra@uah.es
Stephane.Malek@math.univ-lille1.fr

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Abstract

We consider a nonlinear singularly perturbed PDE leaning on a complex perturbation parameter ϵ. The problem possesses an irregular singularity in time at the origin and involves a set of so-called moving turning points merging to 0 with ϵ. We construct outer solutions for time located in complex sectors that are kept away from the origin at a distance equivalent to a positive power of $|\epsilon|$ and we build up a related family of sectorial holomorphic inner solutions for small time inside some boundary layer. We show that both outer and inner solutions have Gevrey asymptotic expansions as ϵ tends to 0 on appropriate sets of sectors that cover a neighborhood of the origin in $\mathbb{C}^*$. We observe that their Gevrey orders are distinct in general.
Key words: asymptotic expansion, Borel-Laplace transform, Fourier transform, Cauchy problem, formal power series, nonlinear integro-differential equation, nonlinear partial differential equation, singular perturbation. 2000 MSC: 35C10, 35C20.

1 Introduction

Within this paper, we focus on a family of nonlinear singularly perturbed equations sharing the shape

$$(1)\quad Q(\partial_z)P(t,\epsilon)u(t,z,\epsilon) + P_1(t,\epsilon)Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon) = f(t,z,\epsilon) + P_2(t,\epsilon,\partial_t,\partial_z)u(t,z,\epsilon)$$

where Q, Q_1, Q_2, P, P_1, P_2 stand for polynomials with complex coefficients and $f(t,z,\epsilon)$ denotes a holomorphic function near the origin regarding t and ϵ in $\mathbb{C}$ and on some horizontal strip $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$ for some $\beta > 0$ w.r.t z.

This work is a continuation of a study initiated in the contribution [14]. Namely, in [14] we considered an equation of the form (1) in the case when $P(0,\epsilon)$ is identically vanishing near 0 that corresponds to a situation which is analog to one of a *turning point* at $t = 0$ (we refer to [21] and [6] for a detailed outline of this terminology in the context of ODEs). The requirements imposed on the main equation forced the polynomial $t \mapsto P(t,\epsilon)$ to have an isolated root at $t=0$ whereas the other moving roots depending upon ϵ stay apart from a fixed disc enclosing the origin. Under suitable constraints, we established the existence of a set of actual holomorphic solutions $y_p(t,z,\epsilon)$, meromorphic at $\epsilon = 0$ and $t = 0$, $0 \leq p \leq \varsigma - 1$, for some integer $\varsigma \geq 2$, defined on domains $\mathcal{T} \times H_\beta \times \mathcal{E}_p$, for some prescribed open bounded sector $\mathcal{T}$ at centered at 0 and $\mathcal{E} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ is some well chosen set of open bounded sectors which covers a neighborhood of 0 in $\mathbb{C}^*$. Furthermore, for convenient integers $a,b \in \mathbb{Z}$ we have shown that all the functions $\epsilon^a t^b y_p(t,z,\epsilon)$ share w.r.t ϵ a common asymptotic expansion $\hat{y}(t,z,\epsilon) = \sum_{n \geq 0} y_n(t,z)\epsilon^n$ with bounded holomorphic coefficients on $\mathcal{T} \times H_\beta$. This asymptotic expansion turns out to be of Gevrey type whose order depends both on data relying on the highest order term of P_2 which is of irregular type in the sense of [15] displayed as $\epsilon^{\Delta} t^{\delta q + m} \partial_t^{\delta} R(\partial_z)$ for some positive integers $\Delta, \delta, q, m > 0$, R some polynomial and on the two polynomial P and P_1 that frame the turning point at $t = 0$.

In this work, we aim attention at a different situation regarding the localization of turning points that is not covered in our previous study. Namely, we assume that $t \mapsto P(t,\epsilon)$ does not vanish at $t = 0$ but possess at least one root leaning on ϵ, a so-called *moving turning point*, which merges to the origin as ϵ tends to 0 (see Lemma 1). Our target is to carry out a comparable statement as in [14] namely the construction of sectorial holomorphic solutions and asymptotic expansions of Gevrey type as ϵ tends to the origin. Nevertheless, the whole picture looks rather different from our previous investigation. More precisely, according to the presence of the shrinking turning points, the solutions we construct by means of Laplace and inverse Fourier transforms are only defined w.r.t t on some boundary layer domains which turn out to be sectors with vertex at 0, with radius that depends on some positive power of $|\epsilon|$ and approaches 0 with ϵ. Besides, we can exhibit another family of solutions of (1) provided that t remains away from the origin on some unbounded sector with inner radius being proportional to some positive power of $|\epsilon|$, tending to 0 with ϵ.

In order to explain the manufacturing of these solutions, we need to specify the nature of the forcing term $f(t,z,\epsilon)$ which is constituted with two terms, one piece is polynomial in t,ϵ with bounded holomorphic coefficients on any strip $H_{\beta'} \subsetneq H_\beta$ with $0 < \beta' < \beta$ and the other part represented as a function $F^{\theta_F}(t,z,\epsilon)$ which solves a singularly perturbed nonhomogeneous linear ODE of the form

$$F_2(-\epsilon^{\gamma}\partial_t)F^{\theta_F}(t,z,\epsilon) = I_{\theta_F}(t,z,\epsilon)$$

for some polynomial $F_2(x)$ with complex coefficients not vanishing at $x = 0$, some real number $\gamma > 1/2$ and I_{θ_F} some rational function in t,ϵ and bounded holomorphic w.r.t z on $H_{\beta'}$ (see Remark 1). This equation is of irregular type at $t = \infty$ and regular at $t = 0$ (we indicate some text book on complex ODEs such as [2], [7] for a definition of these classical notions). According to this last assumption, we stress the fact that the solutions described above actually solve a PDE with rational coefficients in t,ϵ and bounded holomorphic w.r.t z on $H_{\beta'}$ with a shape similar to (1) as displayed in Remark 2.

Our first main construction can be outlined as follows. Under appropriate restriction on the shape of (1), we can select a set $\mathcal{E}^{\infty} = \{\mathcal{E}_j^{\infty}\}_{0 \leq j \leq \iota-1}$ of bounded sectors with aperture slightly larger than π/γ, for some $\iota \geq 2$, which covers a neighborhood of 0 in $\mathbb{C}^*$ and pick up directions $\{\mathfrak{u}_j\}_{0 \leq j \leq \iota-1}$ in $\mathbb{R}$ for which a family of solutions $v^{\mathfrak{u}_j}(t,z,\epsilon)$ of the main equation (1), for a specific choice of $\theta_F = \mathfrak{u}_j$ in the forcing term F^{θ_F} described above, can be built up as a usual Laplace and Fourier inverse transform

$$v^{\mathfrak{u}_j}(t,z,\epsilon) = \frac{\epsilon^{\gamma_0}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\mathfrak{u}_j}} W^{\mathfrak{u}_j}(u,m,\epsilon) \exp(-\frac{t}{\epsilon^{\gamma}}u) e^{izm} du dm$$

along the halfline $L_{\mathfrak{u}_j} = \mathbb{R}_+ e^{i\mathfrak{u}_j}$, for some real number γ_0, where $W^{\mathfrak{u}_j}(u,m,\epsilon)$ represents a function with at most exponential growth of order 1 on a sector enclosing $L_{\mathfrak{u}_j}$ w.r.t u, with exponential decay w.r.t m on $\mathbb{R}$ and analytic dependence on ϵ near 0. In addition, for each fixed $\epsilon \in \mathcal{E}_j^{\infty}$, the restriction $(t,z) \mapsto v^{\mathfrak{u}_j}(t,z,\epsilon)$ is bounded and holomorphic on $\mathcal{T}_{\epsilon}^{\infty} \times H_{\beta'}$, where $\mathcal{T}_{\epsilon}^{\infty}$ stands for an unbounded sector with inner radius proportional to $|\epsilon|^{\gamma-\Gamma}$ for some real number $0 \leq \Gamma < \gamma$ (Theorem 2). Furthermore, we explain why the functions $\epsilon^{-\gamma_0} v^{\mathfrak{u}_j}(t,z,\epsilon)$ own w.r.t ϵ a common asymptotic expansion $\hat{O}_t(\epsilon) = \sum_{k \geq 0} O_{t,k}\epsilon^k$ whose coefficients $O_{t,k}$ represent bounded holomorphic functions on $H_{\beta'}$ which can be called *outer expansions* owing to the fact that it is valid for any fixed value of t in the vicinity of 0 as ϵ tends to 0. Accordingly, we call $v^{\mathfrak{u}_j}(t,z,\epsilon)$ the *outer solutions* of (1). Besides, we can indicate the nature of this asymptotic expansion that turns out to be of Gevrey order (at most) $1/\gamma$, ensuring that $\epsilon^{-\gamma_0} v^{\mathfrak{u}_j}$ can be labeled as $\gamma-$sum of $\hat{O}_t(\epsilon)$ on $\mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$ for some radius σ_t outlined in (146) (Theorem 3). We may notice that this Gevrey order $1/\gamma$ is substantially related to the Stokes phenomena stemming from the solutions $F^{\mathfrak{u}_j}(t,z,\epsilon)$ of the ODE (19).

Now, we proceed to the description of what we call the *inner solutions* of (1). Submitted to additional requirements on the coefficients of (1), we can choose a set $\mathcal{E} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ built up with bounded sectors with opening barely larger than $\frac{\pi}{\chi\kappa}$ for some integer $\kappa \geq 1$ and real number $\chi > \frac{1}{2\kappa}$, for some $\varsigma \geq 2$, which covers a neighborhood of 0 in $\mathbb{C}^*$ and raise a set of real directions $\{\mathfrak{d}_p\}_{0 \leq p \leq \varsigma-1}$ such that for each direction $\mathfrak{u}_j$ coming up from one single outer solution $v^{\mathfrak{u}_j}$, $0 \leq j \leq \iota-1$, one can construct a family of solutions $u^{\mathfrak{d}_p,j}(t,z,\epsilon)$ to (1), $0 \leq p \leq \varsigma-1$, that is represented as a Laplace transform of some order κ and Fourier inverse transform

$$u^{\mathfrak{d}_p,j}(t,z,\epsilon) = \epsilon^{-m_0}\frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\mathfrak{d}_p}} \omega_{\kappa}^{\mathfrak{d}_p,j}(u,m,\epsilon)\exp(-(\frac{u}{\epsilon^{\alpha}t})^{\kappa})e^{izm}\frac{du}{u}dm$$

where the inner integration is performed along the halfline $L_{\mathfrak{d}_p} = \mathbb{R}_+ e^{i\mathfrak{d}_p}$, for some positive integer $m_0 \geq 1$, negative rational number $\alpha < 0$ and where $\omega_{\kappa}^{\mathfrak{d}_p,j}(u,m,\epsilon)$ stands for a function with at most exponential growth of order κ on a sector containing $L_{\mathfrak{d}_p}$ w.r.t u, with exponential decay w.r.t m on $\mathbb{R}$ and analytic dependence w.r.t ϵ in the vicinity of the origin. Besides, for each fixed $\epsilon \in \mathcal{E}_p$, the projection $(t,z) \mapsto u^{\mathfrak{d}_p,j}(t,z,\epsilon)$ is bounded and holomorphic on $\mathcal{T}_{\epsilon,\chi-\alpha} \times H_{\beta'}$, for $\mathcal{T}_{\epsilon,\chi-\alpha} = X\epsilon^{\chi-\alpha}$ where X stands for some fixed bounded sector centered at 0. (Theorem 1). Moreover, we justify why the functions $\epsilon^{m_0}u^{\mathfrak{d}_p,j}(t,z,\epsilon)$ admit w.r.t ϵ a common asymptotic expansion $\hat{I}^j(\epsilon) = \sum_{k \geq 0} I_k^j \epsilon^k$ with coefficients I_k^j belonging to a Banach space of bounded holomorphic functions on $X \times H_{\beta'}$ that may be called *inner expansion* since it is only legitimated for t on the boundary layer set $\mathcal{T}_{\epsilon,\chi-\alpha}$ which shrinks to 0 with ϵ. We can also specify the type of asymptotic expansion which turns out to be of Gevrey order (at most) $\frac{1}{\chi\kappa}$. As a result, $\epsilon^{m_0}u^{\mathfrak{d}_p,j}(t,z,\epsilon)$ can be identified as $\chi\kappa-$sum of $\hat{I}^j(\epsilon)$ on $\mathcal{E}_p$ (Theorem 3). By construction, the integer κ crops up in the highest order term of the operator P_2 which is assumed to be of the form $\epsilon^{\Delta_D}t^{\delta_D(\kappa+1)}\partial_t^{\delta_D}R_D(\partial_z)$ for some integers $\Delta_D \geq 1$, $\delta_D \geq 2$ and a polynomial R_D. The real number χ is in particular related by a set of inequalities to the integers κ,Δ_D,δ_D, the powers of ϵ and t in P, P_1, to the real number γ and the forcing term $f(t,z,\epsilon)$. As outgrowth, we observe that this Gevrey order $\frac{1}{\chi\kappa}$ involves informations emanating from the moving turning points and the irregularity of the operator P_2 at $t=0$.

These so-called inner and outer expansions come into play in vast literature on what is commonly named *matched asymptotic expansions*. For further details on this subject, we refer to classical textbooks such as [3], [5], [8], [16], [17], [21]. We point out the recent work by A. Fruchard and R. Schäfke on composite asymptotic expansions, see [6], which provides a solid bedrock for the method of matching and furnishes hands-on criteria for the study of the nature of these asymptotic expansions which can be shown of Gevrey type with the same order for both inner and outer expansions for several families of singularly perturbed ODEs. In our work, we observe however an interesting situation in which the Gevrey order of the outer and of the inner expansions turn out to be different in general (see the two examples after Theorem 3). Nevertheless, we observe a scaling gap that prevents our inner and outer solutions to share a common domain in time t for all ϵ small enough, see Remark 3. More work is needed if one wants to analytically continue and match our inner and outer solutions. This stays beyond the scope of our approach and we leave it for future inspection.

It is worthwhile noting that a similar phenomenon of parametric multiple scale asymptotics related to moving turning points has been observed in a recent work [18] by K. Suzuki and Y. Takei for singularly perturbed second order ODEs of the form

$$\epsilon^2 \psi''(z,\epsilon) = (z - \epsilon^2 z^2)\psi(z,\epsilon)$$

whose moving turning point $z = 1/\epsilon^2$ tends to infinity which turns out to be an irregular singularity of the equation. In particular, they have shown that the power series part $\hat{\varphi}_{\pm}(z,\epsilon)$ of its WKB solution $\hat{\psi}_{\pm}(z,\epsilon) = \exp(\pm\frac{1}{\epsilon}\int^z \sqrt{z}dz)\hat{\varphi}_{\pm}(z,\epsilon)$ presents a double scale structure of Gevrey order 1/4 and 1 for all fixed z, in being $(4,1)-$multisummable w.r.t ϵ except for a finite number of singular directions. Furthermore, a second example involving three distinct Gevrey levels has been worked out by Y. Takei in [19].

The paper is organized as follows.
In Section 2, after recalling some ground facts about Fourier transforms acting on spaces of functions with exponential decay on $\mathbb{R}$, we disclose the main problem (8) of our study.
In Section 3, we build up our inner solutions. We start by reminding the definition and first properties of our Borel-Laplace transforms of order $k > 0$. Then, we redefine some Banach spaces with exponential growth on sectors of order κ and exponential decay on the real line as introduced in our previous work [14]. In Section 3.3, we search for conjectural time rescaled formal solutions and present the convolution equation satisfied by their Borel transforms. In Section 3.4, we solve this latter convolution problem within the Banach spaces mentioned above with the help of a fixed point procedure. In the last subsection, we construct a family of actual holomorphic solutions to (8) related to a good covering in $\mathbb{C}^*$ w.r.t ϵ, for small time t belonging to an $\epsilon-$depending boundary layer.
In Section 4, we shape our outer solutions. We begin with the description of basic operations on classical Laplace transforms. Then, we introduce Banach spaces of functions with exponential growth of order 1 on sectors and exponential decay on $\mathbb{R}$ which are a slender modification of the ones described in Section 3. In Section 4.3, we seek for speculative solutions of (8) represented as classical Laplace and inverse Fourier transforms and we exhibit a related nonlinear convolution equation (116) in the Borel plane and Fourier space. In Section 4.4, we find solutions of (116) located in the Banach spaces quoted above using again a fixed point argument. In the ending subsection, for a suitable good covering in $\mathbb{C}^*$ w.r.t ϵ we distinguish a set of holomorphic solutions to (8) for both large time and small time t kept distant from the origin by a quantity proportional to a positive power of $|\epsilon|$.
In Section 5, we investigate the asymptotic expansions of the inner and outer solutions. We first bring to mind the Ramis-Sibuya cohomological approach for $k-$summability of formal series. Then, we discuss the existence of a common asymptotic expansion of Gevrey order $\frac{1}{\chi\kappa}$ for the inner solutions and of Gevrey order $1/\gamma$ for the outer solutions on the corresponding coverings w.r.t ϵ.

2 Outline of the main problem

2.1 Fourier transforms

In this subsection, we recall without proofs some properties of the inverse Fourier transform acting on continuous functions with exponential decay on $\mathbb{R}$, see [10], Proposition 7 for more details.

Definition 1 *Let $\beta > 0$ and $\mu > 1$ be real numbers. We denote $E_{(\beta,\mu)}$ the vector space of functions $h : \mathbb{R} \to \mathbb{C}$ such that*

$$||h(m)||_{(\beta,\mu)} = \sup_{m \in \mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)|$$

is finite. The space $E_{(\beta,\mu)}$ endowed with the norm $||.||_{(\beta,\mu)}$ becomes a Banach space.

As stated in Proposition 5 from [10], we notice that

Proposition 1 *Let $Q_1(X), Q_2(X), R(X) \in \mathbb{C}[X]$ be polynomials such that*

$$\deg(R) \geq \deg(Q_1) \quad , \quad \deg(R) \geq \deg(Q_2) \quad , \quad R(im) \neq 0, \tag{2}$$

for all $m \in \mathbb{R}$. Assume that $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. Then, there exists a constant $C_5 > 0$ (depending on Q_1, Q_2, R, μ) such that

$$||\frac{1}{R(im)} \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) f(m-m_1) Q_2(im_1) g(m_1) dm_1||_{(\beta,\mu)} \\ \leq C_5 ||f(m)||_{(\beta,\mu)} ||g(m)||_{(\beta,\mu)} \tag{3}$$

for all $f(m), g(m) \in E_{(\beta,\mu)}$. Therefore, $(E_{(\beta,\mu)}, ||.||_{(\beta,\mu)})$ becomes a Banach algebra for the product $\star$ defined by

$$f \star g(m) = \frac{1}{R(im)} \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) f(m-m_1) Q_2(im_1) g(m_1) dm_1.$$

As a particular case, when $f, g \in E_{(\beta,\mu)}$ with $\beta > 0$ and $\mu > 1$, the classical convolution product

$$f * g(m) = \int_{-\infty}^{+\infty} f(m-m_1) g(m_1) dm_1$$

belongs to $E_{(\beta,\mu)}$.

Proposition 2 *1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and a constant $C > 0$ such that $|f(m)| \leq C \exp(-\beta|m|)$ for all $m \in \mathbb{R}$, for some $\beta > 0$. The inverse Fourier transform of f is defined by the integral representation*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m) \exp(ixm) dm$$

for all $x \in \mathbb{R}$. It turns out that the function $\mathcal{F}^{-1}(f)$ extends to an analytic function on the horizontal strip

$$H_{\beta} = \{z \in \mathbb{C} / |\mathrm{Im}(z)| < \beta\}. \tag{4}$$

Let $\phi(m) = imf(m)$. Then, we have the commuting relation

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z) \tag{5}$$

for all $z \in H_\beta$.
2) Let $f, g \in E_{(\beta,\mu)}$ and let $\psi(m) = \frac{1}{(2\pi)^{1/2}} f * g(m)$, the convolution product of f and g, for all $m \in \mathbb{R}$. From Proposition 1, we know that $\psi \in E_{(\beta,\mu)}$. Moreover, the next formula

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z) \tag{6}$$

holds for all $z \in H_\beta$.

2.2 Display of the main problem

Let $q \geq 1$, $M, Q \geq 0$ and $D \geq 2$ be integers. For $1 \leq l \leq q$, let k_l be a non negative integer such that $1 \leq k_l < k_{l+1}$ for $l \in \{1, \ldots, q-1\}$. For all $0 \leq l \leq q$, let m_l be a non negative integer and a_l be a complex number not equal to 0. For all $0 \leq l \leq M$, we consider non negative integers h_l, μ_l and a complex number c_l such that $1 \leq h_l < h_{l+1}$ for $l \in \{0, \ldots, M-1\}$. For all $0 \leq l \leq Q$, we denote n_l and b_l non negative integers such that $1 \leq b_l < b_{l+1}$ for $l \in \{0, \ldots, Q-1\}$. For $1 \leq l \leq D$, we set nonnegative integers Δ_l, d_l and δ_l such that $1 \leq \delta_l < \delta_{l+1}$ for $l \in \{1, \ldots, D-1\}$.

Let $Q(X), Q_1(X), Q_2(X), R_l(X) \in \mathbb{C}[X]$, $1 \leq l \leq D$, be polynomials that satisfy

$$\deg(Q) = \deg(R_D) \geq \deg(R_l), \quad \deg(R_D) \geq \max(\deg(Q_1), \deg(Q_2)), \qquad Q(im) \neq 0 \quad , \quad R_D(im) \neq 0 \tag{7}$$

for all $m \in \mathbb{R}$, all $1 \leq l \leq D-1$.

We focus on the following nonlinear singularly perturbed PDE

$$\begin{aligned}(\sum_{l=1}^{q} a_l \epsilon^{m_l} t^{k_l} + a_0 \epsilon^{m_0}) Q(\partial_z) u(t,z,\epsilon) + (\sum_{l=0}^{M} c_l \epsilon^{\mu_l} t^{h_l}) Q_1(\partial_z) u(t,z,\epsilon) Q_2(\partial_z) u(t,z,\epsilon) \\ = \sum_{j=0}^{Q} b_j(z) \epsilon^{n_j} t^{b_j} + F^{\theta_F}(t,z,\epsilon) + \sum_{l=1}^{D} \epsilon^{\Delta_l} t^{d_l} \partial_t^{\delta_l} R_l(\partial_z) u(t,z,\epsilon)\end{aligned} \tag{8}$$

We make the following crucial assumption on the integers m_l, $0 \leq l \leq q$. We take for granted that

$$m_0 > m_{l_1} \tag{9}$$

for some $l_1 \in \{1, \ldots, q\}$. Let $P(t,\epsilon) = \sum_{l=1}^{q} a_l \epsilon^{m_l} t^{k_l} + a_0 \epsilon^{m_0}$. The roots of $t \mapsto P(t,\epsilon)$ are called, in our context, *turning points* of the equation (8), with analogy to the situation concerned with ordinary differential equations. See [6], [21] for more details. In the next lemma, we supply some information about the position of some roots of P.

Lemma 1 *Under the constraint (9), the polynomial $t \mapsto P(t,\epsilon)$ has at least one root in the disc $D(0, |\epsilon|^{\mu_P})$ centered at 0 with radius $|\epsilon|^{\mu_P}$ for some small enough real number $\mu_P > 0$ (depending only on m_l, k_l, $1 \leq l \leq q$ and m_0), provided that $|\epsilon|$ is taken small enough.*

Proof According to the restriction (9) we may assume that

$$m_0 > \min_{l=1}^{q} m_l = \{m_{j_1}, \ldots, m_{j_h}\}$$

for some $1 \leq j_s \leq q$, $1 \leq s \leq h$, with $j_1 < \ldots < j_h$. We can rewrite

$$P_1(t,\epsilon) = P(t,\epsilon)/\epsilon^{m_{j_1}} = \sum_{l=1}^{q} a_l \epsilon^{m_l - m_{j_1}} t^{k_l} + a_0 \epsilon^{m_0 - m_{j_1}} = a_{j_1} t^{k_{j_1}} + \sum_{l \in \{1,\ldots,q\}, l \neq j_1} a_l \epsilon^{m_l - m_{j_1}} t^{k_l} + a_0 \epsilon^{m_0 - m_{j_1}}$$

Let $P_0(t) = a_{j_1} t^{k_{j_1}}$. Recall that, by construction, $a_{j_1} \neq 0$. We plan to show that

$$|P_1(t,\epsilon) - P_0(t)| < |P_0(t)| \tag{10}$$

holds for some appropriate $\mu > 0$, for all t in the circle $C(0,|\epsilon|^{\mu})$ centered at 0 with radius $|\epsilon|^{\mu}$, provided that ϵ is small enough. Actually, we will observe that the quantity $\sup_{t \in C(0,|\epsilon|^{\mu})} |P_1(t,\epsilon) - P_0(t)|/|P_0(t)|$ tends to 0 as ϵ tends to 0, which in particular yields the inequality (10). Indeed, we can write

$$\begin{aligned} |P_1(t,\epsilon) - P_0(t)|/|P_0(t)| &\leq \sum_{l \in \{1,\ldots,q\}, l \neq j_1}^{q} |\frac{a_l}{a_{j_1}}||\epsilon|^{m_l - m_{j_1}} |t|^{k_l - k_{j_1}} + |\frac{a_0}{a_{j_1}}||\epsilon|^{m_0 - m_{j_1}} |t|^{-k_{j_1}} \\ &= \sum_{l \in \{1,\ldots,q\}, l \neq j_1}^{q} |\frac{a_l}{a_{j_1}}||\epsilon|^{m_l - m_{j_1} + \mu(k_l - k_{j_1})} + |\frac{a_0}{a_{j_1}}||\epsilon|^{m_0 - m_{j_1} - k_{j_1}\mu} \end{aligned} \tag{11}$$

for all $t \in C(0,|\epsilon|^{\mu})$. We take $\mu > 0$ (which depends only on m_l, k_l, $1 \leq l \leq q$ and m_0) such that

$$m_0 - m_{j_1} - k_{j_1}\mu > 0 \quad , \quad m_l - m_{j_1} + \mu(k_l - k_{j_1}) > 0$$

holds, for all $1 \leq l \leq q$ with $l \neq j_1$. Notice that such a $\mu > 0$ exists since, by construction, $m_0 > m_{j_1}$, hence we can take $\mu < (m_0 - m_{j_1})/k_{j_1}$. Furthermore, for $l \notin \{j_1, \ldots, j_h\}$, we have that $m_l - m_{j_1} > 0$. Hence, if $k_l > k_{j_1}$, then $m_l - m_{j_1} + \mu(k_l - k_{j_1}) > 0$ holds for any $\mu > 0$ and if $k_l < k_{j_1}$, then we may take $0 < \mu < (m_l - m_{j_1})/(k_{j_1} - k_l)$. Likewise, for $l \in \{j_2, \ldots, j_h\}$ (in case $h \geq 2$), $m_l - m_{j_1} = 0$ and $k_l - k_{j_1} > 0$ since $l > j_1$, therefore $m_l - m_{j_1} + \mu(k_l - k_{j_1}) > 0$ holds for any $\mu > 0$.

As a result, for $|\epsilon|$ small enough, the right handside of the inequality (11) can be taken strictly smaller than 1. Hence the inequality (10) holds. Now, we can apply Rouché's theorem which states that $t \mapsto P_1(t,\epsilon)$ and $P_0(t)$ have the same number of roots (counted with multiplicity) inside the disc $D(0,|\epsilon|^{\mu})$. In conclusion, $t \mapsto P_1(t,\epsilon)$ possesses k_{j_1} roots inside $D(0,|\epsilon|^{\mu})$ for $|\epsilon|$ small enough. □

The lemma above ensures that (8) possesses at least one (movable) turning point which tends to 0 as ϵ tends to 0. Notice that this case is not covered by our previous work [14], where $t = 0$ is assumed to be a turning point of the equation and all movable turning points depending on ϵ remain outside a fixed disc enclosing the origin, see Remark 2 therein.

The coefficients $b_j(z)$ are displayed as follows. For all $0 \leq j \leq Q$, we consider functions $m \mapsto B_j(m)$ that belong to the Banach space $E_{(\beta,\mu)}$ for some $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$ and $\beta > 0$. We set

$$b_j(z) = \mathcal{F}^{-1}(m \mapsto B_j(m))(z) \quad , \quad 0 \leq j \leq Q, \tag{12}$$

where $\mathcal{F}^{-1}$ denotes the Fourier inverse transform defined in Proposition 2. By construction, $b_j(z)$ defines a holomorphic function on the horizontal strip $H_\beta = \{z \in \mathbb{C}/|\mathrm{Im}(z)| < \beta\}$ which is bounded on every substrip $H_{\beta'}$ for any given $0 < \beta' < \beta$.

The function $F^{\theta_F}(t,z,\epsilon)$ is a part of the forcing term given as an integral transform

$$F^{\theta_F}(t,z,\epsilon) = \frac{\epsilon^{n_F}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\theta_F}} \omega_F(u,m)(\exp(-\frac{t}{\epsilon^{\gamma}}u) - 1)e^{izm} du dm \tag{13}$$

where $n_F \geq 0$ is some integer, $\gamma > 1/2$ is a real number and $\omega_F(\tau,m)$ is a function defined as

$$\omega_F(\tau,m) = C_F(m)e^{-K_F\tau}\frac{F_1(\tau)}{F_2(\tau)} \tag{14}$$

where $C_F(m)$ belongs to the Banach space $E_{(\beta,\mu)}$, $K_F > 0$ is a real number and $F_1(\tau), F_2(\tau)$ are two polynomials with coefficients in $\mathbb{C}$ such that $\deg(F_1) \leq \deg(F_2)$. The path of integration $L_{\theta_F} = \{ue^{\sqrt{-1}\theta_F}/u \in [0,+\infty)\}$ is chosen in such a way that it avoids the roots of $F_2(\tau)$ and with $\theta_F \in (-\pi/2, \pi/2)$.

We first assert that the function $F^{\theta_F}(t,z,\epsilon)$ is well defined and bounded holomorphic in time t on some $\epsilon-$depending neighborhood of 0, in space z on any strip $H_{\beta'}$ with $0 < \beta' < \beta$, provided that ϵ is not vanishing in the vicinity of the origin. Namely, let us select a real number $\delta_1^0 > 0$ with $\cos(\theta_F) > \delta_1^0/K_F$. We introduce the disc

$$D_{F;\epsilon} = \{t \in \mathbb{C}/|t| < (-\delta_1^0 + K_F\cos(\theta_F))|\epsilon|^{\gamma}\}.$$

According to the assumptions made above, one can sort a constant $C_{F_1,F_2} > 0$ with

$$\left|\frac{F_1(u)}{F_2(u)}\right| \leq C_{F_1,F_2} \tag{15}$$

for $u \in L_{\theta_F}$. Then, the next estimates

$$\begin{aligned} |F^{\theta_F}(t,z,\epsilon)| &\leq \frac{|\epsilon|^{n_F}C_{F_1,F_2}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty}\int_0^{+\infty} |C_F(m)|\exp(-K_F r\cos(\theta_F)) \\ &\times (\exp(|\frac{t}{\epsilon^{\gamma}}|r) + 1)e^{-m\mathrm{Im}(z)} dr dm \leq \frac{|\epsilon|^{n_F}C_{F_1,F_2}||C_F(m)||_{(\beta,\mu)}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} (1+|m|)^{-\mu}e^{-(\beta-\beta')|m|} dm \\ &\times (\int_0^{+\infty} e^{-\delta_1^0 r} + e^{-K_F r\cos(\theta_F)} dr) \end{aligned} \tag{16}$$

hold for all $t \in D_{F;\epsilon}$, $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, for some $\epsilon_0 > 0$ and $z \in H_{\beta'}$, for any $0 < \beta' < \beta$. As a consequence, $(t,z) \mapsto F^{\theta_F}(t,z,\epsilon)$ represents a holomorphic bounded function on $D_{F;\epsilon} \times H_{\beta'}$ for any $0 < \beta' < \beta$ and $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

In a second place, we check that $F^{\theta_F}(t,z,\epsilon)$ represents a holomorphic function w.r.t t on some $\epsilon-$depending unbounded sectorial domain away from the origin, in space z on strips $H_{\beta'}$ with $0 < \beta' < \beta$, for any given ϵ belonging to some suitable bounded sector centered at 0. More specifically, we consider an unbounded open sector U_{θ_F} centered at 0 with an aperture chosen in a manner that it bypasses all the roots of the polynomial $F_2(\tau)$. Let $\delta_2^{\infty} > 0$ be a positive real number. We sort a bounded sector $\mathcal{E}^{\infty}$ centered at 0 with opening contained in the range $(\pi/\gamma, 2\pi)$, a positive real number δ_1^{∞} and suitable directions $\alpha_{\infty} < \beta_{\infty}$ taken in a way that there exists some direction θ_F^{Δ} (that may depend on ϵ and t) satisfying $e^{i\theta_F^{\Delta}} \in U_{\theta_F}$ and fulfills the next demand

$$\theta_F^{\Delta} + \arg(\frac{t}{\epsilon^{\gamma}}) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad , \quad \cos(\theta_F^{\Delta} + \arg(\frac{t}{\epsilon^{\gamma}})) > \delta_1^{\infty} \tag{17}$$

for all $\epsilon \in \mathcal{E}^{\infty}$ and t belonging to the unbounded sector

$$\mathcal{T}_{F;\epsilon}^{\infty} = \{t \in \mathbb{C}^{*}/|t| > \frac{K_F + \delta_2^{\infty}}{\delta_1^{\infty}}|\epsilon|^{\gamma} \quad , \quad \alpha_{\infty} < \arg(t) < \beta_{\infty}\}.$$

Through a path deformation argument, we may observe that $F^{\theta_F}(t,z,\epsilon)$ can be rewritten as the sum

$$F^{\theta_F}(t,z,\epsilon) = \frac{\epsilon^{n_F}}{(2\pi)^{1/2}} \left(\int_{-\infty}^{+\infty} \int_{L_{\theta_F^{\Delta}}} \omega_F(u,m) \exp(-\frac{t}{\epsilon^{\gamma}}u) e^{izm} du dm \right.$$
$$\left. - \int_{-\infty}^{+\infty} \int_{L_{\theta_F}} \omega_F(u,m) e^{izm} du dm \right)$$

for all $t \in \mathcal{T}_{F;\epsilon}^{\infty}$, $z \in H_{\beta'}$ with $0 < \beta' < \beta$ and $\epsilon \in \mathcal{E}^{\infty}$. As above, we can select a constant $C_{F_1,F_2} > 0$ for which (15) holds. Then, from the latter decomposition, we deduce that

$$(18) \quad |F^{\theta_F}(t,z,\epsilon)| \leq \frac{|\epsilon|^{n_F}}{(2\pi)^{1/2}} C_{F_1,F_2} \left(\int_{-\infty}^{+\infty} \int_{0}^{+\infty} |C_F(m)| \right.$$
$$\times \exp(K_F r - \frac{|t|}{|\epsilon|^{\gamma}} r \cos(\theta_F^{\Delta} + \arg(\frac{t}{\epsilon^{\gamma}}))) e^{-m\mathrm{Im}(z)} dr dm$$
$$\left. + \int_{-\infty}^{+\infty} \int_{0}^{+\infty} |C_F(m)| \exp(-K_F r \cos(\theta_F)) e^{-m\mathrm{Im}(z)} dr dm \right)$$
$$\leq \frac{|\epsilon|^{n_F} C_{F_1,F_2} ||C_F(m)||_{(\beta,\mu)}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} (1+|m|)^{-\mu} e^{-(\beta-\beta')|m|} dm$$
$$\times (\int_{0}^{+\infty} e^{-\delta_2^{\infty} r} dr + \int_{0}^{+\infty} e^{-K_F r \cos(\theta_F)} dr)$$

holds for all $t \in \mathcal{T}_{F;\epsilon}^{\infty}$, $z \in H_{\beta'}$ with $0 < \beta' < \beta$ and $\epsilon \in \mathcal{E}^{\infty}$. In particular, we see that $(t,z) \mapsto F^{\theta_F}(t,z,\epsilon)$ represents a holomorphic bounded function on $\mathcal{T}_{F;\epsilon}^{\infty} \times H_{\beta'}$ for any $0 < \beta' < \beta$, when ϵ belongs to $\mathcal{E}^{\infty}$.

Remark 1. Let us set

$$c_F(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_F(m) e^{izm} dm \quad , \quad c_{F_1,F_2,\theta_F} = \int_{L_{\theta_F}} e^{-K_F u} \frac{F_1(u)}{F_2(u)} du$$

and expand $F_1(u) = \sum_{k=0}^{\deg(F_1)} F_{1,k} u^k$. According to the identities concerning the classical Laplace transform coming next in Lemma 4, we can claim that $F^{\theta_F}(t,z,\epsilon)$ solves the next singularly perturbed inhomogeneous linear ODE

$$(19) \qquad F_2(-\epsilon^{\gamma}\partial_t) F^{\theta_F}(t,z,\epsilon) = \epsilon^{n_F} c_F(z) \left(\sum_{k=0}^{\deg(F_1)} F_{1,k} \frac{k!}{(K_F + \frac{t}{\epsilon^{\gamma}})^{k+1}} - F_2(0) c_{F_1,F_2,\theta_F} \right).$$

Our choice for the piece of forcing term $F^{\theta_F}(t,z,\epsilon)$ can be considered as very specific. However, for the sake of simplicity and clarity, we have taken it as an explicit solution of a inhomogeneous basic singularly perturbed ODE which is irregular at $t = \infty$ and regular at $t = 0$. But all the forthcoming results disclosed in this paper may also work for forcing terms being solutions of more general singularly perturbed ODEs sharing the same behaviour at $t = 0$ and $t = \infty$ as our model.

3 Construction of inner solutions to the main problem

Within this section, we build up solutions of the main equation (8) for time t located on small $\epsilon-$depending sectorial domains in the vicinity of the origin, whose radius is proportional to some positive power of $|\epsilon|$.

3.1 Borel-Laplace transforms of order k

In this section, we review some basic statements concerning a $k-$Borel summability method of formal power series which is a slightly modified version of the more classical procedure (see [1], Section 3.2). This novel version has already been used in our most recent works such as [10], [14].

Definition 2 *Let $k \geq 1$ be an integer. Let $(m_k(n))_{n\geq 1}$ be the sequence*

$$m_k(n) = \Gamma\left(\frac{n}{k}\right) = \int_0^{\infty} t^{\frac{n}{k}-1}e^{-t}dt, \qquad n \geq 1.$$

Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space. We say a formal power series

$$\hat{X}(T) = \sum_{n=1}^{\infty} a_n T^n \in T\mathbb{E}[[T]]$$

is m_k-summable with respect to T in the direction $d \in \mathbb{R}$ if the following assertions hold:

1. *There exists $\rho > 0$ such that the m_k-Borel transform of $\hat{X}$, $\mathcal{B}_{m_k}(\hat{X})$, is absolutely convergent for $|\tau| < \rho$, where*

$$\mathcal{B}_{m_k}(\hat{X})(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{\Gamma\left(\frac{n}{k}\right)}\tau^n \in \tau\mathbb{E}[[\tau]].$$

2. *The series $\mathcal{B}_{m_k}(\hat{X})$ can be analytically continued in a sector $S = \{\tau \in \mathbb{C}^{\star} : |d-\arg(\tau)| < \delta\}$ for some $\delta > 0$. In addition to this, the extension is of exponential growth at most k in S, meaning that there exist $C, K > 0$ such that*

$$\left\|\mathcal{B}_{m_k}(\hat{X})(\tau)\right\|_{\mathbb{E}} \leq Ce^{K|\tau|^k}, \quad \tau \in S.$$

Under these assumptions, the vector valued Laplace transform of $\mathcal{B}_{m_k}(\hat{X})$ along direction d is defined by

$$\mathcal{L}^d_{m_k}\left(\mathcal{B}_{m_k}(\hat{X})\right)(T) = k\int_{L_\gamma} \mathcal{B}_{m_k}(\hat{X})(u)e^{-(u/T)^k}\frac{du}{u},$$

where L_γ is the path parametrized by $u \in [0,\infty) \mapsto ue^{i\gamma}$, for some appropriate direction γ depending on T, such that $L_\gamma \subseteq S$ and $\cos(k(\gamma - \arg(T))) \geq \Delta > 0$ for some $\Delta > 0$.

The function $\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}))$ is well defined and turns out to be a holomorphic and bounded function in any sector of the form $S_{d,\theta,R^{1/k}} = \{T \in \mathbb{C}^{\star} : |T| < R^{1/k}, |d-\arg(T)| < \theta/2\}$, for some $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \Delta/K$. This function is known as the m_k-sum of the formal power series $\hat{X}(T)$ in the direction d.

The following are some elementary properties concerning the m_k-sums of formal power series which will be crucial in our procedure.

1) The function $\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}))(T)$ admits $\hat{X}(T)$ as its Gevrey asymptotic expansion of order $1/k$ with respect to T in $S_{d,\theta,R^{1/k}}$. More precisely, for every $\frac{\pi}{k} < \theta_1 < \theta$, there exist $C, M > 0$ such that

$$\left\|\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}))(T) - \sum_{p=1}^{n-1} a_p T^p\right\|_{\mathbb{E}} \leq CM^n\Gamma(1+\frac{n}{k})|T|^n,$$

for every $n \geq 2$ and $T \in S_{d,\theta_1,R^{1/k}}$. Watson's lemma (see Proposition 11 p.75 in [2]) allows us to affirm that $\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}))(T)$ is unique provided that the opening θ_1 is larger than $\frac{\pi}{k}$.

2) Whenever $\mathbb{E}$ is a Banach algebra, the set of holomorphic functions having Gevrey asymptotic expansion of order $1/k$ on a sector with values in $\mathbb{E}$ turns out to be a differential algebra (see Theorem 18, 19 and 20 in [2]). This, and the uniqueness provided by Watson's lemma allow us to obtain some properties on m_k-summable formal power series in direction d.

By $\star$ we denote the product in the Banach algebra and also the Cauchy product of formal power series with coefficients in $\mathbb{E}$. Let $\hat{X}_1$, $\hat{X}_2 \in T\mathbb{E}[[T]]$ be m_k-summable formal power series in direction d. Let $q_1 \geq q_2 \geq 1$ be integers. Then $\hat{X}_1 + \hat{X}_2$, $\hat{X}_1 \star \hat{X}_2$ and $T^{q_1}\partial_T^{q_2}\hat{X}_1$, which are elements of $T\mathbb{E}[[T]]$, are m_k-summable in direction d. Moreover, one has

$$\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_1))(T) + \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_2))(T) = \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_1 + \hat{X}_2))(T),$$

$$\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_1))(T) \star \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_2))(T) = \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_1 \star \hat{X}_2))(T),$$

$$T^{q_1}\partial_T^{q_2}\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}_1))(T) = \mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(T^{q_1}\partial_T^{q_2}\hat{X}_1))(T),$$

for every $T \in S_{d,\theta,R^{1/k}}$.

The next proposition is written without proof for it can be found in [10], Proposition 6.

Proposition 3 *Let $\hat{f}(t) = \sum_{n\geq 1} f_n t^n$ and $\hat{g}(t) = \sum_{n\geq 1} g_n t^n$ that belong to $\mathbb{E}[[t]]$, where $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ is a Banach algebra. Let $k, m \geq 1$ be integers. The following formal identities hold.*

$$\mathcal{B}_{m_k}(t^{k+1}\partial_t \hat{f}(t))(\tau) = k\tau^k \mathcal{B}_{m_k}(\hat{f}(t))(\tau),$$

$$\mathcal{B}_{m_k}(t^m \hat{f}(t))(\tau) = \frac{\tau^k}{\Gamma\left(\frac{m}{k}\right)} \int_0^{\tau^k} (\tau^k - s)^{\frac{m}{k}-1} \mathcal{B}_{m_k}(\hat{f}(t))(s^{1/k}) \frac{ds}{s}$$

and

$$\mathcal{B}_{m_k}(\hat{f}(t) \star \hat{g}(t))(\tau) = \tau^k \int_0^{\tau^k} \mathcal{B}_{m_k}(\hat{f}(t))((\tau^k - s)^{1/k}) \star \mathcal{B}_{m_k}(\hat{g}(t))(s^{1/k}) \frac{1}{(\tau^k - s)s} ds.$$

3.2 Banach spaces with exponential growth and exponential decay

In this section, we recall the definition and display useful properties of Banach spaces as defined in our previous work, [14]. We denote $D(0,\rho)$ the open disc centered at 0 with radius $\rho > 0$ in $\mathbb{C}$ and by $\bar{D}(0,\rho)$ its closure. Let S_d be an open unbounded sector in direction $d \in \mathbb{R}$ and $\mathcal{E}$ be an open sector with finite radius $r_{\mathcal{E}}$, both centered at 0 in $\mathbb{C}$. By convention, these sectors do not contain the origin in $\mathbb{C}$.

Definition 3 *Let $\nu, \rho > 0$ and $\beta > 0, \mu > 1$ be real numbers. Let $\kappa \geq 1$ be an integer and $\chi > 0$ be some real number. Let $\epsilon \in \mathcal{E}$. We denote $F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ the vector space of continuous functions $(\tau, m) \mapsto h(\tau, m)$ on $(\bar{D}(0,\rho) \cup S_d) \times \mathbb{R}$, which are holomorphic w.r.t τ on $D(0,\rho) \cup S_d$ and such that*

$$||h(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} = \sup_{\tau \in \bar{D}(0,\rho)\cup S_d, m \in \mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|) \frac{1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}}{|\frac{\tau}{\epsilon^{\chi}}|} \exp(-\nu|\frac{\tau}{\epsilon^{\chi}}|^{\kappa})|h(\tau,m)|$$

is finite. One can check that the normed space $(F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}, ||.||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)})$ is a Banach space.

Throughout the whole section, we keep the notations of Definitions in this section.

The next lemma and proposition are kept almost unchanged as stated in Section 2 of [14] and we decide to omit their proofs for avoiding overlapping with our previous work.

Lemma 2 *Let $\gamma_1 \geq 0$, $\gamma_2 \geq 1$ be integers and $\gamma_3 \in \mathbb{R}$. Let $R(X)$ be a polynomial that belongs to $\mathbb{C}[X]$ such that $R(im) \neq 0$ for all $m \in \mathbb{R}$. We take a function $B(m)$ located in $E_{(\beta,\mu)}$ and we consider a continuous function $a_{\gamma_1,\kappa}(\tau, m)$ on $(\bar{D}(0,\rho) \cup S_d) \times \mathbb{R}$, holomorphic w.r.t τ on $D(0,\rho) \cup S_d$ such that*

$$|a_{\gamma_1,\kappa}(\tau,m)| \leq \frac{1}{(1+|\tau|^{\kappa})^{\gamma_1}|R(im)|}$$

for all $\tau \in \bar{D}(0,\rho) \cup S_d$, all $m \in \mathbb{R}$.

Then, the function $\epsilon^{-\gamma_3}\tau^{\gamma_2}B(m)a_{\gamma_1,\kappa}(\tau,m)$ belongs to $F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$. Moreover, there exists a constant $C_1 > 0$ (depending on ν,κ and γ_2) such that

$$||\epsilon^{-\gamma_3}\tau^{\gamma_2}B(m)a_{\gamma_1,\kappa}(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq C_1 \frac{||B(m)||_{(\beta,\mu)}}{\inf_{m\in\mathbb{R}}|R(im)|}|\epsilon|^{\chi\gamma_2-\gamma_3} \tag{20}$$

for all $\epsilon \in \mathcal{E}$.

Proposition 4 *Let γ_j, $0 \leq j \leq 3$, be real numbers with $\gamma_1 \geq 0$. Let $R(X), R_D(X)$ be polynomials with complex coefficients such that $\deg(R) \leq \deg(R_D)$ and with $R_D(im) \neq 0$ for all $m \in \mathbb{R}$. We consider a continuous function $a_{\gamma_1,\kappa}(\tau, m)$ on $(\bar{D}(0,\rho) \cup S_d) \times \mathbb{R}$, holomorphic w.r.t τ on $D(0,\rho) \cup S_d$ such that*

$$|a_{\gamma_1,\kappa}(\tau,m)| \leq \frac{1}{(1+|\tau|^{\kappa})^{\gamma_1}|R_D(im)|}$$

for all $\tau \in \bar{D}(0,\rho) \cup S_d$, all $m \in \mathbb{R}$. We make the next assumptions

$$\frac{1}{\kappa} + \gamma_3 + 1 > 0 \quad , \quad \gamma_2 + \gamma_3 + 2 \geq 0 \quad , \quad \gamma_2 > -1. \tag{21}$$

1) If $1+\gamma_3 \leq 0$, then there exists a constant $C_2 > 0$ (depending on $\nu, \kappa, \gamma_2, \gamma_3$ and $R(X), R_D(X)$) such that

$$||\epsilon^{-\gamma_0}a_{\gamma_1,\kappa}(\tau,m)R(im)\tau^{\kappa}\int_0^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\gamma_2}s^{\gamma_3}f(s^{1/\kappa},m)ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq C_2|\epsilon|^{\chi\kappa(\gamma_2+\gamma_3+2)-\gamma_0}||f(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \tag{22}$$

for all $f(\tau,m) \in F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$.
2) If $1+\gamma_3 > 0$ *and* $\gamma_1 \geq 1+\gamma_3$, *then there exists a constant* $C_2' > 0$ *(depending on* $\nu,\kappa,\gamma_1,\gamma_2,\gamma_3$ *and* $R(X), R_D(X)$*) such that*

$$(23) \quad ||\epsilon^{-\gamma_0} a_{\gamma_1,\kappa}(\tau,m) R(im) \tau^{\kappa} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\gamma_2} s^{\gamma_3} f(s^{1/\kappa},m) ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\ \leq C_2' |\epsilon|^{\chi\kappa(\gamma_2+\gamma_3+2)-\gamma_0-\chi\kappa\gamma_1} ||f(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$

for all $f(\tau,m) \in F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$.

The forthcoming proposition presents norms estimates for some bilinear convolution operators acting on the aforementioned Banach spaces.

Proposition 5 *Let* $R_D(X), Q_1(X)$ *and* $Q_2(X)$ *belonging to* $\mathbb{C}[X]$ *such that* $R_D(im) \neq 0$ *for all* $m \in \mathbb{R}$. *Assume that*

$$\deg(R_D) \geq \deg(Q_1), \quad \deg(R_D) \geq \deg(Q_2)$$

and choose the real parameter μ *such that*

$$(24) \qquad \mu > \max(\deg(Q_1)+1, \deg(Q_2)+1).$$

Let $a(m)$ *be a continuous function on* $\mathbb{R}$ *such that*

$$|a(m)| \leq \frac{1}{|R_D(im)|}$$

for all $m \in \mathbb{R}$. *Then, there exists a constant* $C_3 > 0$ *(depending on* Q_1, Q_2, R_D, μ *and* κ*) such that*

$$(25) \quad ||\tau^{\kappa-1} a(m) \int_0^{\tau^\kappa} \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) f((\tau^\kappa - s')^{1/\kappa}, m-m_1) \\ \times Q_2(im_1) g((s')^{1/\kappa}, m_1) \frac{1}{(\tau^\kappa - s')s'} ds' dm_1 ||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\ \leq \frac{C_3}{|\epsilon|^{\chi}} ||f(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} ||g(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$

for all $f(\tau,m), g(\tau,m) \in F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$.

Proof We follow similar steps as in the proof of Proposition 3 from [14]. By definition of the norm, we can write

$$
\begin{aligned}
(26)\quad B &= ||\tau^{\kappa-1}a(m)\int_{0}^{\tau^{\kappa}}\int_{-\infty}^{+\infty}Q_{1}(i(m-m_{1}))f((\tau^{\kappa}-s')^{1/\kappa},m-m_{1})\\
&\quad \times Q_{2}(im_{1})g((s')^{1/\kappa},m_{1})\frac{1}{(\tau^{\kappa}-s')s'}ds'dm_{1}||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\\
&= \sup_{\tau\in\bar{D}(0,\rho)\cup S_{d},m\in\mathbb{R}}(1+|m|)^{\mu}\exp(\beta|m|)\frac{1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}}{|\frac{\tau}{\epsilon^{\chi}}|}\exp(-\nu|\frac{\tau}{\epsilon^{\chi}}|^{\kappa})|a(m)|\\
&\quad \times|\tau^{\kappa-1}\int_{0}^{\tau^{\kappa}}\int_{-\infty}^{+\infty}\{(1+|m-m_{1}|)^{\mu}\exp(\beta|m-m_{1}|)\\
&\quad \times\frac{1+\frac{|\tau^{\kappa}-s'|^{2}}{|\epsilon|^{\chi 2\kappa}}}{\frac{|\tau^{\kappa}-s'|^{1/\kappa}}{|\epsilon|^{\chi}}}\exp(-\nu\frac{|\tau^{\kappa}-s'|}{|\epsilon|^{\chi\kappa}})f((\tau^{\kappa}-s')^{1/\kappa},m-m_{1})\}\\
&\quad \times\{(1+|m_{1}|)^{\mu}\exp(\beta|m_{1}|)\frac{1+\frac{|s'|^{2}}{|\epsilon|^{\chi 2\kappa}}}{\frac{|s'|^{1/\kappa}}{|\epsilon|^{\chi}}}\exp(-\nu\frac{|s'|}{|\epsilon|^{\chi\kappa}})g((s')^{1/\kappa},m_{1})\}\times\mathcal{B}(\tau,s,m,m_{1})ds'dm_{1}|
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{B}(\tau,s,m,m_{1}) &= \frac{\exp(-\beta|m-m_{1}|)\exp(-\beta|m_{1}|)|Q_{1}(i(m-m_{1}))||Q_{2}(im_{1})|}{(1+|m-m_{1}|)^{\mu}(1+|m_{1}|)^{\mu}}\\
&\quad \times\frac{\frac{|s'|^{1/\kappa}|\tau^{\kappa}-s'|^{1/\kappa}}{|\epsilon|^{2\chi}}}{(1+\frac{|\tau^{\kappa}-s'|^{2}}{|\epsilon|^{\chi 2\kappa}})(1+\frac{|s'|^{2}}{|\epsilon|^{\chi 2\kappa}})}\exp(\nu\frac{|\tau^{\kappa}-s'|}{|\epsilon|^{\chi\kappa}})\exp(\nu\frac{|s'|}{|\epsilon|^{\chi\kappa}})\frac{1}{(\tau^{\kappa}-s')s'}.
\end{aligned}
$$

By definition of the norms of f and g and according to the triangular inequality $|m|\leq|m-m_{1}|+|m_{1}|$ for all $m,m_{1}\in\mathbb{R}$, we deduce that

$$
(27)\qquad B\leq C_{3}(\epsilon)||f(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}||g(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}
$$

where

$$
\begin{aligned}
C_{3}(\epsilon) &= \sup_{\tau\in\bar{D}(0,\rho)\cup S_{d},m\in\mathbb{R}}(1+|m|)^{\mu}\frac{1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}}{|\frac{\tau}{\epsilon^{\chi}}|}\exp(-\nu|\frac{\tau}{\epsilon^{\chi}}|^{\kappa})|\tau|^{\kappa-1}|a(m)|\\
&\quad \times\int_{0}^{|\tau|^{\kappa}}\int_{-\infty}^{+\infty}\frac{|Q_{1}(i(m-m_{1}))||Q_{2}(im_{1})|}{(1+|m-m_{1}|)^{\mu}(1+|m_{1}|)^{\mu}}\frac{(h')^{1/\kappa}(|\tau|^{\kappa}-h')^{1/\kappa}}{|\epsilon|^{2\chi}}\frac{1}{(1+\frac{(|\tau|^{\kappa}-h')^{2}}{|\epsilon|^{\chi 2\kappa}})(1+\frac{(h')^{2}}{|\epsilon|^{\chi 2\kappa}})}\\
&\quad \times\exp(\nu\frac{|\tau|^{\kappa}-h'}{|\epsilon|^{\chi\kappa}})\exp(\nu\frac{h'}{|\epsilon|^{\chi\kappa}})\frac{1}{(|\tau|^{\kappa}-h')h'}dh'dm_{1}.
\end{aligned}
$$

We provide upper bounds that can be split in two parts,

$$
(28)\qquad C_{3}(\epsilon)\leq C_{3.1}C_{3.2}(\epsilon)
$$

where

$$
(29)\qquad C_{3.1}=\sup_{m\in\mathbb{R}}\frac{(1+|m|)^{\mu}}{|R_{D}(im)|}\int_{-\infty}^{+\infty}\frac{|Q_{1}(i(m-m_{1}))||Q_{2}(im_{1})|}{(1+|m-m_{1}|)^{\mu}(1+|m_{1}|)^{\mu}}dm_{1}
$$

and

$$C_{3.2}(\epsilon) = \sup_{\tau \in \bar{D}(0,\rho)\cup S_d} \frac{1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}}{|\frac{\tau}{\epsilon^{\chi}}|} |\tau|^{\kappa-1} \int_0^{|\tau|^{\kappa}} \frac{\frac{(h')^{1/\kappa}(|\tau|^{\kappa}-h')^{1/\kappa}}{|\epsilon|^{2\chi}}}{(1+\frac{(|\tau|^{\kappa}-h')^2}{|\epsilon|^{\chi 2\kappa}})(1+\frac{(h')^2}{|\epsilon|^{\chi 2\kappa}})} \frac{1}{(|\tau|^{\kappa}-h')h'} dh'.$$

By construction, we can select three constants $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{R} > 0$ such that

$$(30) \quad |Q_1(i(m-m_1))| \leq \mathfrak{Q}_1(1+|m-m_1|)^{\deg(Q_1)} \quad , \quad |Q_2(im_1)| \leq \mathfrak{Q}_1(1+|m_1|)^{\deg(Q_2)}, \\ |R_D(im)| \geq \mathfrak{R}(1+|m|)^{\deg(R_D)}$$

for all $m, m_1 \in \mathbb{R}$. We deduce that

$$(31) \quad C_{3.1} \leq \frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}} \sup_{m\in\mathbb{R}} (1+|m|)^{\mu-\deg(R_D)} \int_{-\infty}^{+\infty} \frac{1}{(1+|m-m_1|)^{\mu-\deg(Q_1)}(1+|m_1|)^{\mu-\deg(Q_2)}} dm_1$$

which is finite under the condition (24) according to Lemma 4 of [12].

On the other hand, with the help of the estimates (23) and (24) from [14], we conclude that a constant $C_{3.2} > 0$ can be picked out (depending exclusively on κ) with

$$(32) \qquad C_{3.2}(\epsilon) \leq \frac{C_{3.2}}{|\epsilon|^{\chi}}$$

We finish the proof by collecting (26), (27), (28), (29), (31) and (32) which leads to the statement of Proposition 5. □

3.3 Construction of formal solutions

Within this section, we search for time rescaled solutions to (8) of the form

$$u(t,z,\epsilon) = \epsilon^{-m_0} U(\epsilon^{\alpha} t, z, \epsilon)$$

where $\alpha \in \mathbb{Q}$. One can check that the expression $U(T,z,\epsilon)$ formally solves the next nonlinear PDE

$$(33) \quad (\sum_{l=1}^{q} a_l \epsilon^{m_l - m_0 - \alpha k_l} T^{k_l} + a_0) Q(\partial_z) U(T,z,\epsilon) \\ + (\sum_{l=0}^{M} c_l \epsilon^{\mu_l - 2m_0 - \alpha h_l} T^{h_l}) Q_1(\partial_z) U(T,z,\epsilon) Q_2(\partial_z) U(T,z,\epsilon) \\ = \sum_{j=0}^{Q} b_j(z) \epsilon^{n_j - \alpha b_j} T^{b_j} + F^{\theta_F}(\epsilon^{-\alpha} T, z, \epsilon) + \sum_{l=1}^{D} \epsilon^{\Delta_l + \alpha(\delta_l - d_l) - m_0} T^{d_l} R_l(\partial_z) \partial_T^{\delta_l} U(T,z,\epsilon).$$

We make the next further assumptions. We choose α such that

$$(34) \qquad \Delta_D + \alpha(\delta_D - d_D) - m_0 = 0$$

We suppose the existence of a positive integer $\kappa \geq 1$ with

$$(35) \qquad d_D = \delta_D(\kappa+1) \quad , \quad d_l = \delta_l(\kappa+1) + d_{l,\kappa}$$

for some integers $d_{l,\kappa} \geq 1$, for all $1 \leq l \leq D-1$. With the help of the formula (8.7) from [20] p. 3630, we can expand the following pieces of (33) in term of the irregular operator $T^{\kappa+1}\partial_T$,

$$T^{d_D}\partial_T^{\delta_D}R_D(\partial_z)U(T,z,\epsilon) = T^{\delta_D(\kappa+1)}\partial_T^{\delta_D}R_D(\partial_z)U(T,z,\epsilon) = R_D(\partial_z)\left((T^{\kappa+1}\partial_T)^{\delta_D} + \sum_{1\leq p\leq \delta_D-1} A_{\delta_D,p}T^{\kappa(\delta_D-p)}(T^{\kappa+1}\partial_T)^p\right)U(T,z,\epsilon) \tag{36}$$

for some real numbers $A_{\delta_D,p}$, $1 \leq p \leq \delta_D - 1$ and

$$T^{d_l}\partial_T^{\delta_l}R_l(\partial_z)U(T,z,\epsilon) = T^{d_{l,\kappa}}T^{\delta_l(\kappa+1)}\partial_T^{\delta_l}R_l(\partial_z)U(T,z,\epsilon) = R_l(\partial_z)T^{d_{l,\kappa}}\left((T^{\kappa+1}\partial_T)^{\delta_l} + \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p}T^{\kappa(\delta_l-p)}(T^{\kappa+1}\partial_T)^p\right)U(T,z,\epsilon) \tag{37}$$

for well chosen real numbers $A_{\delta_l,p}$, $1 \leq p \leq \delta_l - 1$. Notice that, by convention, the sum $\sum_{1\leq p\leq\delta_l-1}[..]$ appearing in (37) is vanishing provided that $\delta_l = 1$.

We now furnish the formal Taylor expansion of the part of the forcing term $F^{\theta_F}(\epsilon^{-\alpha}T,z,\epsilon)$ with respect to T at $T=0$. Making use of the convergent Taylor expansion of $\exp(-Tu/\epsilon^{\gamma+\alpha})-1$ w.r.t u at $u=0$, we can write

$$F^{\theta_F}(\epsilon^{-\alpha}T,z,\epsilon) = \sum_{n\geq 1} F_n(z,\epsilon)T^n \tag{38}$$

where the coefficients $F_n(z,\epsilon)$ are expressed as an inverse Fourier transform

$$F_n(z,\epsilon) = \mathcal{F}^{-1}(m \mapsto \psi_n(m,\epsilon))(z)$$

$$\psi_n(m,\epsilon) = \epsilon^{n_F}\int_{L_{\theta_F}} e^{-K_F u}\frac{F_1(u)}{F_2(u)}\frac{u^n}{n!}du C_F(m)(-\frac{1}{\epsilon^{\gamma+\alpha}})^n \tag{39}$$

for all $n \geq 1$. Let us assume now, that the expression $U(T,z,\epsilon)$ has a formal power series expansion

$$U(T,z,\epsilon) = \sum_{n\geq 1} U_n(z,\epsilon)T^n \tag{40}$$

where each coefficient $U_n(z,\epsilon)$ is defined as an inverse Fourier transform

$$U_n(z,\epsilon) = \mathcal{F}^{-1}(m \mapsto \omega_n(m,\epsilon))(z)$$

for some function $m \mapsto \omega_n(m,\epsilon)$ belonging to the Banach space $E_{(\beta,\mu)}$ and relying analytically on the parameter ϵ on some punctured disc $D(0,\epsilon_0)\setminus\{0\}$ centered at 0 with radius $\epsilon_0 > 0$. We consider the next formal series

$$\omega_\kappa(\tau,m,\epsilon) = \sum_{n\geq 1}\frac{\omega_n(m,\epsilon)}{\Gamma(n/\kappa)}\tau^n$$

obtained by formally applying a $m_\kappa-$Borel transform w.r.t T and Fourier transform w.r.t z to the formal series (40). We also introduce $\psi_\kappa(\tau,m,\epsilon)$ realized as a $m_\kappa-$Borel transform w.r.t T and Fourier transform w.r.t z of the formal series (38),

$$\Psi_\kappa(\tau,m,\epsilon) = \sum_{n\geq 1}\psi_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{\kappa})}.$$

Under the restrictions (34) and (35), we check that $\omega_{\kappa}(\tau,m,\epsilon)$ must satisfy some nonlinear integral convolution equation by making use of the properties of the $m_{\kappa}-$Borel transform listed in Proposition 3 and Fourier inverse transform discussed in Proposition 2, with the help of the prepared expansions (36), (37). Namely, we get the next problem

$$
\begin{aligned}
(41)\quad & Q(im)\left(\sum_{l=1}^{q} a_{l}\epsilon^{m_{l}-m_{0}-\alpha k_{l}}\frac{\tau^{\kappa}}{\Gamma(\frac{k_l}{\kappa})}\int_{0}^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\frac{k_l}{\kappa}-1}\omega_{\kappa}(s^{1/\kappa},m,\epsilon)\frac{ds}{s} + a_{0}\omega_{\kappa}(\tau,m,\epsilon)\right)\\
& + \sum_{l=0}^{M} c_{l}\epsilon^{\mu_{l}-2m_{0}-\alpha h_{l}}\frac{\tau^{\kappa}}{\Gamma(\frac{h_l}{\kappa})}\int_{0}^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\frac{h_l}{\kappa}-1}\\
& \times\left(s\int_{0}^{s}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}Q_{1}(i(m-m_{1}))\omega_{\kappa}((s-s')^{1/\kappa},m-m_{1},\epsilon)\right.\\
& \left.\times Q_{2}(im_{1})\omega_{\kappa}((s')^{1/\kappa},m_{1},\epsilon)dm_{1}\frac{1}{(s-s')s'}ds'\right)\frac{ds}{s} = \sum_{j=0}^{Q}B_{j}(m)\epsilon^{n_{j}-\alpha b_{j}}\frac{\tau^{b_j}}{\Gamma(\frac{b_j}{\kappa})} + \Psi_{\kappa}(\tau,m,\epsilon)\\
& + R_{D}(im)\left((\kappa\tau^{\kappa})^{\delta_D}\omega_{\kappa}(\tau,m,\epsilon) + \sum_{1\leq p\leq\delta_{D}-1}A_{\delta_{D},p}\frac{\tau^{\kappa}}{\Gamma(\delta_{D}-p)}\right.\\
& \left.\times\int_{0}^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\delta_{D}-p-1}(\kappa s)^{p}\omega_{\kappa}(s^{1/\kappa},m,\epsilon)\frac{ds}{s}\right) + \sum_{l=1}^{D-1}\epsilon^{\Delta_{l}+\alpha(\delta_{l}-d_{l})-m_{0}}R_{l}(im)\\
& \times\left(\frac{\tau^{\kappa}}{\Gamma(\frac{d_{l,\kappa}}{\kappa})}\int_{0}^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\frac{d_{l,\kappa}}{\kappa}-1}(\kappa s)^{\delta_{l}}\omega_{\kappa}(s^{1/\kappa},m,\epsilon)\frac{ds}{s} + \sum_{1\leq p\leq\delta_{l}-1}A_{\delta_{l},p}\right.\\
& \left.\times\frac{\tau^{\kappa}}{\Gamma(\frac{d_{l,\kappa}+\kappa(\delta_{l}-p)}{\kappa})}\int_{0}^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\frac{d_{l,\kappa}+\kappa(\delta_{l}-p)}{\kappa}-1}(\kappa s)^{p}\omega_{\kappa}(s^{1/\kappa},m,\epsilon)\frac{ds}{s}\right)
\end{aligned}
$$

As above, we assume by convention that the sum $\sum_{1\leq p\leq\delta_{l}-1}[..]$ appearing in (41) vanishes whenever $\delta_{l}=1$.

3.4 Analytic and continuous solutions of a nonlinear convolution equation with complex parameter

Our principal aim is the construction of a unique solution of the problem (41) inside the Banach space described in Subsection 3.2.
We make the following further assumptions. The conditions below are very similar to the ones proposed in Section 4 of [10] and in Section 5 of [14]. Namely, we demand that there exists an unbounded sector

$$S_{Q,R_D} = \{z\in\mathbb{C}/|z|\geq r_{Q,R_D}\quad,\quad|\arg(z)-d_{Q,R_D}|\leq\eta_{Q,R_D}\}$$

with direction $d_{Q,R_D}\in\mathbb{R}$, aperture $\eta_{Q,R_D}>0$ for some radius $r_{Q,R_D}>0$ such that

$$(42)\qquad \frac{Q(im)}{R_{D}(im)}\in S_{Q,R_D}$$

for all $m\in\mathbb{R}$. The polynomial $P_{m}(\tau)=Q(im)a_{0}-R_{D}(im)\kappa^{\delta_D}\tau^{\delta_{D}\kappa}$ can be factorized in the form

$$(43)\qquad P_{m}(\tau) = -R_{D}(im)\kappa^{\delta_D}\Pi_{l=0}^{\delta_{D}\kappa-1}(\tau-q_{l}(m))$$

where

$$q_l(m) = (\frac{|a_0Q(im)|}{|R_D(im)|\kappa^{\delta_D}})^{\frac{1}{\delta_D \kappa}} \exp(\sqrt{-1}(\arg(\frac{a_0Q(im)}{R_D(im)\kappa^{\delta_D}})\frac{1}{\delta_D\kappa} + \frac{2\pi l}{\delta_D \kappa})) \tag{44}$$

for all $0 \leq l \leq \delta_D\kappa - 1$, all $m \in \mathbb{R}$.

We select an unbounded sector S_d centered at 0, a small closed disc $\bar{D}(0,\rho)$ and we require the sector S_{Q,R_D} to fulfill the next conditions.

1) There exists a constant $M_1 > 0$ such that

$$|\tau - q_l(m)| \geq M_1(1+|\tau|) \tag{45}$$

for all $0 \leq l \leq \delta_D\kappa - 1$, all $m \in \mathbb{R}$, all $\tau \in S_d \cup \bar{D}(0,\rho)$. Indeed, from (42) and the explicit expression (44) of $q_l(m)$, we first observe that $|q_l(m)| > 2\rho$ for every $m \in \mathbb{R}$, all $0 \leq l \leq \delta_D\kappa - 1$ for an appropriate choice of r_{Q,R_D} and of $\rho > 0$. We also see that for all $m \in \mathbb{R}$, all $0 \leq l \leq \delta_D\kappa - 1$, the roots $q_l(m)$ remain in a union $\mathcal{U}$ of unbounded sectors centered at 0 that do not cover a full neighborhood of the origin in $\mathbb{C}^*$ provided that η_{Q,R_D} is small enough. Therefore, one can choose an adequate sector S_d such that $S_d \cap \mathcal{U} = \emptyset$ with the property that for all $0 \leq l \leq \delta_D\kappa - 1$ the quotients $q_l(m)/\tau$ lay outside some small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_d$, all $m \in \mathbb{R}$. This yields (45) for some small constant $M_1 > 0$.

2) There exists a constant $M_2 > 0$ such that

$$|\tau - q_{l_0}(m)| \geq M_2|q_{l_0}(m)| \tag{46}$$

for some $l_0 \in \{0,\ldots,\delta_D\kappa - 1\}$, all $m \in \mathbb{R}$, all $\tau \in S_d \cup \bar{D}(0,\rho)$. Indeed, for the sector S_d and the disc $\bar{D}(0,\rho)$ chosen as above in 1), we notice that for any fixed $0 \leq l_0 \leq \delta_D\kappa - 1$, the quotient $\tau/q_{l_0}(m)$ stays outside a small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_d \cup \bar{D}(0,\rho)$, all $m \in \mathbb{R}$. Hence (46) must hold for some small constant $M_2 > 0$.

By construction of the roots (44) in the factorization (43) and using the lower bound estimates (45), (46), we get a constant $C_P > 0$ such that

$$\begin{aligned} |P_m(\tau)| &\geq M_1^{\delta_D\kappa-1}M_2|R_D(im)\kappa^{\delta_D}|(\frac{|a_0Q(im)|}{|R_D(im)|\kappa^{\delta_D}})^{\frac{1}{\delta_D\kappa}}(1+|\tau|)^{\delta_D\kappa-1} \\ &\geq M_1^{\delta_D\kappa-1}M_2\frac{\kappa^{\delta_D}|a_0|^{\frac{1}{\delta_D\kappa}}}{(\kappa^{\delta_D})^{\frac{1}{\delta_D\kappa}}}(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}|R_D(im)| \\ &\quad \times (\min_{x\geq 0}\frac{(1+x)^{\delta_D\kappa-1}}{(1+x^{\kappa})^{\delta_D-\frac{1}{\kappa}}})(1+|\tau|^{\kappa})^{\delta_D-\frac{1}{\kappa}} \\ &= C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}|R_D(im)|(1+|\tau|^{\kappa})^{\delta_D-\frac{1}{\kappa}} \end{aligned} \tag{47}$$

for all $\tau \in S_d \cup \bar{D}(0,\rho)$, all $m \in \mathbb{R}$.

In a first step, we show that $\Psi_\kappa(\tau,m,\epsilon)$ belongs to $F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$, for any sector S_d, any fixed disc $D(0,\rho)$, for $\beta > 0, \mu > 1$ set above in (12), for $\kappa \geq 1$ given in (35), for some $\nu > 0$ depending on κ, K_F and θ_F prescribed in (13) and (14), provided that

$$\gamma + \alpha \leq \chi \quad , \quad \chi\kappa > \frac{1}{2} \tag{48}$$

hold. Notice that the second constraint of (48) will only be needed later on in Definition 4. Indeed, since the halfline L_{θ_F} avoids the roots of $F_2(\tau)$, and from the fact that $\deg(F_1) \leq \deg(F_2)$, we get a constant $C_{F_1,F_2} > 0$ such that

$$\left| \frac{F_1(u)}{F_2(u)} \right| \leq C_{F_1,F_2} \tag{49}$$

for all $u \in L_{\theta_F}$. We take a positive real number $\delta_1 > 0$ such that $\cos(\theta_F) > \delta_1$ and deduce the estimates

$$\begin{aligned} \left| \int_{L_{\theta_F}} e^{-K_F u} \frac{F_1(u)}{F_2(u)} \frac{u^n}{n!} du \right| &\leq C_{F_1,F_2} \int_0^{+\infty} \exp(-K_F r \cos(\theta_F)) \frac{r^n}{n!} dr \\ &\leq C_{F_1,F_2} \int_0^{+\infty} \exp(-K_F \delta_1 r) \frac{r^n}{n!} dr = C_{F_1,F_2} (\frac{1}{K_F \delta_1})^{n+1} \end{aligned} \tag{50}$$

by definition of $n! = \int_0^{+\infty} e^{-u} u^n du$, for $n \geq 1$. We deduce that

$$|\Psi_\kappa(\tau, m, \epsilon)| \leq C_{F_1,F_2} |\epsilon|^{n_F} |C_F(m)| E(\tau, \epsilon) \tag{51}$$

where

$$E(\tau, \epsilon) = \sum_{n \geq 1} (\frac{1}{K_F \delta_1})^{n+1} \frac{|\frac{\tau}{\epsilon^{\gamma+\alpha}}|^n}{\Gamma(n/\kappa)} = |\frac{\tau}{\epsilon^{\gamma+\alpha}}| \sum_{n \geq 0} (\frac{1}{K_F \delta_1})^{n+2} \frac{|\frac{\tau}{\epsilon^{\gamma+\alpha}}|^n}{\Gamma(\frac{n+1}{\kappa})}$$

We now provide estimates for the function $E(\tau, \epsilon)$.
We first recall that $\Gamma(a+x) \sim x^a \Gamma(x)$ as $x \to +\infty$, for any real number $a \in \mathbb{R}$, see [2], Appendix B.3. We deduce that

$$\Gamma(\frac{n+1}{\kappa}) \sim (\frac{n}{\kappa} + 1)^{\frac{1}{\kappa} - 1} \Gamma(\frac{n}{\kappa} + 1) \tag{52}$$

as $n \to +\infty$. Furthermore, we take $b > 1$ and a constant $B_\kappa > 0$ with

$$(\frac{n}{\kappa} + 1)^{1 - \frac{1}{\kappa}} \leq B_\kappa b^n \tag{53}$$

for all $n \geq 0$. Gathering (52) and (53), we extract a constant $C_\kappa > 0$ such that

$$E(\tau, \epsilon) \leq B_\kappa C_\kappa (\frac{1}{K_F \delta_1})^2 |\frac{\tau}{\epsilon^{\gamma+\alpha}}| \sum_{n \geq 0} \frac{1}{\Gamma(\frac{n}{\kappa} + 1)} (\frac{b|\tau|}{K_F \delta_1 |\epsilon|^{\gamma+\alpha}})^n \tag{54}$$

for all $\tau \in \mathbb{C}$, all $\epsilon \in \mathbb{C}^*$. At this point, we remind that the Mittag-Leffler's functions $E_\beta(x) = \sum_{n \geq 0} x^n / \Gamma(1 + \beta n)$ with index $\beta > 0$ satisfies the next estimates : there exists a constant $E_\beta > 0$ with

$$E_\beta(x) \leq E_\beta e^{x^{1/\beta}}$$

for all $x \geq 0$, see [2], Appendix B.4. We deduce the next bounds for $E(\tau, \epsilon)$,

$$E(\tau, \epsilon) \leq B_\kappa C_\kappa (\frac{1}{K_F \delta_1})^2 E_{1/\kappa} |\frac{\tau}{\epsilon^{\gamma+\alpha}}| \exp((\frac{b}{K_F \delta_1})^\kappa (\frac{|\tau|}{|\epsilon|^{\gamma+\alpha}})^\kappa) \tag{55}$$

for some constant $E_{1/\kappa} > 0$, for all $\tau \in \mathbb{C}$, all $\epsilon \in \mathbb{C}^*$. Using (48), we get that

$$
\begin{aligned}
(56)\quad E(\tau,\epsilon) \leq B_{\kappa}C_{\kappa}(\frac{1}{K_F\delta_1})^2 E_{1/\kappa}\frac{|\tau|}{|\epsilon|^{\chi}}\exp(2(\frac{b}{K_F\delta_1})^{\kappa}(\frac{|\tau|}{|\epsilon|^{\chi}})^{\kappa})\\
\times \frac{1}{1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}}(1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa})\exp(-(\frac{b}{K_F\delta_1})^{\kappa}(\frac{|\tau|}{|\epsilon|^{\chi}})^{\kappa})\\
\leq B_{\kappa}C_{\kappa}(\frac{1}{K_F\delta_1})^2 E_{1/\kappa}G_{K_F,\delta_1,\kappa}\frac{|\tau|}{|\epsilon|^{\chi}}\frac{1}{1+|\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}}\exp(2(\frac{b}{K_F\delta_1})^{\kappa}(\frac{|\tau|}{|\epsilon|^{\chi}})^{\kappa})
\end{aligned}
$$

for all $\tau \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^*$ with $|\epsilon| < 1$, where

$$G_{K_F,\delta_1,\kappa} = \sup_{x \geq 0}(1+x^{2\kappa})\exp(-(\frac{b}{K_F\delta_1})^{\kappa}x^{\kappa}).$$

Finally, collecting (51) and (56) yields the fact that $\Psi_{\kappa}(\tau,m,\epsilon)$ belongs to $F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ for all the parameters specified as above.

In the next proposition, we disclose sufficient conditions for which the main convolution equation (41) gets a unique solution in the Banach space $F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ described in Section 3.2, for the parameters chosen as above.

Proposition 6 *We take for granted that the next additional assumptions hold,*

$$(57)\qquad \chi k_l + m_l - m_0 - \alpha k_l \geq 0 \quad , \quad \delta_D \geq 1/\kappa$$

for all $1 \leq l \leq q$,

$$(58)\qquad \chi b_j + n_j - \alpha b_j \geq 0 \quad , \quad b_j \geq 1$$

for all $0 \leq j \leq Q$,

$$(59)\qquad \chi\kappa(\frac{d_{l,\kappa}}{\kappa} + \delta_l) + \Delta_l + \alpha(\delta_l - d_l) - m_0 - \chi\kappa(\delta_D - \frac{1}{\kappa}) \geq 0 \quad , \quad \delta_D - \frac{1}{\kappa} \geq \delta_l$$

for $1 \leq l \leq D-1$ *and*

$$(60)\qquad \chi\kappa(\frac{h_l}{\kappa} + \frac{1}{\kappa}) + \mu_l - 2m_0 - \alpha h_l - \chi\kappa(\delta_D - \frac{1}{\kappa}) - \chi \geq 0$$

for all $0 \leq l \leq M$. *Then, there exists a radius* $r_{Q,R_D} > 0$, $\epsilon_0 > 0$ *and a constant* $\varpi > 0$ *such that the equation (41) has a unique solution* $\omega^{d}_{\kappa}(\tau,m,\epsilon)$ *in the Banach space* $F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ *which is subjected to the bounds*

$$||\omega^{d}_{\kappa}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \varpi$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, *for any directions* $d \in \mathbb{R}$ *chosen in such a manner that the sector* S_d *respects the constraints (45) and (46) listed above.*

Proof We enter the proof with a lemma that focuses on some shrinking map upon the Banach spaces mentioned above and scales down the main convolution problem to the construction of a fixed point for this map.

Lemma 3 *Under the approval of the constraints (57), (58), (59), (60) above, one can sort the constant $r_{Q,R_D} > 0$ large enough and a constant $\varpi > 0$ small enough such that for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, the map $\mathcal{H}_\epsilon$ defined as*

$$(61)\quad \mathcal{H}_\epsilon(w(\tau,m)) := -\sum_{l=1}^{q} a_l \epsilon^{m_l - m_0 - \alpha k_l} Q(im) \frac{\tau^\kappa}{P_m(\tau)\Gamma(\frac{k_l}{\kappa})} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{k_l}{\kappa}-1} w(s^{1/\kappa},m) \frac{ds}{s}$$
$$- \sum_{l=0}^{M} c_l \epsilon^{\mu_l - 2m_0 - \alpha h_l} \frac{\tau^\kappa}{P_m(\tau)\Gamma(\frac{h_l}{\kappa})} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{h_l}{\kappa}-1}$$
$$\times \left(s \int_0^s \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) w((s-s')^{1/\kappa}, m-m_1) \right.$$
$$\left. \times Q_2(im_1) w((s')^{1/\kappa}, m_1) dm_1 \frac{1}{(s-s')s'} ds' \right) \frac{ds}{s} + \sum_{j=0}^{Q} B_j(m) \epsilon^{n_j - \alpha b_j} \frac{\tau^{b_j}}{P_m(\tau)\Gamma(\frac{b_j}{\kappa})} + \frac{\Psi_\kappa(\tau,m,\epsilon)}{P_m(\tau)}$$
$$+ R_D(im) \sum_{1 \le p \le \delta_D - 1} A_{\delta_D,p} \frac{\tau^\kappa}{P_m(\tau)\Gamma(\delta_D - p)} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\delta_D - p - 1} (\kappa s)^p w(s^{1/\kappa},m) \frac{ds}{s}$$
$$+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l + \alpha(\delta_l - d_l) - m_0} R_l(im)$$
$$\times \left(\frac{\tau^\kappa}{P_m(\tau)\Gamma(\frac{d_{l,\kappa}}{\kappa})} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{d_{l,\kappa}}{\kappa}-1} (\kappa s)^{\delta_l} w(s^{1/\kappa},m) \frac{ds}{s} + \sum_{1 \le p \le \delta_l - 1} A_{\delta_l,p} \right.$$
$$\left. \times \frac{\tau^\kappa}{P_m(\tau)\Gamma(\frac{d_{l,\kappa}+\kappa(\delta_l - p)}{\kappa})} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{d_{l,\kappa}+\kappa(\delta_l-p)}{\kappa}-1} (\kappa s)^p w(s^{1/\kappa},m) \frac{ds}{s} \right)$$

suffers the next properties.
i) *The following inclusion*

$$(62)\quad \mathcal{H}_\epsilon(\bar{B}(0,\varpi)) \subset \bar{B}(0,\varpi)$$

holds, where $\bar{B}(0,\varpi)$ is the closed ball centered at 0 with radius ϖ in $F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.
ii) *We observe that*

$$(63)\quad ||\mathcal{H}_\epsilon(w_1) - \mathcal{H}_\epsilon(w_2)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \le \frac{1}{2} ||w_1 - w_2||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$

for all $w_1, w_2 \in \bar{B}(0,\varpi)$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

Proof Firstly, we check the property (62). Let $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$ and consider $w(\tau,m) \in F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$. We select $\varpi > 0$ with $||w(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \le \varpi$.
By taking a glance at Proposition 4 1), under (57), we get a constant $C_2 > 0$ (depending on ν, κ, Q, R_D and k_l for $1 \le l \le q$) such that

$$(64)\quad ||\epsilon^{m_l - m_0 - \alpha k_l} Q(im) \frac{\tau^\kappa}{P_m(\tau)} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{k_l}{\kappa}-1} w(s^{1/\kappa},m) \frac{ds}{s} ||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$
$$\le \frac{C_2}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi k_l + m_l - m_0 - \alpha k_l} ||w(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}.$$

Due to Lemma 2, under (58), we get a constant $C_1 > 0$ (depending on ν, κ, α and n_j, b_j for $0 \leq j \leq Q$) with

$$||B_j(m)\epsilon^{n_j-\alpha b_j}\frac{\tau^{b_j}}{P_m(\tau)}||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \frac{C_1}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}}\frac{||B_j(m)||_{(\beta,\mu)}}{\inf_{m\in\mathbb{R}}|R_D(im)|}|\epsilon|^{\chi b_j+n_j-\alpha b_j}. \tag{65}$$

From the estimates (51) and (56), we get a constant C_{Ψ_κ} (depending on $||C_F(m)||_{(\beta,\mu)}$,κ, K_F, θ_F, F_1, F_2) such that

$$||\frac{\Psi_\kappa(\tau,m,\epsilon)}{P_m(\tau)}||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \frac{C_{\Psi_\kappa}}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}\inf_{m\in\mathbb{R}}|R_D(im)|}|\epsilon|^{n_F}. \tag{66}$$

Using Proposition 4 2), we can take a constant $C_2' > 0$ (depending on ν, κ and δ_D) with

$$\begin{aligned}||\frac{R_D(im)}{P_m(\tau)}\tau^\kappa \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\delta_D-p-1}s^{p-1}w(s^{1/\kappa},m)ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\\ \leq \frac{C_2'}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}}|\epsilon|^{\chi}||w(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\end{aligned} \tag{67}$$

for $1 \leq p \leq \delta_D - 1$ and in view of (59), we deduce similarly a constant $C_2' > 0$ (depending on $\nu, \kappa, \delta_D, R_D$ and d_l, δ_l, R_l for $1 \leq l \leq D-1$) such that

$$\begin{aligned}||\epsilon^{\Delta_l+\alpha(\delta_l-d_l)-m_0}R_l(im)\frac{\tau^\kappa}{P_m(\tau)}\int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{d_{l,\kappa}}{\kappa}-1}s^{\delta_l-1}w(s^{1/\kappa},m)ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\\ \leq \frac{C_2'}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}}|\epsilon|^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa}+\delta_l)+\Delta_l+\alpha(\delta_l-d_l)-m_0-\chi\kappa(\delta_D-\frac{1}{\kappa})}||w(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\end{aligned} \tag{68}$$

and

$$\begin{aligned}||\epsilon^{\Delta_l+\alpha(\delta_l-d_l)-m_0}R_l(im)\frac{\tau^\kappa}{P_m(\tau)}\int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{d_{l,\kappa}+\kappa(\delta_l-p)}{\kappa}-1}s^{p-1}w(s^{1/\kappa},m)ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\\ \leq \frac{C_2'}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}}|\epsilon|^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa}+\delta_l)+\Delta_l+\alpha(\delta_l-d_l)-m_0-\chi\kappa(\delta_D-\frac{1}{\kappa})}||w(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}\end{aligned} \tag{69}$$

for all $1 \leq p \leq \delta_l - 1$. Hereafter, we focus on estimates for the nonlinear part of $\mathcal{H}_\epsilon$. Namely, we put

$$\begin{aligned}h(\tau,m) = \tau^{\kappa-1}\frac{1}{R_D(im)}\int_0^{\tau^\kappa}\int_{-\infty}^{+\infty} Q_1(i(m-m_1))w((\tau^\kappa - s')^{1/\kappa}, m-m_1)\\ \times Q_2(im_1)w((s')^{1/\kappa},m_1)\frac{1}{(\tau^\kappa - s')s'}ds'dm_1.\end{aligned}$$

A glimpse into Proposition 5, allows us to catch a constant $C_3 > 0$ (depending on $\mu, \kappa, Q_1, Q_2, R_D$) with

$$||h(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \frac{C_3}{|\epsilon|^\chi}||w(\tau,m)||^2_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \tag{70}$$

On the other hand, bearing in mind Proposition 4 2), it boils down from (60) that there exists a constant $C_2' > 0$ (depending on ν, κ, δ_D and h_l for $0 \leq l \leq M$) with

$$
(71) \quad ||\epsilon^{\mu_l - 2m_0 - \alpha h_l} \frac{\tau^\kappa}{P_m(\tau)} R_D(im) \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{h_l}{\kappa} - 1} s^{\frac{1}{\kappa} - 1} h(s^{1/\kappa}, m) ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\
\leq \frac{C_2'}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi\kappa(\frac{h_l}{\kappa} + \frac{1}{\kappa}) + \mu_l - 2m_0 - \alpha h_l - \chi\kappa(\delta_D - \frac{1}{\kappa})} ||h(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}
$$

Clustering (70) and (71) yields

$$
(72) \quad ||\epsilon^{\mu_l - 2m_0 - \alpha h_l} \frac{\tau^\kappa}{P_m(\tau)} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{h_l}{\kappa} - 1} \\
\times \left(s \int_0^s \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) w((s-s')^{1/\kappa}, m - m_1) \right. \\
\left. \times Q_2(im_1) w((s')^{1/\kappa}, m_1) dm_1 \frac{1}{(s-s')s'} ds' \right) \frac{ds}{s} ||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\
\leq \frac{C_2' C_3}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi\kappa(\frac{h_l}{\kappa} + \frac{1}{\kappa}) + \mu_l - 2m_0 - \alpha h_l - \chi\kappa(\delta_D - \frac{1}{\kappa}) - \chi} ||w(\tau,m)||^2_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}
$$

Finally, we choose $r_{Q,R_D} > 0$ and $\varpi > 0$ in such a way that

$$
(73) \quad \sum_{l=1}^q |a_l| \frac{C_2}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}} \Gamma(\frac{k_l}{\kappa})} \epsilon_0^{\chi k_l + m_l - m_0 - \alpha k_l} \varpi + \sum_{l=0}^M |c_l| \frac{C_2' C_3}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}} \Gamma(\frac{h_l}{\kappa}) (2\pi)^{1/2}} \\
\times \epsilon_0^{\chi\kappa(\frac{h_l}{\kappa} + \frac{1}{\kappa}) + \mu_l - 2m_0 - \alpha h_l - \chi\kappa(\delta_D - \frac{1}{\kappa}) - \chi} \varpi^2 + \sum_{j=0}^Q \frac{C_1}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}} \Gamma(\frac{b_j}{\kappa})} \\
\times \frac{||B_j(m)||_{(\beta,\mu)}}{\inf_{m \in \mathbb{R}} |R_D(im)|} \epsilon_0^{\chi b_j + n_j - \alpha b_j} + \frac{C_{\Psi_\kappa}}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}} \inf_{m \in \mathbb{R}} |R_D(im)|} \epsilon_0^{n_F} + \sum_{1 \leq p \leq \delta_D - 1} |A_{\delta_D, p}| \\
\times \frac{\kappa^p}{\Gamma(\delta_D - p)} \frac{C_2'}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} \epsilon_0^\chi \varpi + \sum_{l=1}^{D-1} \frac{C_2' \kappa^{\delta_l}}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}} \Gamma(\frac{d_{l,\kappa}}{\kappa})} \\
\times \epsilon_0^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa} + \delta_l) + \Delta_l + \alpha(\delta_l - d_l) - m_0 - \chi\kappa(\delta_D - \frac{1}{\kappa})} \varpi + \sum_{1 \leq p \leq \delta_l - 1} |A_{\delta_l, p}| \frac{\kappa^p}{\Gamma(\frac{d_{l,\kappa} + \kappa(\delta_l - p)}{\kappa})} \frac{C_2'}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} \\
\times \epsilon_0^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa} + \delta_l) + \Delta_l + \alpha(\delta_l - d_l) - m_0 - \chi\kappa(\delta_D - \frac{1}{\kappa})} \varpi \leq \varpi
$$

As an issue of the definition of $\mathcal{H}_\epsilon$, by collecting all the bounds (64), (65), (66), (67), (68), (69), (72), we conclude that

$$
(74) \qquad ||\mathcal{H}_\epsilon(w(\tau,m))||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \varpi
$$

and the first claim (62) holds.

In a second part of the proof, we turn our effort to the verification of the affirmation (63). Let $w_1(\tau,m), w_2(\tau,m)$ belong to $F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ with

$$
||w_1(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \varpi \quad , \quad ||w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \varpi
$$

Foremost, according to the estimates (64), we obtain a constant $C_2 > 0$ (depending on ν, κ, Q, R_D and k_l for $1 \leq l \leq q$) such that

$$
(75) \quad ||\epsilon^{m_l - m_0 - \alpha k_l} Q(im) \frac{\tau^\kappa}{P_m(\tau)} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{k_l}{\kappa} - 1} (w_1(s^{1/\kappa}, m) - w_2(s^{1/\kappa}, m)) \frac{ds}{s} ||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\
\leq \frac{C_2}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi k_l + m_l - m_0 - \alpha k_l} ||w_1(\tau,m) - w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}.
$$

Likewise, in agreement with (67), (68), (69) we can select a constant $C_2' > 0$ (depending on $\nu, \kappa, \delta_D, R_D$ and d_l, δ_l, R_l for $1 \leq l \leq D-1$) fulfilling

$$
(76) \quad ||\frac{R_D(im)}{P_m(\tau)} \tau^\kappa \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\delta_D - p - 1} s^{p-1} (w_1(s^{1/\kappa}, m) - w_2(s^{1/\kappa}, m) ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\
\leq \frac{C_2'}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi} ||w_1(\tau,m) - w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}
$$

for $1 \leq p \leq \delta_D - 1$, together with

$$
(77) \quad ||\epsilon^{\Delta_l + \alpha(\delta_l - d_l) - m_0} R_l(im) \frac{\tau^\kappa}{P_m(\tau)} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{d_{l,\kappa}}{\kappa} - 1} s^{\delta_l - 1} \\
\times (w_1(s^{1/\kappa}, m) - w_2(s^{1/\kappa}, m)) ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\
\leq \frac{C_2'}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa} + \delta_l) + \Delta_l + \alpha(\delta_l - d_l) - m_0 - \chi\kappa(\delta_D - \frac{1}{\kappa})} ||w_1(\tau,m) - w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}
$$

and

$$
(78) \quad ||\epsilon^{\Delta_l + \alpha(\delta_l - d_l) - m_0} R_l(im) \frac{\tau^\kappa}{P_m(\tau)} \int_0^{\tau^\kappa} (\tau^\kappa - s)^{\frac{d_{l,\kappa} + \kappa(\delta_l - p)}{\kappa} - 1} s^{p-1} \\
\times (w_1(s^{1/\kappa}, m) - w_2(s^{1/\kappa}, m)) ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \\
\leq \frac{C_2'}{C_P (r_{Q,R_D})^{\frac{1}{\delta_D \kappa}}} |\epsilon|^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa} + \delta_l) + \Delta_l + \alpha(\delta_l - d_l) - m_0 - \chi\kappa(\delta_D - \frac{1}{\kappa})} ||w_1(\tau,m) - w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}
$$

for all $1 \leq p \leq \delta_l - 1$. We turn now to the nonlinear part of $\mathcal{H}_\epsilon$. As a groundwork, let us rewrite

$$
(79) \quad Q_1(i(m - m_1)) w_1((\tau^\kappa - s')^{1/\kappa}, m - m_1) Q_2(im_1) w_1((s')^{1/\kappa}, m_1) \\
- Q_1(i(m - m_1)) w_2((\tau^\kappa - s')^{1/\kappa}, m - m_1) Q_2(im_1) w_2((s')^{1/\kappa}, m_1) \\
= Q_1(i(m - m_1)) \left(w_1((\tau^\kappa - s')^{1/\kappa}, m - m_1) - w_2((\tau^\kappa - s')^{1/\kappa}, m - m_1) \right) \\
\times Q_2(im_1) w_1((s')^{1/\kappa}, m_1) + Q_1(i(m - m_1)) w_2((\tau^\kappa - s')^{1/\kappa}, m - m_1) Q_2(im_1) \\
\times \left(w_1((s')^{1/\kappa}, m_1) - w_2((s')^{1/\kappa}, m_1) \right)
$$

For $j = 1, 2$, we put

$$
h_j(\tau, m) = \frac{\tau^{\kappa - 1}}{R_D(im)} \int_0^{\tau^\kappa} \int_{-\infty}^{+\infty} Q_1(i(m - m_1)) w_j((\tau^\kappa - s')^{1/\kappa}, m - m_1) \\
\times Q_2(im_1) w_j((s')^{1/\kappa}, m_1) \frac{1}{(\tau^\kappa - s') s'} ds' dm_1
$$

Focusing both on the factorization (79) and Proposition 5, we can find a constant $C_3 > 0$ (depending on $\mu, \kappa, Q_1, Q_2, R_D$) with

$$(80)\quad ||h_1(\tau,m) - h_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \frac{C_3}{|\epsilon|^{\chi}}(||w_1(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} + ||w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}) \times ||w_1(\tau,m) - w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$

From (71) together with (80), we can pick up a constant $C_2' > 0$ (depending on ν, κ, δ_D and h_l for $0 \leq l \leq M$) with

(81)

$$||\epsilon^{\mu_l - 2m_0 - \alpha h_l}\frac{\tau^{\kappa}}{P_m(\tau)}R_D(im)\int_0^{\tau^{\kappa}}(\tau^{\kappa}-s)^{\frac{h_l}{\kappa}-1}s^{\frac{1}{\kappa}-1}(h_1(s^{1/\kappa},m) - h_2(s^{1/\kappa},m))ds||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$
$$\leq \frac{C_2'}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}}|\epsilon|^{\chi\kappa(\frac{h_l}{\kappa}+\frac{1}{\kappa})+\mu_l-2m_0-\alpha h_l-\chi\kappa(\delta_D-\frac{1}{\kappa})}||h_1(\tau,m) - h_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$
$$\leq \frac{C_2'C_3}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}}|\epsilon|^{\chi\kappa(\frac{h_l}{\kappa}+\frac{1}{\kappa})+\mu_l-2m_0-\alpha h_l-\chi\kappa(\delta_D-\frac{1}{\kappa})-\chi}$$
$$\times (||w_1(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} + ||w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)})||w_1(\tau,m) - w_2(\tau,m)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$$

As a result, we choose $r_{Q,R_D} > 0$ and $\varpi > 0$ obeying the next inequality

$$(82)\quad \sum_{l=1}^{q}|a_l|\frac{C_2}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}\Gamma(\frac{k_l}{\kappa})}\epsilon_0^{\chi k_l+m_l-m_0-\alpha k_l} + \sum_{l=0}^{M}|c_l|\frac{C_2'C_3}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}\Gamma(\frac{h_l}{\kappa})(2\pi)^{1/2}}$$
$$\times \epsilon_0^{\chi\kappa(\frac{h_l}{\kappa}+\frac{1}{\kappa})+\mu_l-2m_0-\alpha h_l-\chi\kappa(\delta_D-\frac{1}{\kappa})-\chi}2\varpi + \sum_{1\leq p\leq\delta_D-1}|A_{\delta_D,p}|$$
$$\times \frac{\kappa^p}{\Gamma(\delta_D-p)}\frac{C_2'}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}}\epsilon_0^{\chi} + \sum_{l=1}^{D-1}\frac{C_2'\kappa^{\delta_l}}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}\Gamma(\frac{d_{l,\kappa}}{\kappa})}$$
$$\times \epsilon_0^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa}+\delta_l)+\Delta_l+\alpha(\delta_l-d_l)-m_0-\chi\kappa(\delta_D-\frac{1}{\kappa})} + \sum_{1\leq p\leq\delta_l-1}|A_{\delta_l,p}|\frac{\kappa^p}{\Gamma(\frac{d_{l,\kappa}+\kappa(\delta_l-p)}{\kappa})}\frac{C_2'}{C_P(r_{Q,R_D})^{\frac{1}{\delta_D\kappa}}}$$
$$\times \epsilon_0^{\chi\kappa(\frac{d_{l,\kappa}}{\kappa}+\delta_l)+\Delta_l+\alpha(\delta_l-d_l)-m_0-\chi\kappa(\delta_D-\frac{1}{\kappa})} \leq \frac{1}{2}$$

By assembling all the bounds (75), (76), (77), (78), (81), we attain the foreseen estimates (63).

At the very end of the proof, we now take for granted that the two conditions (73) and (82) hold conjointly for the radii r_{Q,R_D} and ϖ. Then both (62) and (63) hold at the same time and the Lemma 3 is shown. □

We consider the ball $\bar{B}(0,\varpi) \subset F^d_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ just built above in Lemma 3 which is actually a complete metric space equipped with the norm $||.||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$. From the lemma above, we get that $\mathcal{H}_{\epsilon}$ is a contractive application from $\bar{B}(0,\varpi)$ into itself. Due to the classical contractive mapping theorem, we deduce that the map $\mathcal{H}_{\epsilon}$ has a unique fixed point denoted $\omega_{\kappa}^d(\tau,m,\epsilon)$ in the ball $\bar{B}(0,\varpi)$, meaning that

$$(83)\quad \mathcal{H}_{\epsilon}(\omega_{\kappa}^d(\tau,m,\epsilon)) = \omega_{\kappa}^d(\tau,m,\epsilon)$$

for a unique $\omega_{\kappa}^{d}(\tau,m,\epsilon) \in F^{d}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ such that $||\omega_{\kappa}^{d}(\tau,m,\epsilon)||_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)} \leq \varpi$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. Moreover, the function $\omega_{\kappa}^{d}(\tau,m,\epsilon)$ depends holomorphically on ϵ in $D(0,\epsilon_0) \setminus \{0\}$.

Now, if one sets apart the terms $Q(im)a_0\omega_{\kappa}(\tau,m,\epsilon)$ in the left handside of (41) and $R_D(im)(\kappa\tau^{\kappa})^{\delta_D}\omega_{\kappa}(\tau,m,\epsilon)$ in the right handside of (41), we recognize by dividing with the polynomial $P_m(\tau)$ given in (43) that (41) can be exactly rewritten as the equation (83) above. Therefore, the unique fixed point $\omega_{\kappa}^{d}(\tau,m,\epsilon)$ of $\mathcal{H}_{\epsilon}$ in $\bar{B}(0,\varpi)$ precisely solves the problem (41). This yields the proposition. □

3.5 Analytic solutions to the main problem on boundary layers $\epsilon-$depending domains in time near the origin

We return to the formal construction of time rescaled solutions to the main equation (8) under the new insight on the main associated convolution equation (41) reached in the previous subsection.

We first recall the definitions of a good covering as introduced in [10].

Definition 4 *Let $\varsigma \geq 2$ be an integer. For all $0 \leq p \leq \varsigma - 1$, we consider open sectors $\mathcal{E}_p$ centered at 0, with radius $\epsilon_0 > 0$ and opening $\frac{\pi}{\chi\kappa} + \xi_p < 2\pi$ with $\xi_p > 0$ small enough such that $\mathcal{E}_p \cap \mathcal{E}_{p+1} \neq \emptyset$, for all $0 \leq p \leq \varsigma - 1$ (with the convention that $\mathcal{E}_{\varsigma} = \mathcal{E}_0$). Moreover, we assume that the intersection of any three different elements in $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ is empty and that $\cup_{p=0}^{\varsigma-1}\mathcal{E}_p = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ is some neighborhood of 0 in $\mathbb{C}$. Such a set of sectors $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ is called a good covering in $\mathbb{C}^*$ with aperture $\frac{\pi}{\chi\kappa}$.*

We now give a definition for a set of $\epsilon-$depending sectors associated to a good covering.

Definition 5 *Let $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ be a good covering with aperture $\frac{\pi}{\chi\kappa}$. Let $\alpha \in \mathbb{Q}$ with $\alpha < \chi$. We choose a fixed open sector X centered at 0 with radius $\varrho_X > 0$ and for each $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, we define the sector*

$$\mathcal{T}_{\epsilon,\chi-\alpha} = \{x\epsilon^{\chi-\alpha}/x \in X\}$$

with radius $\varrho_X|\epsilon|^{\chi-\alpha}$. For each $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, we consider also a family of sectors

$$S_{\mathfrak{d}_p,\theta,\varrho_X|\epsilon|^{\chi}} = \{T \in \mathbb{C}^*/|T| \leq \varrho_X|\epsilon|^{\chi} \quad , \quad |\mathfrak{d}_p - \arg(T)| < \frac{\theta}{2}\}$$

for some aperture $\theta > \frac{\pi}{\kappa}$, where $\mathfrak{d}_p \in \mathbb{R}$, for $0 \leq p \leq \varsigma - 1$ are directions which satisfy the next constraints described below.
Let $q_l(m)$ be the roots of $P_m(\tau)$ defined in (44) and $S_{\mathfrak{d}_p}$, $0 \leq p \leq \varsigma - 1$ be unbounded sectors centered at 0 with direction $\mathfrak{d}_p$ and small aperture. We assume that
1) There exists a constant $M_1 > 0$ such that

$$|\tau - q_l(m)| \geq M_1(1 + |\tau|) \tag{84}$$

for all $0 \leq l \leq \delta_D\kappa - 1$, all $m \in \mathbb{R}$, all $\tau \in S_{\mathfrak{d}_p} \cup \bar{D}(0,\rho)$, for all $0 \leq p \leq \varsigma - 1$.
2) There exists a constant $M_2 > 0$ such that

$$|\tau - q_{l_0}(m)| \geq M_2|q_{l_0}(m)| \tag{85}$$

for some $l_0 \in \{0,\ldots,\delta_D\kappa - 1\}$, all $m \in \mathbb{R}$, all $\tau \in S_{\mathfrak{d}_p} \cup \bar{D}(0,\rho)$, for all $0 \leq p \leq \varsigma - 1$.
3) For all $0 \leq p \leq \varsigma - 1$, all $\epsilon \in \mathcal{E}_p$ and $t \in \mathcal{T}_{\epsilon,\chi-\alpha}$, we have that

$$\epsilon^{\alpha}t \in S_{\mathfrak{d}_p,\theta,\varrho_X|\epsilon|^{\chi}}.$$

We say that the family of sectors $\{(S_{\mathfrak{d}_p,\theta,\varrho_X|\epsilon|^{\chi}})_{0 \leq p \leq \varsigma-1}, \mathcal{T}_{\epsilon,\chi-\alpha}\}$ is associated to the good covering $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$.

In the next main first outcome, we build a family of actual holomorphic solutions to the principal equation (8) that we call *inner solutions*. These solutions are defined on the sectors $\mathcal{E}_p$ of a good covering w.r.t ϵ, on $\epsilon-$depending associated sectors $\mathcal{T}_{\epsilon,\chi-\alpha}$ w.r.t t and on some horizontal strip H_β w.r.t z. Furthermore, we can oversee the difference between any two neighboring solutions on the intersections $\mathcal{E}_p \cap \mathcal{E}_{p+1}$ and ascertain that it is exponentially flat of order at most $\chi\kappa$ w.r.t ϵ.

Theorem 1 *We look at the singularly perturbed PDE (8) and we assume that all the prior constraints (7), (9), (12), (13), (14), (34), (35), (42), (48), (57), (58), (59) and (60) hold. Let* $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ *a good covering in* $\mathbb{C}^*$ *with aperture* $\frac{\pi}{\chi\kappa}$ *be given, for which a family of open sectors* $\{(S_{\mathfrak{d}_p,\theta,\varrho_X|\epsilon|^\chi})_{0\leq p\leq\varsigma-1},\mathcal{T}_{\epsilon,\chi-\alpha}\}$ *associated to this good covering can be distinguished.*

Then, there exist a radius $r_{Q,R_D}>0$ *large enough and* $\epsilon_0>0$ *small enough, for which a family* $\{u^{\mathfrak{d}_p}(t,z,\epsilon)\}_{0\leq p\leq\varsigma-1}$ *of actual solutions of (8) are built up. Furthermore, for each* $\epsilon\in\mathcal{E}_p$*, the function* $(t,z)\mapsto u^{\mathfrak{d}_p}(t,z,\epsilon)$ *defines a bounded holomorphic function on* $\mathcal{T}_{\epsilon,\chi-\alpha}\times H_{\beta'}$ *for any given* $0<\beta'<\beta$*, for all* $0\leq p\leq\varsigma-1$*. Moreover, the functions* $(x,z,\epsilon)\mapsto\epsilon^{m_0}u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha},z,\epsilon)$ *are bounded and holomorphic on* $X\times H_{\beta'}\times\mathcal{E}_p$ *for any given* $0<\beta'<\beta$*,* $0\leq p\leq\varsigma-1$ *and are submitted to the next bounds : there exist constants* $K_p,M_p>0$ *and* $\sigma>0$ *(independent of* ϵ*) such that*

$$\sup_{x\in X\cap D(0,\sigma),z\in H_{\beta'}}|\epsilon^{m_0}u^{\mathfrak{d}_{p+1}}(x\epsilon^{\chi-\alpha},z,\epsilon)-\epsilon^{m_0}u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha},z,\epsilon)|\leq K_p\exp(-\frac{M_p}{|\epsilon|^{\chi\kappa}}) \tag{86}$$

for all $\epsilon\in\mathcal{E}_{p+1}\cap\mathcal{E}_p$*, all* $0\leq p\leq\varsigma-1$ *(where by convention* $u^{\mathfrak{d}_\varsigma}=u^{\mathfrak{d}_0}$*).*

Proof As shown above in Section 3.4, the series

$$\Psi_\kappa(\tau,m,\epsilon)=\sum_{n\geq1}\psi_n(m,\epsilon)\frac{\tau^n}{\Gamma(n/\kappa)}\in E_{(\beta,\mu)}[[\tau]]$$

is convergent for all τ in $\mathbb{C}$, for all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$. Moreover, it is subjected to the next bounds

$$|\Psi_\kappa(\tau,m,\epsilon)|\leq\Psi|\epsilon|^{n_F}(1+|m|)^{-\mu}\exp(-\beta|m|)\frac{|\frac{\tau}{\epsilon^\chi}|}{1+|\frac{\tau}{\epsilon^\chi}|^{2\kappa}}\exp(\nu|\frac{\tau}{\epsilon^\chi}|^\kappa)$$

for some constant $\Psi>0$ (independent of ϵ), for all $\tau\in\mathbb{C}$, all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$. We deduce that the formal series

$$\Phi_\kappa(T,m,\epsilon)=\sum_{n\geq1}\psi_n(m,\epsilon)T^n\in E_{(\beta,\mu)}[[T]]$$

is $m_\kappa-$summable in all directions $d\in\mathbb{R}$ according to Definition 2 and hence defines a convergent series near $T=0$. In order to get its radius of convergence, we can express it as a $m_\kappa-$sum

$$\Phi_\kappa(T,m,\epsilon)=\kappa\int_{L_d}\Psi_\kappa(u,m,\epsilon)\exp(-(\frac{u}{T})^\kappa)\frac{du}{u}$$

for any halfline $L_d=\mathbb{R}_+e^{\sqrt{-1}d}$, with direction $d\in\mathbb{R}$. From Definition 2, we can check that $T\mapsto\Phi_\kappa(T,m,\epsilon)$ defines a $E_{(\beta,\mu)}-$valued holomorphic function on a disc $D(0,\Delta|\epsilon|^\chi)$, for some $\Delta>0$ (independent of ϵ), for all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$.

Now, we select a good covering $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ in $\mathbb{C}^*$ with aperture $\frac{\pi}{\chi\kappa}$ and a family of sectors $\{(S_{\mathfrak{d}_p,\theta,\varrho_X|\epsilon|^\chi})_{0\leq p\leq\varsigma-1},\mathcal{T}_{\epsilon,\chi-\alpha}\}$ associated to this covering according to Definition 5. Proposition

6 allows us to choose for each direction $\mathfrak{d}_p$, a solution $\omega_{\kappa}^{\mathfrak{d}_p}\tau, m, \epsilon)$ of the convolution equation (41) which is located in the Banach space $F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,\chi,\kappa,\epsilon)}$ and thus suffering the next bounds

$$|\omega_{\kappa}^{\mathfrak{d}_p}(\tau, m, \epsilon)| \leq \varpi(1 + |m|)^{-\mu} \exp(-\beta|m|) \frac{|\frac{\tau}{\epsilon^{\chi}}|}{1 + |\frac{\tau}{\epsilon^{\chi}}|^{2\kappa}} \exp(\nu|\frac{\tau}{\epsilon^{\chi}}|^{\kappa}) \tag{87}$$

for all $\tau \in \bar{D}(0, \rho) \cup S_{\mathfrak{d}_p}$, $m \in \mathbb{R}$ and $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, for some suitable $\varpi > 0$. In particular, these functions $\omega_{\kappa}^{\mathfrak{d}_p}(\tau, m, \epsilon)$ are analytic continuation w.r.t τ of a common convergent series

$$\omega_{\kappa}(\tau, m, \epsilon) = \sum_{n \geq 1} \frac{\omega_n(m, \epsilon)}{\Gamma(n/\kappa)} \tau^n \in E_{(\beta,\mu)}\{\tau\}$$

which defines a solution of (41) for $\tau \in D(0, \rho)$. As a result, the formal series

$$\Omega_{\kappa}(T, m, \epsilon) = \sum_{n \geq 1} \omega_n(m, \epsilon) T^n \in E_{(\beta,\mu)}[[T]]$$

turns out to be $m_{\kappa}-$summable in direction $\mathfrak{d}_p$ according to Definition 2, for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. We set

$$\Omega_{\kappa}^{\mathfrak{d}_p}(T, m, \epsilon) = \kappa \int_{L_{\gamma}} \omega_{\kappa}^{\mathfrak{d}_p}(u, m, \epsilon) \exp(-(\frac{u}{T})^{\kappa}) \frac{du}{u} \tag{88}$$

as the $m_{\kappa}-$sum of $\Omega_{\kappa}(T, m, \epsilon)$ in direction $\mathfrak{d}_p$, with $L_{\gamma} = \mathbb{R}_+ e^{\sqrt{-1}\gamma} \subset S_{\mathfrak{d}_p}$. This map defines a $E_{(\beta,\mu)}-$valued holomorphic function w.r.t T on a sector

$$S_{\mathfrak{d}_p,\theta,\Delta|\epsilon|^{\chi}} = \{T \in \mathbb{C}^* : |T| < \Delta|\epsilon|^{\chi} \quad , \quad |\mathfrak{d}_p - \arg(T)| < \theta/2\}$$

for some $\theta \in (\frac{\pi}{\kappa}, \frac{\pi}{\kappa} + \mathrm{Ap}(S_{\mathfrak{d}_p}))$ (where $\mathrm{Ap}(S_{\mathfrak{d}_p})$ stands for the aperture of the sector $S_{\mathfrak{d}_p}$) and some $\Delta > 0$ independent of ϵ, for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$.

According to the formal identities enounced in Proposition 3 and by virtue of the properties of the $m_{\kappa}-$sums with respect to derivatives and product (within the Banach space $E_{(\beta,\mu)}$ endowed with the convolution product $\star$ described in Proposition 1) we notice that the function $\Omega_{\kappa}^{\mathfrak{d}_p}(T, m, \epsilon)$ must solve the next equation

$$\begin{split} Q(im)(\sum_{l=1}^{q} a_l \epsilon^{m_l - m_0 - \alpha k_l} T^{k_l} + a_0)\Omega_{\kappa}^{\mathfrak{d}_p}(T, m, \epsilon) + (\sum_{l=0}^{M} c_l \epsilon^{\mu_l - 2m_0 - \alpha h_l} T^{h_l}) \frac{1}{(2\pi)^{1/2}} \\ \times \int_{-\infty}^{+\infty} Q_1(i(m - m_1))\Omega_{\kappa}^{\mathfrak{d}_p}(T, m - m_1, \epsilon) Q_2(im_1)\Omega_{\kappa}^{\mathfrak{d}_p}(T, m_1, \epsilon) dm_1 = \sum_{j=0}^{Q} B_j(m) \epsilon^{n_j - \alpha b_j} T^{b_j} \\ + \Phi_{\kappa}(T, m, \epsilon) + R_D(im) \left((T^{\kappa+1}\partial_T)^{\delta_D} + \sum_{1 \leq p \leq \delta_D - 1} A_{\delta_D,p} T^{\kappa(\delta_D - p)} (T^{\kappa+1}\partial_T)^p \right) \Omega_{\kappa}^{\mathfrak{d}_p}(T, m, \epsilon) \\ + \sum_{l=1}^{D-1} \epsilon^{\Delta_l + \alpha(\delta_l - d_l) - m_0} R_l(im) T^{d_{l,\kappa}} \Big((T^{\kappa+1}\partial_T)^{\delta_l} \\ + \sum_{1 \leq p \leq \delta_l - 1} A_{\delta_l,p} T^{\kappa(\delta_l - p)} (T^{\kappa+1}\partial_T)^p \Big) \Omega_{\kappa}^{\mathfrak{d}_p}(T, m, \epsilon) \end{split} \tag{89}$$

In the next step, we introduce

$$U^{\mathfrak{d}_p}(T,z,\epsilon) = \mathcal{F}^{-1}(m \mapsto \Omega_{\kappa}^{\mathfrak{d}_p}(T,m,\epsilon))(z)$$

which defines a bounded holomorphic function w.r.t T on $S_{\mathfrak{d}_p,\theta,\Delta|\epsilon|^{\chi}}$, w.r.t z on $H_{\beta'}$ for any $0 < \beta' < \beta$ and all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. Besides, by construction, we observe that the function F^{θ_F} defined in (13) can be expressed as

$$F^{\theta_F}(\epsilon^{-\alpha}T,z,\epsilon) = \mathcal{F}^{-1}(m \mapsto \Phi_{\kappa}(T,m,\epsilon))$$

which represents a bounded holomorphic function w.r.t T on the disc $D(0,\Delta|\epsilon|^{\chi})$, w.r.t z on $H_{\beta'}$ for any $0 < \beta' < \beta$ and all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

Bearing in mind the basic properties of the Fourier inverse transform described in Proposition 2 and taking notice of the expansions (36) and (37), we extract from the latter equality (89) the next problem satisfied by $U^{\mathfrak{d}_p}(T,z,\epsilon)$, namely

$$\begin{aligned}(90)\quad &(\sum_{l=1}^{q} a_l \epsilon^{m_l - m_0 - \alpha k_l} T^{k_l} + a_0) Q(\partial_z) U^{\mathfrak{d}_p}(T,z,\epsilon)\\ &\qquad + (\sum_{l=0}^{M} c_l \epsilon^{\mu_l - 2m_0 - \alpha h_l} T^{h_l}) Q_1(\partial_z) U^{\mathfrak{d}_p}(T,z,\epsilon) Q_2(\partial_z) U^{\mathfrak{d}_p}(T,z,\epsilon)\\ &= \sum_{j=0}^{Q} b_j(z) \epsilon^{n_j - \alpha b_j} T^{b_j} + F^{\theta_F}(\epsilon^{-\alpha}T,z,\epsilon) + \sum_{l=1}^{D} \epsilon^{\Delta_l + \alpha(\delta_l - d_l) - m_0} T^{d_l} R_l(\partial_z) \partial_T^{\delta_l} U^{\mathfrak{d}_p}(T,z,\epsilon)\end{aligned}$$

Finally, we put

$$u^{\mathfrak{d}_p}(t,z,\epsilon) = \epsilon^{-m_0} U^{\mathfrak{d}_p}(\epsilon^{\alpha} t, z, \epsilon) \tag{91}$$

that constitutes a bounded holomorphic function w.r.t t on $\mathcal{T}_{\epsilon,\chi-\alpha}$ and w.r.t z on $H_{\beta'}$ for any $0 < \beta' < \beta$, for each fixed $\epsilon \in \mathcal{E}_p$, according to Definition 5. Furthermore, by direct inspection, one can check that the function $(x,z,\epsilon) \mapsto \epsilon^{m_0} u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha},z,\epsilon)$ is bounded and holomorphic on $X \times \mathcal{E}_p \times H_{\beta'}$ for any given $0 < \beta' < \beta$ and fixed $0 \leq p \leq \varsigma - 1$. Moreover, $u^{\mathfrak{d}_p}(t,z,\epsilon)$ solves the main equation (8) where the piece of forcing term $(t,z) \mapsto F^{\theta_F}(t,z,\epsilon)$ represents in particular a bounded holomorphic function w.r.t t on the disc $D(0,\Delta|\epsilon|^{\chi-\alpha})$, w.r.t z on the strip $H_{\beta'}$ for any given $0 < \beta' < \beta$ and fixed $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

In the last part of the proof, it remains to justify the bounds (86). According to the construction given above, we observe that for each $0 \leq p \leq \varsigma - 1$, the function $\epsilon^{m_0} u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha},z,\epsilon)$ can be written as a $m_{\kappa}-$Laplace and Fourier inverse transform

$$\epsilon^{m_0} u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha},z,\epsilon) = \frac{\kappa}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} \omega_{\kappa}^{\mathfrak{d}_p}(u,m,\epsilon) \exp(-(\frac{u}{x\epsilon^{\chi}})^{\kappa}) e^{izm} \frac{du}{u} dm \tag{92}$$

where $L_{\gamma_p} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_p} \subset S_{\mathfrak{d}_p}$. The steps of the verification are similar to the arguments disclosed in Theorem 1 of [10] but we still decide to present the details for the benefit of clarity. Namely, using the fact that the function $u \mapsto \omega_{\kappa}(u,m,\epsilon)\exp(-(\frac{u}{\epsilon^{\chi}x})^{\kappa})/u$ is holomorphic on $D(0,\rho)$ for all $(m,\epsilon) \in \mathbb{R} \times (D(0,\epsilon_0) \setminus \{0\})$, its integral along the union of a segment starting from 0 to $(\rho/2)e^{i\gamma_{p+1}}$, an arc of circle with radius $\rho/2$ which connects $(\rho/2)e^{i\gamma_{p+1}}$ and $(\rho/2)e^{i\gamma_p}$

and a segment starting from $(\rho/2)e^{i\gamma_p}$ to 0, is vanishing. Therefore, we can write the difference $\epsilon^{m_0}u^{\mathfrak{d}_{p+1}} - \epsilon^{m_0}u^{\mathfrak{d}_p}$ as a sum of three integrals,

$$
\begin{aligned}
(93)\quad & \epsilon^{m_0}u^{\mathfrak{d}_{p+1}}(x\epsilon^{\chi-\alpha},z,\epsilon) - \epsilon^{m_0}u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha},z,\epsilon) \\
&= \frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\rho/2,\gamma_{p+1}}} \omega_\kappa^{\mathfrak{d}_{p+1}}(u,m,\epsilon)e^{-(\frac{u}{\epsilon^\chi x})^\kappa}e^{izm}\frac{du}{u}dm \\
&- \frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\rho/2,\gamma_p}} \omega_\kappa^{\mathfrak{d}_p}(u,m,\epsilon)e^{-(\frac{u}{\epsilon^\chi x})^\kappa}e^{izm}\frac{du}{u}dm \\
&+ \frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{C_{\rho/2,\gamma_p,\gamma_{p+1}}} \omega_\kappa(u,m,\epsilon)e^{-(\frac{u}{\epsilon^\chi x})^\kappa}e^{izm}\frac{du}{u}dm
\end{aligned}
$$

where $L_{\rho/2,\gamma_{p+1}} = [\rho/2,+\infty)e^{i\gamma_{p+1}}$, $L_{\rho/2,\gamma_p} = [\rho/2,+\infty)e^{i\gamma_p}$ and $C_{\rho/2,\gamma_p,\gamma_{p+1}}$ is an arc of circle with radius connecting $(\rho/2)e^{i\gamma_p}$ and $(\rho/2)e^{i\gamma_{p+1}}$ with a well chosen orientation.

We give estimates for the quantity

$$
I_1 = \left|\frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\rho/2,\gamma_{p+1}}} \omega_\kappa^{\mathfrak{d}_{p+1}}(u,m,\epsilon)e^{-(\frac{u}{\epsilon^\chi x})^\kappa}e^{izm}\frac{du}{u}dm\right|.
$$

By construction, the direction γ_{p+1} (which depends on $\epsilon^\chi x$) is chosen in such a way that $\cos(\kappa(\gamma_{p+1} - \arg(\epsilon^\chi x))) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $x \in X$, for some fixed $\delta_1 > 0$. From the estimates (87), we get that

$$
\begin{aligned}
(94)\quad I_1 &\leq \frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{\rho/2}^{+\infty} \varpi(1+|m|)^{-\mu}e^{-\beta|m|}\frac{\frac{r}{|\epsilon|^\chi}}{1+(\frac{r}{|\epsilon|^\chi})^{2\kappa}} \\
&\times \exp(\nu(\frac{r}{|\epsilon|^\chi})^\kappa)\exp(-\frac{\cos(\kappa(\gamma_{p+1}-\arg(\epsilon^\chi x)))}{|\epsilon^\chi x|^\kappa}r^\kappa)e^{-m\mathrm{Im}(z)}\frac{dr}{r}dm \\
&\leq \frac{\kappa\varpi}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}e^{-(\beta-\beta')|m|}dm\int_{\rho/2}^{+\infty}\frac{1}{|\epsilon|^\chi}\exp(-(\frac{\delta_1}{|x|^\kappa}-\nu)(\frac{r}{|\epsilon|^\chi})^\kappa)dr \\
&\leq \frac{2\kappa\varpi}{(2\pi)^{1/2}}\int_0^{+\infty}e^{-(\beta-\beta')m}dm\int_{\rho/2}^{+\infty}\frac{|\epsilon|^{\chi(\kappa-1)}}{(\frac{\delta_1}{|x|^\kappa}-\nu)\kappa(\frac{\rho}{2})^{\kappa-1}}\times\frac{(\frac{\delta_1}{|x|^\kappa}-\nu)\kappa r^{\kappa-1}}{|\epsilon|^{\chi\kappa}}\exp(-(\frac{\delta_1}{|x|^\kappa}-\nu)(\frac{r}{|\epsilon|^\chi})^\kappa)dr \\
&\leq \frac{2\kappa\varpi}{(2\pi)^{1/2}}\frac{|\epsilon|^{\chi(\kappa-1)}}{(\beta-\beta')(\frac{\delta_1}{|x|^\kappa}-\nu)\kappa(\frac{\rho}{2})^{\kappa-1}}\exp(-(\frac{\delta_1}{|x|^\kappa}-\nu)\frac{(\rho/2)^\kappa}{|\epsilon|^{\chi\kappa}}) \\
&\leq \frac{2\kappa\varpi}{(2\pi)^{1/2}}\frac{|\epsilon|^{\chi(\kappa-1)}}{(\beta-\beta')\delta_2\kappa(\frac{\rho}{2})^{\kappa-1}}\exp(-\delta_2\frac{(\rho/2)^\kappa}{|\epsilon|^{\chi\kappa}})
\end{aligned}
$$

for all $x \in X$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|x| < (\frac{\delta_1}{\delta_2+\nu})^{1/\kappa}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

In the same way, we also give estimates for the integral

$$
I_2 = \left|\frac{\kappa}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\rho/2,\gamma_p}} \omega_\kappa^{\mathfrak{d}_p}(u,m,\epsilon)e^{-(\frac{u}{\epsilon^\chi x})^\kappa}e^{izm}\frac{du}{u}dm\right|.
$$

Namely, the direction γ_p (which depends on $\epsilon^\chi x$) is chosen in such a way that $\cos(\kappa(\gamma_p - \arg(\epsilon^\chi x))) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $x \in X$, for some fixed $\delta_1 > 0$. Again from the estimates

(87) and following the same steps as in (94), we deduce that

$$I_2 \leq \frac{2\kappa\varpi}{(2\pi)^{1/2}} \frac{|\epsilon|^{\chi(\kappa-1)}}{(\beta-\beta')\delta_2\kappa(\frac{\rho}{2})^{\kappa-1}} \exp(-\delta_2 \frac{(\rho/2)^{\kappa}}{|\epsilon|^{\chi\kappa}}) \tag{95}$$

for all $x \in X$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|x| < (\frac{\delta_1}{\delta_2+\nu})^{1/\kappa}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Finally, we give upper bound estimates for the integral

$$I_3 = \left| \frac{\kappa}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho/2,\gamma_p,\gamma_{p+1}}} \omega_\kappa(u,m,\epsilon) e^{-(\frac{u}{\epsilon^{\chi}x})^{\kappa}} e^{izm} \frac{du}{u} dm \right|.$$

By construction, the arc of circle $C_{\rho/2,\gamma_p,\gamma_{p+1}}$ is chosen in such a way that $\cos(\kappa(\theta - \arg(\epsilon^{\chi}x))) \geq \delta_1$, for all $\theta \in [\gamma_p, \gamma_{p+1}]$ (if $\gamma_p < \gamma_{p+1}$), $\theta \in [\gamma_{p+1}, \gamma_p]$ (if $\gamma_{p+1} < \gamma_p$), for all $x \in X$, all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, for some fixed $\delta_1 > 0$. Bearing in mind (87) and the classical estimates

$$\sup_{s \geq 0} s^{m_1} \exp(-m_2 s) = (\frac{m_1}{m_2})^{m_1} e^{-m_1}$$

for any $m_1 \geq 0$, $m_2 > 0$, we get that

$$\begin{aligned} I_3 \leq{} & \frac{\kappa}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \left| \int_{\gamma_p}^{\gamma_{p+1}} \varpi(1+|m|)^{-\mu} e^{-\beta|m|} \frac{\frac{\rho/2}{|\epsilon|^{\chi}}}{1+(\frac{\rho/2}{|\epsilon|^{\chi}})^{2\kappa}} \right. \\ & \left. \times \exp(\nu(\frac{\rho/2}{|\epsilon|^{\chi}})^{\kappa}) \exp(-\frac{\cos(\kappa(\theta - \arg(\epsilon^{\chi}x)))}{|\epsilon^{\chi}x|^{\kappa}}(\frac{\rho}{2})^{\kappa})\, e^{-m\mathrm{Im}(z)} d\theta \right| dm \\ \leq{} & \frac{\kappa\varpi}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \times |\gamma_p - \gamma_{p+1}| \frac{\rho/2}{|\epsilon|^{\chi}} \exp(-\frac{(\frac{\delta_1}{|x|^{\kappa}} - \nu)}{2}(\frac{\rho/2}{|\epsilon|^{\chi}})^{\kappa}) \\ & \times \exp(-\frac{(\frac{\delta_1}{|x|^{\kappa}} - \nu)}{2}(\frac{\rho/2}{|\epsilon|^{\chi}})^{\kappa}) \\ \leq{} & \frac{2\kappa\varpi|\gamma_p - \gamma_{p+1}|}{(2\pi)^{1/2}(\beta-\beta')} \sup_{s \geq 0} s^{1/\kappa} e^{-\frac{1}{2}(\frac{\delta_1}{|x|^{\kappa}} - \nu)s} \times \exp(-\frac{(\frac{\delta_1}{|x|^{\kappa}} - \nu)}{2}(\frac{\rho/2}{|\epsilon|^{\chi}})^{\kappa}) \\ \leq{} & \frac{2\kappa\varpi|\gamma_p - \gamma_{p+1}|}{(2\pi)^{1/2}(\beta-\beta')} (\frac{2/\kappa}{\delta_2})^{1/\kappa} e^{-1/\kappa} \exp(-\frac{\delta_2}{2}(\frac{\rho/2}{|\epsilon|^{\chi}})^{\kappa}) \end{aligned} \tag{96}$$

for all $x \in X$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|x| < (\frac{\delta_1}{\delta_2+\nu})^{1/\kappa}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Finally, gathering the three above inequalities (94), (95) and (96), we deduce from the decomposition (93) that

$$\begin{aligned} |\epsilon^{m_0} u^{\mathfrak{d}_{p+1}}(x\epsilon^{\chi-\alpha}, z, \epsilon) - \epsilon^{m_0} u^{\mathfrak{d}_p}(x\epsilon^{\chi-\alpha}, z, \epsilon)| \leq{} & \frac{4\kappa\varpi}{(2\pi)^{1/2}} \frac{|\epsilon|^{\chi(\kappa-1)}}{(\beta-\beta')\delta_2\kappa(\frac{\rho}{2})^{\kappa-1}} \exp(-\delta_2 \frac{(\rho/2)^{\kappa}}{|\epsilon|^{\chi\kappa}}) \\ & + \frac{2\kappa\varpi|\gamma_p - \gamma_{p+1}|}{(2\pi)^{1/2}(\beta-\beta')} (\frac{2/\kappa}{\delta_2})^{1/\kappa} e^{-1/\kappa} \exp(-\frac{\delta_2}{2}(\frac{\rho/2}{|\epsilon|^{\chi}})^{\kappa}) \end{aligned}$$

for all $x \in X$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|x| < (\frac{\delta_1}{\delta_2+\nu})^{1/k}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Therefore, the inequality (86) holds. □

4 Construction of outer solutions to the main problem

In this section, we construct solutions of the main equation (8) for t in a large sectorial domain outside the origin and we provide constraints under which their domain of holomorphy in time can be extended to some $\epsilon-$depending domains in the vicinity of the origin.

4.1 Classical Laplace transforms

In this little subsection, we report some identities for the usual Laplace transform of holomorphic functions on unbounded sectors involving convolution products and derivations. The next lemma has already appeared in our previous work [13] and is classical in reference textbooks such as [1].

Lemma 4 *Let $m \geq 0$ be an integer. Let $w_1(\tau)$, $w_2(\tau)$ be holomorphic functions on an unbounded open sector U_d centered at 0 with bisecting direction $d \in \mathbb{R}$ such that there exist $C, K > 0$ with*

$$|w_j(\tau)| \leq C \exp(K|\tau|) \quad , \quad j = 1, 2$$

for all $\tau \in U_d$. We denote

$$w_1 * w_2(\tau) = \int_0^{\tau} w_1(\tau - s) w_2(s) ds$$

their convolution product on U_d. We pick up an unbounded sector $\mathcal{D}$ centered at 0 for which there exists $\delta_1 > 0$ with

$$d + \arg(t) \in (-\pi/2, \pi/2) \quad , \quad \cos(d + \arg(t)) \geq \delta_1,$$

for all $t \in \mathcal{D}$. Then the following identities hold for the Laplace transforms

$$\int_{L_d} \tau^m \exp(-t\tau) d\tau = \frac{m!}{t^{m+1}} \quad , \quad \partial_t (\int_{L_d} w_1(\tau) \exp(-t\tau) d\tau) = \int_{L_d} (-\tau) w_1(\tau) \exp(-t\tau) d\tau,$$

$$\int_{L_d} w_1 * w_2(\tau) \exp(-t\tau) d\tau = (\int_{L_d} w_1(\tau) \exp(-t\tau) d\tau)(\int_{L_d} w_2(\tau) \exp(-t\tau) d\tau)$$

where $L_d = \mathbb{R}_+ e^{id} \subset U_d \cup \{0\}$, for all $t \in \mathcal{D} \cap \{|t| > K/\delta_1\}$.

4.2 Sets of Banach spaces with exponential growth and decay of order 1

In this subsection, we study a slightly modified version of the Banach spaces mentioned in subsection 3.2 of this work in the particular situation of functions with exponential growth of order 1 on unbounded sectors in $\mathbb{C}$ and exponential decay on $\mathbb{R}$. Although the proofs of the next lemma are proximate to the ones of the statements disclosed in Subsection 3.2, we decide to present them for the sake of clarity and convenience for the reader.

Definition 6 *Let U_d be an open unbounded sector with bisecting direction $d \in \mathbb{R}$ and $\mathcal{E}$ be an open sector with finite radius $r_{\mathcal{E}}$, both centered at 0 in $\mathbb{C}$. Let $\nu, \rho > 0$ and $\beta > 0, \Gamma \geq 0, \mu > 1$ be real numbers and let $\epsilon \in \mathcal{E}$. We define $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ as the space of continuous functions $(\tau, m) \mapsto f(\tau, m)$ on $(\bar{D}(0,\rho) \cup U_d) \times \mathbb{R}$ with values in $\mathbb{C}$, holomorphic w.r.t τ on $D(0,\rho) \cup U_d$, with*

$$||f(\tau, m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} = \sup_{\tau \in \bar{D}(0,\rho) \cup U_d, m \in \mathbb{R}} (1 + |m|)^{\mu} e^{\beta|m|} (1 + |\frac{\tau}{\epsilon^{\Gamma}}|^2) \exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|) |f(\tau, m)|$$

is finite. It turns out that the normed space $(E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}, ||.||_{(\nu,\beta,\mu,\Gamma,\epsilon)})$ is a Banach space.

Remark: Compared to the space $F^d_{(\nu,\beta,\mu,\chi,1,\epsilon)}$ mentioned above, the functions from $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ do not need to vanish at $\tau = 0$.

Lemma 5 *Let $\gamma_2 \geq 0$ be an integer. Take $B(m) \in E_{(\beta,\mu)}$ for some real numbers $\beta > 0$ and $\mu > 1$. Then, $\tau^{\gamma_2}B(m)$ belongs to $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ for any real numbers $\nu > 0$,$\Gamma \geq 0$ and $\epsilon \in \mathcal{E}$. Moreover, there exists a constant $B_1 > 0$ (depending on γ_2, ν) such that*

$$||\tau^{\gamma_2}B(m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_1||B(m)||_{(\beta,\mu)}|\epsilon|^{\Gamma\gamma_2}. \tag{97}$$

Proof By definition, we can write

$$\begin{aligned} ||\tau^{\gamma_2}B(m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} &= \sup_{\tau \in \bar{D}(0,\rho)\cup U_d, m\in\mathbb{R}} (1+|m|)^{\mu}e^{\beta|m|}|B(m)|(1+|\frac{\tau}{\epsilon^{\Gamma}}|^2) \\ &\times \exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|)|\frac{\tau}{\epsilon^{\Gamma}}|^{\gamma_2}|\epsilon|^{\Gamma\gamma_2} \leq ||B(m)||_{(\beta,\mu)}(\sup_{x\geq 0}(1+x^2)e^{-\nu x}x^{\gamma_2})|\epsilon|^{\Gamma\gamma_2} \end{aligned} \tag{98}$$

from which the lemma follows owing to the fact that an exponential function grows faster than any polynomial. □

Lemma 6 *Let $\gamma_1, \gamma_2, \gamma_3 \geq 0$ be real numbers. We assume that*

$$\gamma_1 \leq \gamma_2 + \gamma_3 + 1 \quad , \quad \gamma_1 \geq \gamma_3. \tag{99}$$

Then, there exists a constant $B_2 > 0$ (depending on $\gamma_1, \gamma_2, \gamma_3, \nu$) such that

$$||\frac{1}{\tau^{\gamma_1}}\int_0^{\tau}(\tau-s)^{\gamma_2}s^{\gamma_3}f(s,m)ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_2||f(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}|\epsilon|^{\Gamma(\gamma_2+\gamma_3+1)-\Gamma\gamma_1} \tag{100}$$

for all $f(\tau,m) \in E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$.

Proof By factoring out the pieces composing the norm of $f(\tau,m)$, we can rewrite the left handside of (100) as

$$\begin{aligned} A &= ||\frac{1}{\tau^{\gamma_1}}\int_0^{\tau}(\tau-s)^{\gamma_2}s^{\gamma_3}f(s,m)ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ &= \sup_{\tau \in \bar{D}(0,\rho)\cup U_d, m\in\mathbb{R}} (1+|m|)^{\mu}e^{\beta|m|}(1+|\frac{\tau}{\epsilon^{\Gamma}}|^2)\exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|) \\ &\times \left|\frac{1}{\tau^{\gamma_1}}\int_0^{\tau}\left\{(1+|m|)^{\mu}e^{\beta|m|}(1+|\frac{s}{\epsilon^{\Gamma}}|^2)\exp(-\nu|\frac{s}{\epsilon^{\Gamma}}|)f(s,m)\right\}\mathcal{A}(\tau,s,m,\epsilon)ds\right| \end{aligned} \tag{101}$$

where

$$\mathcal{A}(\tau,s,m,\epsilon) = \frac{1}{(1+|m|)^{\mu}e^{\beta|m|}}\frac{\exp(\nu|\frac{s}{\epsilon^{\Gamma}}|)}{1+|\frac{s}{\epsilon^{\Gamma}}|^2}(\tau-s)^{\gamma_2}s^{\gamma_3}.$$

As a result, we obtain

$$A \leq B_{2.1}(\epsilon)||f(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \tag{102}$$

where

$$B_{2.1}(\epsilon) = \sup_{\tau\in\bar{D}(0,\rho)\cup U_d}(1+|\frac{\tau}{\epsilon^{\Gamma}}|^2)\exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|)\frac{1}{|\tau|^{\gamma_1}}\int_0^{|\tau|}\frac{\exp(\nu\frac{h}{|\epsilon|^{\Gamma}})}{1+(\frac{h}{|\epsilon|^{\Gamma}})^2}(|\tau|-h)^{\gamma_2}h^{\gamma_3}dh.$$

We perform the change of variable $h = |\epsilon|^{\Gamma}h'$ inside the integral part of $B_{2.1}(\epsilon)$ and get the next bounds

$$(103)\quad B_{2.1}(\epsilon) = \sup_{\tau \in \bar{D}(0,\rho)\cup U_d} (1 + |\frac{\tau}{\epsilon^{\Gamma}}|^2) \exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|) \frac{1}{(\frac{|\tau|}{|\epsilon|^{\Gamma}}|\epsilon|^{\Gamma})^{\gamma_1}} |\epsilon|^{\Gamma(\gamma_2+\gamma_3+1)}$$
$$\times \int_0^{\frac{|\tau|}{|\epsilon|^{\Gamma}}} \frac{e^{\nu h'}}{1+(h')^2} (\frac{|\tau|}{|\epsilon|^{\Gamma}} - h')^{\gamma_2}(h')^{\gamma_3} dh' \leq |\epsilon|^{\Gamma(\gamma_2+\gamma_3+1)-\Gamma\gamma_1} \sup_{x \geq 0} (1+x^2)e^{-\nu x}\frac{1}{x^{\gamma_1}}G(x)$$

where

$$G(x) = \int_0^x \frac{e^{\nu h'}}{1+(h')^2}(x-h')^{\gamma_2}(h')^{\gamma_3}dh'.$$

In the last part of the proof, we need to study the function $G(x)$ near 0 and $+\infty$. In order to investigate its behaviour in the vicinity of the origin, we make the change of variable $h' = xu$ inside $G(x)$, getting

$$(104)\qquad G(x) = x^{\gamma_2+\gamma_3+1} \int_0^1 \frac{e^{\nu x u}}{1+(xu)^2}(1-u)^{\gamma_2}u^{\gamma_3}du.$$

From the first constraint in (99), we deduce that $G(x)/x^{\gamma_1}$ is bounded near 0. For large values of x, we proceed as in Proposition 1 of [11] and split $G(x)$ into two pieces

$$G(x) = G_1(x) + G_2(x)$$

where

$$G_1(x) = \int_0^{x/2} \frac{e^{\nu h'}}{1+(h')^2}(x-h')^{\gamma_2}(h')^{\gamma_3}dh' \quad , \quad G_2(x) = \int_{x/2}^{x} \frac{e^{\nu h'}}{1+(h')^2}(x-h')^{\gamma_2}(h')^{\gamma_3}dh'$$

Since $\gamma_2 \geq 0$, we notice that $(x-h')^{\gamma_2} \leq x^{\gamma_2}$ for $0 \leq h' \leq x/2$. Therefore,

$$G_1(x) \leq x^{\gamma_2}e^{\nu x/2} \int_0^{x/2} (h')^{\gamma_3}dh' = x^{\gamma_2}e^{\nu x/2}\frac{(x/2)^{\gamma_3+1}}{\gamma_3+1}.$$

Accordingly, we get that

$$(105)\qquad \sup_{x \geq 1} (1+x^2)e^{-\nu x}\frac{1}{x^{\gamma_1}}G_1(x)$$

is finite. On the other hand, we check that $1+(h')^2 \geq 1+(x/2)^2$ for $x/2 \leq h' \leq x$. Hence,

$$G_2(x) \leq \frac{1}{1+(x/2)^2}G_{2.1}(x)$$

where

$$G_{2.1}(x) = \int_{x/2}^{x} e^{\nu h'}(h')^{\gamma_3}(x-h')^{\gamma_2}dh'$$

Bestowing the estimates (18) in [11], we get a constant $K_{2.1} > 0$ (depending on ν, γ_2, γ_3) such that

$$G_{2.1}(x) \leq K_{2.1}x^{\gamma_3}e^{\nu x}$$

for all $x \geq 1$. It follows that

$$(106)\qquad \sup_{x \geq 1} (1+x^2)e^{-\nu x}\frac{1}{x^{\gamma_1}}G_2(x)$$

is finite provided that the second constraint from (99) holds. Finally, collecting (101), (102), (103), (104), (105) and (106) yields the estimates (100). □

Lemma 7 *Let $Q_1(X), Q_2(X)$ and $R_D(X)$ belonging to $\mathbb{C}[X]$ with $R_D(im) \neq 0$ for all $m \in \mathbb{R}$ and*

$$\deg(R_D) \geq \deg(Q_1) \quad , \quad \deg(R_D) \geq \deg(Q_2). \tag{107}$$

Besides, we choose the real parameter $\mu > 1$ with $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. Then, there exists a constant $B_3 > 0$ (depending on μ, Q_1, Q_2, R_D) such that

$$||\frac{1}{R_D(im)} \int_0^\tau \int_{-\infty}^{+\infty} Q_1(i(m-m_1))f(\tau-s, m-m_1)Q_2(im_1)g(s,m_1)dsdm_1||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ \leq B_3|\epsilon|^\Gamma ||f(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}||g(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \tag{108}$$

for all $f(\tau,m), g(\tau,m) \in E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$.

Proof As above, by setting apart the terms stemming from the norms of f and g, we can reorganize the left handside of (108) as follows

$$K = ||\frac{1}{R_D(im)} \int_0^\tau \int_{-\infty}^{+\infty} Q_1(i(m-m_1))f(\tau-s,m-m_1) \\ \times Q_2(im_1)g(s,m_1)dsdm_1||_{(\nu,\beta,\mu,\Gamma,\epsilon)} = \sup_{\tau \in \bar{D}(0,\rho)\cup U_d, m \in \mathbb{R}} (1+|m|)^\mu e^{\beta|m|}(1+|\frac{\tau}{\epsilon^\Gamma}|^2)\exp(-\nu|\frac{\tau}{\epsilon^\Gamma}|) \\ \times |\frac{1}{R_D(im)} \int_0^\tau \int_{-\infty}^{+\infty} \{(1+|m-m_1|)^\mu e^{\beta|m-m_1|}(1+(\frac{|\tau-s|}{|\epsilon|^\Gamma})^2)\exp(-\nu|\frac{\tau-s}{\epsilon^\Gamma}|)f(\tau-s,m-m_1)\} \\ \times \{(1+|m_1|)^\mu e^{\beta|m_1|}(1+|\frac{s}{\epsilon^\Gamma}|^2)\exp(-\nu|\frac{s}{\epsilon^\Gamma}|)g(s,m_1)\}\mathcal{K}(\tau,s,m,m_1)dsdm_1| \tag{109}$$

where

$$\mathcal{K}(\tau,s,m,m_1) = \\ \frac{e^{-\beta|m-m_1|}e^{-\beta|m_1|}}{(1+|m-m_1|)^\mu(1+|m_1|)^\mu} Q_1(i(m-m_1))Q_2(im_1) \frac{\exp(\nu|\frac{\tau-s}{\epsilon^\Gamma}|)\exp(\nu|\frac{s}{\epsilon^\Gamma}|)}{(1+(\frac{|\tau-s|}{|\epsilon|^\Gamma})^2)(1+|\frac{s}{\epsilon^\Gamma}|^2)}$$

According to the triangular inequality, $|m| \leq |m-m_1| + |m_1|$ for all $m, m_1 \in \mathbb{R}$, we get that

$$K \leq B_{3.1}B_{3.2}(\epsilon)||f(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}||g(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \tag{110}$$

where

$$B_{3.1} = \sup_{m \in \mathbb{R}} \frac{(1+|m|)^\mu}{|R_D(im)|} \int_{-\infty}^{+\infty} \frac{|Q_1(i(m-m_1))||Q_2(im_1)|}{(1+|m-m_1|)^\mu(1+|m_1|)^\mu}dm_1$$

and

$$B_{3.2}(\epsilon) = \sup_{\tau \in \bar{D}(0,\rho)\cup U_d} (1+|\frac{\tau}{\epsilon^\Gamma}|^2) \int_0^{|\tau|} \frac{1}{1+\frac{(|\tau|-h')^2}{|\epsilon|^{2\Gamma}}} \frac{1}{1+\frac{(h')^2}{|\epsilon|^{2\Gamma}}}dh'.$$

Since $B_{3.1} = C_{3.1}$ in formula (29), we deduce from the bounds (31), that $B_{3.1}$ is finite. Besides, by operating the change of variable $h' = |\epsilon|^\Gamma h$ inside the integral piece of $B_{3.2}(\epsilon)$, we observe that

$$B_{3.2}(\epsilon) = \sup_{\tau \in \bar{D}(0,\rho)\cup U_d} (1+|\frac{\tau}{\epsilon^\Gamma}|^2)|\epsilon|^\Gamma \int_0^{\frac{|\tau|}{|\epsilon|^\Gamma}} \frac{1}{1+(\frac{|\tau|}{|\epsilon|^\Gamma}-h)^2}\frac{1}{1+h^2}dh \leq |\epsilon|^\Gamma \sup_{x\geq 0} \tilde{B}_{3.2}(x) \tag{111}$$

where

$$\tilde{B}_{3.2}(x) = (1+x^2)\int_0^x \frac{1}{1+(x-h)^2}\frac{1}{1+h^2}dh.$$

In accordance with Corollary 4.9 of [4], we get that $\sup_{x\geq 0}\tilde{B}_{3.2}(x)$ is finite. Gathering (109), (110) and (111) furnishes the result. □

4.3 Construction of formal expressions solutions of the main equation as classical Laplace and Fourier inverse transforms

Within this subsection, we search for solutions of the main equation (8) expressed as integral representations through classical Laplace and Fourier inverse transforms

$$v(t,z,\epsilon) = \frac{\epsilon^{\gamma_0}}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\mathfrak{u}}} W(u,m,\epsilon)\exp(-(\frac{t}{\epsilon^{\gamma}})u)e^{izm}dudm \tag{112}$$

for some real number $\gamma_0 \in \mathbb{R}$, where $\gamma > 1/2$ is the positive real number introduced in formula (13) and $L_{\mathfrak{u}} = \mathbb{R}_+ e^{\sqrt{-1}\mathfrak{u}}$ is a halfline with direction $\mathfrak{u} \in \mathbb{R}$. Our prominent goal is the presentation of a related problem satisfied by the expression $W(u,m,\epsilon)$ that is planned to be solved in the next subsection among the Banach spaces introduced in the previous subsection.

Overall this subsection, let us assume that the function $W(\tau,m,\epsilon)$ belongs to the Banach space $E^{d}_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ for some positive real numbers $\nu,\beta > 0$, $\mu > 1$ and $0 \leq \Gamma < \gamma$, with ϵ belonging to some punctured disc $D(0,\epsilon_0) \setminus \{0\}$. The unbounded sector U_d is properly chosen in a way that it avoids the roots of the polynomial $F_2(\tau)$ introduced in the expression (14). According to Lemma 4 and Proposition 2, we can check that the expression $v(t,z,\epsilon)$ given in (112) is well defined for all $t \in \mathbb{C}$, $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$ and $\mathfrak{u} \in \mathbb{R}$ such that

$$\mathfrak{u} + \arg(t/\epsilon^{\gamma}) \in (-\pi/2,\pi/2) \quad , \quad \cos(\mathfrak{u} + \arg(t/\epsilon^{\gamma})) \geq \delta_1$$

for some $\delta_1 > 0$, provided that $|t| > \frac{\nu}{\delta_1}|\epsilon|^{\gamma-\Gamma}$ and $z \in H_{\beta}$.

We make the following assumption

$$d_D \geq d_i \quad , \quad d_D \geq k_j \quad , \quad d_D \geq b_k \quad , \quad d_D \geq h_l \tag{113}$$

for $1 \leq i \leq D-1$, $1 \leq j \leq q$, $0 \leq k \leq Q$ and $0 \leq l \leq M$. Moreover, the real numbers γ,γ_0 are selected in such a way that

$$\Delta_D = \gamma\delta_D - \gamma_0. \tag{114}$$

We divide (8) by t^{d_D} and we focus our attention on the next problem

$$\begin{aligned}
&(\sum_{l=1}^{q} a_l\epsilon^{m_l}t^{k_l-d_D} + a_0\epsilon^{m_0}t^{-d_D})Q(\partial_z)v(t,z,\epsilon)\\
&\qquad + (\sum_{l=0}^{M} c_l\epsilon^{\mu_l}t^{h_l-d_D})Q_1(\partial_z)v(t,z,\epsilon)Q_2(\partial_z)v(t,z,\epsilon)\\
&= \sum_{j=0}^{Q} b_j(z)\epsilon^{n_j}t^{b_j-d_D} + t^{-d_D}F^{\theta_F}(t,z,\epsilon) + \epsilon^{\gamma\delta_D-\gamma_0}\partial_t^{\delta_D}R_D(\partial_z)v(t,z,\epsilon)\\
&\qquad + \sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l-d_D}\partial_t^{\delta_l}R_l(\partial_z)v(t,z,\epsilon)
\end{aligned} \tag{115}$$

By means of the identities displayed in Lemma 4 for the classical Laplace transform and in Proposition 2 for the Fourier inverse transform, we see that $v(t,z,\epsilon)$ given by (112) solves the equation (115) if the related function $W(\tau,m,\epsilon)$ solves the next nonlinear convolution equation

$$
\begin{aligned}
(116)\quad & \sum_{l=1}^{q} \frac{a_l}{(d_D-k_l-1)!} \frac{\epsilon^{m_l+\gamma_0}}{\epsilon^{\gamma(d_D-k_l)}} Q(im) \int_0^{\tau} (\tau-s)^{d_D-k_l-1} W(s,m,\epsilon) ds \\
&+ \frac{a_0}{(d_D-1)!} \frac{\epsilon^{m_0+\gamma_0}}{\epsilon^{\gamma d_D}} Q(im) \int_0^{\tau} (\tau-s)^{d_D-1} W(s,m,\epsilon) ds + \sum_{l=0}^{M} \frac{c_l}{(d_D-h_l-1)!} \frac{\epsilon^{\mu_l+2\gamma_0}}{\epsilon^{\gamma(d_D-h_l)}} \\
&\times \int_0^{\tau} (\tau-s)^{d_D-h_l-1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_0^{s} Q_1(i(m-m_1)) W(s-s',m-m_1,\epsilon) \\
&\times Q_2(im_1) W(s',m_1,\epsilon) ds' dm_1 ds = \sum_{j=0}^{Q} \frac{1}{(d_D-b_j-1)!} \frac{\epsilon^{n_j}}{\epsilon^{\gamma(d_D-b_j)}} B_j(m) \tau^{d_D-b_j-1} \\
&+ \Upsilon(\tau,m,\epsilon) + (-\tau)^{\delta_D} R_D(im) W(\tau,m,\epsilon) + \sum_{l=1}^{D-1} \frac{1}{(d_D-d_l-1)!} \frac{\epsilon^{\Delta_l+\gamma_0}}{\epsilon^{\gamma(d_D-d_l+\delta_l)}} R_l(im) \\
&\times \int_0^{\tau} (\tau-s)^{d_D-d_l-1} (-s)^{\delta_l} W(s,m,\epsilon) ds
\end{aligned}
$$

where

$$
\Upsilon(\tau,m,\epsilon) = \frac{1}{(d_D-1)!} \frac{\epsilon^{n_F}}{\epsilon^{\gamma d_D}} \left(\int_0^{\tau} (\tau-s)^{d_D-1} \omega_F(s,m) ds - \tau^{d_D-1} \int_{L_{\theta_F}} \omega_F(u,m) du \right)
$$

4.4 Construction of actual solutions of some auxiliary nonlinear convolution equation with complex parameter

The major purpose of this subsection is the construction of a unique solution of the problem (116) located in the Banach spaces introduced in Subsection 4.2.
We first select an unbounded open sector U_d with bisecting direction $d \in \mathbb{R}$ taken in a way that it does not contain any root of the polynomial $F_2(\tau)$ appearing in the expression (14).

In an initial step, we prove that $\Upsilon(\tau,m,\epsilon)/\tau^{\delta_D} R_D(im)$ belongs to $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$, for $\beta > 0$ and $\mu > 1$ set above in (12), for some $\nu > 0$ (depending on K_F, Γ and ϵ_0), with $0 \leq \Gamma < \gamma$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$ granting that

$$
(117)\qquad d_D \geq 1 + \delta_D \quad , \quad \delta_D \geq 0 \quad , \quad n_F + \Gamma(d_D - 1 - \delta_D) - \gamma d_D \geq 0
$$

hold. As a primary task, we check that the function $\omega_F(\tau,m)$ belongs to $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$. Indeed, since U_d is taken as above and from the fact that $\deg(F_1) \leq \deg(F_2)$, we get a constant $C_{F_1,F_2} > 0$ with

$$
|\frac{F_1(\tau)}{F_2(\tau)}| \leq C_{F_1,F_2}
$$

for all $U_d \cup D(0,\rho)$, for some $\rho > 0$ selected small enough. We deduce the next estimates

$$
\begin{aligned}
||\omega_F(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} &= \sup_{\tau \in U_d \cup \bar{D}(0,\rho), m \in \mathbb{R}} (1+|m|)^{\mu} e^{\beta|m|} (1 + |\frac{\tau}{\epsilon^{\Gamma}}|^2) \exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|) \\
&\qquad \times |C_F(m) e^{-K_F \tau} \frac{F_1(\tau)}{F_2(\tau)}| \\
&\leq ||C_F(m)||_{(\beta,\mu)} C_{F_1,F_2} \sup_{\tau \in U_d \cup \bar{D}(0,\rho)} (1 + |\frac{\tau}{\epsilon^{\Gamma}}|^2) \exp(-\nu|\frac{\tau}{\epsilon^{\Gamma}}|) \exp(K_F|\epsilon|^{\Gamma}|\frac{\tau}{\epsilon^{\Gamma}}|) \\
&\leq ||C_F(m)||_{(\beta,\mu)} C_{F_1,F_2} \sup_{x \geq 0} (1+x^2) \exp((-\nu + K_F|\epsilon|^{\Gamma})x)
\end{aligned}
$$

which is finite accepting that $|\epsilon|^{\Gamma} < \nu/K_F$. Next in order, from Lemma 6, we get a constant $B_2 > 0$ (depending on δ_D, d_D, ν) with

$$
\begin{aligned}
(118) \quad & ||\frac{\epsilon^{n_F}}{\epsilon^{\gamma d_D}} \frac{1}{\tau^{\delta_D} R_D(im)} \int_0^{\tau} (\tau - s)^{d_D - 1} \omega_F(s,m) ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\
& \qquad \leq B_2 \frac{1}{\inf_{m \in \mathbb{R}} |R_D(im)|} \frac{|\epsilon|^{n_F}}{|\epsilon|^{\gamma d_D}} |\epsilon^{\Gamma d_D - \Gamma \delta_D}| ||\omega_F(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}
\end{aligned}
$$

taking into account that $d_D \geq \delta_D$ and $\delta_D \geq 0$ which follows from (117). In order to keep the norm in (118) bounded w.r.t ϵ near 0, we make the assumption that $n_F + \Gamma(d_D - \delta_D) - \gamma d_D \geq 0$ which again results from (117).

Now, we focus on the second piece of $\Upsilon(\tau,m,\epsilon)$. Namely, using Lemma 5, we obtain a constant $B_1 > 0$ (depending on d_D, δ_D, ν) such that

$$
\begin{aligned}
(119) \quad & ||\frac{\epsilon^{n_F}}{\epsilon^{\gamma d_D}} \frac{\tau^{d_D - 1 - \delta_D}}{R_D(im)} \int_{L_{\theta_F}} \omega_F(u,m) du||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_1 |\int_{L_{\theta_F}} e^{-K_F u} \frac{F_1(u)}{F_2(u)} du| \\
& \qquad \times \frac{||C_F(m)||_{(\beta,\mu)}}{\inf_{m \in \mathbb{R}} |R_D(im)|} \frac{|\epsilon|^{n_F}}{|\epsilon|^{\gamma d_D}} |\epsilon|^{\Gamma(d_D - 1 - \delta_D)}
\end{aligned}
$$

when $d_D - 1 - \delta_D \geq 0$ which is part of (117). Besides, we ask the norm in (119) to be bounded w.r.t ϵ in the vicinity of the origin which turns out to be an effect of (117).

In the forthcoming proposition, we display suitable conditions under which the main convolution equation (116) possesses a unique solution rooted in the Banach space $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ described in Subsection 4.2, for a convenient choice of its parameters ν, β, μ, Γ given just above.

Proposition 7 *We accredit that the next further constraints hold*

$$
(120) \quad d_D - k_l - 1 \geq 0 \quad , \quad \delta_D \geq 0 \quad , \quad \delta_D \leq d_D - k_l \quad , \quad m_l + \gamma_0 + (\Gamma - \gamma)(d_D - k_l) - \Gamma \delta_D \geq 0
$$

for all $1 \leq l \leq q$,

$$
(121) \qquad d_D \geq 1 \quad , \quad \delta_D \geq 0 \quad , \quad \delta_D \leq d_D \quad , \quad m_0 + \gamma_0 + (\Gamma - \gamma) d_D - \Gamma \delta_D \geq 0,
$$

$$
(122) \qquad \delta_D \leq d_D - h_l \quad , \quad \delta_D \geq 0 \quad , \quad \mu_l + 2\gamma_0 + (\Gamma - \gamma)(d_D - h_l) - \Gamma(\delta_D - 1) \geq 0,
$$

for all $0 \leq l \leq M$,

$$
(123) \qquad d_D - b_j - 1 \geq \delta_D \quad , \quad n_j - \gamma(d_D - b_j) + \Gamma(d_D - bj - 1 - \delta_D) \geq 0
$$

for all $0 \leq j \leq Q$,

$$\delta_D \leq d_D - d_l + \delta_l \quad , \quad \delta_D \geq \delta_l \quad , \quad \Delta_l + \gamma_0 + (\Gamma - \gamma)(d_D - d_l + \delta_l) - \Gamma\delta_D \geq 0 \tag{124}$$

as long as $1 \leq l \leq D-1$.

Then, there exist two constants $\varpi_1 > 0$ *and* $\zeta_1 > 0$ *small enough, such that if*

$$|a_i| \leq \zeta_1 \quad , \quad |c_j| \leq \zeta_1 \quad , \quad ||B_k(m)||_{(\beta,\mu)} \leq \zeta_1 \quad , \quad ||C_F(m)||_{(\beta,\mu)} \leq \zeta_1 \quad , \quad \sup_{m \in \mathbb{R}} \frac{|R_l(im)|}{|R_D(im)|} \leq \zeta_1, \tag{125}$$

for $0 \leq i \leq q$, $0 \leq j \leq M$, $0 \leq k \leq Q$ *and* $1 \leq l \leq D-1$, *then, the equation (116) has a unique solution* $W^d(\tau,m,\epsilon)$ *stemming from the Banach space* $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ *which is governed by the bounds*

$$||W^d(\tau,m,\epsilon)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq \varpi_1 \tag{126}$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, *for any directions* $d \in \mathbb{R}$ *taken in such a manner that the sector* U_d *fulfills the constraint proposed at the beginning of this subsection.*

Proof We depart from a lemma that aims attention at a shrinking map acting on the Banach spaces quoted above and downsizes our main convolution problem to the existence and unicity of a fixed point for this map.

Lemma 8 *Taking for granted the constraints (120), (121), (122), (123), (124) presented above, one can adjust a constant* $\varpi_1 > 0$ *small enough and a constant* $\zeta_1 > 0$ *taken in a way that if the smallness condition (125) hold, then for all* $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, *the map* $\mathcal{G}_\epsilon$ *prescribed as*

$$\begin{aligned}
\mathcal{G}_\epsilon(w(\tau,m)) := &\sum_{l=1}^{q} \frac{a_l}{(d_D - k_l - 1)!} \frac{\epsilon^{m_l+\gamma_0}}{\epsilon^{\gamma(d_D-k_l)}} \frac{Q(im)}{R_D(im)(-\tau)^{\delta_D}} \int_0^\tau (\tau-s)^{d_D-k_l-1} w(s,m)ds \\
&+ \frac{a_0}{(d_D-1)!} \frac{\epsilon^{m_0+\gamma_0}}{\epsilon^{\gamma d_D}} \frac{Q(im)}{R_D(im)(-\tau)^{\delta_D}} \int_0^\tau (\tau-s)^{d_D-1} w(s,m)ds \\
&+ \sum_{l=0}^{M} \frac{c_l}{(d_D-h_l-1)!} \frac{\epsilon^{\mu_l+2\gamma_0}}{\epsilon^{\gamma(d_D-h_l)}} \\
&\times \frac{1}{R_D(im)(-\tau)^{\delta_D}} \int_0^\tau (\tau-s)^{d_D-h_l-1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_0^s Q_1(i(m-m_1))w(s-s',m-m_1) \\
&\times Q_2(im_1)w(s',m_1)ds'dm_1ds \\
&- \sum_{j=0}^{Q} \frac{1}{(d_D-b_j-1)!} \frac{\epsilon^{n_j}}{\epsilon^{\gamma(d_D-b_j)}} \frac{B_j(m)}{R_D(im)} \frac{\tau^{d_D-b_j-1}}{(-\tau)^{\delta_D}} - \frac{\Upsilon(\tau,m,\epsilon)}{R_D(im)(-\tau)^{\delta_D}} \\
&- \sum_{l=1}^{D-1} \frac{1}{(d_D-d_l-1)!} \frac{\epsilon^{\Delta_l+\gamma_0}}{\epsilon^{\gamma(d_D-d_l+\delta_l)}} \frac{R_l(im)}{R_D(im)} \frac{1}{(-\tau)^{\delta_D}} \\
&\times \int_0^\tau (\tau-s)^{d_D-d_l-1}(-s)^{\delta_l} w(s,m)ds
\end{aligned} \tag{127}$$

undergo the next properties.
i) *The next inclusion*

$$\mathcal{G}_\epsilon(\bar{B}(0,\varpi_1)) \subset \bar{B}(0,\varpi_1) \tag{128}$$

takes place, where $\bar{B}(0,\varpi_1)$ *stands for the closed ball of radius* ϖ_1 *centered at 0 in the space* $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$, *for any* $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.
ii) *The ensuing shrinking constraint*

$$||\mathcal{G}_\epsilon(w_1) - \mathcal{G}_\epsilon(w_2)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq \frac{1}{2}||w_1 - w_2||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \tag{129}$$

holds for all $w_1, w_2 \in \bar{B}(0,\varpi_1)$, *all* $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

Proof Foremost, we focus on the first property (128). Namely, let $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$ and consider $w(\tau,m) \in E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$. We take $\varpi_1 > 0$ with $||w(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq \varpi_1$.
Bearing in mind Lemma 6, we get a constant $B_2 > 0$ (depending on ν, δ_D, d_D, k_l, for $1 \leq l \leq q$) with

$$\begin{aligned} ||\frac{\epsilon^{m_l+\gamma_0}}{\epsilon^{\gamma(d_D-k_l)}} \frac{Q(im)}{R_D(im)\tau^{\delta_D}} \int_0^\tau (\tau-s)^{d_D-k_l-1} w(s,m) ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ \leq B_2 \sup_{m\in\mathbb{R}} \frac{|Q(im)|}{|R_D(im)|} \frac{|\epsilon|^{m_l+\gamma_0}}{|\epsilon|^{\gamma(d_D-k_l)}} |\epsilon|^{\Gamma(d_D-k_l)-\Gamma\delta_D} \varpi_1 \end{aligned} \tag{130}$$

for all $1 \leq l \leq q$, submitted to (120). Likewise, we get a constant $B_2 > 0$ (depending on ν, δ_D, d_D) with

$$\begin{aligned} ||\frac{\epsilon^{m_0+\gamma_0}}{\epsilon^{\gamma d_D}} \frac{Q(im)}{R_D(im)\tau^{\delta_D}} \int_0^\tau (\tau-s)^{d_D-1} w(s,m) ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ \leq B_2 \sup_{m\in\mathbb{R}} \frac{|Q(im)|}{|R_D(im)|} \frac{|\epsilon|^{m_0+\gamma_0}}{|\epsilon|^{\gamma d_D}} |\epsilon|^{\Gamma d_D-\Gamma\delta_D} \varpi_1 \end{aligned} \tag{131}$$

counting on (121). Now, we put

$$h(\tau,m) = \frac{1}{R_D(im)} \int_0^\tau \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) w(\tau-s',m-m_1) Q_2(im_1) w(s',m_1) ds' dm_1.$$

From Lemma 7, under the constraint (7), we get a constant $B_3 > 0$ (depending on μ, Q_1, Q_2, R_D) such that

$$||h(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_3 |\epsilon|^\Gamma ||w(\tau,m)||^2_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$

In accordance with Lemma 6, we deduce a constant $B_2 > 0$ (depending on ν, δ_D, d_D, h_l for $0 \leq l \leq M$) with

$$\begin{aligned} ||\frac{\epsilon^{\mu_l+2\gamma_0}}{\epsilon^{\gamma(d_D-h_l)}} \frac{1}{\tau^{\delta_D}} \int_0^\tau (\tau-s)^{d_D-h_l-1} h(s,m) ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ \leq B_2 \frac{|\epsilon|^{\mu_l+2\gamma_0}}{|\epsilon|^{\gamma(d_D-h_l)}} |\epsilon|^{\Gamma(d_D-h_l)-\Gamma\delta_D} ||h(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_2 B_3 \frac{|\epsilon|^{\mu_l+2\gamma_0}}{|\epsilon|^{\gamma(d_D-h_l)}} |\epsilon|^{\Gamma(d_D-h_l)-\Gamma\delta_D} |\epsilon|^\Gamma \varpi_1^2 \end{aligned} \tag{132}$$

in agreement with (122). Hereafter, we concentrate on the inhomogeneous terms. Namely, according to Lemma 5, we get a constant $B_1 > 0$ (depending on ν, d_D, δ_D, b_j for $0 \leq j \leq Q$) such that

$$\begin{aligned} ||\frac{\epsilon^{n_j}}{\epsilon^{\gamma(d_D-b_j)}} \frac{B_j(m)}{R_D(im)} \frac{\tau^{d_D-b_j-1}}{\tau^{\delta_D}}||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ \leq B_1 \sup_{m\in\mathbb{R}} |\frac{1}{R_D(im)}| ||B_j(m)||_{(\beta,\mu)} \frac{|\epsilon|^{n_j}}{|\epsilon|^{\gamma(d_D-b_j)}} |\epsilon|^{\Gamma(d_D-b_j-1-\delta_D)} \end{aligned} \tag{133}$$

taking notice that (123) happens. On the other hand, under the assumptions (117), we have seen above that there exists a constant $B_{\Upsilon} > 0$ (depending on $\omega_F, \theta_F, \nu, \Gamma, \gamma, n_F, d_D, \delta_D, R_D(im)$) for which

$$||\frac{\Upsilon(\tau,m,\epsilon)}{R_D(im)\tau^{\delta_D}}||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_{\Upsilon}||C_F(m)||_{(\beta,\mu)} \tag{134}$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. At last, we provide estimates for the remaining convolution terms. Specifically, Lemma 6 yields a constant $B_2 > 0$ (depending on $\nu, \delta_D, d_D, d_l, \delta_l$ for $1 \leq l \leq D-1$) such that

$$\begin{aligned} &||\frac{\epsilon^{\Delta_l+\gamma_0}}{\epsilon^{\gamma(d_D-d_l+\delta_l)}}\frac{R_l(im)}{R_D(im)}\frac{1}{\tau^{\delta_D}}\int_0^\tau (\tau-s)^{d_D-d_l-1}s^{\delta_l}w(s,m)ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ &\qquad \leq B_2 \sup_{m\in\mathbb{R}} \frac{|R_l(im)|}{|R_D(im)|}\frac{|\epsilon|^{\Delta_l+\gamma_0}}{|\epsilon|^{\gamma(d_D-d_l+\delta_l)}}|\epsilon|^{\Gamma(d_D-d_l+\delta_l)-\Gamma\delta_D}\varpi_1 \end{aligned} \tag{135}$$

for $1 \leq l \leq D-1$, under the requirement that (124) holds. Finally, we select both $\varpi_1 > 0$ and $\zeta_1 > 0$ satisfying (125) in such a way that

$$\begin{aligned} &\sum_{l=1}^{q} \frac{|a_l|}{(d_D-k_l-1)!}B_2 \sup_{m\in\mathbb{R}}\frac{|Q(im)|}{|R_D(im)|}\frac{\epsilon_0^{m_l+\gamma_0}}{\epsilon_0^{\gamma(d_D-k_l)}}\epsilon_0^{\Gamma(d_D-k_l)-\Gamma\delta_D}\varpi_1 \\ &\quad + \frac{|a_0|}{(d_D-1)!}B_2 \sup_{m\in\mathbb{R}}\frac{|Q(im)|}{|R_D(im)|}\frac{\epsilon_0^{m_0+\gamma_0}}{\epsilon_0^{\gamma d_D}}\epsilon_0^{\Gamma d_D-\Gamma\delta_D}\varpi_1 \\ &\quad + \sum_{l=0}^{M}\frac{|c_l|}{(d_D-h_l-1)!(2\pi)^{1/2}}B_2B_3\frac{\epsilon_0^{\mu_l+2\gamma_0}}{\epsilon_0^{\gamma(d_D-h_l)}}\epsilon_0^{\Gamma(d_D-h_l)-\Gamma\delta_D}\epsilon_0^{\Gamma}\varpi_1^2 \\ &\quad + \sum_{j=0}^{Q}\frac{1}{(d_D-b_j-1)!}B_1\sup_{m\in\mathbb{R}}|\frac{1}{R_D(im)}|||B_j(m)||_{(\beta,\mu)}\frac{\epsilon_0^{n_j}}{\epsilon_0^{\gamma(d_D-b_j)}}\epsilon_0^{\Gamma(d_D-b_j-1-\delta_D)} \\ &\quad + B_{\Upsilon}||C_F(m)||_{(\beta,\mu)} + \sum_{l=1}^{D-1}\frac{1}{(d_D-d_l-1)!}B_2\sup_{m\in\mathbb{R}}\frac{|R_l(im)|}{|R_D(im)|} \\ &\quad \times \frac{\epsilon_0^{\Delta_l+\gamma_0}}{\epsilon_0^{\gamma(d_D-d_l+\delta_l)}}\epsilon_0^{\Gamma(d_D-d_l+\delta_l)-\Gamma\delta_D}\varpi_1 \leq \varpi_1. \end{aligned} \tag{136}$$

From the very definition of $\mathcal{G}_\epsilon$, by compiling the bounds (130), (131), (132), (133), (134), (135), we recover the inclusion announced in (128).

In the next part of the proof, we target the shrinking restriction (129). Namely, let us choose $w_1(\tau,m)$ and $w_2(\tau,m)$ in the space $E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ inside the ball $\bar{B}(0,\varpi_1)$.

Ahead in position, according to the bounds (130), (131) and (135), we get a constant $B_2 > 0$ (depending on ν, δ_D, d_D, k_l for $1 \leq l \leq q$ and d_l, δ_l for $1 \leq l \leq D-1$) for which

$$\begin{aligned} &||\frac{\epsilon^{m_l+\gamma_0}}{\epsilon^{\gamma(d_D-k_l)}}\frac{Q(im)}{R_D(im)\tau^{\delta_D}}\int_0^\tau (\tau-s)^{d_D-k_l-1}(w_1(s,m)-w_2(s,m))ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \\ &\qquad \leq B_2\sup_{m\in\mathbb{R}}\frac{|Q(im)|}{|R_D(im)|}\frac{|\epsilon|^{m_l+\gamma_0}}{|\epsilon|^{\gamma(d_D-k_l)}}|\epsilon|^{\Gamma(d_D-k_l)-\Gamma\delta_D}||w_1(\tau,m)-w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \end{aligned} \tag{137}$$

holds for all $1 \leq l \leq q$, together with

$$(138)\quad ||\frac{\epsilon^{m_0+\gamma_0}}{\epsilon^{\gamma d_D}}\frac{Q(im)}{R_D(im)\tau^{\delta_D}}\int_0^\tau (\tau-s)^{d_D-1}(w_1(s,m)-w_2(s,m))ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$
$$\leq B_2 \sup_{m\in\mathbb{R}} \frac{|Q(im)|}{|R_D(im)|}\frac{|\epsilon|^{m_0+\gamma_0}}{|\epsilon|^{\gamma d_D}}|\epsilon|^{\Gamma d_D - \Gamma\delta_D}||w_1(\tau,m)-w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$

and

$$(139)\quad ||\frac{\epsilon^{\Delta_l+\gamma_0}}{\epsilon^{\gamma(d_D-d_l+\delta_l)}}\frac{R_l(im)}{R_D(im)}\frac{1}{\tau^{\delta_D}}\int_0^\tau (\tau-s)^{d_D-d_l-1}s^{\delta_l}(w_1(s,m)-w_2(s,m))ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$
$$\leq B_2 \sup_{m\in\mathbb{R}} \frac{|R_l(im)|}{|R_D(im)|}\frac{|\epsilon|^{\Delta_l+\gamma_0}}{|\epsilon|^{\gamma(d_D-d_l+\delta_l)}}|\epsilon|^{\Gamma(d_D-d_l+\delta_l)-\Gamma\delta_D}||w_1(\tau,m)-w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$

for $1 \leq l \leq D-1$. We concentrate now on the nonlinear part of $\mathcal{G}_\epsilon$. In a similar way as we have proceed for the map $\mathcal{H}_\epsilon$ in the proof of Lemma 3, we may write as a preparation the next identity

$$(140)\quad Q_1(i(m-m_1))w_1(s-s',m-m_1)Q_2(im_1)w_1(s',m_1)$$
$$- Q_1(i(m-m_1))w_2(s-s',m-m_1)Q_2(im_1)w_2(s',m_1)$$
$$= Q_1(i(m-m_1))\left(w_1(s-s',m-m_1)-w_2(s-s',m-m_1)\right)Q_2(im_1)w_1(s',m_1)$$
$$+ Q_1(i(m-m_1))w_2(s-s',m-m_1)Q_2(im_1)\left(w_1(s',m_1)-w_2(s',m_1)\right)$$

For $j=1,2$, we assign

$$h_j(\tau,m) = \frac{1}{R_D(im)}\int_0^\tau\int_{-\infty}^{+\infty} Q_1(i(m-m_1))w_j(\tau-s',m-m_1)Q_2(im_1)w_j(s',m_1)ds'dm_1.$$

Keeping in view the latter factorization (140), accordingly to Lemma 7, under the assumption (7), we get a constant $B_3 > 0$ (depending on μ, Q_1, Q_2, R_D) with

$$||h_1(\tau,m)-h_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq B_3|\epsilon|^\Gamma||w_1(\tau,m)-w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$
$$\times\left(||w_1(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} + ||w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}\right).$$

As a result, with the help of the first inequality of (132), we can select a constant $B_2 > 0$ (depending on ν, δ_D, d_D, h_l for $0 \leq l \leq M$) such that

$$(141)\quad ||\frac{\epsilon^{\mu_l+2\gamma_0}}{\epsilon^{\gamma(d_D-h_l)}}\frac{1}{\tau^{\delta_D}}\int_0^\tau (\tau-s)^{d_D-h_l-1}(h_1(s,m)-h_2(s,m))ds||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$
$$\leq B_2\frac{|\epsilon|^{\mu_l+2\gamma_0}}{|\epsilon|^{\gamma(d_D-h_l)}}|\epsilon|^{\Gamma(d_D-h_l)-\Gamma\delta_D}||h_1(\tau,m)-h_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$
$$\leq B_2B_3\frac{|\epsilon|^{\mu_l+2\gamma_0}}{|\epsilon|^{\gamma(d_D-h_l)}}|\epsilon|^{\Gamma(d_D-h_l)-\Gamma\delta_D}|\epsilon|^\Gamma||w_1(\tau,m)-w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$$
$$\times\left(||w_1(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} + ||w_2(\tau,m)||_{(\nu,\beta,\mu,\Gamma,\epsilon)}\right).$$

We adjust both $\varpi_1 > 0$ and $\zeta_1 > 0$ with (125) in a manner that

$$\sum_{l=1}^{q} \frac{|a_l|}{(d_D - k_l - 1)!} B_2 \sup_{m\in\mathbb{R}} \frac{|Q(im)|}{|R_D(im)|} \frac{\epsilon_0^{m_l+\gamma_0}}{\epsilon_0^{\gamma(d_D-k_l)}} \epsilon_0^{\Gamma(d_D-k_l)-\Gamma\delta_D} + \frac{|a_0|}{(d_D-1)!} B_2 \sup_{m\in\mathbb{R}} \frac{|Q(im)|}{|R_D(im)|} \frac{\epsilon_0^{m_0+\gamma_0}}{\epsilon_0^{\gamma d_D}} \epsilon_0^{\Gamma d_D - \Gamma\delta_D} + \sum_{l=0}^{M} \frac{|c_l|}{(d_D-h_l-1)!(2\pi)^{1/2}} B_2 B_3 \frac{\epsilon_0^{\mu_l+2\gamma_0}}{\epsilon_0^{\gamma(d_D-h_l)}} \epsilon_0^{\Gamma(d_D-h_l)-\Gamma\delta_D} \epsilon_0^{\Gamma} 2\varpi_1 + \sum_{l=1}^{D-1} \frac{1}{(d_D-d_l-1)!} B_2 \sup_{m\in\mathbb{R}} \frac{|R_l(im)|}{|R_D(im)|} \times \frac{\epsilon_0^{\Delta_l+\gamma_0}}{\epsilon_0^{\gamma(d_D-d_l+\delta_l)}} \epsilon_0^{\Gamma(d_D-d_l+\delta_l)-\Gamma\delta_D} \leq 1/2. \quad (142)$$

By grouping the above estimates (137), (138), (139) and (141), we are led to the shrinking constraints (129).

In order to complete the proof, let us allow the two conditions (136) and (142) to mutually occur for well selected $\varpi_1 > 0$ and $\zeta_1 > 0$. Then, both (128) and (129) are conjointly verified. □

Take the ball $\bar{B}(0,\varpi_1) \subset E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ just built above in Lemma 8 which furnishes a complete metric space endowed with the norm $||.||_{(\nu,\beta,\mu,\Gamma,\epsilon)}$. From the lemma above, we get that $\mathcal{G}_\epsilon$ is a contractive map from $\bar{B}(0,\varpi_1)$ into itself. Due to the classical contractive mapping theorem, we deduce that the map $\mathcal{G}_\epsilon$ has a unique fixed point denoted $W^d(\tau,m,\epsilon)$ in the ball $\bar{B}(0,\varpi_1)$, meaning that

$$\mathcal{G}_\epsilon(W^d(\tau,m,\epsilon)) = W^d(\tau,m,\epsilon) \quad (143)$$

for a unique solution $W^d(\tau,m,\epsilon) \in E^d_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ such that $||W^d(\tau,m,\epsilon)||_{(\nu,\beta,\mu,\Gamma,\epsilon)} \leq \varpi_1$, for all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. Moreover, the function $W^d(\tau,m,\epsilon)$ depends holomorphically on ϵ in $D(0,\epsilon_0)\setminus\{0\}$.

If one sets apart the term $(-\tau)^{\delta_D} R_D(im) W(\tau,m,\epsilon)$ in the right handside of (116), we notice that (116) can be scaled down to the equation (143) above using a mere division by $(-\tau)^{\delta_D} R_D(im)$. As a result, the unique fixed point $W^d(\tau,m,\epsilon)$ of $\mathcal{G}_\epsilon$ in $\bar{B}(0,\varpi_1)$ precisely solves the problem (116). The proposition 7 follows. □

4.5 Analytic solutions to the main problem on large $\epsilon-$depending sectorial domains in time

We go back to the speculative solutions to the main equation (8) displayed in Section 4.3 under the new light shed on the related nonlinear convolution equation (116) in Section 4.4. We first provide the definition of the set of $\epsilon-$depending associated sector and directions to a good covering.

Definition 7 *Let $\iota \geq 2$ be an integer. For all $0 \leq j \leq \iota - 1$, we consider an open sector $\mathcal{E}_j^\infty$ centered at 0, with radius $\epsilon_0^\infty > 0$ and opening $\frac{\pi}{\gamma} + \xi_j < 2\pi$ for some real number $\xi_j > 0$. We assume that the family $\{\mathcal{E}_j^\infty\}_{0\leq j\leq\iota-1}$ forms a good covering in $\mathbb{C}^*$ with aperture π/γ. For all $0 \leq j \leq \iota - 1$, let $\mathfrak{u}_j$ be a real number belonging to $(-\pi/2,\pi/2)$ sorted in a way that there exists* *an unbounded sector $U_{\mathfrak{u}_j}$ centered at 0 with bisecting direction $\mathfrak{u}_j$ and appropriate aperture not containing any root of the polynomial $F_2(\tau)$ introduced in the formula (14). Let $\nu > 0$ be fixed as above at the beginning of Section 4.4.*

We assume that one can select a real number $\delta_1^{\infty} > 0$ and $\Delta_{\nu,\delta_1^{\infty}} > \nu/\delta_1^{\infty}$ such that for all $0 \leq j \leq \iota - 1$, all $\epsilon \in \mathcal{E}_j^{\infty}$, all $t \in \mathcal{T}_{\epsilon}^{\infty}$, where

$$\mathcal{T}_{\epsilon}^{\infty} = \{t \in \mathbb{C}/|t| > \Delta_{\nu,\delta_1^{\infty}}|\epsilon|^{\gamma-\Gamma} \quad , \quad \alpha_{\infty} < \arg(t) < \beta_{\infty}\}$$

there exists some direction $\mathfrak{u}_j^{\Delta} \in \mathbb{R}$ (that may depend on ϵ and t) with $\exp(i\mathfrak{u}_j^{\Delta}) \in U_{\mathfrak{u}_j}$ that satisfies the next requirement

$$\mathfrak{u}_j^{\Delta} + \arg(\frac{t}{\epsilon^{\gamma}}) \in (-\frac{\pi}{2},\frac{\pi}{2}) \quad , \quad \cos(\mathfrak{u}_j^{\Delta} + \arg(\frac{t}{\epsilon^{\gamma}})) \geq \delta_1^{\infty},$$

for suitable fixed angles $\alpha_{\infty} < \beta_{\infty}$.

If the above constraints hold, we claim that the family of $\epsilon-$depending sector and directions $\{\mathcal{T}_{\epsilon}^{\infty}, \{\mathfrak{u}_j\}_{0 \leq j \leq \iota-1}\}$ is associated to the good covering $\{\mathcal{E}_j^{\infty}\}_{0 \leq j \leq \iota-1}$.

In the forthcoming second main outcome of this work, we construct a set of actual holomorphic solutions to the principal equation (8) which we name *outer solutions*. These solutions are well defined on the sectors $\mathcal{E}_j^{\infty}$ of a good covering w.r.t ϵ, on an associated sector $\mathcal{T}_{\epsilon}^{\infty}$ w.r.t t and on an horizontal strip H_{β} w.r.t z. Moreover, we can control the difference between any two consecutive solutions on the crossing sector $\mathcal{E}_j^{\infty} \cap \mathcal{E}_{j+1}^{\infty}$ and confirm that it is exponentially flat of order at most γ w.r.t ϵ.

Theorem 2 *We focus on the singularly perturbed equation (8) and we take for granted that all the aforementioned constraints (7), (9), (12), (13), (14), (113), (114), (117), (120), (121), (122), (123), (124) hold. Besides, we choose a good covering $\{\mathcal{E}_j^{\infty}\}_{0 \leq j \leq \iota-1}$ with aperture $\frac{\pi}{\gamma}$ for which an associated family of a sector $\mathcal{T}_{\epsilon}^{\infty}$ and directions $\{\mathfrak{u}_j\}_{0 \leq j \leq \iota-1}$ can be singled out.*

Then, there exists a constant $\zeta_1 > 0$ for which we assume the restriction (125) to take place. As a result, for each $0 \leq j \leq \iota - 1$, one can build up an actual solution $v^{\mathfrak{u}_j}(t,z,\epsilon)$ of (8), where the piece of forcing term $(t,z) \mapsto F^{\theta_F}(t,z,\epsilon)$ needs to be specified for $\theta_F = \mathfrak{u}_j$ and represents a bounded holomorphic function denoted $(t,z) \mapsto F^{\mathfrak{u}_j}(t,z,\epsilon)$ w.r.t t on $\mathcal{T}_{\epsilon}^{\infty}$, w.r.t z on a strip $H_{\beta'}$, for any given $0 < \beta' < \beta$, when ϵ belongs to $\mathcal{E}_j^{\infty}$.

Moreover, for each $\epsilon \in \mathcal{E}_j^{\infty}$, the function $(t,z) \mapsto v^{\mathfrak{u}_j}(t,z,\epsilon)$ is bounded and holomorphic on $\mathcal{T}_{\epsilon}^{\infty} \times H_{\beta'}$ for any given $0 < \beta' < \beta$, $0 \leq j \leq \iota - 1$. Besides, for each prescribed $t \in \mathcal{T}^{\infty}$, where

$$\mathcal{T}^{\infty} = \{t \in \mathbb{C}^{*}/\alpha_{\infty} < \arg(t) < \beta_{\infty}\}, \tag{144}$$

the function $(z,\epsilon) \mapsto \epsilon^{-\gamma_0}v^{\mathfrak{u}_j}(t,z,\epsilon)$ is bounded holomorphic on $(\mathcal{E}_j^{\infty} \cap D(0,\sigma_t)) \times H_{\beta'}$, for any given $0 < \beta' < \beta$, $0 \leq j \leq \iota - 1$ and suffers the next upper bounds: there exist $K_j, M_j > 0$ (independent of ϵ) such that

$$\sup_{z \in H_{\beta'}} |\epsilon^{-\gamma_0}v^{\mathfrak{u}_{j+1}}(t,z,\epsilon) - \epsilon^{-\gamma_0}v^{\mathfrak{u}_j}(t,z,\epsilon)| \leq K_j \exp(-\frac{M_j|t|}{|\epsilon|^{\gamma}}) \tag{145}$$

for all $\epsilon \in \mathcal{E}_{j+1}^{\infty} \cap \mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$, for $0 \leq j \leq \iota - 1$ (where by convention $v^{\mathfrak{u}_{\iota}} = v^{\mathfrak{u}_0}$), where

$$\sigma_t = (\frac{\delta_1^{\infty} - \delta_2^{\infty}}{\nu})^{\frac{1}{\gamma-\Gamma}}|t|^{\frac{1}{\gamma-\Gamma}} \tag{146}$$

for some positive real number $\delta_2^{\infty} > 0$ chosen in a way that $\delta_2^{\infty} < \delta_1^{\infty}$ holds.

Proof We select a good covering $\{\mathcal{E}_j^{\infty}\}_{0 \leq j \leq \iota-1}$ in $\mathbb{C}^*$ with aperture $\frac{\pi}{\gamma}$ and a family of sectors and directions $\{\mathcal{T}_{\epsilon}^{\infty}, \{\mathfrak{u}_j\}_{0 \leq j \leq \iota-1}\}$ associated to this covering according to Definition 7.

As a result of the estimates (134), we observe that the function $\Upsilon(\tau, m, \epsilon)$ must be governed by the next bounds

$$|\Upsilon(\tau, m, \epsilon)| \leq B_{\Upsilon}||C_F(m)||_{(\beta,\mu)}(1+|m|)^{-\mu}\exp(-\beta|m|)|R_D(im)|\frac{|\tau|^{\delta_D}}{1+|\frac{\tau}{\epsilon^{\Gamma}}|^2}\exp(\nu|\frac{\tau}{\epsilon^{\Gamma}}|) \tag{147}$$

for all $\tau \in U_{\mathfrak{u}_j} \cup D(0, \rho)$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. By construction, we observe that the piece of forcing term $t^{-d_D}F^{\mathfrak{u}_j}(t, z, \epsilon)$ for the specific value $\theta_F = \mathfrak{u}_j$ as described in Subsection 2.2 may be written as a usual Laplace/Fourier inverse transform along the halfline $L_{\mathfrak{u}_j^{\Delta}}$ of $\Upsilon(\tau, m, \epsilon)$ as follows

$$t^{-d_D}F^{\mathfrak{u}_j}(t, z, \epsilon) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\mathfrak{u}_j^{\Delta}}}\Upsilon(u, m, \epsilon)\exp(-(\frac{t}{\epsilon^{\gamma}})u)e^{izm}dudm.$$

Furthermore, Proposition 7 permits us, for each direction $\mathfrak{u}_j$, to build a solution named $W^{\mathfrak{u}_j}(\tau, m, \epsilon)$ of the convolution equation (116) which is stemming from the Banach space $E^{\mathfrak{u}_j}_{(\nu,\beta,\mu,\Gamma,\epsilon)}$ and is therefore submitted to the next bounds

$$|W^{\mathfrak{u}_j}(\tau, m, \epsilon)| \leq \varpi_1(1+|m|)^{-\mu}e^{-\beta|m|}\frac{1}{(1+|\frac{\tau}{\epsilon^{\Gamma}}|^2)}\exp(\nu|\frac{\tau}{\epsilon^{\Gamma}}|) \tag{148}$$

for all $\tau \in \bar{D}(0, \rho) \cup U_{\mathfrak{u}_j}$, $m \in \mathbb{R}$, $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$, for some well chosen $\varpi_1 > 0$. In particular, these functions $W^{\mathfrak{u}_j}(\tau, m, \epsilon)$ are analytic continuations w.r.t τ of a common function set as $\tau \mapsto W(\tau, m, \epsilon)$ on $D(0, \rho)$. We define $v^{\mathfrak{u}_j}(t, z, \epsilon)$ as a usual Laplace and Fourier inverse transform

$$v^{\mathfrak{u}_j}(t, z, \epsilon) = \frac{\epsilon^{\gamma_0}}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\mathfrak{u}_j^{\Delta}}}W^{\mathfrak{u}_j}(u, m, \epsilon)\exp(-(\frac{t}{\epsilon^{\gamma}})u)e^{izm}dudm.$$

By construction, each function $(t, z) \mapsto t^{-d_D}F^{\mathfrak{u}_j}(t, z, \epsilon)$ and $(t, z) \mapsto v^{\mathfrak{u}_j}(t, z, \epsilon)$ represents a bounded and holomorphic map on the domain $\mathcal{T}_{\epsilon}^{\infty} \times H_{\beta'}$, for any given $0 < \beta' < \beta$, for any fixed $\epsilon \in \mathcal{E}_j^{\infty}$, according to Definition 7.

Refering to the basic properties of the classical Laplace and Fourier inverse transforms disclosed in Proposition 2 and Lemma 4, we notice that the function $(t, z) \mapsto v^{\mathfrak{u}_j}(t, z, \epsilon)$ actually solves the equation (115) and hence the equation (8) after multiplication by t^{d_D}, where the expression $F^{\theta_F}(t, z, \epsilon)$ needs to be specialized for $\theta_F = \mathfrak{u}_j$ and subsequently be replaced by the function $F^{\mathfrak{u}_j}(t, z, \epsilon)$, for all $\epsilon \in \mathcal{E}_j^{\infty}$ and $(t, z) \in \mathcal{T}_{\epsilon}^{\infty} \times H_{\beta'}$. Furthermore, by direct inspection, we can check that for each $t \in \mathcal{T}^{\infty}$, the function $(\epsilon, z) \mapsto \epsilon^{-\gamma_0}v^{\mathfrak{u}_j}(t, z, \epsilon)$ is bounded holomorphic on $\mathcal{E}_j^{\infty} \times H_{\beta'}$, for any $0 < \beta' < \beta$ provided that $|\epsilon| < \sigma_t$, for σ_t defined in (146).

In the remaining part of the proof, we aim attention at the bounds (145). The lines of arguments are bordering those given in Theorem 1 in order to yield the estimates (86). Namely, the first task consists in splitting the difference $\epsilon^{-\gamma_0}v^{\mathfrak{u}_{j+1}} - \epsilon^{-\gamma_0}v^{\mathfrak{u}_{j+1}}$ into a sum of three integrals that are easier to handle. More precisely, owing to the fact that the function $u \mapsto W(u, m, \epsilon)\exp(-(\frac{tu}{\epsilon^{\gamma}}))$ is holomorphic on $D(0, \rho)$, for all $(m, \epsilon) \in \mathbb{R} \times (D(0, \epsilon_0) \setminus \{0\})$, its integral along a segment connecting 0 and $(\rho/2)e^{i\mathfrak{u}_{j+1}}$, followed by an arc of circle with radius $\rho/2$ joining $(\rho/2)e^{i\mathfrak{u}_{j+1}}$ and $(\rho/2)e^{i\mathfrak{u}_j}$ and ending with a segment with edges located at $(\rho/2)e^{i\mathfrak{u}_j}$

and 0, is vanishing. As a result, we can expand the next difference

$$(149)\quad \epsilon^{-\gamma_0} v^{\mathfrak{u}_{j+1}}(t,z,\epsilon) - \epsilon^{-\gamma_0} v^{\mathfrak{u}_j}(t,z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\mathfrak{u}_{j+1}^{\Delta}}} W^{\mathfrak{u}_{j+1}}(u,m,\epsilon) \times \exp(-(\frac{t}{\epsilon^{\gamma}})u) e^{izm} du dm - \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\mathfrak{u}_j^{\Delta}}} W^{\mathfrak{u}_j}(u,m,\epsilon) \exp(-(\frac{t}{\epsilon^{\gamma}})u) e^{izm} du dm + \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho/2,\mathfrak{u}_j^{\Delta},\mathfrak{u}_{j+1}^{\Delta}}} W(u,m,\epsilon) \exp(-(\frac{t}{\epsilon^{\gamma}})u) e^{izm} du dm$$

where $L_{\rho/2,\mathfrak{u}_{j+1}^{\Delta}} = [\rho/2,+\infty)e^{i\mathfrak{u}_{j+1}^{\Delta}}$, $L_{\rho/2,\mathfrak{u}_j^{\Delta}} = [\rho/2,+\infty)e^{i\mathfrak{u}_j^{\Delta}}$ and $C_{\rho/2,\mathfrak{u}_j^{\Delta},\mathfrak{u}_{j+1}^{\Delta}}$ stands for an arc of circle with radius joining $(\rho/2)e^{i\mathfrak{u}_j^{\Delta}}$ and $(\rho/2)e^{i\mathfrak{u}_{j+1}^{\Delta}}$ with an appropriate orientation.

We provide upper bounds for the first integral

$$J_1 = \left| \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\mathfrak{u}_{j+1}^{\Delta}}} W^{\mathfrak{u}_{j+1}}(u,m,\epsilon) \exp(-(\frac{t}{\epsilon^{\gamma}})u) e^{izm} du dm \right|$$

In accordance with the above estimates (148) and with the constraints disclosed in Definition 7, we check that

$$(150)\quad J_1 \leq \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{\rho/2}^{+\infty} \varpi_1 (1+|m|)^{-\mu} e^{-\beta|m|} \frac{1}{1+(\frac{r}{|\epsilon|^{\Gamma}})^2} \exp(\nu \frac{r}{|\epsilon|^{\Gamma}}) \times \exp(-\frac{|t|}{|\epsilon|^{\gamma}} r \cos(\mathfrak{u}_j^{\Delta} + \arg(\frac{t}{\epsilon^{\gamma}}))) \exp(-m\mathrm{Im}(z)) dr dm$$
$$\leq \frac{\varpi_1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \int_{\rho/2}^{+\infty} \exp(-r(-\frac{\nu}{|\epsilon|^{\Gamma}} + \frac{|t|}{|\epsilon|^{\gamma}} \delta_1^{\infty})) dr$$
$$= \frac{2\varpi_1}{(2\pi)^{1/2}(\beta-\beta')} \frac{1}{-\frac{\nu}{|\epsilon|^{\Gamma}} + \frac{|t|}{|\epsilon|^{\gamma}}\delta_1^{\infty}} \exp(-\frac{\rho}{2}(-\frac{\nu}{|\epsilon|^{\Gamma}} + \frac{|t|}{|\epsilon|^{\gamma}}\delta_1^{\infty}))$$
$$\leq \frac{2\varpi_1}{(2\pi)^{1/2}(\beta-\beta')} \frac{|\epsilon|^{\gamma}}{\delta_2^{\infty}|t|} \exp(-\frac{\rho}{2}\delta_2^{\infty}\frac{|t|}{|\epsilon|^{\gamma}})$$

for all $\epsilon \in \mathcal{E}_{j+1}^{\infty} \cap \mathcal{E}_j^{\infty}$, with $|\epsilon| < \sigma_t$. In a similar manner, we can furnish estimates for the second integral

$$J_2 = \left| \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\mathfrak{u}_j^{\Delta}}} W^{\mathfrak{u}_j}(u,m,\epsilon) \exp(-(\frac{t}{\epsilon^{\gamma}})u) e^{izm} du dm \right|.$$

Namely, we can show that

$$(151)\quad J_2 \leq \frac{2\varpi_1}{(2\pi)^{1/2}(\beta-\beta')} \frac{|\epsilon|^{\gamma}}{\delta_2^{\infty}|t|} \exp(-\frac{\rho}{2}\delta_2^{\infty}\frac{|t|}{|\epsilon|^{\gamma}})$$

for all $\epsilon \in \mathcal{E}_{j+1}^{\infty} \cap \mathcal{E}_j^{\infty}$, assuming that $|\epsilon| < \sigma_t$.

At last, we target the third integral along an arc of circle

$$J_3 = \left| \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho/2,\mathfrak{u}_j^{\Delta},\mathfrak{u}_{j+1}^{\Delta}}} W(u,m,\epsilon) \exp(-(\frac{t}{\epsilon^{\gamma}})u) e^{izm} du dm \right|$$

Calling again to mind the bounds (148) and the constraints discussed in Definition 7, we observe that

$$J_3 \leq \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \left| \int_{\mathfrak{u}_j^{\Delta}}^{\mathfrak{u}_{j+1}^{\Delta}} \varpi_1 (1+|m|)^{-\mu} e^{-\beta|m|} \frac{1}{1+(\frac{\rho/2}{|\epsilon|^{\Gamma}})^2} \exp(\nu \frac{\rho/2}{|\epsilon|^{\Gamma}}) \right.$$
$$\left. \times \exp(-(\frac{|t|}{|\epsilon|^{\gamma}} \frac{\rho}{2}) \cos(\theta + \arg(\frac{t}{\epsilon^{\gamma}}))) e^{-m\mathrm{Im}(z)} \frac{\rho}{2} d\theta \right| dm$$
$$\leq \frac{\varpi_1 \rho/2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm |\mathfrak{u}_{j+1}^{\Delta} - \mathfrak{u}_j^{\Delta}| \exp(-\frac{\rho}{2}(-\frac{\nu}{|\epsilon|^{\Gamma}} + \frac{|t|}{|\epsilon|^{\gamma}} \delta_1^{\infty}))$$
$$\leq \frac{\varpi_1 \rho}{(2\pi)^{1/2}(\beta-\beta')} |\mathfrak{u}_{j+1}^{\Delta} - \mathfrak{u}_j^{\Delta}| \exp(-\frac{\rho}{2} \frac{\delta_2^{\infty}}{|\epsilon|^{\gamma}} |t|) \tag{152}$$

for all $\epsilon \in \mathcal{E}_{j+1}^{\infty} \cap \mathcal{E}_j^{\infty}$, when $|\epsilon| < \sigma_t$.

In an ultimate step, we gather the three above inequalities (150), (151), (152) and conclude from the splitting (149) that

$$|\epsilon^{-\gamma_0} v^{\mathfrak{u}_{j+1}}(t,z,\epsilon) - \epsilon^{-\gamma_0} v^{\mathfrak{u}_j}(t,z,\epsilon)| \leq \frac{4\varpi_1}{(2\pi)^{1/2}(\beta-\beta')} \frac{|\epsilon|^{\gamma}}{\delta_2^{\infty}|t|} \exp(-\frac{\rho}{2} \delta_2^{\infty} \frac{|t|}{|\epsilon|^{\gamma}})$$
$$+ \frac{\varpi_1 \rho}{(2\pi)^{1/2}(\beta-\beta')} |\mathfrak{u}_{j+1}^{\Delta} - \mathfrak{u}_j^{\Delta}| \exp(-\frac{\rho}{2} \frac{\delta_2^{\infty}}{|\epsilon|^{\gamma}} |t|)$$

for all $\epsilon \in \mathcal{E}_{j+1}^{\infty} \cap \mathcal{E}_j^{\infty}$, granting that $|\epsilon| < \sigma_t$. As a result, the inequality (145) shows up. □

5 Gevrey asymptotic expansions of the inner and outer solutions

5.1 The Ramis-Sibuya approach for the $k-$summability of formal series

We first remind the reader the notion of $k-$summability as defined in classical textbooks such as [1], [2].

Definition 8 *Let $k > 1/2$ be a real number. A formal series*

$$\hat{a}(\epsilon) = \sum_{j=0}^{\infty} a_j \epsilon^j \in \mathbb{F}[[\epsilon]]$$

whose coefficients belong to the Banach space $(\mathbb{F}, ||.||_{\mathbb{F}})$ is called $k-$summable with respect to ϵ in the direction $d \in \mathbb{R}$ if

i) *one can choose a radius $\rho \in \mathbb{R}_+$ such that the following formal series, called formal Borel transform of $\hat{a}$ of order k*

$$\mathcal{B}_k(\hat{a})(\tau) = \sum_{j=0}^{\infty} \frac{a_j \tau^j}{\Gamma(1+\frac{j}{k})} \in \mathbb{F}[[\tau]],$$

is absolutely convergent for $|\tau| < \rho$*,*

ii) *there exists an aperture* $\delta > 0$ *such that the series* $\mathcal{B}_k(\hat{a})(\tau)$ *can be analytically continued w.r.t* τ *in a sector* $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$*. Moreover, there exist two constants* $C, K > 0$ *with*

$$||\mathcal{B}_k(\hat{a})(\tau)||_{\mathbb{F}} \leq Ce^{K|\tau|^k}$$

for all $\tau \in S_{d,\delta}$*.*

If these constraints are fulfilled, the vector valued Laplace transform of order k of $\mathcal{B}_k(\hat{a})(\tau)$ in the direction d is introduced as

$$\mathcal{L}_k^d(\mathcal{B}_k(\hat{a}))(\epsilon) = \epsilon^{-k} \int_{L_\gamma} \mathcal{B}_k(\hat{a})(u) e^{-(u/\epsilon)^k} k u^{k-1} du,$$

along any half-line $L_\gamma = \mathbb{R}_+ e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ may depend on ϵ and is chosen in such a way that $\cos(k(\gamma - \arg(\epsilon))) \geq \delta_1 > 0$, for some fixed δ_1, for all ϵ in a sector

$$S_{d,\theta,R^{1/k}} = \{\epsilon \in \mathbb{C}^* : |\epsilon| < R^{1/k} \quad , \quad |d - \arg(\epsilon)| < \theta/2\},$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$. The function $\mathcal{L}_k^d(\mathcal{B}_k(\hat{a}))(\epsilon)$ is then named the $k-$sum of the formal series $\hat{a}(t)$ in the direction d. It turns out to be bounded and holomorphic on the sector $S_{d,\theta,R^{1/k}}$ and has the formal series $\hat{a}(\epsilon)$ as Gevrey asymptotic expansion of order $1/k$ with respect to ϵ on $S_{d,\theta,R^{1/k}}$. In other words, for all $\frac{\pi}{k} < \theta_1 < \theta$, there exist $C, M > 0$ such that

$$||\mathcal{L}_k^d(\mathcal{B}_k(\hat{a}))(\epsilon) - \sum_{p=0}^{n-1} a_p \epsilon^p||_{\mathbb{F}} \leq CM^n \Gamma(1 + \frac{n}{k})|\epsilon|^n$$

for all $n \geq 1$, all $\epsilon \in S_{d,\theta_1,R^{1/k}}$.

The next cohomological criterion for $k-$summability of formal series with coefficients in Banach spaces (see [2], p. 121 or [7], Lemma XI-2-6) is accustomed to be called Ramis-Sibuya Theorem in the literature. This result appears as a fundamental tool in the proof of our third main result (Theorem 3).

Theorem (R.S.) *Let* $(\mathbb{F}, ||.||_{\mathbb{F}})$ *be a Banach space over* $\mathbb{C}$ *and* $\{\mathcal{E}_p\}_{0 \leq i \leq \varsigma - 1}$ *be a good covering in* $\mathbb{C}^*$ *with aperture* $\pi/k < 2\pi$ *(as displayed in Definition 4). For all* $0 \leq p \leq \varsigma - 1$*, let* G_p *be a holomorphic function from* $\mathcal{E}_p$ *into the Banach space* $(\mathbb{F}, ||.||_{\mathbb{F}})$ *and let the cocycle* $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ *be a holomorphic function from the sector* $Z_p = \mathcal{E}_{p+1} \cap \mathcal{E}_p$ *into* $\mathbb{F}$ *(with the convention that* $\mathcal{E}_\varsigma = \mathcal{E}_0$ *and* $G_\varsigma = G_0$*). We make the following assumptions.*

1) *The functions* $G_p(\epsilon)$ *are bounded as* $\epsilon \in \mathcal{E}_p$ *tends to the origin in* $\mathbb{C}$*, for all* $0 \leq p \leq \varsigma - 1$*.*

2) *The functions* $\Theta_p(\epsilon)$ *are exponentially flat of order* k *on* Z_p*, for all* $0 \leq p \leq \varsigma - 1$*. More specifically, there exist constants* $C_p, A_p > 0$ *such that*

$$||\Theta_p(\epsilon)||_{\mathbb{F}} \leq C_p e^{-A_p/|\epsilon|^k}$$

for all $\epsilon \in Z_p$*, all* $0 \leq p \leq \varsigma - 1$*.*

Then, for all $0 \leq p \leq \nu - 1$*, the functions* $G_p(\epsilon)$ *represent the* $k-$*sums on* $\mathcal{E}_p$ *of a common* $k-$*summable formal series* $\hat{G}(\epsilon) \in \mathbb{F}[[\epsilon]]$*.*

5.2 Parametric Gevrey asymptotic expansions of the inner and outer solutions of our main problem

In this last subsection, we show that both inner solutions (constructed in Section 3) and outer solutions (built up in Section 4) of our main equation (8) have asymptotic expansions in the small parameter ϵ near the origin. These asymptotic expansions have the special feature to be of some Gevrey types relying on data involved in the shape of equation (8) and it is worthwhile noting that these two types turn out to be distinct in general. We are now in position to state the third main result of our work.

Theorem 3 *We consider the singularly perturbed PDE (8) and we pretend that all the foregoing constraints, by merging the ones from both Theorem 1 and Theorem 2, listed as follows (7), (9), (12), (13), (14), (34), (35), (42), (48), (57), (58), (59), (60) and (113), (114), (117), (120), (121), (122), (123), (124), (125) hold conjointly.*
1) Let us consider a good covering $\{\mathcal{E}_j^{\infty}\}_{0 \leq j \leq \iota-1}$ *with aperture* $\frac{\pi}{\gamma}$ *for which an associated family of sector* $\mathcal{T}_{\epsilon}^{\infty}$ *and directions* $\{\mathfrak{u}_j\}_{0 \leq j \leq \iota-1}$ *can be picked up. For each fixed value of* t *belonging to the sector* $\mathcal{T}^{\infty}$ *(see (144)) and* $0 \leq j \leq \iota-1$*, we view the function* $(z,\epsilon) \mapsto \epsilon^{-\gamma_0} v^{\mathfrak{u}_j}(t,z,\epsilon)$*, built up in Theorem 2, as a bounded holomorphic function named* $O_t^j(\epsilon)$ *from* $\mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$ *into* $\mathcal{O}_b(H_{\beta'})$ *(which stands for the Banach space of bounded holomorphic functions on* $H_{\beta'}$ *equipped with the sup norm). Then, for each* $0 \leq j \leq \iota-1$, $O_t^j(\epsilon)$ *is the* $\gamma-$*sum on* $\mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$ *of a common formal series*

$$\hat{O}_t(\epsilon) = \sum_{k \geq 0} O_{t,k} \epsilon^k \in \mathcal{O}_b(H_{\beta'})[[\epsilon]].$$

In other words, for all $0 \leq j \leq \iota-1$*, there exist two constants* $C_j, M_j > 0$ *such that*

$$\sup_{z \in H_{\beta'}} |\epsilon^{-\gamma_0} v^{\mathfrak{u}_j}(t,z,\epsilon) - \sum_{k=0}^{n-1} O_{t,k}\epsilon^k| \leq C_j M_j^n \Gamma(1+\frac{n}{\gamma})|\epsilon|^n \tag{153}$$

for all $n \geq 1$*, all* $\epsilon \in \mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$*.*
2) We select a good covering $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ *with aperture* $\frac{\pi}{\chi\kappa}$ *for which a family of open sectors* $\{(S_{\mathfrak{d}_p,\theta,\rho_X|\epsilon|^{\chi}})_{0 \leq p \leq \varsigma-1}, \mathcal{T}_{\epsilon,\chi-\alpha}\}$ *associated to it can be singled out. Then, Theorem 1 asserts that for each direction* $\mathfrak{u}_j$, $0 \leq j \leq \iota-1$*, one can construct a family of holomorphic functions* $\{u^{\mathfrak{d}_p,j}(t,z,\epsilon)\}_{0 \leq p \leq \varsigma-1}$ *solving the main equation (8) where the piece of forcing term* $F^{\theta_F}(t,z,\epsilon)$ *is asked to be specialized for* $\theta_F = \mathfrak{u}_j$*. For all* $0 \leq p \leq \varsigma-1$*, we regard the map* $(x,z,\epsilon) \mapsto \epsilon^{m_0} u^{\mathfrak{d}_p,j}(x\epsilon^{\chi-\alpha},z,\epsilon)$ *as a bounded holomorphic function called* $I^{p,j}(\epsilon)$ *from* $\mathcal{E}_p$ *into* $\mathcal{O}_b((X \cap D(0,\sigma)) \times H_{\beta'})$ *(which represents the Banach space of bounded holomorphic functions on* $(X \cap D(0,\sigma)) \times H_{\beta'}$ *endowed with the sup norm). Then, for all* $0 \leq p \leq \varsigma-1$*, each* $I^{p,j}(\epsilon)$ *is the* $\chi\kappa$*-sum on* $\mathcal{E}_p$ *of a common formal series*

$$\hat{I}^j(\epsilon) = \sum_{k \geq 0} I_k^j \epsilon^k \in \mathcal{O}_b((X \cap D(0,\sigma)) \times H_{\beta'})[[\epsilon]].$$

Equivalently, for each $0 \leq p \leq \varsigma-1$*, there exist two constants* $C_p, M_p > 0$ *such that*

$$\sup_{x \in X \cap D(0,\sigma), z \in H_{\beta'}} |\epsilon^{m_0} u^{\mathfrak{d}_p,j}(x\epsilon^{\chi-\alpha},z,\epsilon) - \sum_{k=0}^{n-1} I_k^j \epsilon^k| \leq C_p M_p^n \Gamma(1+\frac{n}{\chi\kappa})|\epsilon|^n \tag{154}$$

for all $n \geq 1$*, all* $\epsilon \in \mathcal{E}_p$*.*

Proof Let us concentrate on the first item. We consider the family of functions $\{v^{\mathfrak{u}_j}(t,z,\epsilon)\}_{0 \leq j \leq \iota-1}$ constructed in Theorem 2. For each prescribed value of t inside the sector $\mathcal{T}^{\infty}$ and all $0 \leq j \leq \iota - 1$, we set $G_j(\epsilon) := z \mapsto \epsilon^{-\gamma_0} v^{\mathfrak{u}_j}(t,z,\epsilon)$ which defines a bounded holomorphic function from $\mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$ into the Banach space $\mathbb{F}$ of bounded holomorphic functions on $H_{\beta'}$ outfitted with the sup norm. Bearing in mind the estimates (145), we deduce that the cocycle $\Theta_j(\epsilon) = G_{j+1}(\epsilon) - G_j(\epsilon)$ is exponentially flat of order γ on $Z_j = \mathcal{E}_j^{\infty} \cap \mathcal{E}_{j+1}^{\infty} \cap D(0,\sigma_t)$. According to Theorem (R.S.) stated in Section 5.1, there exists a formal power series $\hat{G}(\epsilon) \in \mathbb{F}[[\epsilon]]$ for which the functions $G_j(\epsilon)$ are the $\gamma-$sums on $\mathcal{E}_j^{\infty} \cap D(0,\sigma_t)$, for all $0 \leq j \leq \iota - 1$.

We next focus on the second item. For each fixed direction $\mathfrak{u}_j$, $0 \leq j \leq \iota - 1$, we consider the set of functions $\{u^{\mathfrak{d}_p,j}(t,z,\epsilon)\}_{0 \leq p \leq \varsigma-1}$ introduced in Theorem 1 for the choice of the forcing term $F^{\mathfrak{u}_j}(t,z,\epsilon)$ in the main equation (8). For all $0 \leq p \leq \varsigma - 1$, we define this time $\tilde{G}_p(\epsilon) := (x,z) \mapsto \epsilon^{m_0} u^{\mathfrak{d}_p,j}(x\epsilon^{\chi-\alpha},z,\epsilon)$ that represents a bounded holomorphic function from $\mathcal{E}_p$ into $\mathbb{F}$ which stands now for the Banach space of bounded holomorphic functions on $(X \cap D(0,\sigma)) \times H_{\beta'}$ supplied with the sup norm. Keeping in view the estimates (86), we find out that the cocycle $\tilde{\Theta}_p(\epsilon) := \tilde{G}_{p+1}(\epsilon) - \tilde{G}_p(\epsilon)$ decays exponentially with order $\chi\kappa$ on the crossing section $Z_p = \mathcal{E}_{p+1} \cap \mathcal{E}_p$. In agreement with Theorem (R.S.) outlined above, there exists a formal power series $\hat{\tilde{G}}(\epsilon) \in \mathbb{F}[[\epsilon]]$ admitting the maps $\tilde{G}_p(\epsilon)$ as its $\chi\kappa-$sums on $\mathcal{E}_p$, for all $0 \leq p \leq \varsigma - 1$. This ends the proof of Theorem 3. $\square$

In order to illustrate the theorem enounced above, we provide two examples of the main equation (8) satisfying conjointly the constraints outlined in Theorem 1 and Theorem 2.

Examples. We take $q = 1$,$M = 0$,$Q = 0$ and $D = 2$.
1) We first consider a situation for which $\kappa = 1$. We select the powers of t and ϵ in the coefficients of (8) as follows

$$m_0 = 5, m_1 = 4, k_1 = 2, \mu_0 = 2, h_0 = 2, n_0 = 5, b_0 = 1, \Delta_2 = 3, d_2 = 4, \delta_2 = 2,$$
$$\Delta_1 = 3, d_1 = 3, \delta_1 = 1.$$

In this setting, we choose $\kappa = 1$,$\chi = 6$,$\Gamma = 1$,$\gamma_0 = 0$,$\gamma = 3/2$,$\alpha = -1$ and $n_F = 5$. Notice that we cannot sort χ smaller than 6 due to the inequality (60). For these data, one can check that all the constraints asked in Theorem 3 (combining the ones of Theorem 1 and 2) on the coefficients of (8) w.r.t t and ϵ are fulfilled. For this special situation, the main equation (8) is displayed as follows

$$(a_1\epsilon^4 t^2 + a_0\epsilon^5)Q(\partial_z)u(t,z,\epsilon) + c_0\epsilon^2 t^2 Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon)$$
$$= b_0(z)\epsilon^5 t + \frac{\epsilon^5}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\theta_F}} \omega_F(u,m)(\exp(-\frac{t}{\epsilon^{3/2}}u) - 1)e^{izm} du dm$$
$$+ \epsilon^3 t^4 \partial_t^2 R_2(\partial_z)u(t,z,\epsilon) + \epsilon^3 t^3 \partial_t R_1(\partial_z)u(t,z,\epsilon)$$

This last equation can be divided by ϵ^2 but not by any positive power of t. The resulting equation is still singularly perturbed with an irregular singularity at $t = 0$ and carries two movable turning points which coalesce to 0 as ϵ tends to the origin.
2) The second example concerns the case $\kappa = 2$. Namely, let us pick out the powers of t and ϵ in the following manner,

$$m_0 = 9, m_1 = 8, k_1 = 4, \mu_0 = 2, h_0 = 4, n_0 = 9, b_0 = 3, \Delta_2 = 5, d_2 = 6, \delta_2 = 2,$$
$$\Delta_1 = 6, d_1 = 4, \delta_1 = 1.$$

Under this choice, we set $\kappa = 2$,$\chi = 12$,$\Gamma = 5/2$,$\gamma_0 = 1$,$\gamma = 3$,$\alpha = -1$ and $n_F = 11$. As in the first example, there is some lower bound for χ that cannot be taken less than 12 according to (60). Under these conditions, all the requirements needed on the coefficients of (8) w.r.t t and ϵ demanded in Theorem 3 are favorably completed. In this case, (8) is written as follows

$$(a_1\epsilon^8 t^4 + a_0\epsilon^9)Q(\partial_z)u(t,z,\epsilon) + c_0\epsilon^2 t^4 Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon)$$
$$= b_0(z)\epsilon^9 t^3 + \frac{\epsilon^{11}}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\theta_F}} \omega_F(u,m)(\exp(-\frac{t}{\epsilon^3}u)-1)e^{izm}dudm$$
$$+ \epsilon^5 t^6 \partial_t^2 R_2(\partial_z)u(t,z,\epsilon) + \epsilon^6 t^4 \partial_t R_1(\partial_z)u(t,z,\epsilon)$$

As above, one can factor out the power ϵ^2 from the equation but not any power of t. The new equation obtained remains singularly perturbed in ϵ with an irregular singularity at $t = 0$ and still possess four turning points which merge at 0 as ϵ reachs the origin.

Remark 2. According to the first remark disclosed in Section 2.2 which states that the forcing term $F^{\mathfrak{u}_j}(t,z,\epsilon)$ solves the special ODE (19), we observe that for all $0 \le j \le \iota-1$ and $0 \le p \le \varsigma-1$ both outer solution $v^{\mathfrak{u}_j}(t,z,\epsilon)$ constructed in Theorem 2 and related inner solution $u^{\mathfrak{d}_p,j}(t,z,\epsilon)$ defined in Theorem 1 actually solve the next singularly PDE which displays a similar shape as the main equation (8) but possesses rational coefficients in time t and parameter ϵ,

$$F_2(-\epsilon^{\gamma}\partial_t)\left((\sum_{l=1}^{q} a_l\epsilon^{m_l}t^{k_l} + a_0\epsilon^{m_0})Q(\partial_z)u(t,z,\epsilon)\right)$$
$$+ F_2(-\epsilon^{\gamma}\partial_t)\left((\sum_{l=0}^{M} c_l\epsilon^{\mu_l}t^{h_l})Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon)\right)$$
$$= \sum_{j=0}^{Q} b_j(z)\epsilon^{n_j}F_2(-\epsilon^{\gamma}\partial_t)t^{b_j} + \epsilon^{n_F}c_F(z)\left(\sum_{k=0}^{\deg(F_1)} F_{1,k}\frac{k!}{(K_F+\frac{t}{\epsilon^{\gamma}})^{k+1}} - F_2(0)c_{F_1,F_2,\mathfrak{u}_j}\right)$$
$$+ \sum_{l=1}^{D}\epsilon^{\Delta_l}F_2(-\epsilon^{\gamma}\partial_t)\left(t^{d_l}\partial_t^{\delta_l}R_l(\partial_z)u(t,z,\epsilon)\right).$$

Remark 3. Let the constraints of Theorem 1 and Theorem 2 hold mutually. Select some integers $0 \le j \le \iota-1$ and $0 \le p \le \varsigma-1$ for which $\mathcal{E}_j^{\infty} \cap \mathcal{E}_p \neq \emptyset$. Then, for any fixed $\epsilon \in \mathcal{E}_j^{\infty} \cap \mathcal{E}_p$, the outer solution $v^{\mathfrak{u}_j}(t,z,\epsilon)$ is well defined for all $t \in \mathcal{T}_{\epsilon}^{\infty}$ and the inner solution $u^{\mathfrak{d}_p,j}(t,z,\epsilon)$ for all $t \in \mathcal{T}_{\epsilon,\chi-\alpha}$, provided that $z \in H_{\beta'}$. But it turns out that

$$\mathcal{T}_{\epsilon}^{\infty} \cap \mathcal{T}_{\epsilon,\chi-\alpha} = \emptyset \tag{155}$$

subjected to the fact that $|\epsilon|$ is taken small enough. Namely, let us first notice that

$$\chi - \alpha > \gamma - \Gamma. \tag{156}$$

According to (114), (34) and (35), we get that

$$\gamma\delta_D - \gamma_0 = \alpha\delta_D\kappa + m_0. \tag{157}$$

Bearing in mind (121) and (35), we deduce

$$m_0 + \gamma_0 \ge (\gamma - \Gamma)\delta_D(\kappa+1) + \Gamma\delta_D. \tag{158}$$

As an offshoot of (157) and (158), we obtain

$$\gamma\delta_D \geq \alpha\delta_D\kappa + (\gamma - \Gamma)\delta_D(\kappa + 1) + \Gamma\delta_D.$$

Since $\delta_D, \kappa \geq 1$, we can factor out δ_D and κ in this last inequality in order to obtain

$$0 \geq \alpha + \gamma - \Gamma.$$

Since χ is assumed to be a real number larger than $\frac{1}{2\kappa}$, we deduce that (156) must hold. In particular, we deduce that

$$\rho_X|\epsilon|^{\chi-\alpha} < \frac{\Delta_{\nu,\delta_1^\infty}}{2}|\epsilon|^{\gamma-\Gamma} < \Delta_{\nu,\delta_1^\infty}|\epsilon|^{\gamma-\Gamma}$$

for all $|\epsilon|$ small enough. This implies the empty intersection (155).

As a result, we observe some *scaling gap* in time t between these two families of solutions. We postpone for future investigations the study of possible analytic continuation in time t and matching properties between the inner and outer solutions constructed above.

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