The global bifurcation picture for ground states in nonlinear Schrödinger equations

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Abstract

In this paper, we propose a method of finding all coherent structures supported by a given nonlinear wave equation. It relies on enhancing the recent global bifurcation theory as developed by Dancer, Toland, Buffoni and others, by determining all the limit points of the coherent structure manifolds at the boundary of the Fredholm domain. Local bifurcation theory is then used to trace back these manifolds from their limit points into the interior of the Fredholm domain identifying the singularities along them. This way all coherent structure manifold are discovered except may be the ones which form loops and hence never reach the boundary. The method is then applied to the Schrödinger equation with a power nonlinearity for which all ground states are identified.

1 Introduction

The long standing conjecture in dispersive wave equations is the asymptotic decomposition of any initial data into coherent structures. The conjecture is true in the linear case, in nonlinear integrable systems and also certain particular cases of small amplitude regimes, see for example [33, 32, 34, 11, 21, 22]. However, the set of all coherent structures is not completely, which is a big obstacle in even starting to approach the conjecture in large amplitude regimes for non-integrable systems. In this paper we propose an original method for finding all coherent structures of a given equation, see Section 6.4. We apply it to the case of nonlinear Schrödinger equation with potential, which is sometimes called Gross-Pitaevski equation:

$$iu_t = -\Delta u + V(x)u + \sigma|u|^{2p}u, \tag{1.1}$$

where $u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is the wave function, Δ is the Laplacian, $0 < p < 2/(n-2)$ is the nonlinearity power, $\sigma \in \mathbb{R}$ determines the defocusing (repulsive), respectively focusing (attractive), character of the nonlinearity when $\sigma > 0$, respectively $\sigma < 0$, and $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is an external, real-valued potential satisfying:

(H1) $V(x) \in L^\infty(\mathbb{R}^n)$,

(H2) $\lim_{|x|\to\infty} V(x) = 0$,

Hypothesis (H1) implies that $-\Delta + V(x)$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$, while (H1)-(H2) make V a relatively compact perturbation of $-\Delta$, see [30]. At the expense of complicating some of the proofs without adding new ideas we can recover all our result by just assuming V is a relative compact perturbation of the Laplacian, for example $V \in L^q(\mathbb{R}^n)$, $q \geq 1$, $q > n/2$, see the discussion in [15]. In certain instances we will also make the following spectral assumption:

(H3) $L_0 = -\Delta + V(x)$ has the lowest eigenvalue $-E_0 < 0$.

It is well known from second order elliptic theory that the lowest eigenvalue of L_0 is simple, the corresponding eigenfunction, ψ_0, can be chosen to be real valued and normalized in L^2 :

$$-\Delta\psi_0(x) + V(x)\psi_0(x) + E_0\psi_0(x) = 0, \quad \psi_0(x) > 0,\ x \in \mathbb{R}^n,\ \|\psi_0\|_{L^2} = 1. \tag{1.2}$$

The equation is an ubiquitous model for wave propagation in optics, fluids and statistical quantum physics. Its most striking features are the coherent structures (bound states) which propagate without changing shape leading to important application such as propagating modes in optical wave guides, solitary waves at the interfaces between fluids, Bose-Einstein condensates. This paper focuses on the bound states of (1.1) which are solutions of the form

$$u(t,x) = e^{iEt}\psi(x).$$

By plugging in we get

$$-\Delta\psi(x) + V(x)\psi(x) + \sigma|\psi(x)|^{2p}\psi(x) + E\psi(x) = 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

The latter is usually studied in the weak (distributional) sense i.e., a solution $\psi \in H^1(\mathbb{R}^n, \mathbb{C})$ satisfies:

$$\langle \nabla\psi, \nabla\phi\rangle + \langle (V+E)\psi + \sigma|\psi|^{2p}\psi, \phi\rangle = 0 \qquad \forall\ \phi \in H^1(\mathbb{R}^n). \tag{1.4}$$

Remark 1.1. *A standard elliptic regularity argument, see the Appendix, shows that any H^1 solution of* (1.4) *is a strong, H^2, solution of* (1.3) *i.e., the latter holds for almost all $x \in \mathbb{R}^n$.*

This paper is inspired by the fairly recent theory of global bifurcations, see for example [29, 3], which states that, in this particular case, the solutions of (1.3) bifurcating from the trivial solution at E_0 form a C^1 manifold which can be continued until $E = 0$ or $E = \infty$ or H^2 norm of the solutions blows up at finite $E > 0$. We develop new techniques to study the limit points of this manifold at the above boundary sets. Our results show that in the attractive case, $\sigma < 0$, there are no limit points at $E = 0$ or at $0 < E < \infty$, $\|\psi\|_{H^2} = \infty$, hence this manifold can be extended for all $E > E_0$. Moreover, we show that the limit point at the $E = \infty$ boundary must be a subset of a rather precise set defined with the help of solutions of a limiting equation

$$-\Delta\Psi + \Psi + |\Psi|^{2p}\Psi = 0. \tag{1.5}$$

In the case of the ground states, see Definition 2.1, we actually fully determine their limit points, see Remark 4.1. Since the limit points depend on the critical points of the potential, we analyze a few well known examples such as one-well and double-well potentials, where we determine the rather complicated bifurcation diagram of the ground states, see Figure 6.5.

Our point is that the techniques we have developed can be extended to other equations in order to determine, as in the above mentioned examples, all coherent structures supported by them, see Section 6.4. There are no such results that we know of in the existing literature. However, the existence of unique ground state branch in one space dimension ($n = 1$) in the case of an even and strictly monotonic potential (for $x > 0$) has been proven in [12] using monotonicity techniques for ordinary differential equations. Arguments using the behavior of a branch of ground states approaching the $E = \infty$ boundary have again been used in one space dimension in [18], where, along the branch starting at E_0 a symmetry-breaking bifurcation point is found and analyzed. We actually generalize that result to any space dimension and more complicated symmetries of the potential in Section 5. The result in this section is related to the one in [1], where the authors studied Hartree equation using variational methods. We think that our techniques can be extended to the latter equation in which case they prove not only that the shape of ground states change from small amplitudes to large amplitudes, but also that there is a bifurcation point that can be analyzed.

One of the first results showing that ground states for our problem (1.3) with attractive nonlinearity ($\sigma < 0$) exists for all $E > E_0$ is [31]. In it the authors used variational methods and conjectured that the profile of the ground state will converge to the ground state of the limiting equation (1.5). In this paper, we not only rigorously prove their conjecture but we also show that depending on the number of critical points of the potential there are many other "ground states" in the limit $E \to \infty$, with complicated bifurcation diagram, see for example Figure 6.5, where by ground states we no longer mean the global minimizer of a certain functional, but also other local minimizers or saddle points.

There are similarities between our equation (1.3) as $E \to \infty$ and the semi-classical limit for nonlinear Schrödinger equation studied in [6] and [27]. Their approach which combines Lyapunov-Schmidt decomposition with degree theory for the reduced finite-dimensional system leaves open the question of uniqueness of the solutions they find. However, our arguments in Theorem 4.3 can be adapted to their problem to show not only that the single profile ground states in [6] and the multiprofile ground states in [27] exist but also that they are unique at each value of the parameter h (the Planck constant).

Our the paper is organized as follows. In Section 2, we recall a few results on small amplitudes nonlinear bound states of (1.3) which, it turns out, bifurcate from the zero solutions at eigenvalues of the linear part $-\Delta + V$ and organize themselves in C^1 branches. Then we study how far these branches can be extended without passing through singular points i.e., we determine their maximal extension. We also generalize these results to arbitrary i.e., not necessarily small amplitude, branches of bound states. In Section 3 we study the finite end points of these maximal branches and via a compactness result we show that they are actually bound states themselves. In particular, these allows us to determine all the ground states in the defocusing case ($\sigma > 0$), see Theorem 3.2. In doing this we generalize global bifurcation result due to Jeanjean, Lucia and Stuart, see [14] and Remark 3.1. In the focusing case ($\sigma < 0$), we prove for the first time to the best of our knowledge, the compactness result given by Theorem 3.3. It is a delicate result which involves concentration compactness and spectral properties of Schrödinger type linear operators with potentials separated by a large distance, but it allows us not only to identify the limit points of the maximal branches, see Corollary 3.1, but also to apply global bifurcation techniques to continue these branches until they reach the $E = \infty$ boundary. In Section 4, we study the behavior of bound state branches in the limit $E \to \infty$. In particular, we show that there exists a unique renormalization under which each of these branches converges to a superposition of solutions (profiles) of the limiting equation (1.5), see Theorem 4.1. Since the ground state of the latter is unique up to translations, we fully determine all the ground state branches at this limit, see Theorems 4.2 and 4.3. Similar to the compactness result at finite E, Theorem 4.2 uses an original argument based on local bifurcation tools to prevent the drifting of profiles to $x = \infty$.

In Section 5, we combine all the results in the previous sections to generalize the symmetry-breaking bifurcation result in [18] to any dimensions and more complicated symmetries. Moreover, in Section 6 we show how our results can be combined with global bifurcation tools to determine all ground states of our problem. Certain implications about bound states are also discussed. This section concludes by showing that the algorithm for identifying all coherent structures of a large class of dispersive wave equations, proposed by one of the authors in [15], is successful for the nonlinear Schrödinger equation with a power nonlinearity and by discussing possible generalizations.

In the Appendix, we recall or prove slight generalizations of known results about regularity and exponential decay of solutions of elliptic equations and about compactness in Sobolev spaces. We end this section by proving the existence of a natural decomposition of branches of solutions approaching a singularity or the limit $E = \infty$, based on invariant subspaces of the linearized operator. In comparison to similar decompositions, for example the one in [27], this one enables us to refine the analysis of the limit point and establish the uniqueness of the multi profile solution.

2 Maximal branches of solutions

In this section we show that the set of solutions of (1.3) can be organized in C^1 manifolds and we identify some of their properties near singular points. In particular we construct the manifolds of nontrivial solutions bifurcating from zero at the isolated eigenvalues of $-\Delta + V$.

We rewrite (1.3) as the problem of finding the zeroes of the functional

$$F : H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n, \mathbb{C})$$

$$F(\psi, E) = (-\Delta + V + E)\psi + \sigma|\psi|^{2p}\psi \tag{2.1}$$

where $L^2(\mathbb{R}^n, \mathbb{C})$ is the Hilbert space of the complex valued square integrable functions on $\mathbb{R}^n$ and $H^2(\mathbb{R}^n, \mathbb{C}) \hookrightarrow L^2(\mathbb{R}^n, \mathbb{C})$, is the Hilbert space of the complex valued functions on $\mathbb{R}^n$ which are square integrable together with their first and second weak derivatives. Note that the functional F is not Fréchet differentiable over the natural complex structure of the Hilbert spaces $H^2(\mathbb{R}^n, \mathbb{C})$, $L^2(\mathbb{R}^n, \mathbb{C})$ due to the absolute value term. It is however differentiable over the reals, hence, in what follows, the above Hilbert spaces will be considered over the real field. In particular the scalar product in $L^2(\mathbb{R}^n, \mathbb{C})$ is:

$$\langle \phi, \psi \rangle = \Re \int_{\mathbb{R}^n} \overline{\phi}(x)\psi(x)dx = \int_{\mathbb{R}^n} \phi_1(x)\psi_1(x)dx + \int_{\mathbb{R}^n} \phi_2(x)\psi_2(x)dx,\ \phi = \phi_1 + i\phi_2,\ \psi = \psi_1 + i\psi_2,$$

and any orthogonality statement will refer to the above scalar product unless specified otherwise. In some situations it is also useful to separate the real and imaginary parts of the functions involved $\psi = \psi_1 + i\psi_2 = [\psi_1 \ \psi_2]^\top$ and identify the following isomorphic *real* Banach spaces:

$$L^2 = L^2(\mathbb{R}^n, \mathbb{C}) \cong L^2(\mathbb{R}^n, \mathbb{R}) \times L^2(\mathbb{R}^n, \mathbb{R}), \qquad H^2 = H^2(\mathbb{R}^n, \mathbb{C}) \cong H^2(\mathbb{R}^n, \mathbb{R}) \times H^2(\mathbb{R}^n, \mathbb{R}).$$

Then (1.3) is equivalent to:

$$(-\Delta + V + E)\psi + \sigma|\psi|^{2p}\psi = 0 \quad \Leftrightarrow \quad F(\psi, E) = 0,$$

where now

$$F(\psi = \psi_1 + i\psi_2, E) = [F_1 \ F_2]^\top = \begin{bmatrix} (-\Delta + V + E)\psi_1 + \sigma(|\psi_1|^2 + |\psi_2|^2)^p\psi_1 \\ (-\Delta + V + E)\psi_2 + \sigma(|\psi_1|^2 + |\psi_2|^2)^p\psi_2 \end{bmatrix}$$

Remark 2.1. *F is equivariant with respect to rotations in the complex plane and complex conjugation:*

$$\begin{aligned} F(e^{i\theta}\psi, E) &= e^{i\theta}F(\psi, E), \qquad \text{for } 0 \le \theta < 2\pi \\ F(\overline{\psi}, E) &= \overline{F(\psi, E)}. \end{aligned} \tag{2.2}$$

Note that F is C^1 i.e., continuous with continuous Frechet derivative. For real valued $\psi \in H^2$ we have

$$D_\psi F(\psi, E) = L_+(\psi, E) + iL_-(\psi, E) \equiv \begin{bmatrix} L_+(\psi, E) & 0 \\ 0 & L_-(\psi, E) \end{bmatrix} \tag{2.3}$$

where,

$$L_+(\psi, E) = -\Delta + V + E + (2p+1)\sigma|\psi|^{2p} \tag{2.4}$$

$$L_-(\psi, E) = -\Delta + V + E + \sigma|\psi|^{2p} \tag{2.5}$$

Remark 2.2. *For $\psi \in H^2$ (actually H^1 suffices) and $0 < 2p < 4/(n-2)$ we have $|\psi|^{2p} \in L^q$ for some $q > \max\{1, n/2\}$. Hence the effective potentials $V + (2p+1)\sigma|\psi|^{2p}$, respectively $V + \sigma|\psi|^{2p}$, are relatively compact, symmetric perturbations of $-\Delta + E$. Therefore, $L_\pm$ remain self adjoint operators on L^2 with domain H^2 and, Via Weyl's Theorem, their essential spectrum is $[E, \infty)$, see [30]. Moreover, by differentiating the symmetry relation (2.2) with respect to θ at $\theta = 0$, we get*

$$D_\psi F(\psi, E)[i\psi] = iF(\psi, E).$$

Therefore, at zeroes (ψ_E, E) of F i.e., solutions of (1.3), zero is an e-value of the linearized operator with corresponding eigenfunction $i\psi_E$. In the case ψ_E is real valued, zero becomes an e-value of $L_-(\psi_E, E)$ with corresponding eigenfunction ψ_E.

Since $(0, E)$, $E \in \mathbb{R}$ is always a zero of F and the Fréchet derivative in the first variable at these solutions is:

$$D_\psi F(0, E) = \begin{bmatrix} L_+(0, E) & 0 \\ 0 & L_-(0, E) \end{bmatrix} = \begin{bmatrix} (-\Delta + V + E) & 0 \\ 0 & (-\Delta + V + E) \end{bmatrix},$$

the implicit function theorem states:

Proposition 2.1. *If $-E_*$ is not in the spectrum of the self adjoint operator $-\Delta + V$ then there exist ϵ, $\delta > 0$ such that for $|E - E_*| < \delta$ and $\|\psi\|_{H^2} < \epsilon$, (1.3) has only the trivial solution.*

Next subsection discusses small, nontrivial zeroes of F i.e., solutions of equation (1.3) as they bifurcate from the trivial ones at eigenvalues of $-\Delta + V$.

2.1 Bifurcations from zero

The following local bifurcation result describes the existence of solutions of the stationary problem (1.3) near E_0 defined by (1.2) or any other negative, simple eigenvalue of $-\Delta + V$.

Proposition 2.2. *Let $-E_0 < 0$ be a simple eigenvalue of $-\Delta + V$ and let ψ_0 with $\|\psi_0\|_{L^2} = 1$ be a corresponding real valued eigenfunction. Then there exists $\epsilon, \delta > 0$ such that:*

(i) If $\sigma > 0$, for each $E \in \mathcal{I} \equiv (E_0 - \delta, E_0)$, the stationary problem (1.3) *has exactly two, nonzero, real valued solutions $\pm\psi_E$ satisfying $\| \pm \psi_E\|_{H^2} < \epsilon$, and we choose ψ_E to be the one with positive L^2 projection on ψ_0. If $\sigma < 0$, for each $E \in \mathcal{I} \equiv (E_0, E_0 + \delta)$, the stationary problem* (1.3) *has exactly two, nonzero, real valued solutions $\pm\psi_E$ satisfying $\|\psi_E\|_{H^2} < \epsilon$, and ψ_E is chosen as above.*

(ii) Any solution (ψ, E) of (1.3) *satisfying $|E - E_0| < \delta$ and $\|\psi\|_{H^2} < \epsilon$ is of the form $(\psi, E) = (e^{i\theta}\psi_E, E)$ where E and ψ_E are given in (i) and $0 \le \theta < 2\pi$.*

Moreover, the function $E \mapsto \psi_E$ is C^1 from $\mathcal{I}$ to H^2, and $\lim_{E\to E_0} \|\psi_E\|_{H^2} = 0$.

Proof. Variants of this theorem have already appeared in Pillet-Wayne [28], as well as in many recent publications. We sketch the main steps for the proof of the result in this form.

Let ψ_0 be the eigenfunction of $-\Delta + V$ corresponding to the simple eigenvalue $-E_0$. Because of invariance of $(-\Delta + V)\psi_0 = -E_0\psi_0$ under complex conjugation we can choose ψ_0 to be real valued and of L^2 norm 1. Then $D_\psi F(0, E_0)$ is Fredholm of index zero with kernel$=\mathrm{span}_{\mathbb{R}}\{\psi_0 \equiv [\psi_0\ 0]^\top, i\psi_0 \equiv [0\ \psi_0]^\top\}$ and range equal to the orthogonal complement in L^2 of its kernel. Let

$$P_{\|}\phi = \langle\psi_0, \phi\rangle\psi_0 + \langle i\psi_0, \phi\rangle i\psi_0 = \langle\psi_0, \phi\rangle_{\mathbb{C}}\psi_0, \qquad P_\perp = I - P_{\|}$$

be the two projections associated to this Lyapunov-Schmidt decomposition where, depending on the context, I is the identity on H^2 respectively on L^2. Then

$$F(\psi, E) = 0$$

is equivalent to

$$P_\perp F(P_{\|}\psi + P_\perp\psi, E) = 0, \tag{2.6}$$
$$P_{\|} F(P_{\|}\psi + P_\perp\psi, E) = 0. \tag{2.7}$$

But the implicit function theorem can now be applied to (2.6) which will have a unique solution:

$$P_\perp\psi = h(P_{\|}\psi, E)$$

for $|E - E_0|$ and $P_{\|}\psi$ sufficiently small. Hence the system (2.6)-(2.7) is now equivalent with:

$$P_{\|}F(P_{\|}\psi + h(P_{\|}\psi, E), E) = 0.$$

Identifying now the range of $P_{\|}$ with $\mathbb{R}^2 = \mathbb{C}$ via its basis $\{\psi_0, i\psi_{02}\}$, the above equation has the form $G(a, E) = 0 \in \mathbb{C},\ a \in \mathbb{C},\ E \in \mathbb{R}$ Moreover, from the implicit function theorem $h : \mathbb{C} \times \mathbb{R} \mapsto H^2$

is a C^1 map which inherits the equivariance properties of F, see Remark 2.1, because the two projections are also equivariant with respect to these symmetries. Consequently G is C^1 and $G(\cdot, E)$ is equivariant with respect to rotations in complex plane and complex conjugation. In particular if (a, E) is a zero for G so is $(e^{i\theta}a, E)$, $0 \leq \theta < 2\pi$. Hence we can first find solutions with a real, nonzero, and obtain all the others by rotations. Now:

$$\tilde{G}(a, E) = G(a, E)/a = (E - E_0) + \sigma\langle\psi_0, \frac{1}{a}(a\psi_0 + h(a, E))^{2p+1}\rangle = 0, \; a \in \mathbb{R}\backslash 0, \; E \in \mathbb{R},$$

can be extended to a C^1 function at $a = 0$ and

$$\tilde{G}(0, E_0) = 0 \qquad \partial_E\tilde{G}(0, E_0) = 1.$$

Via implicit function theorem the above equation has a unique solution $E = E(a)$ for $a \in \mathbb{R}$ sufficiently close to zero and $E \in \mathbb{R}$ sufficiently close to E_0. From it we get the C^1 curve of solutions:

$$E = E(a), \qquad \psi_E = a\psi_0 + h(a, E(a)) \equiv a\psi_0 + h(a),$$

where ψ_E is real valued, since ψ_0 is and $\overline{h(a)} = h(\overline{a}) = h(a)$, $a \in \mathbb{R}$. All other solutions are obtained by rotating a, i.e. are of the form $(e^{i\theta}\psi_E, E)$. $\square$

The above Proposition combined with regular perturbation theory allows us to determine the spectrum of the linearization $D_\psi F(\psi_E, E)$ along the nontrivial branch bifurcating from $-E_0$.

Remark 2.3. *On the C^1 curve of bound states $E \mapsto \psi_E$ given by* Proposition 2.2 *the $L_\pm$ operators depend continuously on E, hence their eigenvalues depend continuously on E. Consequently* 0 *is a simple eigenvalue of $L_-(\psi_E, E)$ at least on a subinterval containing zero: $E \in \mathcal{I}_\epsilon \subseteq \mathcal{I}$. Moreover, from $L_+ - L_- = 2p\sigma|\psi_E|^{2p}$, we obtain that $L_+ < L_-$ in the focusing case $\sigma < 0$. Via eigenvalue comparison principle, L_+ has its eigenvalues smaller than the corresponding eigenvalues of L_-, in particular* 0 *is not an eigenvalue of L_+ which has the number of strictly negative eigenvalues increased by one compared to $-\Delta + V + E_0$. If $-E_0$ is the lowest eigenvalue of $-\Delta + V$ then* 0 *is the lowest eigenvalue of L_-, and L_+ has exactly one negative eigenvalue. The situation is reversed in the defocusing case $\sigma > 0$. More precisely $L_+ > L_-$, L_+ has the number of strictly positive eigenvalues increased by one compared to $-\Delta + V + E_0$, and if $-E_0$ is the lowest eigenvalue of $-\Delta + V$ then the lowest eigenvalue of L_+ is strictly positive. In both cases $L_\pm$ are relatively compact perturbations of $-\Delta + E$ hence via Weyl's Theorem their essential spectrum is the $[E, \infty)$ interval.*

Of particular interest are the ground states of (1.3):

Definition 2.1. *Solutions (ψ, E) of* (1.3) *which can written as $\psi = e^{i\theta}\tilde{\psi}$ with $\tilde{\psi} > 0$ and some $\theta \in [0, 2\pi)$ will be called ground state.*

Linearization at ground states has the additional spectral property:

Remark 2.4. *Standard properties of second order, self adjoint elliptic operators show that 0 must be the lowest e-value of $L_-(\tilde{\psi}, E)$ as it corresponds, see* Remark 2.2, *to the positive eigenfunction $\tilde{\psi}$. Moreover, it must be a simple e-value.*

In fact small amplitude ground states emerge from the lowest e-value of the linearization at zero $-\Delta + V$:

Corollary 2.1. *The C^1 manifold of zeroes of F i.e., solutions of* (1.3)*, bifurcating from* $(0, E_0)$ *where $-E_0$ is the lowest e-value of $-\Delta + V$ is made of ground states.*

Proof. Standard properties of the second order, self adjoint, elliptic operator $-\Delta+V$ on L^2 implies that its lowest e-value is simple with corresponding real eigenfunction ψ_0 which does not change sign. The existence of zeroes of F in a neighborhood of $(0, E_0)$ follows from Proposition 2.2 and we know that they form a unique, two dimensional, C^1 manifold obtained via rotations of a curve of real valued solutions. We first restrict our analysis to this curve given by part (i) in Proposition 2.2. Note that for $L_-(0, E_0) = -\Delta + V + E_0$ zero is the lowest e-value. By Remark 2.3 it will remain the lowest at least on a small neighborhood of $(0, E_0)$ in this curve. If a second eigenvalue of L_- crosses zero on a point on this curve then L_- will have its lowest e-value at least of multiplicity two in contradiction with L_- being a second order, elliptic, self adjoint operator. Hence, zero remains the lowest eigenvalue of $L_-(\psi_E, E)$ at all points (ψ_E, E) on this curve, and, consequently, its real valued eigenfunction ψ_E *cannot change sign.* Therefore, at each point (ψ_E, E) on this curve we have either $\psi_E > 0$ or $e^{i\pi}\psi_E = -\psi_E > 0$ Using now part (ii) of Proposition 2.2 we get that any solution in a neighborhood of $(0, E_0)$ can be rotated into a positive one hence it is a ground state. $\square$

Even though the kernel of $D_\psi F(\psi_E, E)$ is nontrivial along manifolds of solutions, see Remark 2.2, the next two theorems show that, by moding out rotations, the curves of real valued solutions can be uniquely continued until L_+ does not have a continuous inverse. Moreover, all solutions are rotations of this curve until the kernel of L_- is at least two dimensional.

2.2 Arbitrary branches of solutions and their maximal extension

Theorem 2.1. *If (ψ_{E_1}, E_1) is a nonzero, real valued, H^2 solution of* (1.3) *with the properties that L_+ is invertible with continuous inverse at this solution, then there exist $\epsilon,\ \delta > 0$ such that:*

(i) for each $E \in (E_1 - \delta, E_1 + \delta)$, (1.3) *has a unique real valued solution ψ_E satisfying $\|\psi_E - \psi_{E_1}\|_{H^2} < \epsilon$;*

(ii) if in addition L_- at (ψ_{E_1}, E_1) has kernel=span$\{\psi_{E_1}\}$ then any solution (ψ, E) of (1.3) *satisfying $|E - E_1| < \delta$ and $\min_{0\le\theta<2\pi} \|\psi - e^{i\theta}\psi_{E_1}\|_{H^2} < \epsilon$ is of the form $(\psi, E) = (e^{i\theta}\psi_E, E)$ where E and ψ_E are given in (i) and $0 \le \theta < 2\pi$.*

Moreover the function $E \mapsto \psi_E$ is C^1 from $(E_1 - \delta, E_1 + \delta)$ to H^2.

Proof. By the symmetries of F, see remark 2.1, we have that $F(\psi, E)$ is real valued if ψ is real valued. We can now apply the implicit function theorem for

$$F(\psi, E) = 0$$

at (ψ_{E_1}, E_1) where F is restricted to real valued functions: $F : H^2(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n, \mathbb{R})$. Note that under this restriction:

$$D_\psi F(\psi_{E_1}, E_1) = L_+,$$

see (2.3), hence, by hypothesis, it is an isomorphism. Implicit function theorem now implies (i).

For (ii) we remark that if (ψ, E) is a solution of $F(\psi, E) = 0$ then so are $(e^{-i\theta}\psi, E)$, $0 \leq \theta < 2\pi$. If we choose θ_0 such that it minimizes $\|\psi - e^{i\theta}\psi_{E_1}\|^2_{H^2}$, then

$$\tilde{\psi} = e^{-i\theta_0}\psi - \psi_{E_1}$$

is orthogonal to $i\psi_{E_1}$ in H^2 and satisfies:

$$F(\psi_{E_1} + \tilde{\psi}, E) = 0$$

in particular $\tilde{\psi}$ solves:

$$P_\perp F(\psi_{E_1} + h, E) = 0, \qquad h \in \{i\psi_{E_1}\}^{\perp_{H^2}}.$$

where $P_\perp$ is the projection onto the space orthogonal to $i\psi_{E_1}$ in L^2. But, via implicit function theorem, the latter has a unique branch of solutions near (ψ_{E_1}, E_1), and, by part (i) $(h = \psi_E - \psi_{E_1}, E)$ is such a branch. Uniqueness of the branch implies $\tilde{\psi} = \psi_E - \psi_{E_1}$ or, equivalently, $\psi = e^{i\theta_0}\psi_E$. □

Remark 2.5. *As in Remark 2.3 the linearized operators $L_\pm$ along the curve $E \mapsto \psi_E$ given by Theorem 2.1,see* (2.3)*, depend continuously on E and are relatively compact perturbations of $-\Delta + E$. Hence their discrete spectrum depends continuously on E and the essential spectrum is $[E, \infty)$. In particular, along this curve* 0 *is not in the spectrum of L_+ and, in case (ii),* 0 *remains a simple eigenvalue of L_-.*

We can actually continue the curve given by the previous theorem to a maximal one:

Theorem 2.2. *(i) If $\sigma < 0$, then the curve (ψ_E, E) of real valued solutions of* (1.3) *given in Theorem 2.1 can be uniquely continued to a maximal interval (E_-, E_+) such that either:*

(a+) $E_+ = \infty$;

or

(b+) $E_+ < \infty$ *and there exists a sequence $\{E_m\}_{m\in\mathbb{N}} \subset (E_1, E_+)$ such that* $\lim_{m\to\infty} E_m = E_+$ *and* $L_+(\psi_{E_m}, E_m)$ *has an eigenvalue λ_m satisfying* $\lim_{m\to\infty} \lambda_m = 0$*;*

and

(a-) $E_- = 0$;

or

(b-) $E_- > 0$ *and there exists a sequence $\{E_m\}_{m\in\mathbb{N}} \subset (E_-, E_1)$ such that* $\lim_{m\to\infty} E_m = E_-$ *and* $L_+(\psi_{E_m}, E_m)$ *has an eigenvalue λ_m satisfying* $\lim_{m\to\infty} \lambda_m = 0$*;*

(ii) If $\sigma > 0$, then the curve (ψ_E, E) of real valued solutions of (1.3) *given in Theorem 2.1 can be uniquely continued to a maximal interval (E_-, E_+) such that either:*

(a-) $E_- = 0$;

or

(b-) $0 < E_-$, *and there exists a sequence* $\{E_m\}_{m\in\mathbb{N}} \subset (E_-, E_1)$ *such that* $\lim_{m\to\infty} E_m = E_-$ *and* $L_+(\psi_{E_m}, E_m)$ *has an eigenvalue* λ_m *satisfying* $\lim_{m\to\infty} \lambda_m = 0$*;*

and

(b+) $0 < E_+$, *and there exists a sequence* $\{E_m\}_{m\in\mathbb{N}} \subset (E_1, E_+)$ *such that* $\lim_{m\to\infty} E_m = E_+$ *and* $L_+(\psi_{E_m}, E_m)$ *has an eigenvalue* λ_n *satisfying* $\lim_{m\to\infty} \lambda_m = 0$.

(iii) In both cases $\sigma < 0$, *or* $\sigma > 0$, *if* (ψ_E, E) *is a real valued solution on such a maximal curve with* $\ker L_- = \mathrm{span}\{\psi_E\}$ *then there exist* $\epsilon,\ \delta > 0$ *such that any solution* (ψ, E_*) *of* (1.3) *satisfying* $|E_* - E| < \delta$ *and* $\min_{0\le\theta<2\pi} \|\psi - e^{i\theta}\psi_E\|_{H^2} < \epsilon$ *is of the form* $(\psi, E_*) = (e^{i\theta}\psi_{E_*}, E_*)$ *where* (ψ_{E_*}, E_*) *is on the same maximal curve as* (ψ_E, E) *and* $0 \le \theta < 2\pi$.

Proof. Consider first the case $\sigma < 0$. Define:

$$\begin{aligned} E_+ \;=\; & \sup\{\tilde E : \quad \tilde E > E_1,\ E \mapsto \psi_E \text{ is a } C^1 \text{ extension on } [E_1, \tilde E) \text{ of the curve in Theorem 2.1,} \\ & \text{for which 0 is not in the spectrum of } L_+\} \end{aligned}$$

Theorem 2.1 and the Remark following it guarantees that the set above is not empty. Assume neither (a+) nor (b+) hold for $E_+ > E_1$ defined above. Then for all $E \in [E_1, E_+)$, the spectrum of $L_+(\psi_E, E)$ (which is real valued since L_+ is self-adjoint) has no points in the interval $[-d, d]$ for some $0 < d < E_1$. Indeed, as discussed in Remark 2.5 the essential spectrum of L_+ at E is $[E, \infty)$, and if no $d > 0$ exists, there must be a sequence of eigenvalues λ_m for L_+ at $E_m \in [E_1, E_+)$ such that $\lim_{m\to\infty} \lambda_m = 0$. But $[E_1, E_+]$ is compact since (a+) does not hold, hence there exists a subsequence E_{m_k} of E_m converging to $E_2 \in [E_1, E_+]$. However, $E_2 \ne E_+$ because we assumed that (b+) does not hold, so $E_2 \in [E_1, E_+)$ and by continuous dependence of the eigenvalues of L_+ on $E \in [E_1, E_+)$ we get that 0 is an eigenvalue of L_+ at $E_2 < E_+$ which contradicts the choice of E_+.

Consequently $L_+^{-1} : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ is bounded with uniform bound $K = 1/d$ on $[E_1, E_+)$. Moreover, by differentiating (1.3) with respect to E we have:

$$L_+\partial_E\psi_E = -\psi_E \quad \Rightarrow \quad \partial_E\psi_E = -L_+^{-1}\psi_E, \quad E \in (E_0, E_+), \tag{2.8}$$

hence

$$\|\partial_E\psi_E\|_{L^2} \le K\|\psi_E\|_{L^2}, \quad E \in [E_1, E_+). \tag{2.9}$$

and, by Cauchy-Schwarz inequality:

$$\frac{d}{dE}\|\psi_E\|_{L^2}^2 = 2\langle\partial_E\psi_E, \psi_E\rangle \le 2K\|\psi_E\|_{L^2}^2, \quad E \in [E_1, E_+).$$

The latter implies

$$\|\psi_E\|_{L^2}^2 \le \|\psi_{E_1}\|_{L^2}^2 e^{2K(E-E_1)}, \quad E \in [E_1, E_+), \tag{2.10}$$

which combined with $E_+ < \infty$ and bound (2.9) gives that both $\partial_E\psi_E$ and ψ_E have uniformly bounded L^2 norms on $[E_1, E_+)$. By the Mean Value Theorem there exists $\psi_{E_+}(x) \in L^2(\mathbb{R})$ such that

$$\lim_{E \nearrow E_+} \|\psi_E - \psi_{E_+}\|_{L^2} = 0. \tag{2.11}$$

We claim that $\psi_{E_+}(\mathbb{R}^n) \in H^1(\mathbb{R}^n)$ is a weak solution of the stationary equation (1.3) with $E = E_+$. Indeed, consider the energy functional

$$\mathcal{E}(E) = \int_{\mathbb{R}^n} |\nabla\psi_E(x)|^2 dx + \int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 dx + \frac{\sigma}{p+1}\int_{\mathbb{R}^n} |\psi_E(x)|^{2p+2} dx. \tag{2.12}$$

Note that because, ψ_E is a weak solution of the stationary equation (1.3) for any $E \in [E_1, E_+)$ we have

$$\frac{d\mathcal{E}}{dE} = -2E\langle\psi_E, \partial_E\psi_E\rangle_{L^2} = -E\frac{d\|\psi_E\|^2_{L^2}}{dE} \tag{2.13}$$

which integrated from E_1 to $E < E_+$ and using one integration by parts gives:

$$\mathcal{E}(E) - \mathcal{E}(E_1) = E_1\|\psi_{E_1}\|^2_{L^2} - E\|\psi_E\|^2_{L^2} + \int_{E_1}^{E} \|\psi_E\|^2_{L^2} dE. \tag{2.14}$$

Hence, from the uniform bounds for $E \in [E_1, E_+)$ and $\|\psi_E\|_{L^2},\ E \in [E_1, E_+)$ see (2.10), the energy $\mathcal{E}(E)$, is uniformly bounded on $[E_1, E_+)$. On the other hand, from the weak formulation of solutions of (1.3), we get

$$\|\nabla\psi_E\|^2_{L^2} + \int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 dx + \sigma\|\psi_E\|^{2p+2}_{L^{2p+2}} + E\|\psi_E\|^2_{L^2} = 0 \tag{2.15}$$

Subtracting the latter from $(p+1)\mathcal{E}(E)$ we get that there exists an $M > 0$ such that

$$\left| p\|\nabla\psi_E\|^2_{L^2} + p\int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 dx \right| \le M, \qquad \text{for all } E \in [E_1, E_+). \tag{2.16}$$

From Hölder inequality we obtain:

$$\left|\int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 dx\right| \le \|V\|_{L^\infty}\|\psi_E\|^2_{L^2}.$$

Using the inequality in (2.16) we deduce that $\|\nabla\psi_E\|_{L^2}$ has to be uniformly bounded. Consequently, there exists $M > 0$ such that

$$\|\psi_E\|_{H^1} \le M, \qquad \text{for all } E \in [E_1, E_+). \tag{2.17}$$

Because of the embedding of $H^1(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, $2 \le q \le 2n/(n-2)$, and the Riesz-Thorin interpolation

$$\|f\|_{L^q} \le \|f\|^{1+n/q-n/2}_{L^2}\|f\|^{n/2-n/q}_{L^{2n/(n-2)}}, \quad 2 \le q \le 2n/(n-2),$$

bound (2.17) together with convergence (2.11) imply that as $E \nearrow E_+$ we have:

$$\left.\begin{array}{ll} \psi_E \to \psi_{E_+}, & \text{in } L^2(\mathbb{R}^n) \\ \psi_E \rightharpoonup \psi_{E_+}, & \text{in } H^1(\mathbb{R}^n) \end{array}\right\} \quad \Rightarrow \quad \psi_E \to \psi_{E_+}, \quad \text{in } L^q(\mathbb{R}^n),\ 2 \le q < 2n/(n-2).$$

Now, by passing to the limit in the weak formulation of the stationary equation (1.3), we conclude that $\psi_{E_+}(x) \in H^1(\mathbb{R}^n)$ is a weak solution and hence also a strong H^2 solution, see Remark 1.1. Moreover, the linearized operator L_+ depends continuously on E on the interval $[E_1, E_+]$. By the standard perturbation theory, the discrete spectrum of L_+ depends continuously on E. Since 0 is not in the spectrum of L_+ for $E \in [E_1, E_+)$ and we assumed that (b+) does not hold we deduce that 0 is not an eigenvalue of L_+ at E_+. Moreover, since the essential spectrum of L_+ is $[E, \infty)$, 0 is not in the spectrum of L_+ at $E = E_+$. Applying now Theorem 2.1 at $(\psi_{E_+},\ E_+)$ we can continue the C^1 branch (ψ_E, E) past $E = E_+$ which contradicts the choice of E_+.

The dichotomy (a+) - (b+) is now proven. The above argument holds for extensions to the left of E_1 with the only modification that E_- is defined as:

$$\begin{aligned} E_- \quad = \quad & \inf\{\tilde{E} : \quad \tilde{E} < E_1,\ E \mapsto \psi_E \text{ is a } C^1 \text{ extension on } (\tilde{E}, E_1] \text{ of the curve in Theorem 2.1,} \\ & \text{for which 0 is not in the spectrum of } L_+\} \end{aligned}$$

the uniform estimates are obtained on an interval of the form $(E_-, E_1]$ and we study convergence as $E \searrow E_-$. Obviously $E_- \geq 0$ because the essential spectrum of L_+ at (ψ_E, E) is $[E, \infty)$ and 0 would be in the spectrum of L_+ for $E \leq 0$.

(ii) The proof for $\sigma > 0$ is analogous to the one for $\sigma < 0$. Part (iii) is a direct consequence of part (ii) in Theorem 2.1. The theorem is now completely proven. $\square$

Since the spectral assumption in part (iii) of Theorem 2.2 is satisfied for ground states, we have the following corollary:

Corollary 2.2. *(i) If $\sigma < 0$, then the curve (ψ_E, E) of solutions of* (1.3) *given in Theorem 2.2, which bifurcates from the lowest eigenvalue $-E_0$ of $-\Delta + V$, can be uniquely continued to a maximal interval (E_0, E_+) such that either:*

(a) $E_+ = \infty$;

or

(b) $E_+ < \infty$ *and there exists a sequence* $\{E_m\}_{m\in\mathbb{N}} \subset (E_0, E_+)$ *such that* $\lim_{m\to\infty} E_m = E_+$ *and* $L_+(\psi_{E_m}, E_m)$ *has an eigenvalue* λ_m *satisfying* $\lim_{m\to\infty} \lambda_m = 0$.

(ii) If $\sigma > 0$, then the curve (ψ_E, E) of solutions of (1.3) *given in Theorem 2.2 which bifurcates from the lowest eigenvalue $-E_0$ of $-\Delta + V$, can be uniquely continued to a maximal interval (E_-, E_0) such that either:*

(a) $E_- = 0$;

or

(b) $0 < E_- < E_0$ *and there exists a sequence* $\{E_m\}_{m\in\mathbb{N}} \subset (E_-, E_0)$ *such that* $\lim_{m\to\infty} E_m = E_-$ *and* $L_+(\psi_{E_m}, E_m)$ *has an eigenvalue* λ_m *satisfying* $\lim_{m\to\infty} \lambda_m = 0$.

(iii) In both cases $\sigma < 0$, or $\sigma > 0$, if (ψ_E, E) is a real valued solution on such a maximal curve then there exists $\epsilon,\ \delta > 0$ such that any solution (ψ, E_*) of (1.3) *satisfying $|E_* - E| < \delta$ and* $\min_{0\leq\theta<2\pi} \|\psi - e^{i\theta}\psi_E\|_{H^2} < \epsilon$ *is of the form $(\psi, E_*) = (e^{i\theta}\psi_{E_*}, E_*)$ where (ψ_{E_*}, E_*) is on the same maximal curve as (ψ_E, E) and $0 \leq \theta < 2\pi$.*

Proof. Fix $E_1 \in \mathcal{I}$ given by Proposition 2.2. Parts (i) and (ii) follow directly from Theorem 2.2 applied to the curve of solutions of (1.3) starting from (ψ_{E_1}, E_1). Part (iii) uses the fact that the lowest eigenvalue of L_- must be simple according to the general theory of self adjoint, second order elliptic operators. Note that 0 is always eigenvalue of L_- along the maximal curve of solutions, and it is the lowest one on $(E_0, E_1]$, see Remark 2.3. If 0 is not the lowest eigenvalue of L_- at some $E_2,\ E_1 < E_2 < E_+$ then, from continuous dependence of the discrete spectrum of L_- on E, we deduce that there is $E_3,\ E_1 < E_3 < E_2$, such that 0 is the lowest eigenvalue of L_- at E_3 and has multiplicity at least two, contradiction. So 0 is a simple eigenvalue of L_- on the entire maximal curve, and part (iii) follows from part (iii) in Theorem 2.2. $\square$

Remark 2.6. *Corollary 2.2 is also valid for the (excited state) manifolds bifurcating from the second or higher negative eigenvalues of $-\Delta + V$ provided we are in one space dimension $n = 1$. The above proof needs no changes because Sturm-Liouville Theory implies that all eigenvalues of $-\Delta+V$ and $L_\pm$ are simple. In particular Theorem 2.2 applies to all negative eigenvalues of $-\Delta+V$ and the kernel of L_- is always one dimensional.*

Note that along this maximal curves of solutions the number of negative eigenvalues of L_+ does not change. Using this information and the following proposition based on comparison principles for the linearized operators, the estimates for the end points of the maximal curves of solutions can be improved:

Proposition 2.3. *(i) If $\sigma < 0$ and $(\psi_E, E),\ E > 0$, is a nontrivial, real valued solution of (1.3), then, at this solution, L_+ has $k \geq 1$ negative eigenvalues counting multiplicity and $E \geq E_{k-1}$, where $-E_{k-1} < 0$ is the k^{th} discrete eigenvalue of $-\Delta + V$ counting multiplicity. If the latter does not exist $-E_{k-1}$ is replaced with zero, the edge of the essential spectrum.*

(ii) If $\sigma > 0$ and $(\psi_E, E),\ E > 0$, is a nontrivial, real valued solution of (1.3) then the number $k \geq 0$ of negative eigenvalues of L_+ counting multiplicity is strictly less than the number of negative eigenvalues of $-\Delta + V$ counting multiplicity and $E \leq E_k$, where $-E_k < 0$ is the $(k+1)^{th}$ discrete eigenvalue of $-\Delta + V$.

The proposition immediately implies:

Corollary 2.3. *(i) If $\sigma < 0$ and $(\psi_E, E),\ E \in (E_-, E_+)$, is a maximal curve of real valued solutions of (1.3), then, along it, L_+ has $k \geq 1$ negative eigenvalues counting multiplicity and $E_- \geq E_{k-1}$, where $-E_{k-1} < 0$ is the k^{th} discrete eigenvalue of $-\Delta + V$ counting multiplicity. If the latter does not exist $-E_{k-1}$ is replaced with zero, the edge of the essential spectrum.*

(ii) If $\sigma > 0$ and $(\psi_E, E),\ E \in (E_-, E_+)$, is a maximal curve of real valued solutions of (1.3) then the number $k \geq 0$ of negative eigenvalues of L_+ counting multiplicity is less than the number of negative eigenvalues of $-\Delta + V$ counting multiplicity and $E_+ \leq E_k$, where $-E_k < 0$ is the $(k+1)^{th}$ discrete eigenvalue of $-\Delta + V$.

Proof. Indeed, for each $(\psi_E, E),\ E \in (E_-, E_+)$, on the curve we have by Proposition 2.3 and the remark above it that in case (i) $E_{k-1} \leq E$, while in case (ii) $E_k \geq E$. Consequently $E_{k-1} \leq E_-$ in case (i), while in case (ii) $E_k \geq E_+$. $\square$

We now prove Proposition 2.3

Proof. In case (i) we first have

$$\langle L_+\psi_E, \psi_E\rangle = \langle L_-\psi_E, \psi_E\rangle + 2p\sigma\|\psi_E\|_{L^{2p+2}}^{2p+2} = 2p\sigma\|\psi_E\|_{L^{2p+2}}^{2p+2} < 0$$

hence the lowest eigenvalue of L_+ :

$$\lambda_0 = \inf_{\|\phi\|_{L^2}=1} \langle L_+\phi, \phi\rangle < 0.$$

Then we use use $L_+ \leq L_-$ to deduce that zero is at most the k^{th} eigenvalue of L_- counting multiplicity. Let ϕ be the linear combination of the first k eigenvectors of $-\Delta + V$ which is orthogonal on each of the first $k-1$ eigenvectors of L_-. Then we have:

$$0 \leq \langle L_-\phi, \phi\rangle \leq \langle(-\Delta + V + E)\phi, \phi\rangle \leq (-E_{k-1} + E)\|\phi\|_{L^2}^2$$

which implies $E_{k-1} \leq E$.

For part (ii) we use $L_+ \geq L_-$ to deduce that zero is at least the $(k+1)^{th}$ eigenvalue of $L_- \geq -\Delta + V + E$. In particular $-\Delta + V$ has at least $k+1$ negative eigenvalue since $E > 0$. Moreover, if ϕ is the linear combination of the first $k+1$ eigenvectors of L_- which is orthogonal on each of the first k eigenvectors of $-\Delta + V$, then we have:

$$0 \geq \langle L_-\phi, \phi\rangle \geq \langle(-\Delta + V + E)\phi, \phi\rangle \geq (-E_k + E)\|\phi\|_{L^2}^2$$

which implies $E_k \geq E$. □

Remark 2.7. *Since along ground state branches zero is the lowest eigenvalue of L_-, the above argument shows that, in the focusing case $\sigma < 0$, their left endpoint E_- always satisfies $E_- \geq E_0$, where E_0 is the lowest eigenvalue of $-\Delta + V$, while in the defocusing case $\sigma > 0$ their right endpoint E_+ always satisfies $E_+ \leq E_0$.*

3 Bifurcations at the endpoints of maximal branches

In this section we first employ differential inequalities involving the terms in the energy to show that if (ψ_E, E) is a C^1 branch of solutions of (1.3) then $\|\psi_E\|_{H^2}$ is bounded as long as E does not approach infinity or zero. In other words, such branches have no limit points at the $\{\|\cdot\|_{H^2} = \infty, E_* > 0\}$ boundary of the Fredholm domain. This result already has important consequences for the maximal branches of bound states in the defocusing $\sigma > 0$ case which we discuss in Theorem 3.2 and Remark 3.1. For the focusing case, we employ concentration compactness and spectral properties of the linearized operator L_+ along the manifold of solutions to show, for the first time to our knowledge, that the ground state maximal branches are compact. In other words, as E approaches one of the endpoints, say E_*, ψ_E has at least one limit point ψ_{E_*} in H^2 and (ψ_{E_*}, E_*) is also a solution of (1.3). This is essential in "continuing" the branch past the bifurcation point (ψ_{E_*}, E_*). For example, if the nonlinearity is real analytic (p is an integer) then the limit point is unique and the branch has a unique analytic continuation, see [3, Corollary 7.5.3]. For non-analytic nonlinearities we can obtain similar results when we have certain information on the eigenvalue(s) crossing zero, see the examples in Section 6. Eventually, we will combine the results of this section with the ones analyzing branches with one end point at infinity, see next section, to obtain information on the global bifurcation properties of the ground state branches.

Theorem 3.1. *Consider a C^1 curve (ψ_E, E), $E \in (E_-, E_+)$, of solutions of* (1.3) *(for example the ones given by Theorem 2.2). Assume that $0 < E_- < E_+ < \infty$, then: $\|\psi_E\|_{L^2}$, $\|\psi_E\|_{H^1}$ and $\|\psi_E\|_{H^2}$ are uniformly bounded on (E_-, E_+).*

Proof. Since $E \mapsto \psi_E$ is a C^1 map from an interval to the normed space H^2, we get that $\|\psi_E\|_{H^2}$ is bounded on compact subintervals of (E_-, E_+). It remains to get bounds as E approaches one of the endpoints $E_-,\ E_+$.

Consider the energy (2.12):

$$\mathcal{E}(E) = \int_{\mathbb{R}^n} |\nabla\psi_E(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 \,\mathrm{d}x + \frac{\sigma}{p+1}\int_{\mathbb{R}^n} |\psi_E(x)|^{2p+2} \,\mathrm{d}x,$$

and mass functional:

$$\mathcal{N}(E) = \|\psi_E\|^2_{L^2}, \tag{3.1}$$

along this curve. Then from (2.13) we have

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}E} = -E\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}E} \tag{3.2}$$

and from (2.15) we have

$$\|\nabla\psi_E\|^2_{L^2} + \int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 \mathrm{d}x + \sigma\|\psi_E\|^{2p+2}_{L^{2p+2}} + E\|\psi_E\|^2_{L^2} = 0, \tag{3.3}$$

which can be rewritten as

$$\mathcal{E}(E) + \frac{\sigma p}{p+1}\|\psi_E\|^{2p+2}_{L^{2p+2}} = -E\mathcal{N}(E).$$

Differentiating the above equation with respect to E gives

$$\frac{dQ}{dE}(E) = \frac{\mathrm{d}}{\mathrm{d}E}\|\psi_E\|^{2p+2}_{L^{2p+2}} = \frac{p+1}{-\sigma p}\mathcal{N}(E), \tag{3.4}$$

where

$$Q(E) = \|\psi_E\|^{2p+2}_{L^{2p+2}}.$$

We first show that $\mathcal{N}(E) = \|\psi_E\|^2_{L^2}$ is uniformly bounded in (E_-, E_+).

For focussing $\sigma < 0$ nonlinearity, it follows from the above differential equation that $Q(E)$ i.e., the L^{2p+2} norm of the solutions, is increasing with E and obviously bounded below by zero. Consequently it is uniformly bounded on intervals $(E_-, \tilde{E}]$, $\tilde{E} < E_+$. If we assume:

$$\limsup_{E \to E_-} \mathcal{N}(E) = \infty,$$

we have at least on a sequence $E_m \searrow E_-$ as $m \to \infty$ that

$$\lim_{m\to\infty} \frac{Q(E_m)}{\mathcal{N}(E_m)} = 0.$$

Then for

$$\Psi_m = \frac{\psi_{E_m}}{\|\psi_{E_m}\|_{L^2}}$$

we have by plugging in (3.3) and dividing by $\|\psi_{E_m}\|_{L^2}^2 = \mathcal{N}(E_m)$:

$$\lim_{m\to\infty}\left(\|\nabla\Psi_m\|_{L^2}^2 + \int_{\mathbb{R}^n} V(x)|\Psi_m(x)|^2dx\right) = -E_-$$

Since $\|\Psi_m\|_{L^{2p+2}} \to 0$, it can be shown that $\lim_{m\to\infty}\int_{\mathbb{R}^n} V(x)|\Psi_m(x)|^2dx = 0$ Therefore:

$$\lim_{m\to\infty}\|\nabla\Psi_m\|_{L^2}^2 = -E_- < 0$$

contradiction. So $\mathcal{N}(E) = \|\psi_E\|_{L^2}^2$ is uniformly bounded in a neighborhood of E_-. At the E_+ endpoint we first assume

$$\liminf_{E\to E_+}\frac{Q(E)}{\mathcal{N}(E)} > 0 \quad\Leftrightarrow\quad \exists E_- < \tilde{E} < E_+,\ C > 0 :\ Q(E) > C\mathcal{N}(E)\ \forall E\in[\tilde{E}, E_+).$$

Plugging the inequality in (3.4) and integrating we get:

$$Q(E) \le Q(\tilde{E})e^{\frac{p+1}{-\sigma pC}(E-\tilde{E})}, \qquad \forall E\in[\tilde{E}, E_+),$$

which again shows $Q(E)$ is uniformly bounded in a neighborhood of E_+ since $[\tilde{E}, E_+)$ has finite length. As above this will imply $\mathcal{N}(E) = \|\psi_E\|_{L^2}^2$ is uniformly bounded in a neighborhood of E_+. Now, if we assume

$$\liminf_{E\to E_+}\frac{Q(E)}{\mathcal{N}(E)} = 0 \quad\Leftrightarrow \exists E_m \nearrow E_+ :\ \lim_{m\to\infty}\frac{Q(E_m)}{\mathcal{N}(E_m)} = 0,$$

we have, in the case $\lim_{m\to\infty}\mathcal{N}(E_m) = 0$, that $\lim_{m\to\infty}Q(E_m) = 0$ which contradicts the fact that $Q(E)$ is an increasing function of E. In the case $\limsup_{m\to\infty}\mathcal{N}(E_m) > 0$ we get by possibly passing to a subsequence that for $\Psi_m = \frac{\psi_{E_m}}{\|\psi_{E_m}\|_{L^2}}$ we have $\|\Psi_m\|_{L^{2p+2}} \to 0$, hence $\lim_{m\to\infty}\int_{\mathbb{R}^n} V(x)|\Psi_m(x)|^2dx = 0$ and by plugging in (3.3) and dividing by $\|\psi_{E_m}\|_{L^2}^2 = \mathcal{N}(E_m)$ we get the contradiction

$$\lim_{m\to\infty}\|\nabla\Psi_m\|_{L^2}^2 = -E_+.$$

All in all, for $\sigma < 0$, $\mathcal{N}(E) = \|\psi_E\|_{L^2}^2$ is uniformly bounded in (E_-, E_+).

For defocussing $\sigma > 0$ nonlinearity, it follows from the differential equation (3.4) that $Q(E)$ i.e., the L^{2p+2} norm of the solutions, is decreasing with E and obviously bounded below by zero. Consequently it is uniformly bounded on intervals $[\tilde{E}, E_+),\ \tilde{E} > E_-$. Uniform bounds for $\mathcal{N}(E) = \|\psi_E\|_{L^2}^2$ at this endpoint follow as above. At the E_- endpoint we again split the analysis based on whether $\liminf_{E\to E_-}\frac{Q(E)}{\mathcal{N}(E)}$ is zero or not and with obvious modifications of the above argument we get uniform bounds on the L^2 norm.

Now, by integrating (3.2) and doing an integration by parts on the right hand side we get that uniform bounds for $\mathcal{N}(E),\ E\in(E_-, E_+)$ and the finite length of (E_-, E_+) imply uniform bounds for $\mathcal{E}(E),\ E\in(E_-, E_+)$. This together with (3.3) implies H^1 uniform bounds, see (2.14)-(2.17). Finally, standard regularity theory and H^1 uniform bounds implies H^2 uniform bounds, see Remark 7.1 in the Appendix.

The theorem is now completely proven. $\square$

We first apply the above result to solutions (ψ_E, E) of (1.3) with $\sigma > 0$ (the defocusing case). From Corollary 2.3 we get $0 < E < E_0$, and the above result shows that the maximal branches remain bounded as long as E stays away from zero. Therefore, these branches can either be extended to $E_- = 0$ or they undergo a bifurcation at their left end $0 < E_- < E_0$. The statement can be made much more precise for the ground state branch:

Theorem 3.2. *If $\sigma > 0$, then the ground states (ψ_E, E) of (1.3) form a unique C^1 manifold, namely the one bifurcating from $(\psi = 0, E = E_0)$, see Proposition 2.2, where $-E_0$ is the lowest e-value of $-\Delta + V$. Moreover, this manifold can be uniquely extended to the maximal interval $E \in (0, E_0)$.*

Proof. Consider a nontrivial ground state solution $(\psi_E, E),\ E > 0$ of (1.3) and, if necessary, rotate it such that $\psi_E > 0$, see Definition 2.1. Consequently, zero is the lowest e-value of $L_-(\psi_E, E)$ and it is simple with corresponding eigenvector ψ_E. As for the spectrum of $L_+(\psi_E, E)$ we already know that the essential part is $[E, \infty)$, see Remark 2.2, and, for its lowest e-value λ_0 with corresponding real valued, L^2 normalized eigenfunction ψ_0 we can apply the following comparison principle:

$$\lambda_0 = \langle \psi_0, L_+(\psi_E, E)\psi_0\rangle = \langle \psi_0, 2p\sigma|\psi_E|^{2p}\rangle + \langle \psi_0, L_-(\psi_E, E)\psi_0\rangle \geq 2p\sigma \int_{\mathbb{R}^n} |\psi_E|^{2p}|\psi_0|^2 dx > 0,$$

where we used

$$L_+(\psi_E, E) - L_-(\psi_E, E) = 2p\sigma|\psi_E|^{2p} > 0,$$

see (2.4)-(2.5), and $\langle \psi_0, L_-(\psi_E, E)\psi_0\rangle \geq 0$ due to the L^2 spectrum of $L_-(\psi_E, E)$ being included in $[0, \infty)$. Consequently, zero is not in the spectrum of $L_+(\psi_E, E)$ so this is an invertible operator with continuous inverse. We can then apply Theorems 2.1-2.2 to construct a maximal C^1 curve of ground states $E \mapsto \psi_E,\ E \in (E_-, E_+)$ passing through our initial ground state. Note that due to Theorem 2.2 and Corollary 2.3 we have

$$0 \leq E_- < E_+ \leq E_0, \tag{3.5}$$

where $-E_0$ is the lowest e-value of $-\Delta + V$, and that zero remains the lowest e-value of $L_-(\psi_E, E),\ E \in (E_-, E_+)$.

We claim that $E_- = 0$. Indeed, if we assume contrary we get, by Theorem 2.2 part (b-), that on any sequence (ψ_{E_n}, E_n) on the above curve such that $\lim_{n\to\infty} E_n = E_-$ we have

$$\lim_{n\to\infty} \lambda_n = 0 \text{ where } \lambda_n \text{ is the lowest e-value of } L_+(\psi_{E_n}, E_n). \tag{3.6}$$

By Theorem 3.1 the sequence $(\psi_{E_n})_{n\in\mathbb{N}}$ is bounded in H^2, hence, using the compactness result in the Appendix Corollary 7.1, there is a subsequence re-denoted by $(\psi_{E_n})_{n\in N}$ and $\psi_{E_-} \in H^2$ such that

$$\lim_{n\to\infty} \|\psi_{E_n} - \psi_{E_-}\|_{H^2} = 0, \qquad F(\psi_{E_-}, E_-) = 0.$$

Moreover, by continuity, zero will be the lowest e-value of $L_-(\psi_{E_-}, E_-)$ with corresponding eigenfunction ψ_{E_-} provided the latter is not the zero function. In this case ψ_{E_-} does not change sign and, by repeating the argument at the beginning of this proof, the lowest e-value of $L_+(\psi_{E_-}, E_-)$ must be strictly positive contradicting its continuity with respect to E, see (3.6). Therefore, $\psi_{E_-} = 0$ and $L_-(\psi_{E_-}, E_-) = -\Delta + V + E_-$ must have zero as its lowest e-value i.e., $E_- = E_0$, which contradicts (3.5).

It remains to show that $E_+ = E_0$ and

$$\lim_{E\to E_+} \|\psi_E\|_{H^2} = 0,$$

hence the maximal curve through our initial, arbitrary, ground state actually bifurcates from $(0, E_0)$. The fact that $\lim_{n\to\infty} \|\psi_{E_n}\|_{H^2} = 0$ on any sequence (ψ_{E_n}, E_n) on the above curve with $\lim_{n\to\infty} E_n = E_+$ follows exactly as in the previous paragraph by simply changing any minus index to a plus. Since the sequence is arbitrary we get

$$\lim_{E\to E_+} \|\psi_E\|_{H^2} = 0$$

But now, instead of a contradiction we have: $L_-(0, E_+) = -\Delta + V + E_+$ must have zero as its lowest e-value i.e., $E_+ = E_0$. This proves that the curve bifurcates from $(0, E_0)$.

The theorem is now completely proven. □

Remark 3.1. *Theorem 3.2 completely describes all ground states of* (1.3) *in the defocusing case* $\sigma > 0$. *It generalizes the result in [14] for this particular nonlinearity. We are currently extending Theorem 3.1 to general nonlinearities, see [20], in which case Theorem 3.2 will be valid for general, repelling type nonlinearities. For a discussion on the excited states of this problem, see Section 6.1.*

In what follows we focus on the attractive nonlinearity $\sigma < 0$. As in the proof of the previous theorem the existence of limit points at the end of the maximal branches plays an important role in our analysis. This compactness type result is not at all trivial, and we prove it for the first time to our knowledge in the following theorem. As opposed to the defocusing case, uniform decay as $|x| \to \infty$ for solutions $\psi_E(x)$ as E approaches the endpoints of the maximal branches is no longer guaranteed. We use concentration compactness and further assumptions on the behavior of the potential as $|x| \to \infty$ to prevent splitting of solutions into profiles that drift to infinity.

We will need the following lemma through out this paper. Recall the map $F(U, E) : H^2(\mathbb{R}^n)\times\mathbb{R} \mapsto L^2(\mathbb{R}^n)$ defined as

$$F(U, E) = (-\Delta + V + E)U + \sigma|U|^{2p}U$$

and restricted to real valued U. For any $U, v \in H^2(\mathbb{R}^n)$, we have

$$F(U + v, E) = F(U, E) + D_U F(U, E)v + N(U, v),$$

where $D_U F(U, E)$ is the Fréchet derivative of F with respect to U at (U, E) and the nonlinear function N has the following properties.

Lemma 3.1. *Consider the nonlinear operator* $N(U, v) : H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ *defined as*

$$N(U, v) = \sigma|U + v|^{2p}(U + v) - \sigma|U|^{2p}U - \sigma(2p + 1)|U|^{2p}v, \tag{3.7}$$

where $\sigma \in \mathbb{R}$ *and* $p \in (0, 2n/(n - 2))$. *Then there exists* $C > 0$ *such that*

$$\|N(U, v_1) - N(U, v_2)\|_{L^2} \le C\|v_1 - v_2\|_{H^2}^{2p+1} \quad \text{if} \quad p \le 1/2, \tag{3.8}$$

$$\|N(U, v_1) - N(U, v_2)\|_{L^2} \le C\|U\|_{H^2}^{2p-1}\|v_1 - v_2\|_{H^2}^2 + C\|v_1 - v_2\|_{H^2}^{2p+1} \quad \text{if} \quad p > 1/2. \tag{3.9}$$

Furthermore, for any bounded sequences $\{U_k\}_{k\in\mathbb{N}},\ \{v_k\}_{k\in\mathbb{N}} \subset H^2(\mathbb{R}^n)$ *with* $\lim_{k\to\infty}\|v_k\|_{L^q} = 0$ *for some* $2 < q < 2n/(n-2)$ *(*$2 < q < \infty$ *if* $n = 2$*,* $2 < q \le \infty$ *if* $n = 1$*), we have*

$$\lim_{k\to\infty} \|N(U_k, v_k)\|_{L^2} = 0. \tag{3.10}$$

Proof. For each $x \in \mathbb{R}^n$, using the mean value theorem, we get

$$N(U(x), v_1(x)) - N(U(x), v_2(x)) = \sigma(2p+1)|U+\tilde{v}|^{2p}(v_1(x)-v_2(x)) - \sigma(2p+1)|U|^{2p}(v_1(x)-v_2(x))$$

with $|\tilde{v}(x)| \le |v_1(x) - v_2(x)|$.

Suppose that $p \le 1/2$. Then we have

$$|U(x) + \tilde{v}(x)| \le (|\tilde{v}(x)|^{2p} + |U(x)|^{2p})^{1/2p}$$

and therefore for some $C > 0$ and all $x \in \mathbb{R}^n$,

$$|N(U(x), v_1(x)) - N(U(x), v_2(x))| \le C|v_1(x) - v_2(x)|^{2p+1}.$$

Therefore

$$\|N(U, v_1) - N(U, v_2)\|_{L^2} \le C\|v_1 - v_2\|^{2p+1}_{L^{4p+2}}, \tag{3.11}$$

which, using Sobolev embedding and the constraint $p \in (0, 2n/(n-2))$, gives (3.8). Now let $(U_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ be as in the lemma. Then $\lim_{k\to\infty}\|v_k\|_{L^q} = 0$ for some $2 < q < 2n/(n-4)$ ($2 < q < \infty$ if $n < 5$) and $\sup_{k\in\mathbb{N}}\|v_k\|_{L^{q'}} < \infty$ for every $2 < q' < 2n/(n-4)$ ($2 < q' < \infty$ if $n < 5$). Since $p \in (0, 2n/(n-2))$ implies that $2 < 4p+2 < 2n/(n-4)$ ($2 < 4p+2 < \infty$ if $n < 5$), we can apply the Riesz-Thorin theorem with the appropriate q' to interpolate at the intermediate space L^{4p+2} to get $\|v_k\|_{L^{4p+2}} \le \|v_k\|^{\theta}_{L^q}\|v_k\|^{1-\theta}_{q'}$ for some $\theta \in (0,1)$. Hence $\lim_{k\to\infty}\|v_k\|_{L^{4p+2}} = 0$. It now follows from (3.11) (with $U = U_k$, $v_1 = v_k$ and $v_2 = 0$) that (3.10) holds.

Suppose that $p > 1/2$. In this case, using the Taylor's formula with second order remainder for (3.7), we have

$$N(U(x), v_1(x)) - N(U(x), v_2(x)) = \sigma p(2p+1)|U + \tilde{v}|^{2p-1}(v_1(x) - v_2(x))^2$$

with $|\tilde{v}(x)| \le |v_1(x) - v_2(x)|$. In the region $|U(x)| \ge |v_1(x) - v_2(x)|$ we have

$$|N(U(x), v_1(x)) - N(U(x), v_2(x))| \le 2^{2p-1}|\sigma|p(2p+1)|U(x)|^{2p-1}|v_1(x) - v_2(x)|^2$$

and in the region $|U(x)| < |v_1(x) - v_2(x)|$ we have

$$|N(U(x), v_1(x)) - N(U(x), v_2(x))| \le 2^{2p-1}|\sigma|p(2p+1)|v_1(x) - v_2(x)|^{2p+1}.$$

From this we get that for all $x \in \mathbb{R}^n$

$$|N(U(x), v_1(x)) - N(U(x), v_2(x))| \le C\left[|U(x)|^{2p-1}|v_1(x) - v_2(x)|^2 + |v_1(x) - v_2(x)|^{2p+1}\right]$$

and therefore

$$\|N(U, v_1) - N(U, v_2)\|_{L^2} \le C\left(\int_{\mathbb{R}^n} |U(x)|^{4p-2}|v_1(x) - v_2(x)|^4 \mathrm{d}x\right)^{\frac{1}{2}} + C\|v_1 - v_2\|^{2p+1}_{L^{4p+2}}. \tag{3.12}$$

Since $p \in (0, 2n/(n-2))$, $p > 1/2$ implies that $n \leq 5$. In this case, there exist $\alpha, \alpha' > 1$ with $1/\alpha + 1/\alpha' = 1$ such that $2 \leq (4p-2)\alpha < 2n/(n-4)$ ($2 \leq (4p-2)\alpha < \infty$ if $n < 5$), $2 < 4\alpha' < 2n/(n-4)$ ($2 < 4\alpha' < \infty$ if $n < 5$). Indeed if $n = 1, 2, 3, 4$ let $\alpha = \max\{2, 1/(2p-1)\}$ and if $n = 5$ let $\max\{5/3, 1/(2p-1)\} < \alpha < 5/(2p-1)$. (Note that if $n = 5$, $p < 2/3$.) We then get using Hölder's inequality that

$$\int_{\mathbb{R}^n} |U(x)|^{4p-2} |v_1(x) - v_2(x)|^4 \mathrm{d}x \leq \|U\|^{4p-2}_{L^{(4p-2)\alpha}} \|v_1 - v_2\|^4_{L^{4\alpha'}} \tag{3.13}$$

which, along with (3.12), implies using Sobolev embedding that (3.9) holds i.e.,

$$\|N(U, v_1) - N(U, v_2)\|_{L^2} \leq C\|U\|^{2p-1}_{H^2} \|v_1 - v_2\|^2_{H^2} + C\|v_1 - v_2\|^{2p+1}_{H^2}.$$

Let $(U_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ be as in the lemma. Then like in the case of $p \leq 1/2$, we can apply the Riesz-Thorin theorem to interpolate at the intermediate spaces L^{4p+2} and $L^{4\alpha'}$ to conclude that $\lim_{k\to\infty} \|v_k\|_{L^{4p+2}} = 0$ and $\lim_{k\to\infty} \|v_k\|_{L^{4\alpha'}} = 0$. It now follows from (3.12) and (3.13) (with $U = U_k$, $v_1 = v_k$ and $v_2 = 0$) and our choice of α that (3.10) holds. □

For $p \geq 1/2$, it follows from the above lemma that for each $U, v_1, v_2 \in H^2(\mathbb{R}^n)$ satisfying $\|U\|_{H^2} \leq L$, $\|v_1\|_{H^2} \leq L$ and $\|v_2\|_{H^2} \leq L$ there exists $C(L) > 0$, independent of U, v_1 and v_2, such that

$$\|N(U, v_1) - N(U, v_2)\|_{L^2} \leq C(L)\|v_1 - v_2\|^2_{H^2}. \tag{3.14}$$

Now we present our compactness result.

Theorem 3.3. *Let $\sigma < 0$, $p \geq 1/2$. Assume that the potential $V \in W^{1,\infty}(\mathbb{R}^n)$ satisfies Hypothesis (H2). Furthermore, let the radial derivative $\partial_r V(x)$ of V and a set of its tangential derivatives $\{\partial_{s_i} V(x), i = 1, 2, \ldots n-1\}$, where $s_1, s_2, \ldots s_{n-1}$ are a set of mutually orthogonal directions which are also orthogonal to the radial direction, satisfy the following hypotheses: There exists a set $\mathcal{M}$ of measure zero such that*

$$\lim_{x\in\mathbb{R}^n\setminus\mathcal{M},\ |x|\to\infty} \frac{V^2(x)}{|\partial_r V(x)|} = 0 \tag{3.15}$$

and there exist constants $\mathcal{C}, \beta, \rho > 0$ such that for any $x \in \mathbb{R}^n \setminus \mathcal{M}$ with $|x| > \rho$

$$\mathcal{C}|\partial_r V(x)| > \frac{1}{|x|^\beta}, \tag{3.16}$$

$$\mathcal{C}|\partial_r V(x)| > \Big(\sum_{i=1}^{n-1} |\partial_{s_i} V(x)|^2\Big)^{\frac{1}{2}}. \tag{3.17}$$

Under the above assumptions, consider a set $\mathcal{G}$ of real-valued, positive ground states of (1.3)*, see Definition 2.1, bounded in $H^2(\mathbb{R}^n) \times (0, \infty)$ and with E bounded away from zero i.e., there exist $E_-, E_+, M > 0$, such that*

$$\mathcal{G} \subseteq \{(\psi, E) \in H^2 \times (0, \infty) \,\big|\, \|\psi\|_{H^2} < M, E \in [E_-, E_+], F(\psi, E) = 0, \psi > 0\}.$$

Then the set $\mathcal{G}$ is relatively compact in $H^2(\mathbb{R}^n) \times (0, \infty)$. Therefore for any sequence $(\psi_m, E_m)_{m=1}^\infty$ in $\mathcal{G}$, there exists a subsequence $(\psi_{m_k}, E_{m_k})_{k=1}^\infty$, an $E_* \in [E_-, E_+]$ and a function $\psi_{E_*} \in H^2(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} E_{m_k} = E_*, \qquad \lim_{k\to\infty} \|\psi_{E_{m_k}} - \psi_{E_*}\|_{H^2} = 0.$$

Furthermore (ψ_{E_*}, E_*) solves (1.3).

Before presenting a proof of Theorem 3.3, we will state and prove a corollary to this theorem.

Corollary 3.1. *Under the hypothesis on the potential V in Theorem 3.3, consider a maximal C^1 curve (ψ_E, E), $E \in (E_-, E_+)$ of real-valued, positive ground states of (1.3), see Definition 2.1, with $0 < E_- < E_+ < \infty$. Then for any sequence $(E_m)_{m=1}^\infty$ in (E_-, E_+) with $\lim_{m\to\infty} E_m = E_*$, there exists a subsequence $(E_{m_k})_{k=1}^\infty$ and a function $\psi_{E_*} \in H^2(\mathbb{R}^n)$ such that (ψ_{E_*}, E_*) solves (1.3) and*

$$\lim_{k\to\infty} \|\psi_{E_{m_k}} - \psi_{E_*}\|_{H^2} = 0.$$

Proof. From Theorem 3.1 we have $M = \sup_{E\in(E_-,E_+)} \|\psi_E\|_{H^2} < \infty$. Hence for any sequence $(E_m)_{m=1}^\infty$ in (E_-, E_+) with $\lim_{m\to\infty} E_m = E_*$, applying Theorem 3.3 to the set

$$\mathcal{G} = \{(\psi_{E_m}, E_m) \big| m \in \mathbb{N}\} \subset H^2(\mathbb{R}^n, \mathbb{C}) \times (0, \infty),$$

it follows that there exists a subsequence $(E_{m_k})_{k=1}^\infty$ and a function $\psi_{E_*} \in H^2(\mathbb{R}^n)$ such that (ψ_{E_*}, E_*) solves (1.3) and $\lim_{k\to\infty} \|\psi_{E_{m_k}} - \psi_{E_*}\|_{H^2} = 0$. □

We now present the proof of Theorem 3.3.

Proof. Consider a sequence $(\psi_{E_m}, E_m)_{m=1}^\infty$ in $\mathcal{G}$. Since $(\psi_{E_m})_{m=1}^\infty$ is bounded in $H^2(\mathbb{R}^n)$ and $(E_m)_{m=1}^\infty$ is bounded in $\mathbb{R}$, we can construct a subsequence, denoted again by $(\psi_m, E_m)_{m=1}^\infty$, such that $(\psi_{E_m})_{m=1}^\infty$ is weakly convergent in $H^1(\mathbb{R}^n)$ to a $\psi_* \in H^1(\mathbb{R}^n)$ and $\lim_{m\to\infty} E_m = E_*$. Proposition 7.3 in the Appendix completes the proof of the theorem provided we show that at least on a subsequence of $(\psi_{E_m})_{m\in\mathbb{N}}$ we have

$$\psi_{E_m} \xrightarrow{L^q} \tilde{\psi} \quad \text{for some} \quad 2 \le q < \frac{2n}{n-2} \ (2 \le q < \infty \ \text{ if } \ n = 2, \ 2 \le q \le \infty \ \text{ if } \ n = 1). \tag{3.18}$$

If $\liminf_{m\to\infty} \|\psi_{E_m}\|_{L^2} = 0$, then there is a subsequence of $(\psi_{E_m})_{m\in\mathbb{N}}$ converging to 0 in L^2 and hence (3.18) is satisfied with $q = 2$. If $\liminf_{m\to\infty} \|\psi_{E_m}\|_{L^2} = a > 0$, then consider a subsequence of $(E_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} \|\psi_{E_m}\|_{L^2} = a$ and

$$\Psi_m = \frac{\psi_{E_m}}{\|\psi_{E_m}\|_{L^2(\mathbb{R}^n)}} \ , \tag{3.19}$$

which is a sequence normalized in L^2 and bounded in H^1.

Next we will apply concentration compactness theory, see for example [4, Section 1.7] and our Appendix 7, to $(\Psi_m)_{m\in\mathbb{N}}$. Consider the concentration function $\rho(\phi, r)$ defined on $L^2(\mathbb{R}^n) \times [0, \infty)$ as

$$\rho(\phi, r) = \sup_{y\in\mathbb{R}^n} \int_{|x-y|<r} |\phi(x)|^2 \mathrm{d}x$$

and let

$$\mu = \lim_{r\to\infty} \liminf_{m\to\infty} \rho(\Psi_m, r).$$

Out of three possible cases in the concentration compactness theory we will show that vanishing a splitting cannot occur and that compactness implies (3.18) for a subsequence of Ψ_m, hence for ψ_{E_m}.

Vanishing ($\mu = 0$): In this case there is a subsequence Ψ_{m_k} of Ψ_m convergent to zero in $L^q(\mathbb{R}^n)$ for each $2 < q < 2n/(n-2)$ ($2 < q \le \infty$ if $n = 1$). Since

$$\psi_{E_{m_k}} = \|\psi_{E_{m_k}}\|_{L^2} \Psi_{m_k}$$

and $\|\psi_{E_{m_k}}\|_{L^2}$ is bounded, see Theorem 3.1, we get that

$$\psi_{E_{m_k}} \stackrel{L^q}{\to} 0.$$

Hence, by Lemma 7.3, we deduce that $\psi_{E_{m_k}}$ converges to 0 in $H^1(\mathbb{R}^n)$ in contradiction with

$$\lim_{k\to\infty} \|\psi_{E_{m_k}}\|_{L^2} = a > 0.$$

Hence vanishing cannot occur.

Compactness ($\mu = 1$): Since $\psi_{E_m} = \|\psi_{E_m}\|_{L^2}\Psi_m$ and $\lim_{m\to\infty} \|\psi_{E_m}\|_{L^2} = a > 0$, it follows from concentration compactness theory that if $\mu = 1$, then there exists a sequence $(y_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n$ and a $\hat{\psi}_* \in H^1(\mathbb{R}^n)$ such that on a subsequence of E_m (again denoted as E_m) we can decompose ψ_{E_m} as

$$\psi_{E_m} = \hat{\psi}_m(\cdot - y_m) + \hat{v}_m, \tag{3.20}$$

such that $\hat{\psi}_m, \hat{v}_m \in H^1(\mathbb{R}^n)$ and as $E_m \to E_*$ we have

(i) $\hat{\psi}_m \stackrel{H^1}{\rightharpoonup} \hat{\psi}_*$ and $\hat{\psi}_m \stackrel{L^q}{\to} \hat{\psi}_*$ for $2 \le q < 2n/(n-2)$ ($2 \le q < \infty$ if $n = 2$, $2 \le q \le \infty$ if $n = 1$),

(ii) $\hat{v}_m \stackrel{H^1}{\rightharpoonup} 0$ and $\hat{v}_m \stackrel{L^q}{\to} 0$ for $2 \le q < 2n/(n-2)$ ($2 \le q < \infty$ if $n = 2$, $2 \le q \le \infty$ if $n = 1$).

We will show that $(y_m)_{m\in\mathbb{N}}$ has a subsequence converging to some $y_* \in \mathbb{R}^n$. Therefore we can conclude that ψ_{E_m} converges weakly in H^1 to a ψ_* and satisfies (3.18) (at least on a subsequence) with $\psi_* = \hat{\psi}_*(\cdot - y_*)$. Now Proposition 7.3 finishes the proof of the theorem.

By contradiction assume that

$$\lim_{m\to\infty} |y_m| = \infty. \tag{3.21}$$

Using the translation operator $T_y u(x) = u(x-y)$, the equation (1.3) satisfied by the pair (ψ_{E_m}, E_m) can be translated to

$$(-\Delta + T_{-y_m}V + E_m)\, T_{-y_m}\psi_{E_m} + \sigma T_{-y_m}|\psi_{E_m}|^{2p}\psi_{E_m} = 0.$$

Taking the L^2 scalar product of the above equation with $\phi \in C_0^\infty(\mathbb{R}^n)$ we get

$$\langle T_{-y_m}\nabla\psi_{E_m}, \nabla\phi\rangle + \langle (T_{-y_m}V + E_m)T_{-y_m}\psi_{E_m}, \phi\rangle + \sigma\left\langle T_{-y_m}|\psi_{E_m}|^{2p}\psi_{E_m}, \phi\right\rangle = 0.$$

Substituting for ψ_{E_m} from (3.20) into the last equation and taking the limit as $E_m \to E_*$, we get using (3.21) that

$$\big\langle \nabla\hat\psi_*, \nabla\phi\big\rangle + \big\langle E_*\hat\psi_*, \phi\big\rangle + \sigma\big\langle |\hat\psi_*|^{2p}\hat\psi_*, \phi\big\rangle = 0.$$

Therefore $\hat\psi_*$ is a weak solution (and hence a strong solution) to the equation

$$(-\Delta + E)\psi + \sigma|\psi|^{2p}\psi = 0 \tag{3.22}$$

at $E = E_*$. Since $\psi_{E_m} > 0$ for all m, it follows using (3.20) that for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^n)$

$$0 \le \liminf_{m\to\infty}\langle \psi_{E_m}, T_{y_m}\phi\rangle \,=\, \langle\hat\psi_*, \phi\rangle + \liminf_{m\to\infty}\langle \hat v_m, T_{y_m}\phi\rangle \,=\, \langle\hat\psi_*, \phi\rangle\,,$$

which implies that $\hat\psi_*$ is a nonnegative function. Since every non-negative solution of (3.22) for any $E > 0$ is a translation of the unique non-negative radially symmetric solution U_E to (3.22) [23], and so U_E satisfies

$$(-\Delta + E)U_E + \sigma|U_E|^{2p}U_E = 0, \qquad U_E > 0, \tag{3.23}$$

it follows that $\hat\psi_* = T_{y_*}U_{E_*}$ for some $y_* \in \mathbb{R}^n$. By redefining $\hat v_m$ to be $\hat v_m - T_{y_m+y_*}U_{E_m} + T_{y_m}\hat\psi_m$ and then redefining y_m to be $y_m + y_*$ we obtain the following different decomposition for ψ_{E_m} in place of (3.20):

$$\psi_{E_m} = T_{y_m}U_m + \hat v_m, \qquad U_m = U_{E_m}. \tag{3.24}$$

Clearly, the redefined y_m satisfies (3.21) and the limits in (ii) continue to hold for the redefined $\hat v_m$.

We need the following properties of U_E in (3.23) in the sequel. From Proposition 7.2 in the Appendix we get that for every $\gamma \in (0, E)$, there exists a $C(\gamma, E) > 0$ such that

$$|U_E(x)| \le C(\gamma, E)e^{-\sqrt{E-\gamma}|x|}, \qquad |\nabla U_E(x)| \le C(\gamma, E)e^{-\sqrt{E-\gamma}|x|} \qquad \forall x \in \mathbb{R}^n. \tag{3.25}$$

Furthermore,

$$U_E(x) \,=\, (E)^{\frac{1}{2p}}\, U_1(\sqrt{E}x) \qquad \forall\, x \in \mathbb{R}^n\,, \qquad \forall\, E > 0\,, \tag{3.26}$$

where U_1 is the unique positive radially symmetric solution to (3.23) with $E = 1$.

Since (ψ_{E_m}, E_m) solves (1.3), by substituting (3.24) we obtain

$$(-\Delta + V + E_m)(T_{y_m}U_m + \hat v_m) + \sigma\,|T_{y_m}U_m + \hat v_m|^{2p}\,(T_{y_m}U_m + \hat v_m) = 0. \tag{3.27}$$

Define the linear operator $L_+(U, E) : H^2(\mathbb{R}^n) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n)$ as follows:

$$L_+(U, E)[v] = (-\Delta + V + E)v + \sigma(2p + 1)|U|^{2p}v. \tag{3.28}$$

Recall the nonlinear operator $N(U, v) : H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ from (3.7):

$$N(U, v) = \sigma|U + v|^{2p}(U + v) - \sigma|U|^{2p}U - \sigma(2p + 1)|U|^{2p}v.$$

We can rewrite (3.27) as

$$L_+(T_{y_m}U_m, E_m)[\hat v_m] + VT_{y_m}U_m + N(T_{y_m}U_m, \hat v_m) = 0.$$

From (3.27) we get that

$$\hat{v}_m = -(-\Delta + E_*)^{-1}\left[VT_{y_m}U_m + (E_m - E_* + V + \sigma(2p+1)|T_{y_m}U_m|^{2p})\hat{v}_m + N(T_{y_m}U_m, \hat{v}_m)\right]. \tag{3.29}$$

We will now show that

$$\lim_{m\to\infty} \|\hat{v}_m\|_{H^2} = 0. \tag{3.30}$$

Recall the limits $\hat{v}_m \overset{L^q}{\to} 0$ for $2 \le q < 2n/(n-2)$ ($2 \le q < \infty$ if $n = 2$, $2 \le q \le \infty$ if $n = 1$) as $m \to \infty$ from (ii). Note that from (3.25), (3.26) and $\lim_{m\to\infty} E_m = E_*$ we get that $U_m \in H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\|U_m\|_{H^2}$ and $\|U_m\|_{L^\infty}$ can be bounded by a constant independent of m. This, along with the limit $\lim_{m\to\infty} \|\hat{v}_m\|_{L^2} = 0$ in (ii), the limit $\lim_{m\to\infty} E_m = E_*$ and $V \in L^\infty(\mathbb{R}^n)$, implies that

$$\lim_{m\to\infty} \|(E_m - E_* + V + \sigma(2p+1)|T_{y_m}U_m|^{2p})\hat{v}_m\|_{L^2} = 0. \tag{3.31}$$

Also, using $\lim_{|x|\to\infty} V(x) = 0$, (3.21), (3.25), (3.26) and $\lim_{m\to\infty} E_m = E_*$ we get that

$$\lim_{m\to\infty} \|VT_{y_m}U_m\|_{L^2} \le \lim_{m\to\infty} \|VT_{y_m}(U_m - U_{E_*})\|_{L^2} + \lim_{m\to\infty} \|VT_{y_m}U_{E_*}\|_{L^2} = 0. \tag{3.32}$$

From Lemma 3.1 we get that

$$\lim_{m\to\infty} \|N(T_{y_m}U_m, \hat{v}_m)\|_{L^2} = 0, \tag{3.33}$$

The limit in (3.30) follows from (3.29) and the limits in (3.31), (3.32) and (3.33).

Note that the decomposition

$$\psi_{E_m} = U_{E_m}(\cdot - y_m) + \hat{v}_m$$

together with $\hat{v}_m \overset{H^2}{\to} 0$ and the assumptions $|y_m| \to \infty$, $E_m \to E_*$ imply that our solution bifurcates from U_{E_*} at $|x| = \infty$, see Figure 3.1. While this bifurcation point is inaccessible in the Banach spaces we are working with, we will use a related Lyapunov-Schmidt like decomposition. In fact $L_+(T_{y_m}U_m, E_m)$ is the linearization at the profiles $T_{y_m}U_m = U_{E_m}(\cdot - y_m)$. Each can be viewed as a Schrödinger operator with potential given by a sum of two potentials V and $(2p+1)\sigma|U_m(\cdot - y_m)|^{2p}$. Since $\lim_{m\to\infty} |y_m| = \infty$ the distance between the two potentials eventually becomes very large. From the spectral theory for operators with potentials separated by a large distance [25], as $|y_m| \to \infty$ every eigenvalue of $L_+(T_{y_m}U_m, E_m)$ tends to either an eigenvalue of $(-\Delta + V + E_m)$ or an eigenvalue of $-\Delta + E_m + (2p+1)\sigma|U_m|^{2p}$. The eigenvalues of the former operator are bounded away from zero by $E_m - E_0 > 0$ and those of the latter operator are also bounded away from zero if we remove the projection onto its kernel.

Therefore we now move to a decomposition with respect to the invariant subspaces of $-\Delta + E_m + \sigma(2p+1)T_{y_m}|U_m|^{2p}$. Recall that its kernel is span $\{T_{y_m}\partial_{x_k}U_m | k = 1, 2, \dots n\}$, which are infinitesimal generators of translations of U_m in $\mathbb{R}^n$. Hence the projection onto this subspace can be absorbed into translations of the profile U_m. This is established in the following lemma.

Lemma 3.2. *There exists a constant $\varepsilon > 0$ and a function $\mathcal{S}$ defined on the set of all $y \in \mathbb{R}^n$, all $E \in (E_0, E_*)$ and all $\psi \in H^1(\mathbb{R}^n)$ which satisfy the condition*

$$\|\psi - T_yU_E\|_{L^2} < \varepsilon,$$

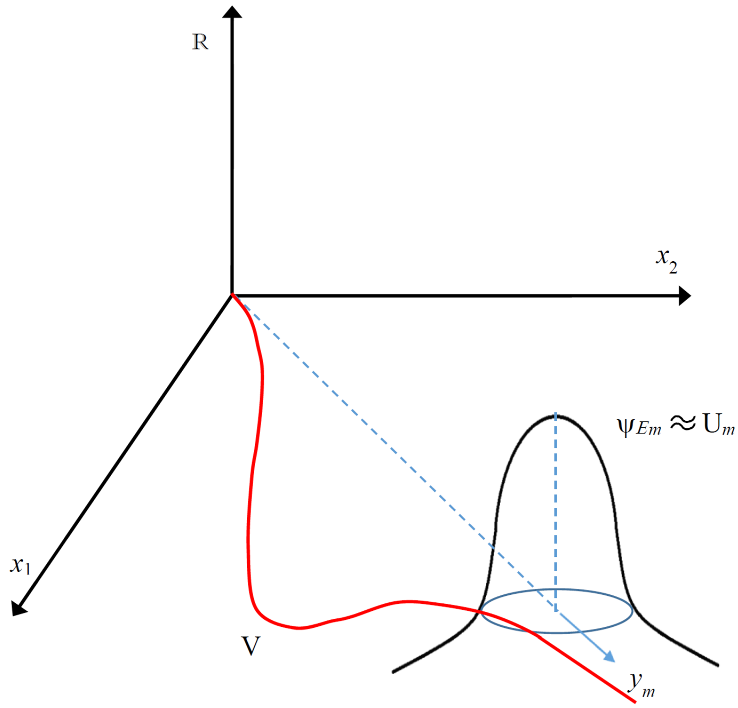

Figure 3.1: A schematic representation showing that a "drifting" ψ_{E_m} experiences only the effect of the tails of the potential as it bifurcates from U_{E_*} shifted to infinity.

such that using $s = \mathcal{S}(\psi, E, y) \in \mathbb{R}^n$, ψ can be decomposed as follows:

$$\psi = T_{y+s} U_E + v, \tag{3.34}$$

where v satisfies

$$\langle v, T_{y+s} \partial_{x_k} U_E \rangle = 0 \qquad \forall\, k \in \{1, 2, \ldots n\}.$$

In addition,

$$\lim_{\|\psi - T_y U_E\|_{L^2} \to 0} \mathcal{S}(\psi, E, y) = 0, \tag{3.35}$$

uniformly in y and E.

Proof. Define the map $\mathbf{F} : L^2(\mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ as follows:

$$\mathbf{F}(\psi, E, y, s) = [F_1(\psi, E, y, s), F_2(\psi, E, y, s), \ldots F_n(\psi, E, y, s)]^\top,$$

where

$$F_k(\psi, E, y, s) = \left\langle \frac{T_{y+s} \partial_{x_k} U_E}{\|\partial_{x_k} U_E\|_{L^2}^2}, \psi - T_{y+s} U_E \right\rangle_{L^2} \qquad \forall\, k \in \{1, 2, \ldots n\}.$$

We will establish the proposition by solving

$$\mathbf{F}(\psi, E, y, s) = 0 \tag{3.36}$$

for $s \in \mathbb{R}^n$, given ψ, E and y. Note that an s that solves (3.36) for a given ψ, y and E implies the decomposition in (3.34). The function $\mathbf{F}$ is C^1 on $L^2(\mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. Its *Fréchet* derivative with respect to s, denoted as $D_s\mathbf{F}(\psi, E, y, s)$, is a $\mathbb{R}^n \times \mathbb{R}^n$ matrix whose (k,i) entry is

$$\frac{\partial F_k}{\partial s_i} = \left\langle \frac{T_{y+s}\partial_{x_k} U_E}{\|\partial_{x_k} U_E\|_{L^2}^2}, T_{y+s}\partial_{x_i} U_E \right\rangle - \left\langle \frac{T_{y+s}\partial^2_{x_k x_i} U_E}{\|\partial_{x_k} U_E\|_{L^2}^2}, \psi - T_{y+s} U_E \right\rangle.$$

Here s_i is the i^{th} entry of the vector s. Using the above expression and the radial symmetry of U_E it follows that if ψ, E and y satisfy the conditions in the lemma for some ε, then

$$\left|\frac{\partial F_k}{\partial s_i}\right| \le \delta_{ik} + \left(\varepsilon + \|T_{y+s}U_E - T_y U_E\|_{L^2}\right) \frac{\|U_E\|_{H^2}}{\|\partial_{x_k} U_E\|_{L^2}^2}.$$

Choose a $\delta > 0$. Fix $\varepsilon > 0$ and $\gamma \in (0, \delta)$ such that the following estimates hold: for any $E \in (E_0, E_*)$, $y \in \mathbb{R}^n$, ψ that satisfies the condition in the lemma and s that satisfies $|s| < \gamma$,

$$|\mathbf{F}(\psi, E, y, 0)| < \frac{\gamma}{4}, \qquad \|[D_s\mathbf{F}(T_y U_E, E, y, 0)]^{-1}\| < 2, \tag{3.37}$$

$$\|D_s\mathbf{F}(T_y U_E, E, y, 0) - D_s\mathbf{F}(\psi, E, y, s)\| < \frac{1}{4}. \tag{3.38}$$

The existence of the constants ε and γ follows from the definition of $\mathbf{F}$ and the estimates for the elements of $D_s\mathbf{F}$. Given ψ, E and y that satisfy the conditions in the lemma, we now apply the implicit function theorem to (3.36) at its solution $(T_y U_E, E, y, 0)$ and conclude, using the estimates in (3.37) and (3.38), that there exists a unique s satisfying $|s| < \gamma$ such that $\mathbf{F}(\psi, E, y, s) = 0$. We define $\mathcal{S}(\psi, E, y) = s$. The estimate in (3.35) follows from the fact that we can choose $\delta > 0$ to be arbitrarily small, in which case the corresponding ε must be sufficiently small. $\square$

Remark 3.2. *In the proof of Lemma 3.2, for some fixed $E \in (E_0, E_*)$ and y, we could have applied the implicit function theorem at $(T_y U_E, E, y, 0)$ to conclude the existence of $\varepsilon(E, y) > 0$ and $\gamma(E, y) > 0$ such that if ψ satisfies the condition in the statement of the lemma with $\varepsilon = \varepsilon_y$, then there is a unique s with $|s| < \gamma(E, y)$ for which $\mathbf{F}(\psi, E, y, s) = 0$. But to establish the existence of ε and γ independent of $E \in (E_0, E_*)$ and y, we have appealed to the estimates in* (3.37) *and* (3.38) *which are uniform in E and y. These estimates can be used (in some form) in the proof of the implicit function theorem to construct closed balls for applying the contraction mapping principle.*

Applying Lemma 3.2 to (3.24) we obtain the following decomposition for ψ_{E_m}:

$$\psi_{E_m} = T_{y_m + s_m} U_m + v_m = T_{z_m} U_m + v_m = U_m(\cdot - z_m) + v_m,$$

where $z_m = y_m + s_m$ and v_m satisfies

$$\langle v_m, T_{y_m+s_m} \partial_{x_k} U_m \rangle = 0 \qquad \forall\, k \in \{1, 2, \ldots n\}.$$

Moreover as $E_m \to E_*$ we have $|s_m| \to 0$ and $|z_m| \to \infty$. Furthermore $v_m \xrightarrow{H^2} 0$ as $m \to \infty$ because $v_m = \hat{v}_m + T_{y_m} U_m - T_{y_m+s_m} U_m$ and both $\hat{v}_m \xrightarrow{H^2} 0$ (see (3.30)) and, due to $s_m \to 0$, $T_{y_m} U_m - T_{y_m+s_m} U_m \xrightarrow{H^2} 0$.

Equation (1.3) can now be rewritten as

$$(-\Delta + V + E_m)(T_{z_m}U_m + v_m) + \sigma\,|T_{z_m}U_m + v_m|^{2p}\,(T_{z_m}U_m + v_m) = 0 \qquad (3.39)$$

or, equivalently:

$$L_+(T_{z_m}U_m, E_m)[v_m] + VT_{z_m}U_m + N(T_{z_m}U_m, v_m) = 0, \qquad (3.40)$$

for which the Lyapunov-Schmidt procedure can be applied as follows. Define the operator $P_m^\perp$ on $L^2(\mathbb{R}^n)$ as

$$P_m^\perp \phi = \phi - \sum_{k=1}^{n} \langle \phi, T_{z_m}\partial_{x_k}U_m\rangle \frac{T_{z_m}\partial_{x_k}U_m}{\|T_{z_m}\partial_{x_k}U_m\|_{L^2}^2} \qquad \forall \phi \in L^2(\mathbb{R}^n). \qquad (3.41)$$

Then (3.40) is equivalent to the set of equations

$$P_m^\perp L_+(T_{z_m}U_m, E_m)[v_m] + P_m^\perp VT_{z_m}U_m + P_m^\perp N(T_{z_m}U_m, v_m) = 0, \qquad (3.42)$$

which is an infinite dimensional equation, and

$$\langle \partial_{x_k}T_{z_m}U_m, VT_{z_m}U_m + Vv_m + N(T_{z_m}U_m, v_m)\rangle = 0, \quad k = 1\ldots n, \qquad (3.43)$$

which is a set of finite dimensional equations.

We can write $L_+(T_{z_m}U_m, E_m) = \widetilde{L}_m + \widetilde{W}_m$, where

$$\widetilde{L}_m = -\Delta + V + E_* + \sigma(2p+1)\left|T_{z_m}U_{E_*}\right|^{2p}, \qquad \widetilde{W}_m = E_m - E_* + \sigma(2p+1)\Big(\left|T_{z_m}U_m\right|^{2p} - \left|T_{z_m}U_{E_*}\right|^{2p}\Big).$$

Since $\lim_{m\to\infty} E_m = E_*$, we have $\|\widetilde{W}_m\|_{H^2\mapsto L^2} \to 0$ as $m \to \infty$. Clearly, $\widetilde{L}_m$ is a linear operator with two potentials $V_1 = V$ and $V_2 = \sigma(2p+1)|U_{E_*}|^{2p}$ separated by a large distance. Let $\widetilde{P}_m$ be the projection operator in $L^2(\mathbb{R}^n)$ onto the set $\{\ker(-\Delta + V_1 + E_*)\} \cup \{\ker(-\Delta + V_2 + E_*)\}$. Note that $\ker(-\Delta + V_1 + E_*)$ is an empty set since the eigenvalues of $(-\Delta + V_1 + E_*)$ are bounded away from zero by $E_* - E_0 > 0$ and $\ker(-\Delta + V_2 + E_*) = \{\partial_{x_k}U_{E_*}\,\big|\, k = 1, 2, \ldots n\}$. Hence

$$\widetilde{P}_m^\perp \phi = \phi - \sum_{k=1}^{n} \langle \phi, T_{z_m}\partial_{x_k}U_{E_*}\rangle \frac{T_{z_m}\partial_{x_k}U_{E_*}}{\|T_{z_m}\partial_{x_k}U_{E_*}\|_{L^2}^2} \qquad \forall \phi \in L^2(\mathbb{R}^n). \qquad (3.44)$$

Let $R_m = 1/E_m$. We can apply Proposition 7.5 in the Appendix to $\widetilde{L}_m$, with $R = R_m$, $L_{R_m} = \widetilde{L}_m$, $E(R_m) = E_*$, $s_1(R_m) = 0$, $s_2(R_m) = z_m$ and V_1 and V_2 as defined above, and then apply the spectral perturbation theory to $\widetilde{L}_m + \widetilde{W}_m$, by regarding $\widetilde{W}_m$ as a perturbation, to conclude that the operator $\widetilde{P}_m^\perp L_+(T_{z_m}U_m, E_m)\widetilde{P}_m^\perp : H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ has a bounded inverse and the norm of the inverse operator can be bounded uniformly in m, for large m. The same conclusion holds for the operator $P_m^\perp L_+(T_{z_m}U_m, E_m)P_m^\perp$, since (3.41), (3.44) and $\lim_{m\to\infty} E_m = E_*$ imply that $\|P_m^\perp - \widetilde{P}_m^\perp\|_{L^2\mapsto L^2} \to 0$ and $\|P_m^\perp - \widetilde{P}_m^\perp\|_{H^2\mapsto H^2} \to 0$ as $m \to \infty$. Hence for all m large and some $K > 0$,

$$\|[P_m^\perp L_+(T_{z_m}U_m, E_m)P_m^\perp]^{-1}\|_{L^2\mapsto H^2} \le K. \qquad (3.45)$$

We estimate v_m from (3.42) which can be rewritten as

$$v_m = [P_m^\perp L_+(T_{z_m}U_m, E_m)P_m^\perp]^{-1} P_m^\perp (VT_{z_m}U_m + N(T_{z_m}U_m, v_m)), \qquad (3.46)$$

where we have used $P_m^{\perp} v_m = v_m$. From the estimate on the nonlinear term (3.14) and the convergence of v_m to zero in $H^2(\mathbb{R}^n)$, it follows that there exist $L,\, \delta > 0$ and $M \in \mathbb{N}$ such that for all $m > M$, (3.45) holds and

$$\|U_m\|_{H^2} \le L, \quad \|v_m\|_{H^2} \le \delta \quad \text{and} \quad KC(L)2\delta \le \frac{1}{2},$$

where $C(L)$ is defined as in (3.14). Then applying the contraction principle to (3.46) it follows that for all $m > M$ there exists an unique solution v_m to (3.46) with the bound

$$\|v_m\|_{H^2} \le 2K\|VU_m(\cdot - z_m)\|_{L^2}. \tag{3.47}$$

Our assumption (3.21):

$$\lim_{m\to\infty} |y_m| = \infty \equiv \lim_{m\to\infty} |z_m| = \infty$$

will now be contradicted when we show that no such sequence z_m can solve the remaining, finite-dimensional part (3.43) of (3.40).

Let x_m denote the radial direction at z_m. For each $m \in \mathbb{N}$ consider the following finite dimensional equation obtained as a linear combination of the equations in (3.43):

$$\langle T_{z_m}\partial_{x_m}U_m, VT_{z_m}U_m + Vv_m + N(T_{z_m}U_m, v_m)\rangle = 0. \tag{3.48}$$

We claim that, under our hypothesis on the behavior of the potential $V(x)$ as $|x| \to \infty$, the dominant term in (3.48) as $z_m \to \infty$ is $\langle T_{z_m}\partial_{x_m}U_m, VT_{z_m}U_m\rangle = 0$ and by dividing the right hand side of (3.48) with it and taking the limit as $m \to \infty$, we get the contradiction that $1 = 0$.

Fix $0 < \theta < \pi/2$ such that $2\mathcal{C}\tan\theta < 1$ where $\mathcal{C}$ is the constant in (3.17). Let B_m be closed ball of radius $|z_m|\sin\theta$ in $\mathbb{R}^n$ centered at z_m, i.e. B_m is the closure of $B(z_m, |z_m|\sin\theta)$, see Figure 3.2. Clearly

$$\begin{aligned}-2\left\langle \partial_{x_m}U_m(x - z_m), V(x)U_m(x - z_m)\right\rangle &= \left\langle U_m^2(x - z_m), \partial_{x_m}V(x)\right\rangle \\ &= \int\limits_{B_m\cup\mathbb{R}^n\setminus B_m} U_m^2(x - z_m)\partial_{x_m}V(x)\,\mathrm{d}x.\end{aligned}$$

Since $E_m \to E_*$ as $m \to \infty$, there exist $C, \epsilon > 0$ and $M \in \mathbb{N}$ such that for all $m > M$ and each $x \in \mathbb{R}^n$ we have $|U_m(x)| < Ce^{-\sqrt{E_*-\epsilon}|x|}$ and $|\partial_{x_m}U_m| < Ce^{-\sqrt{E_*-\epsilon}|x|}$, see (3.25). Using this we get for $m > M$

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^n\setminus B_m} U_m^2(x - z_m)\partial_{x_m}V(x)\,\mathrm{d}x\right| &\le C\|\nabla V\|_{L_\infty} \int\limits_{\mathbb{R}^n\setminus B_m} e^{-2\sqrt{E_*-\epsilon}|x-z_m|}\,\mathrm{d}x \\ &\le C\|\nabla V\|_{L_\infty} \int\limits_{r>|z_m|\sin\theta} e^{-2\sqrt{E_*-\epsilon}\,r} r^{n-1}\,\mathrm{d}r \\ &\le Ce^{-2\sqrt{E_*-2\epsilon}|z_m|\sin\theta}.\end{aligned} \tag{3.49}$$

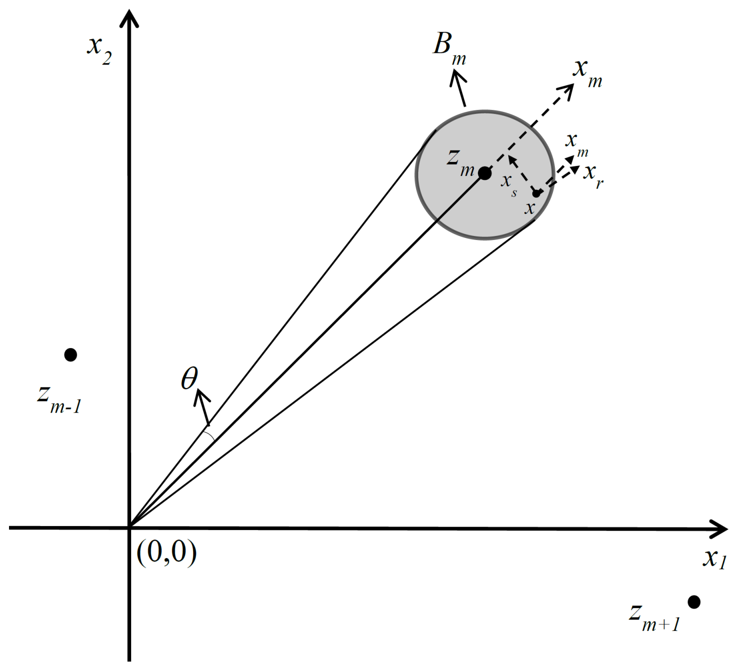

Figure 3.2: This picture depicts the ideas used while computing the integrals in the compactness case when the space dimension $n = 2$. The radially symmetric function U_m is centered at z_m and x_m denotes the radial direction at z_m. As $m \to \infty$, $|z_m| \to \infty$. The z_m's can be in any quadrant (see for example z_{m-1} and z_{m+1}). When m is large, since U_m decays exponentially (uniformly in m), most of the value of the integrals involving U_m can be obtained by integrating over the ball B_m of radius $z_m \sin\theta$ centered at z_m. The ball B_m lies within a cone of half-angle θ. At each $x \in B_m$, the radial direction x_r and the tangential direction x_s are at an angle of $\pi/2 - \theta_1$ and θ_1, respectively, to the direction x_m with $\theta_1 < \theta$. Hence for each $x \in B_m$ and sufficiently small θ, under the hypothesis in (3.17), $\partial_{x_m} V(x)$ behaves like $\partial_r V(x)$ on which we have imposed the hypothesis in (3.15) and (3.16).

For each $x \in B_m$, let $\{\theta_i(x) \big| i = 1, 2, \ldots n\}$ with $|\theta_i(x)| < \theta$ be the set of angles using which the derivative of the potential along the direction x_m can be written as a linear combination of the derivatives along the radial and tangential directions at x, i.e.

$$\partial_{x_m} V(x) = \partial_r V(x) \cos\theta_n(x) + \sum_{i=1}^{n-1} \partial_{s_i} V(x) \sin\theta_i(x),$$

where $\partial_r V$ and $\partial_{s_i} V$ are the radial and tangential derivatives introduced in the theorem. The function $\partial_{x_m} V(x)$ has the same sign a.e. in B_m for large m since for each $x \in B_m$, using (3.16) and

(3.17), we have

$$
\begin{aligned}
&|\partial_{x_m}V(x)| = \Big|\partial_r V(x)\cos\theta_n(x) + \sum_{i=1}^{n-1}\partial_{s_i}V(x)\sin\theta_i(x)\Big| \\
\Longrightarrow\quad & |\partial_{x_m}V(x)| \geq \cos\theta\,|\partial_r V(x)| - \Big(\sum_{i=1}^{n-1}|\partial_{s_i}V(x)|^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{n-1}\sin^2\theta_i(x)\Big)^{\frac{1}{2}} \\
\Longrightarrow\quad & |\partial_{x_m}V(x)| \geq \cos\theta\,|\partial_r V(x)| - \mathcal{C}\sin\theta\,|\partial_r V(x)| \qquad (3.50) \\
\Longrightarrow\quad & |\partial_{x_m}V(x)| > \frac{C}{|z_m|^\beta}\ .
\end{aligned}
$$

Therefore it follows that for m large

$$
\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m}V(x)\,\mathrm{d}x\right| \geq \frac{C}{|z_m|^\beta}\int\limits_{B_m} U_m^2(x-z_m)\,\mathrm{d}x \geq \frac{C}{|z_m|^\beta}
$$

which along with (3.49) gives

$$
\lim_{m\to\infty}\frac{\left|\int\limits_{\mathbb{R}^n\setminus B_m} U_m^2(x-z_m)\partial_{x_m}V(x)\,\mathrm{d}x\right|}{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m}V(x)\,\mathrm{d}x\right|} = 0. \tag{3.51}
$$

For the remaining terms in (3.48) we have the following estimates using (3.14) and (3.47):

$$
\begin{aligned}
2\,&|\langle T_{z_m}\partial_{x_m}U_m, Vv_m + N(T_{z_m}U_m, v_m)\rangle| \\
&\leq\ C\,\|VT_{z_m}\partial_{x_m}U_m\|_{L^2}\,\|v_m\|_{L^2} + C\,\|\partial_{x_m}U_m\|_{L^2}\,\|N(T_{z_m}U_m, v_m)\|_{L^2} \\
&\leq\ C\left(\|VT_{z_m}\partial_{x_m}U_m\|_{L^2}\,\|VT_{z_m}U_m\|_{L^2} + \|VT_{z_m}U_m\|_{L^2}^2\right).
\end{aligned}
$$

Now as in (3.49), using the decay estimates in (3.25) for $|U_m(x)|$ and $|\partial_{x_i}U_m(x)|$, we obtain

$$
\left|\int\limits_{\mathbb{R}^n\setminus B_m} U_m^2(x-z_m)V^2(x)\,\mathrm{d}x\right| \leq Ce^{-2\sqrt{E_*-2\epsilon}|z_m|\sin\theta}, \tag{3.52}
$$

$$
\left|\int\limits_{\mathbb{R}^n\setminus B_m} (\partial_{x_m}U_m(x-z_m))^2V^2(x)\,\mathrm{d}x\right| \leq Ce^{-2\sqrt{E_*-2\epsilon}|z_m|\sin\theta}. \tag{3.53}
$$

Since $\partial_{x_m}V(x)$ does not change sign in B_m, using (3.15) along with (3.50) gives us the following estimates:

$$
\lim_{m\to\infty}\frac{\left|\int\limits_{B_m} U_m^2(x-z_m)V^2(x)\,\mathrm{d}x\right|}{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m}V(x)\,\mathrm{d}x\right|} = \lim_{m\to\infty}\frac{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m}V(x)\frac{V^2(x)}{\partial_{x_m}V(x)}\,\mathrm{d}x\right|}{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m}V(x)\,\mathrm{d}x\right|} = 0 \tag{3.54}
$$

and

$$\lim_{m\to\infty} \frac{\left|\int\limits_{B_m} (\partial_{x_m} U_m(x-z_m))^2 V^2(x)\,\mathrm{d}x\right|}{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m} V(x)\,\mathrm{d}x\right|}$$

$$= \lim_{m\to\infty} \frac{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m} V(x)\frac{(\partial_{x_m} U_m(x-z_m))^2}{U_m^2(x-z_m)}\frac{V^2(x)}{\partial_{x_m} V(x)}\,\mathrm{d}x\right|}{\left|\int\limits_{B_m} U_m^2(x-z_m)\partial_{x_m} V(x)\,\mathrm{d}x\right|} = 0. \tag{3.55}$$

In deriving the last limit we have used the estimate

$$\sup_{x\in\mathbb{R}^n} \left|\frac{\partial_{x_m} U_m(x)}{U_m(x)}\right| < C$$

for some $C > 0$ and each $m \in \mathbb{N}$. This follows directly from the relation

$$U_E(x) = (E)^{\frac{1}{2p}}\, u_\infty(\sqrt{E}x) \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, E > 0,$$

where U_E is the unique positive radially symmetric solution to (3.23) and $u_\infty = U_1$, and the estimate

$$\sup_{x\in\mathbb{R}^n} \left|\frac{\partial_{x_m} u_\infty(x)}{u_\infty(x)}\right| < C.$$

for some $C > 0$. While this estimate appears to be known, see [26] who state that this estimate is established in [23], we could not locate its proof in the literature. Hence in the next lemma we present a short proof. We will denote the derivatives of functions on $\mathbb{R}$ using *primes*.

Lemma 3.3. *Define the function $v : [0,\infty) \to \mathbb{R}$ as follows: $v(|x|) = u_\infty(x)$ for each $x \in \mathbb{R}^n$. Then there exists $C > 0$ such that for each $r \in [0,\infty)$*

$$\left|\frac{v'(r)}{v(r)}\right| < C.$$

Proof. The radially symmetric function u_∞ satisfies (3.23) with $E = 1$. It follows from the properties of solutions to (3.23) that v is a C^2 function on the interval $(0,\infty)$ which satisfies

$$-v'' - \frac{n-1}{r}v' + v + \sigma|v|^{2p}v = 0 \tag{3.56}$$

and there exist constants $\gamma, C_1(\gamma), C_2(\gamma) > 0$ such that $\sqrt{1+\gamma} - \sqrt{1-\gamma} < 1$ and for each $r \in [0,\infty)$

$$v(r) < C_1(\gamma)e^{-\sqrt{1-\gamma}\,r}, \quad v(r) > C_2(\gamma)e^{-\sqrt{1+\gamma}\,r} \quad \text{and} \quad |v'(r)| < C_1(\gamma)e^{-\sqrt{1-\gamma}\,r}. \tag{3.57}$$

From these estimates we can infer that there exists a $\widetilde{L} > 0$ such that $v'(\widetilde{L}) \le 0$ and $v(r) + \sigma|v|^{2p}v(r) > 0$ for each $r > \widetilde{L}$. It now follows from (3.56) that v cannot have a local maxima on the interval $(\widetilde{L},\infty)$ which in turn implies that, on the same interval, $v'(r) \le 0$.

Since v and v' are continuous functions on $[0,\infty)$ and $v>0$, we only need to show that $|v'(r)/v(r)|$ is uniformly bounded for r large. Define the function $f(r)\geq 0$ on $(\widetilde{L},\infty)$ as

$$f(r) = -\frac{v'(r)}{v(r)}.$$

Then it is easy to verify using (3.56) that

$$f'(r) \,=\, -\frac{n-1}{r}f(r) - 1 - \sigma|v|^{2p}(r) + f^2(r). \tag{3.58}$$

Assume that

$$f(L) \,>\, 1+\frac{n-1}{L} \qquad \text{and} \qquad -\frac{n-1}{L}f(L) - 1 + f^2(L) \,>\, f(L) \tag{3.59}$$

for some $L>\widetilde{L}$ (if no such L exists, we are done). Then clearly $f'(L)>f(L)$. We claim that $f'(r)>f(r)$ for each $r\in(L,\infty)$. Suppose this is false. Then there exists a $\hat{L}\in(L,\infty)$ such that $f'(r)>f(r)\geq 0$ for $r\in(L,\hat{L})$ and $f'(\hat{L})=f(\hat{L})$. From (3.59) we have

$$\Big(1+\frac{n-1}{L}\Big)f(L) \,<\, -1+f^2(L).$$

Since $\hat{L}>L$ and $f(\hat{L})>f(L)$, the above inequality implies that

$$\Big(1+\frac{n-1}{\hat{L}}\Big)f(\hat{L}) \,<\, -1+f^2(\hat{L})$$

which, using (3.58) with $r=\hat{L}$, gives the contradiction $f'(\hat{L})>f(\hat{L})$. Hence

$$f'(r) \,>\, f(r) \qquad \forall\, r\in(L,\infty).$$

Therefore comparing (3.58) with the differential equation

$$y'(r) \,=\, y(r), \qquad y(L) \,=\, f(L),$$

we get that

$$f(r) \,\geq\, y(r) \,=\, f(L)e^{r-L} \qquad \forall\, r\in(L,\infty). \tag{3.60}$$

From the estimates in (3.57) we obtain

$$f(r) \,=\, -\frac{v'(r)}{v(r)} \leq \frac{C_1(\gamma)}{C_2(\gamma)}e^{(\sqrt{1+\gamma}-\sqrt{1-\gamma})r} \qquad \forall\, r\in(0,\infty). \tag{3.61}$$

The estimates in (3.60) and (3.61) give the following contradiction

$$1 \,=\, \lim_{r\to\infty}\frac{f(r)}{f(r)} \leq \lim_{r\to\infty}\frac{C_1(\gamma)e^{(\sqrt{1+\gamma}-\sqrt{1-\gamma})r}}{C_2(\gamma)f(L)e^{r-L}} = 0.$$

Therefore no $L>\widetilde{L}$ exists such that (3.59) holds. Hence f is a bounded function on $(0,\infty)$. This completes the proof of the lemma. $\square$

Using the estimates in (3.49), (3.51), (3.52), (3.53), (3.54) and (3.55) we see that the term

$$\int\limits_{\mathbb{R}^n} U_m^2(x - z_m)\partial_{x_m} V(x)\,\mathrm{d}x$$

cannot be canceled by the remaining terms in (3.48) when m is sufficiently large. Therefore it follows that there are no solutions to (3.39) such that the sequence z_m, and consequently y_m, has an unbounded subsequence. Hence y_m must be bounded. This completes the proof of the theorem in the compactness case (see discussion above (3.21)).

Splitting ($0 < \mu < 1$): Since $\psi_{E_m} = \|\psi_{E_k}\|_{L^2}\Psi_m$ and $\lim_{m\to\infty}\|\psi_{E_m}\|_{L^2} = a > 0$, it follows from concentration compactness theory, see Proposition 7.4, that if $0 < \mu < 1$, then there exists a subsequence of $(E_m)_{m\in N}$, again denoted as $(E_m)_{m\in\mathbb{N}}$, on which we can decompose ψ_{E_m} as follows:

$$\psi_{E_m} = \sum_{j=1}^{d} T_{y_m^j}\psi_m^j + \psi_m^0 + \hat{v}_m. \tag{3.62}$$

Here $\psi_m^j, \hat{v}_m \in H^1(\mathbb{R}^n)$ for $j \in \mathcal{J} = \{0, 1, 2, \dots d\}$ and there exist non-zero functions ψ_*^j such that as $m \to \infty$

(i) $\psi_m^j \overset{H^1}{\rightharpoonup} \psi_*^j$ and $\psi_m^j \overset{L^q}{\to} \psi_*^j$ for $2 \le q < 2n/(n-2)$ ($2 \le q < \infty$ if $n = 2$, $2 \le q \le \infty$ if $n = 1$),

(ii) $|y_m^j| \to \infty$ and $|y_m^j - y_m^k| \to \infty$ for $j, k \in \{1, 2, \dots d\}$ and $j \ne k$,

(iii) $\hat{v}_m \overset{H^1}{\rightharpoonup} 0$ and $\hat{v}_m \overset{L^q}{\to} 0$ for $2 < q < 2n/(n-2)$ ($2 < q < \infty$ if $n = 2$, $2 < q \le \infty$ if $n = 1$).

We will show that splitting cannot occur. We will obtain the contradiction that if the y_m^js satisfy the limits in (ii) above, then (ψ_{E_m}, E_m) cannot be a solution of (1.3).

Concentration compactness theory does not imply that $\psi_*^0 \ne 0$, but we assume it only to simplify our presentation. All the arguments in this section remain valid if $\psi_*^0 = 0$. Let $\psi = \psi_{E_m}$ and $E = E_m$ in (1.3) and take the L^2 scalar product of the resulting equation with $\phi \in C_0^\infty(\mathbb{R}^n)$ to obtain

$$\langle\nabla\psi_{E_m}, \nabla\phi\rangle + \langle(V + E_m)\psi_{E_m}, \phi\rangle + \sigma\left\langle|\psi_{E_m}|^{2p}\psi_{E_m}, \phi\right\rangle = 0.$$

Substituting for ψ_{E_m} from (3.62) and taking the limit along $E_m \to E_*$ we get,

$$\left\langle\nabla\psi_*^0, \nabla\phi\right\rangle + \left\langle(V + E_*)\psi_*^0, \phi\right\rangle + \sigma\left\langle|\psi_*^0|^{2p}\psi_*^0, \phi\right\rangle = 0.$$

Therefore ψ_*^0 is a weak solution (and hence a strong solution) of (1.3) at $E = E_*$ and hence

$$(-\Delta + V + E_*)\psi_{E_*}^0 + \sigma|\psi_{E_*}^0|^{2p}\psi_{E_*}^0 = 0.$$

Similarly for any $j \in \mathcal{J}$, $j \ne 0$, taking the L^2 scalar product of the equation

$$(-\Delta + T_{-y_m^j}V + E_m)T_{-y_m^j}\psi_{E_m} + \sigma T_{-y_m^j}|\psi_{E_m}|^{2p}\psi_{E_m} = 0$$

(which is nothing but a translated version of (1.3) with $E = E_m$ and $\psi = \psi_{E_m}$) with $\phi \in C_0^\infty(\mathbb{R}^n)$ and taking the limit as $m \to \infty$ we can conclude that ψ_*^j is a weak solution (and hence a strong solution) to

$$(-\Delta + E)\psi + \sigma|\psi|^{2p}\psi = 0 \tag{3.63}$$

at $E = E_*$. It follows from Proposition 7.2 in the Appendix that there exists $C > 0$ and $\gamma \in (0, E_*)$ such that

$$|\psi_*^j(x)| \le Ce^{-\sqrt{E_*-\gamma}|x|} \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, j \in \mathcal{J}\,. \tag{3.64}$$

Since (ψ_{E_m}, E_m) is a ground state solution for (1.3), we have $\psi_{E_m} > 0$ for all m. It now follows using (3.62) and (3.64) and the limits in (i), (ii) and (iii) above that for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^n)$ and any $j \in \mathcal{J}$, using the notation $y_m^0 = 0$, we have

$$0 \le \liminf_{m\to\infty} \langle \psi_{E_m}, T_{y_m^j}\phi\rangle \,=\, \langle \psi_*^j, \phi\rangle + \liminf_{m\to\infty}\langle \hat{v}_m, T_{y_m}\phi\rangle \,=\, \langle \psi_*^j, \phi\rangle\,,$$

which implies that ψ_*^j is a nonnegative function. Since any non-negative solution to (3.63) must be a translation of the radially symmetric nonnegative function U_E introduced in (3.23), it follows that for each $j \in \mathcal{J}$, $j \ne 0$, the function ψ_*^j is a translation of U_{E_*} and so $\psi_*^j = T_{y_*^j} U_{E_*}$. By redefining $\hat{v}_m$ to be $\hat{v}_m + \psi_m^0 - \psi_*^0 + \sum_{j=1}^d T_{y_m^j}\psi_m^j - \sum_{j=1}^d T_{y_m^j + y_*^j} U_m$ and then redefining $y_m^j + y_*^j$ to be y_m^j, we obtain the following decomposition instead of (3.62) for ψ_{E_m}:

$$\psi_{E_m} = \sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0 + \hat{v}_m. \tag{3.65}$$

Here we have denoted U_{E_m} by U_m. Using $\lim_{m\to\infty} E_m = E_*$, (3.25), (3.26) and the fact that ψ_m^js, y_m^js and $\hat{v}_m$ satisfy the limits in (i), (ii) and (iii) above, it is easy to verify that the redefined y_m^js and $\hat{v}_m$ also satisfy the limits in (ii) and (iii).

Since (ψ_{E_m}, E_m) solves (1.3), by substituting (3.65) we obtain

$$\begin{aligned}(-\Delta + V + E_m)\Big(\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0 + \hat{v}_m\Big)& \\ + \sigma\left|\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0 + \hat{v}_m\right|^{2p} \sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0 + \hat{v}_m) &= 0.\end{aligned} \tag{3.66}$$

Recall the nonlinear operator $N(U, v)$ from (3.7):

$$N(U, v) = \sigma|U + v|^{2p}(U + v) - \sigma|U|^{2p}U - \sigma(2p + 1)|U|^{2p}v.$$

Using this, (3.66) can be rewritten as

$$\begin{aligned}\hat{v}_m = (-\Delta + E_m)^{-1}\Bigg[- V\sum_{j=1}^d T_{y_m^j} U_m - \Big(E_m - E_* + V + \sigma(2p+1)\Big|\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0\Big|^{2p}\Big)\hat{v}_m \\ -N\Big(\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0, \hat{v}_m\Big) - \sigma\Big|\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0\Big|^{2p}\Big(\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0\Big) + \sigma\sum_{j=1}^d T_{y_m^j}|U_m|^{2p}U_m + \sigma|\psi_*^0|^{2p}\psi_*^0\Bigg].\end{aligned}$$

We will now show that the term in the bracket in the above expression (to which $-(-\Delta + E_*)^{-1}$ is applied) converges to 0 in $L^2(\mathbb{R}^n)$ as $m \to \infty$. This will imply that

$$\lim_{m\to\infty} \|\hat{v}_m\|_{H^2} = 0. \tag{3.67}$$

Recall the limits $|y_m^j| \to \infty$ and $|y_m^j - y_m^k| \to \infty$ for $j,k \in \{1,2,\dots d\}$ and $j \neq k$ from (ii) and $\hat{v}_m \overset{L^q}{\to} 0$ for $2 < q < 2n/(n-2)$ ($2 < q < \infty$ if $n = 2$, $2 < q \leq \infty$ if $n = 1$) as $m \to \infty$ from (iii). Note that from (3.25), (3.26), $\lim_{m\to\infty} E_m = E_*$ and (3.64) we get that $U_m, \psi_*^0 \in L^q(\mathbb{R}^n)$ and $\|U_m\|_{L^q}$ can be bounded by a constant independent of m for each $q \in [1,\infty]$. This, along with the limit for $\hat{v}_m$ in (iii), the limit $\lim_{m\to\infty} E_m = E_*$ and the fact that $\sup_{m\in\mathbb{N}} \|\hat{v}_m\|_{L^2} = M_v < \infty$, implies that

$$\lim_{m\to\infty} \Big\| \Big(E_m - E_* + \sigma(2p+1) \Big| \sum_{j=1}^{d} T_{y_m^j} U_m + \psi_*^0 \Big|^{2p} \Big) \hat{v}_m \Big\|_{L^2} = 0. \tag{3.68}$$

Also, using $\lim_{|x|\to\infty} V(x) = 0$, the limit for y_m^j from (ii), (3.25), (3.26) and $\lim_{m\to\infty} E_m = E_*$ we get that

$$\lim_{m\to\infty} \Big\| V \sum_{j=1}^{d} T_{y_m^j} U_m \Big\|_{L^2} \leq \lim_{m\to\infty} \Big\| V \sum_{j=1}^{d} T_{y_m^j} (U_m - U_{E_*}) \Big\|_{L^2} + \lim_{m\to\infty} \Big\| V \sum_{j=1}^{d} T_{y_m}^j U_{E_*} \Big\|_{L^2} = 0. \tag{3.69}$$

From the exponential decay of U_m (uniform in m) and ψ_*^0 (see (3.25), (3.26) and (3.64)) it follows that

$$\lim_{m\to\infty} \Big\| \Big| \sum_{j=1}^{d} T_{y_m^j} U_m + \psi_*^0 \Big|^{2p} \Big(\sum_{j=1}^{d} T_{y_m^j} U_m + \psi_*^0 \Big) - \sum_{j=1}^{d} T_{y_m^j} |U_m|^{2p} U_m - |\psi_*^0|^{2p} \psi_*^0 \Big\|_{L^2} = 0. \tag{3.70}$$

Note that from (3.25), (3.26), $\lim_{m\to\infty} E_m = E_*$ and (3.64) we get that $U_m, \psi_*^0 \in H^2(\mathbb{R}^n)$ and $\|U_m\|_{H^2}$ is bounded uniformly in m. Since $\|\psi_{E_m}\|_{H^2}$ is bounded in uniformly in m, it follows that $\|\hat{v}_m\|_{H^2}$ is bounded uniformly in m. It now follows from Lemma 3.1 and (iii) that

$$\lim_{m\to\infty} \Big\| N \Big(\sum_{j=1}^{d} T_{y_m^j} U_m + \psi_*^0, \hat{v}_m \Big) \Big\|_{L^2} = 0, \tag{3.71}$$

To show that $\|V\hat{v}_m\|_{L^2}$ converges to 0, we use the following argument. Fix $\epsilon > 0$ and choose $L_\varepsilon > 0$ sufficiently large such that $\|V\|_{L^\infty(\{|x|\geq L_\varepsilon\})} < \epsilon/(2M)$. Here $M = \sup_{m\in\mathbb{N}} \|\hat{v}_m\|_{L^2}$. Using $V \in L^\infty(\mathbb{R}^n)$ and the limits for $\hat{v}_m$ in (iii) it follows via Hölder's inequality that $\|\hat{v}_m V\|_{L^2(\{|x|\leq L_\varepsilon\})} < \epsilon/2$ for all $m > m_\varepsilon$ and some m_ε. Hence for all $m > m_\varepsilon$, we have $\|V\hat{v}_m\|_{L^2} < \varepsilon$. Since $\varepsilon > 0$ is arbitrary we get that

$$\lim_{m\to\infty} \|V\hat{v}_m\|_{L^2} = 0. \tag{3.72}$$

It now follows from (3.68)-(3.72) that (3.67) holds.

Like in the compactness case, the decomposition

$$\psi_{E_m} = \sum_{j=1}^{d} T_{y_m^j} U_m + \psi_*^0 + \hat{v}_m,$$

together with $\hat{v}_m \overset{H^2}{\to} 0$ and the assumptions $|y_m^j| \to \infty, |y_m^j - y_m^k| \to \infty,\ E_m \to E_*$ imply that our solution bifurcates from a sum of ψ_*^0 and d copies of U_{E_*} located at $|x| = \infty$. While this bifurcation point is inaccessible in the Banach spaces we are working with, we will use a related Lyapunov-Schmidt like decomposition. Recall the linear operator $L_+(U,E)$ from (3.28):

$$L_+(U,E)[v] = (-\Delta + V + E)v + \sigma(2p+1)|U|^{2p}v.$$

Note that $L_+(\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0, E_m)$, the linearization of (3.66) at the profiles $\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0$, can be viewed as a Schrödinger operator with $d+1$ potentials, $V + \sigma(2p+1)|\psi_*^0|^{2p}$ and $\sigma(2p+1)|T_{y_m^j} U_m|^{2p}$ for $j \in \{1,2,\dots d\}$. Since $|y_m^j| \to \infty$ and $|y_m^j - y_m^k| \to \infty$ as $m \to \infty$, the distance between the potentials eventually becomes very large. From the spectral theory for operators with potentials separated by a large distance [25], as $m \to \infty$ the projection operator associated with the kernel of $L_+(\sum_{j=1}^d T_{y_m^j} U_m + \psi_*^0, E_m)$ converges to the projection operator associated with the set containing the kernel of $(-\Delta+V+E_*+(2p+1)\sigma|\psi_{E_*}^0|^{2p})$ and the kernel of $(-\Delta+E_m+\sigma|T_{y_m^j} U_m|^{2p})$ for all $j \in \{1,2,\dots d\}$. Therefore we now move to a decomposition with respect to the kernels of the operators $(-\Delta+V+E_*+(2p+1)\sigma|\psi_{E_*}^0|^{2p})$ and $(-\Delta+E_m+\sigma|T_{y_m^j} U_m|^{2p})$. Since the kernel of the latter operator is $\mathrm{span}\{T_{y_m^j} \partial_{x_k} U_m \,\big|\, i = 1,2,\dots n\}$, which are infinitesimal generators of translations of U_m in $\mathbb{R}^n$, the projection onto this kernel can be absorbed into translations of the profile U_m. This is established in a general setting in Proposition 7.6 in the Appendix. Let

$$\ker\left(-\Delta + V + E_* + (2p+1)\sigma|\psi_*^0|^{2p}\right) = \mathrm{span}\{\phi^i \,\big|\, i = 1,2,\dots n_0\}.$$

Applying Proposition 7.6 (also see Remark 7.3) with u_i, ψ, ϕ_i in the proposition being U_m (independent of i), ψ_*^0 and ϕ^i, respectively, it follows from the limits $\hat{v}_m \overset{H^2}{\to} 0$, $|y_m^j| \to \infty$ and $|y_m^j - y_m^k| \to \infty$ as $m \to \infty$ that (3.65) can be rewritten (for large m) as

$$\psi_{E_m} = \sum_{j=1}^d T_{y_m^j + s_m^j} U_m + \psi_*^0 + \sum_{i=1}^{n_0} a_m^i \phi^i + v_m\,, \tag{3.73}$$

where v_m satisfies the following orthogonality relations:

$$\left\langle v_m, T_{y_m^j + s_m^j} \partial_{x_k} U_m \right\rangle = 0, \qquad \forall\, j \in \{1,2,\dots d\}, \qquad \forall\, k \in \{1,2,\dots n\},$$

$$\left\langle v_m, \phi^i \right\rangle = 0 \qquad \forall\, i \in \{1,2,\dots n_0\}.$$

Moreover, as $m \to \infty$ we have $|s_m^j| \to 0$ for $j \in \{1,2,\dots d\}$ and $|a_m^i| \to 0$ for $i \in \{1,2,\dots n_0\}$. Furthermore, since

$$v_m = \hat{v}_m + \sum_{j=1}^d T_{y_m^j} U_m - \sum_{j=1}^d T_{y_m^j + s_m^j} U_m - \sum_{i=1}^{n_0} a_m^i \phi^i$$

and (3.67) holds, $a_m^i \to 0$ and $\sum_{j=1}^d T_{y_m^j} U_m - \sum_{j=1}^d T_{y_m^j + s_m^j} U_m \overset{H^2}{\to} 0$ (because $s_m^j \to 0$ and $E_m \to E_*$), we have

$$\lim_{m\to\infty} \|v_m\|_{H^2} = 0.$$

We assume, with no loss of generality, that the decomposition of ψ_{E_m} in (3.73) holds for all m.

Define

$$z_m^j = y_m^j + s_m^j \qquad \forall\, j \in \{1, 2, \dots d\}, \qquad \widetilde{U}_m = \sum_{j=1}^{d} T_{z_m^j} U_m + \psi_*^0, \qquad \phi_m^a = \sum_{i=1}^{n_0} a_m^i \phi^i.$$

Using (3.73), we can rewrite (1.3) as

$$(-\Delta + V + E_m)(\widetilde{U}_m + \phi_m^a + v_m) + \sigma|\widetilde{U}_m + \phi_m^a + v_m|^{2p}(\widetilde{U}_m + \phi_m^a + v_m) = 0, \tag{3.74}$$

which can equivalently be written using the operators $L_+(U, E)$ and $N(U, v)$, defined in (3.28) and (3.7), as follows:

$$\begin{aligned} &L_+(\widetilde{U}_m + \phi_m^a, E_m)[v_m] + N(\widetilde{U}_m + \phi_m^a, v_m) + V\sum_{j=1}^{d} T_{z_m^j} U_m + \sigma|\widetilde{U}_m + \phi_m^a|^{2p}(\widetilde{U}_m + \phi_m^a) \\ &- \sigma \sum_{j=1}^{d} T_{z_m^j} |U_m|^{2p} U_m - \sigma|\psi_*^0|^{2p}\psi_*^0 + (E_m - E_*)(\psi_*^0 + \phi_m^a) - \sigma(2p+1)|\psi_*^0|^{2p}\phi_m^a = 0. \end{aligned} \tag{3.75}$$

We will apply the Lyapunov-Schmidt procedure to the above equation. Note that using the exponential decays of U_m and ψ_*^0 and the fact that $E_m \to E_*$ we can write the above equation as

$$\begin{aligned} &L_+(\widetilde{U}_m + \phi_m^a, E_m)[v_m] + N(\widetilde{U}_m + \phi_m^a, v_m) + V\sum_{j=1}^{d} T_{z_m^j} U_m + O(|a_m|) \\ &+ O(E_m - E_*) + \sum_{j=1}^{d} O\left(e^{-\sqrt{E_* - \gamma}|z_m^j|}\right) + \sum_{i \neq j,\ i,j=1}^{d} O\left(e^{-\sqrt{E_* - \gamma}|z_m^i - z_m^j|}\right) = 0, \end{aligned} \tag{3.76}$$

for a constant $\gamma > 0$. Define the operator $P_m^\perp$ on $L^2(\mathbb{R}^n)$ to be the orthogonal projection onto its subspace

$$\left\{\phi^i, T_{z_m^j}\partial_{x_k} U_m \,\middle|\, i = 1, 2, \dots n_0,\ k = 1, 2, \dots n,\ j = 1, 2, \dots d\right\}^\perp.$$

For simplicity of notation, let us write (3.75) as

$$F(v_m, z_m, a_m) = 0,$$

where z_m is the set $\{z_m^1, z_m^2, \dots z_m^d\}$ and a_m is the set $\{a_m^1, a_m^2, \dots a_m^{n_0}\}$. This equation is equivalent to the following set of equations:

$$P_m^\perp F(v_m, z_m, a_m) = 0, \tag{3.77}$$

which is an infinite dimensional equation to be solved for v_m as a function of z_m and a_m, and

$$\begin{aligned} \langle F(v_m, z_m, a_m), \phi^i\rangle &= 0, \qquad \forall\, i \in \{1, 2, \dots n_0\}, \\ \langle F(v_m, z_m, a_m), T_{z_m^j}\partial_{x_k} U_m\rangle &= 0, \qquad \forall\, j \in \{1, 2, \dots d\}, \qquad \forall\, k \in \{1, 2, \dots n\}, \end{aligned} \tag{3.78}$$

which is a set of finite dimensional equations to be solved for z_m and a_m using the solution v_m of (3.77).

We can write $L_+(\widetilde{U}_m+\phi_m^a,E_m)=\widetilde{L}_m+\widetilde{W}_m$, where

$$\widetilde{L}_m=-\Delta+V+E_*+\sigma(2p+1)\Big(\sum_{j=1}^{d}\big|T_{z_m^j}U_{E_*}\big|^{2p}+|\psi_*^0|^{2p}\Big),$$

$$\widetilde{W}_m=E_m-E_*+\sigma(2p+1)\Big(\Big|\sum_{j=1}^{d}T_{z_m^j}U_m+\psi_*^0+\phi_m^a\Big|^{2p}-\sum_{j=1}^{d}\big|T_{z_m^j}U_{E_*}\big|^{2p}-|\psi_*^0|^{2p}\Big).$$

Since $E_m\to E_*$, $a_m^i\to 0$, $|y_m^j|\to\infty$ and $|y_m^j-y_m^k|\to\infty$ as $m\to\infty$, we get $\|\widetilde{W}_m\|_{H^2\mapsto L^2}\to 0$ as $m\to\infty$. Clearly, $\widetilde{L}_m$ is a Schrödinger operator with $d+1$ potentials,

$$V_j=\sigma(2p+1)|T_{y_m^j}U_{E_*}|^{2p}\qquad\forall\, j\in\{1,2,\dots d\},\qquad V_{d+1}=V+\sigma(2p+1)|\psi_*^0|^{2p},$$

which are separated by a large distance. Let $\widetilde{P}_m^\perp$ be the projection operator in $L^2(\mathbb{R}^n)$ onto the set

$$\Big\{\phi^i,T_{z_m^j}\partial_{x_k}U_{E_*}\,\big|\,i=1,2,\dots n_0,\ k=1,2,\dots n,\ j=1,2,\dots d\Big\}^\perp.$$

This is the set containing the kernel of the operators $(-\Delta+V+\sigma(2p+1)|\psi_*^0|^{2p}+E_*)$ and $(-\Delta+E_*+\sigma(2p+1)|T_{y_m^j}U_{E_*}|^{2p})$. Let $R_m=1/E_m$. We can apply Proposition 7.5 in the Appendix to $\widetilde{L}_m$, with $R=R_m$, $L_{R_m}=\widetilde{L}_m$, $E(R_m)=E_*$, $s_j(R_m)=z_m^j$ for $j\in\{1,2,\dots d\}$, $s_{d+1}(R_m)=0$ and the potential $V_1,V_2,\dots V_{d+1}$ as defined above, and then apply the spectral perturbation theory to $\widetilde{L}_m+\widetilde{W}_m$, by regarding $\widetilde{W}_m$ as a perturbation, to conclude that the operator $\widetilde{P}_m^\perp L_+(\widetilde{U}_m+\phi_m^a,E_m)\widetilde{P}_m^\perp : H^2(\mathbb{R}^n)\mapsto L^2(\mathbb{R}^n)$ has a bounded inverse and the norm of the inverse operator can be bounded uniformly in m, for large m. The same conclusion holds for the operator $P_m^\perp L_+(\widetilde{U}_m+\phi_m^a,E_m)P_m^\perp$ since the definitions of $P_m^\perp$ and $\widetilde{P}_m^\perp$ and the limit $\lim_{m\to\infty}E_m=E_*$ imply that $\|P_m^\perp-\widetilde{P}_m^\perp\|_{L^2\mapsto L^2}\to 0$ and $\|P_m^\perp-\widetilde{P}_m^\perp\|_{H^2\mapsto H^2}\to 0$ as $m\to\infty$. Hence for all m large and some $K>0$,

$$\|[P_m^\perp L_+(\widetilde{U}_m+\phi_m^a,E_m)P_m^\perp]^{-1}\|_{L^2\mapsto H^2}\le K.$$

Hence for large m, using (3.76) and the relation $P_m^\perp v_m=v_m$, we can rewrite the infinite dimensional equation (3.77) as

$$\begin{aligned}v_m=-[P_m^\perp L_+(\widetilde{U}_m+\phi_m^a,E_m)P_m^\perp]^{-1}\Big(&N(\widetilde{U}_m+\phi_m^a,v_m)+V\sum_{j=1}^{d}T_{z_m^j}U_m\\&+(E_m-E_*)\psi_*^0+O(|a_m|)+\sum_{\substack{j,k=1\\j\neq k}}^{d}O\left(e^{-\sqrt{E_*-\gamma}|z_m^j-z_m^k|}\right)\Big).\end{aligned}\tag{3.79}$$

Like in the case of compactness, existence of an unique solution v_m to this equation can established by applying contraction mapping principle using (3.14) and the smallness of $\|v_m\|_{H^2}$.

While analyzing the finite dimensional equations (3.78) it would be useful to have estimates for the decay of $\|v_m\|_{H^2}$, explicitly in terms of the z_m^js, similar to (3.47) in the case of compactness. But due to the presence of the non-drifting terms ψ_*^0 and ϕ_m^a in (3.73), that remain localized near the origin in $\mathbb{R}^n$, we will not be able to get such estimates. Indeed, by adopting the approach in the compactness case, we can derive from (3.79) that for some $C > 0$

$$\|v_m\|_{H^2} \le C \sum_{j=1}^{d} \|VT_{z_m^j} U_m\|_{L^2} + C(E_m - E_*)\|\psi_*^0\|_{L^2} + O(|a_m|).$$

But without knowing the rate of convergence of the terms $(E_m - E_*)\|\psi_*^0\|_{L^2} + O(|a_m|)$ to 0, this estimate is not useful. Hence we will express v_m as

$$v_m = P_m^{\perp} v_m^0 + v_m^1.$$

Here v_m^0 will be associated with the non-drifting terms in (3.73) while v_m^1 will be associated with the drifting terms. For v_m^0 we will derive certain exponential in space decay estimates (see (3.84)) so that its scalar product with the drifting terms $T_{z_m^j}\partial_{x_k} U_m$ in (3.78), which is a quantity of interest, decays at a rate which depends explicitly on the z_m^js. For $\|v_m^1\|_{H^2}$ we will derive an decay estimate explicitly in terms of the z_m^js, like (3.47) in the compactness case (see (3.86)). Using these estimates we will analyze the equations (3.78) to obtain a contradiction implying that splitting cannot occur.

We first derive the estimate (3.84). Let P_0 be the projection operator in $L^2(\mathbb{R}^n)$ onto its subspace

$$S = \ker\left(-\Delta + V + E_* + (2p+1)\sigma|\psi_*^0|^{2p}\right) = \{\phi^1, \phi^2, \dots \phi^{n_0}\}.$$

Consider the equation

$$P_0^{\perp}\left[(-\Delta + V + E_m)(\psi_*^0 + \phi_m^a + v_m^0) + \sigma|\psi_*^0 + \phi_m^a + v_m^0|^{2p}(\psi_*^0 + \phi_m^a + v_m^0)\right] = 0, \tag{3.80}$$

We wish to find a $v_m^0 \in H^2(\mathbb{R}^n)$ that solves this equation. Clearly the operator

$$F_L(\cdot) = P_0^{\perp}\Big[(-\Delta + V + E_*)(\cdot) + \sigma(2p+1)|\psi_*^0|^{2p}(\cdot)\Big],$$

which maps $S^{\perp} \cap H^2(\mathbb{R}^n)$ to $S^{\perp} \cap L^2(\mathbb{R}^n)$, has a bounded inverse. Hence we can write (3.80) as

$$\begin{aligned} v_m^0 =& F_L^{-1} P_0^{\perp}\Big[(E_* - E_m)(\psi_*^0 + \phi_m^a + v_m^0) + \sigma(2p+1)(|\psi_0^*|^{2p} - |\psi_0^* + \phi_m^a|^{2p})v_m^0 \\ &+ \sigma(2p+1)|\psi_*^0|^{2p}\phi_m^a + \sigma(|\psi_*^0|^{2p}\psi_*^0 - |\psi_*^0 + \phi_m^a|^{2p}(\psi_*^0 + \phi_m^a))\Big]. \end{aligned}$$

By using the contraction mapping principle it follows from the above equation that for m sufficiently large (so that $|E_m - E_*|$ and $|a_m|$ are sufficiently small) there exists a unique $v_m^0 \in S^{\perp} \cap H^2(\mathbb{R}^n)$ that solves this equation with $\|v_m^0\|_{H^2} = O(|E_m - E_*|) + O(|a_m|)$. For $n \le 3$, this implies directly that $v_m^0 \stackrel{L^\infty}{\to} 0$ as $m \to \infty$. For $n \ge 4$, the same conclusion can be arrived at by first using regularity to show that v_m^0 converges to 0 in $W^{2,q}$ for all $q \ge 2$, see the Appendix 7. We will now show that v_m^0 decays exponentially.

From (3.80) we get that there exist $\beta_m^i \in \mathbb{R}$, for each $i \in \{1, 2, \dots n_0\}$, such that

$$\Delta(\psi_*^0 + \phi_m^a + v_m^0) = (V + E_m + \sigma|\psi_*^0 + \phi_m^a + v_m^0|^{2p})(\psi_*^0 + \phi_m^a + v_m^0) - \sum_{i=1}^{n_0} \beta_m^i \phi^i. \tag{3.81}$$

Note that for each $i \in \{1, 2, \ldots n_0\}$ we have

$$\begin{aligned}\beta_m^i \langle \phi^i, \phi^i \rangle &= \big\langle \phi^i, (-\Delta + V + E_m)(\psi_*^0 + \phi_m^a + v_m^0) + \sigma|\psi_*^0 + \phi_m^a + v_m^0|^{2p}(\psi_*^0 + \phi_m^a + v_m^0)\big\rangle \\ &= \big\langle \phi^i, (L_* + E_m - E_*)(\phi_m^a + v_m^0) + (E_m - E_*)\psi_*^0 + N(\psi_*^0, \phi_m^a + v_m^0)\big\rangle \\ &= \big\langle \phi^i, (L_* + E_m - E_*)(\phi_m^a + v_m^0) + (E_m - E_*)\psi_*^0\big\rangle + O(\|\phi_m^a + v_m^0\|_{H^2}^2),\end{aligned}$$

where we have used (3.14) to get the last equality and $L_*[v] = (-\Delta + V + E_*)v + (2p+1)\sigma|\psi_{E_*}^0|^{2p}v$. It now follows using $L_*\phi^i = 0$ and the estimate for $\|v_m^0\|_{H^2}$ discussed earlier that

$$|\beta_m^i| = O(|E_m - E_*|) + O(|a_m|).$$

Define the function $W_m = \psi_*^0 + \phi_m^a + v_m^0$. From the estimates on ψ_*^0, ϕ_m^a and v_m^0 we get that $W_m \in H^2(\mathbb{R}^n)$ and, again using the regularity result in the Appendix 7, $W_m \in L^\infty(\mathbb{R}^n)$. Consider the set

$$K_+ = \Big\{x \in \mathbb{R}^n \,\big|\, W_m(x) > \max_{i=1,2,\ldots n_0} |\phi^i(x)|\Big\}.$$

On K_+ we have

$$\Delta W_m(x) \geq \Big(E_m + \sigma|W_m(x)|^{2p} - |V(x)| - \sum_{i=1}^{n_0} |\beta_m^i|\Big) W_m(x).$$

Fix $0 < \gamma < E_*$. Let $M \in \mathbb{N}$ and $L > 0$ be sufficiently large so that for all $m > M$

$$\sum_{i=1}^{n_0} |\beta_m^i| \leq \frac{\gamma}{4}, \quad |E_m - E_*| \leq \frac{\gamma}{4}, \quad \|V\|_{L^\infty} \leq \frac{\gamma}{4}, \quad |\sigma||W_m(x)|^{2p} \leq \frac{\gamma}{4} \qquad \forall\, x \in \mathbb{R}^n \setminus B(0, L).$$

Here $B(0, L)$ is the closed ball of radius L centered at the origin in $\mathbb{R}^n$ and the last inequality follows from the inequalities $\psi_*^0(x) \leq Ce^{-\sqrt{E_* - \gamma}|x|}$ and $\phi_m^a(x) \leq Ce^{-\sqrt{E_* - \gamma}|x|}$ for some $C > 0$ and the limit $v_m^0 \stackrel{L^\infty}{\to} 0$ as $m \to \infty$. Therefore

$$\Delta W_m(x) \geq (E_* - \gamma) W_m(x) \qquad \forall\, x \in K_+ \cap \mathbb{R}^n \setminus B(0, L). \tag{3.82}$$

Consider the function $\Phi(x) = C_0 e^{-\sqrt{E_* - \gamma}|x|}$ with $C_0 > 0$ being a constant (to be chosen later). This function satisfies

$$\Delta \Phi(x) \leq (E_* - \gamma)\Phi(x) \qquad \forall\, x \in \mathbb{R}^n \setminus 0. \tag{3.83}$$

It follows from (3.82) and (3.83) that

$$\Delta(W_m - \Phi)(x) \geq (E_* - \gamma)(W_m - \Phi)(x) \qquad \forall\, x \in K_+ \cap \mathbb{R}^n \setminus B(0, L).$$

Hence there exists no positive maxima for the function $W_m - \Phi$ on the set $K_+ \cap B(0, L)$. Choose C_0 sufficiently large so that

$$\Phi(x) \geq W_m(x) \qquad \forall\, x \in \partial B(0, L), \qquad \Phi(x) \geq \max_{i=1,2,\ldots n_0} |\phi^i(x)| \qquad \forall\, x \in \mathbb{R}^n.$$

Here $\partial B(0, L) = \{x \in \mathbb{R}^n \,\big|\, |x| = L\}$. Also, on the boundary of the set K_+, we have

$$W_m(x) = \max_{i=1,\ldots,n_0} |\phi^i(x)| \leq \Phi(x).$$

Moreover, since $v_m^0 \in W^{2,q}$ with $q \geq 2$, we have $v_m^0(x) \to 0$ as $|x| \to \infty$ which implies that $W_m(x) \to 0$ as $|x| \to \infty$. It now follows from the above discussion that

$$W_m(x) \leq \Phi(x) \qquad \forall\, x \in K_+ \cap \mathbb{R}^n \setminus B(0,L).$$

A similar argument applied to the function $-W_m$ on the set

$$K_- = \Big\{ x \in \mathbb{R}^n \,\Big|\, -W_m(x) > \max_{i=1,2\ldots n_0} |\phi^i(x)| \Big\}$$

gives

$$-W_m(x) \leq \Phi(x) \qquad \forall\, x \in K_- \cap \mathbb{R}^n \setminus B(0,L).$$

Therefore we can conclude that for each $x \in \mathbb{R}^n \setminus B(0,L)$ and some $C > 0$

$$\begin{aligned} & |\psi_*^0(x) + \phi_m^a(x) + v_m^0(x)| \leq \max\{\phi^1(x), \phi^2(x), \ldots \phi^{n_0}(x), \Phi(x)\} \leq Ce^{-\sqrt{E_*-\gamma}|x|} \\ \implies & |v_m^0(x)| \leq Ce^{-\sqrt{E_*-\gamma}|x|}. \end{aligned} \tag{3.84}$$

For $x \in B(0,L)$, we have $|v_m^0(x)| \leq \|v_m^0\|_{L^\infty}$ which tends to zero as $m \to \infty$.

We will next derive estimates for the H^2 norm of $v_m^1 = v_m - P_m^\perp v_m^0$. This function is the part of v_m associated with the drifting terms in (3.73). Since $v_m^0 \in S^\perp$ we have

$$P_m^\perp v_m^0 = v_m^0 - \sum_{k=1}^n \sum_{j=1}^d \langle T_{z_m^j} \partial_{x_k} U_m, v_m^0 \rangle T_{z_m^j} \partial_{x_k} U_m = v_m^0 - P_z v_m^0.$$

Here P_z is the projection operator in $L^2(\mathbb{R}^n)$ onto its subspace

$$\mathrm{span}\{T_{z_m^j} \partial_{x_k} U_m \,|\, k = 1,2,\ldots n,\ j = 1,2,\ldots d\}.$$

To simplify the notation, we do not explicitly indicate the dependence of P_z on m. Define

$$\overline{U}_m = \sum_{j=1}^d T_{z_m^j} U_m.$$

Then (3.74) can be rewritten as

$$(-\Delta + V + E_m)(\overline{U}_m + W_m + v_m^1 - P_z v_m^0) + \sigma |\overline{U}_m + W_m + v_m^1 - P_z v_m^0|^{2p} (\overline{U}_m + W_m + v_m^1 - P_z v_m^0) = 0,$$

where $W_m = \psi_*^0 + \phi_m^a + v_m^0$. Using (3.81) it follows that the above equation is equivalent to

$$\begin{aligned} 0 = & (-\Delta + V + E_m) v_m^1 + V\overline{U}_m - \sigma \sum_{j=1}^d T_{z_m^j} |U_m|^{2p} U_m - \sigma |W_m|^{2p} W_m + \sum_{i=1}^{n_0} \beta_m^i \phi^i \\ & - V \sum_{k=1}^n \sum_{j=1}^d \langle T_{z_m^j} \partial_{x_k} U_m, v_m^0 \rangle T_{z_m^j} \partial_{x_k} U_m + \sigma(2p+1) \sum_{k=1}^n \sum_{j=1}^d \langle T_{z_m^j} \partial_{x_k} U_m, v_m^0 \rangle T_{z_m^j} |U_m|^{2p} \partial_{x_k} U_m \\ & + \sigma |\overline{U}_m + W_m + v_m^1 - P_z v_m^0|^{2p} (\overline{U}_m + W_m + v_m^1 - P_z v_m^0), \end{aligned}$$

which can be rewritten using the linear operator $L_+(\overline{U}_m + W_m - P_z v_m^0, E_m)$ and the nonlinear operator $N(\overline{U}_m + W_m - P_z v_m^0, v_m^1)$ as follows:

$$\begin{aligned}
0 = {} & L_+(\overline{U}_m + W_m - P_z v_m^0, E_m) v_m^1 + N(\overline{U}_m + W_m - P_z v_m^0, v_m^1) + V\overline{U}_m \\
& + \sigma|\overline{U}_m + W_m - P_z v_m^0|^{2p}(\overline{U}_m + W_m - P_z v_m^0) - \sigma \sum_{j=1}^{d} T_{z_m^j}|U_m|^{2p}U_m - \sigma|W_m|^{2p}W_m + \sum_{i=1}^{n_0} \beta_m^i \phi^i \\
& - V \sum_{k=1}^{n}\sum_{j=1}^{d} \left\langle T_{z_m^j}\partial_{x_k}U_m, v_m^0 \right\rangle T_{z_m^j}\partial_{x_k}U_m + \sigma(2p+1)\sum_{k=1}^{n}\sum_{j=1}^{d}\left\langle T_{z_m^j}\partial_{x_k}U_m, v_m^0\right\rangle T_{z_m^j}|U_m|^{2p}\partial_{x_k}U_m.
\end{aligned} \tag{3.85}$$

The argument based on Proposition 7.5 and spectral perturbation theory (see discussion above (3.79)) used to show that $P_m^{\perp} L_+(\widetilde{U}_m + \phi_m^a, E_m) P_m^{\perp}$ is invertible, with a uniform in m bound for its inverse, can be used to show that the same is true for $P_m^{\perp} L_+(\overline{U}_m + W_m - P_z v_m^0, E_m) P_m^{\perp}$. Indeed, let $L_+(\overline{U}_m + W_m - P_z v_m^0, E_m) = \widetilde{L}_m + \widetilde{W}_m$, where

$$\widetilde{L}_m = -\Delta + V + E_* + \sigma(2p+1)\Big(\sum_{j=1}^{d} \big|T_{z_m^j} U_{E_*}\big|^{2p} + |\psi_*^0|^{2p}\Big),$$

$$\widetilde{W}_m = E_m - E_* + \sigma(2p+1)\Big(\Big|\overline{U}_m + W_m - P_z v_m^0\Big|^{2p} - \sum_{j=1}^{d}\big|T_{z_m^j} U_{E_*}\big|^{2p} - |\psi_*^0|^{2p}\Big).$$

Since $E_m \to E_*$, $a_m^i \to 0$, $|y_m^j| \to \infty$ and $|y_m^j - y_m^k| \to \infty$ as $m \to \infty$, we get $\|\widetilde{W}_m\|_{H^2 \mapsto L^2} \to 0$ as $m \to \infty$. Applying the argument above (3.79) to $\widetilde{L}_m$ and $\widetilde{W}_m$ defined here, we get that $L_m = P_m^{\perp} L_+(\overline{U}_m + W_m - P_z v_m^0, E_m) P_m^{\perp}$ is invertible for large m and the norm of its inverse operator can be uniformly bounded. Therefore applying $P_m^{\perp}$ to (3.85) and rewriting it as $v_m^1 = L_m^{-1}$(remaining terms), and using $P_m^{\perp} \sum_{i=1}^{n_0} \beta_m^i \phi^i = 0$ and deriving estimates for terms containing product of exponentially decaying functions separated by large distances, we can deduce via contraction principle that

$$\|v_m^1\|_{H^2} \le C\|V\overline{U}_m\|_{L^2} + \sum_{j=1}^{d} O\Big(e^{-\sqrt{E_*-\gamma}|z_m^j|}\Big) + \sum_{i\neq j,\ i,j=1}^{d} O\Big(e^{-\sqrt{E_*-\gamma}|z_m^i - z_m^j|}\Big). \tag{3.86}$$

In summary, we have

$$v_m = P_m^{\perp} v_m^0 + v_m^1 = v_m^0 - P_z v_m^0 + v_m^1$$

with $v_m^0(x) \le Ce^{-\sqrt{E_*-\gamma}|x|}$ for all $x \in \mathbb{R}^n$ which, using the definition of P_z, gives

$$\|P_z v_m^0\|_{H^2} + \|v_m^1\|_{H^2} \le C\|V\overline{U}_m\|_{L^2} + \sum_{i\neq j,\ i,j=1}^{d} Ce^{-\sqrt{E_*-\gamma}|z_m^i - z_m^j|}\ .$$

We will now study the set of finite dimensional equations in (3.78) obtained by projecting (3.75) on the functions $T_{z_m^j}\partial_{x_k}U_m$. Explicitly, the set of equations are

$$
\begin{aligned}
0 = &\Big\langle T_{z_m^j}\partial_{x_k}U_m, V\sum_{i=1}^{d} T_{z_m^i}U_m + N(\widetilde{U}_m+\phi_m^a, v_m)\Big\rangle + (E_m - E_*)\Big\langle T_{z_m^j}\partial_{x_k}U_m, \psi^0_{E_*}+\phi_m^a\Big\rangle \\
&+\sigma\Big\langle T_{z_m^j}\partial_{x_k}U_m, |\widetilde{U}_m+\phi_m^a|^{2p}(\widetilde{U}_m+\phi_m^a) - \sum_{i=1}^{d} T_{z_m^i}|U_m|^{2p}U_m - |\psi^0_*|^{2p}\psi^0_* - (2p+1)|\psi^0_*|^{2p}\phi_m^a\Big\rangle \\
&+\Big\langle\Big(V+\sigma(2p+1)\big[|\widetilde{U}_m+\phi_m^a|^{2p} - T_{z_m^j}|U_m|^{2p}\big]\Big)T_{z_m^j}\partial_{x_k}U_m, v_m\Big\rangle \qquad (3.87)
\end{aligned}
$$

for $j = 1, 2, \dots d$ and $k = 1, 2, \dots n$. We will show that the set of equations in (3.87) has no solution. Our analysis will focus on two types of terms in (3.87) - terms involving interaction between $T_{z_m^j}U_m$ and $T_{z_m^k}U_m$ for $j \neq k$ (such terms arise from the second line of (3.87)) and terms involving interaction between the various $T_{z_m^i}U_m$s and the potential V. We will consider two cases, one of which must occur. In the first case we assume that the former terms, which decay like $e^{-\sqrt{E_*-\gamma}|z_m^j - z_m^k|}$, dominate the latter terms. Then, by combining the equations in (3.87) appropriately we construct a new equation in which the former terms dominate all the other terms. This will imply that there is no solution to (3.87). In the second case, we will combine the equations in (3.87) appropriately to obtain a new equation in which the former terms (at least the important ones) are eliminated. Under the assumption that the terms involving the potential dominate terms decaying like $e^{-2\sqrt{E_*-\gamma}|z_m^j - z_m^k|}$, we can apply the argument from the proof of the compactness case to conclude that (3.87) has no solution. We will now discuss the two cases rigorously.

Let $M_m = \min\{|z_m^j - z_m^k|\,\big|\, j,k = 1,2,\dots d,\ j\neq k\}$. We assume without loss of generality, by considering subsequences if necessary, that each of the bounded sequences $(M_m/|z_m^j - z_m^k|)_{m\in\mathbb{N}}$ is converging and that $\|VT_{z_m^1}U_m\|_{L^2}$ is the maximum of the set $\{\|VT_{z_m^j}U_m\|_{L^2}\,\big|\, j = 1,2,\dots d\}$ for each m and also that each of the sequences $(|z_m^1 - z_m^k|/|z_m^1|)_{m\in\mathbb{N}}$ converges either to a finite limit or to ∞. Partition $\{z_m^1, z_m^2, \dots z_m^d\}$ into sets Z_m^1 and Z_m^2 such that for each $z_m^k \in Z_m^1$ and no $z_m^k \in Z_m^2$ we have

$$\lim_{m\to\infty}\frac{|z_m^k - z_m^1|}{|z_m^1|} = 0.$$

Define

$$\widehat{U}_m = \sum_{z_m^k\in Z_m^1} T_{z_m^k}U_m \quad\text{and}\quad \widehat{W}_m = \sum_{z_m^k\in Z_m^2} T_{z_m^k}U_m.$$

In Case (i) we suppose that

$$\sqrt{E_*} < \limsup_{m\to\infty}\frac{-\ln\|VT_{z_m^j}U_m\|_{L^2}}{M_m} \qquad \forall\, j\in\{1,2,\dots d\} \qquad (3.88)$$

holds and in Case (ii) we suppose that

$$2\sqrt{E_*} > \liminf_{m\to\infty}\frac{-\ln\Big|\Big\langle\partial_{x_m}V, \widehat{U}_m^2\Big\rangle\Big|}{M_m}, \qquad (3.89)$$

where x_m is the direction along the vector z_m^1, holds. One of the above conditions must hold. Indeed if (3.88) does not hold for any $j\in\{1,2,\dots d\}$ then, since we have assumed that $\|VT_{z_m^1}U_m\|_{L^2}$ is the maximum of the set $\{\|VT_{z_m^j}U_m\|_{L^2}\,\big|\,j=1,2,\dots d\}$ for each m, it follows that

$$\limsup_{m\to\infty}\frac{-\ln\|VT_{z_m^1}U_m\|_{L^2}}{M_m}\le\sqrt{E_*}. \tag{3.90}$$

Then

$$\begin{aligned}&\liminf_{m\to\infty}\frac{-\ln\left|\left\langle\partial_{x_m}V,\widehat{U}_m^2\right\rangle\right|}{M_m}\\ =\;&\liminf_{m\to\infty}\frac{-\ln\|VT_{z_m^1}U_m\|_{L^2}^2}{M_m}\frac{-\ln\left|\left\langle\partial_{x_m}V,\widehat{U}_m^2\right\rangle\right|}{-\ln\|VT_{z_m^1}U_m\|_{L^2}^2}\\ <\;&2\sqrt{E_*},\end{aligned}$$

where we have used (3.90) and the fact that the second ratio in the second line above is less than 1 because for large m

$$1>\left|\left\langle\partial_{x_m}V,V_{R_m}\widehat{U}_m^2\right\rangle\right|>\|VT_{z_m^1}U_m\|_{L^2}^2.$$

Hence if (3.88) does not hold, then (3.89) holds, i.e. either Case (i) or Case (ii) must occur.

We say that the sequences $(z_m^j)_{m\in\mathbb{N}}$ and $(z_m^k)_{m\in N}$ are connected if and only if

$$\lim_{m\to\infty}\frac{M_m}{|z_m^j-z_m^k|}=1.$$

Case (i): Assume that (3.88) holds. We say that the sequences $(z_m^{a_1})_{m\in\mathbb{N}},(z_m^{a_2})_{m\in\mathbb{N}},\dots(z_m^{a_r})_{m\in\mathbb{N}}$, where each $a_i\in\{1,2,\dots d\}$, form a connected component if two conditions are satisfied: (a) For any $i,j\in\{a_1,a_2,\dots a_r\}$ with $i\ne j$, there exists a chain $(b_1,b_2,\dots b_t)$, where each $b_k\in\{a_1,a_2,\dots a_r\}$, $b_1=i$ and $b_t=j$, such that the sequences $(z_m^{b_k})_{m\in\mathbb{N}}$ and $(z_m^{b_{k+1}})_{m\in\mathbb{N}}$ are connected for each $k\in\{1,2,\dots r-1\}$ and (b) For any $i\notin\{a_1,a_2,\dots a_r\}$, there exists no $j\in\{a_1,a_2,\dots a_r\}$ such that $(z_m^i)_{m\in\mathbb{N}}$ is connected to $(z_m^j)_{m\in\mathbb{N}}$.

Consider a connected component with at least two sequences, say $(z_m^1)_{m\in\mathbb{N}},(z_m^2)_{m\in\mathbb{N}},\dots(z_m^r)_{m\in\mathbb{N}}$, see Figure 3.3. For each m, the baricenter of these z_m^js is

$$\bar{z}_m=\frac{z_m^1+z_m^2\dots+z_m^r}{r}.$$

By going to a subsequence if necessary, suppose that $q\in\{1,2,\dots r\}$, is such that $|z_m^q-\bar{z}_m|\ge|z_m^j-\bar{z}_m|$ for all $j\in\{1,2,\dots r\}$ and all m. For each z_m^l connected to z_m^q it follows that

$$\cos(\alpha_m^l)\ge\frac{|z_m^l-z_m^q|}{2|\bar{z}_m-z_m^q|},$$

where α_m^l is the angle between the vectors $z_m^l-z_m^q$ and $\bar{z}_m-z_m^q$. Also, we have

$$\begin{aligned}\lim_{m\to\infty}\frac{|\bar{z}_m-z_m^q|}{M_m}&=\lim_{m\to\infty}\frac{|z_m^1-z_m^q+\dots+z_m^{q-1}-z_m^q+z_m^{q+1}-z_m^q+\dots+z_m^r-z_m^q|}{rM_m}\\ &\le\lim_{m\to\infty}\sum_{i=1}^{r-1}\frac{|z_m^i-z_m^q|}{rM_m}\le\frac{(r-1)^2}{r}\le\frac{(d-1)^2}{d}.\end{aligned}$$

which implies that for each z_m^l connected to z_m^q

$$\lim_{m\to\infty} \frac{|z_m^l - z_m^q|}{2|\bar{z}_m - z_m^q|} \geq \liminf_{m\to\infty} \frac{r|z_m^l - z_m^q|}{2(r-1)\sum_{i=1}^{r-1}|z_m^i - z_m^{i+1}|} \geq \frac{r}{2(r-1)^2} \geq \frac{d}{2(d-1)^2}.$$

By letting $j = q$ in (3.87), we get that for some $\epsilon > 0$

$$\begin{aligned} 0 = &\Big\langle T_{z_m^q}\partial_{x_k}U_m, \sigma(2p+1)T_{z_m^q}U_m^{2p} \sum_{\substack{l: z_m^l, z_m^q \text{ is} \\ \text{connected}}} T_{z_m^l}U_m \Big\rangle \\ &+ C\sum_{j=1}^{d} \|VT_{z_m^j}U_m\|_{L^2} + \sum_{i\neq j,\ i,j=1}^{d} O\Big(e^{-(1+\epsilon)\sqrt{E_*-\gamma}|z_m^i - z_m^j|}\Big). \end{aligned} \tag{3.91}$$

Here we have used (3.16), which implies that $|V(x)| > C|x|^{-\beta+1}$ for large $|x|$, to dominate some exponentially decaying terms (for instance we have used that $\|VT_{z_m^j}U_m\|_{L^2}$ dominates terms with decay $e^{-\sqrt{E_*-\gamma}|z_m^j|}$). By choosing $x_k \parallel \bar{z}_m - z_m^q$ in (3.91) and by denoting the direction parallel to $(z_m^l - z_m^q)$ by x_{lq}, we get

$$\begin{aligned} &\Big|\Big\langle T_{z_m^q}\partial_{x_k}U_m, \sigma(2p+1)T_{z_m^q}U_m^{2p} \sum_{\substack{l: z_m^l, z_m^q \text{ is} \\ \text{connected}}} T_{z_m^l}U_m \Big\rangle\Big| \\ &= \Big|\sigma(2p+1) \sum_{\substack{l: z_m^l, z_m^q \text{ is} \\ \text{connected}}} \cos\alpha_m^l \big\langle T_{z_m^q}\partial_{x_{lq}}U_m, T_{z_m^q}U_m^{2p}T_{z_m^l}U_m\big\rangle\Big| \\ &\geq \frac{C(\gamma)d}{2(d-1)^2}\exp^{-\sqrt{E_*+\gamma}M_m}, \end{aligned} \tag{3.92}$$

for any $\gamma > 0$. While deriving the above expression, we have used the following fact: if $x_i \perp (z_m^l - z_m^q)$ and z_m^l and z_m^q are connected, then

$$\big\langle T_{z_m^q}\partial_{x_i}U_m, \sigma(2p+1)T_{z_m^q}U_m^{2p}T_{z_m^l}U_m\big\rangle = 0.$$

It now follows from (3.88) that as $m \to \infty$, for a sufficiently small $\gamma > 0$ and each $j \in \{1, 2, \ldots d\}$, we have (at least on a subsequence) that

$$\begin{aligned} &M_m\left[\sqrt{E_*+\gamma} + \frac{\ln\|VT_{z_m^j}U_m\|_{L^2}}{M_m}\right] \to -\infty \\ \iff\ &\sqrt{E_*+\gamma}M_m + \ln\|VT_{z_m^j}U_m\|_{L^2} \to -\infty, \\ \iff\ &e^{\sqrt{E_*+\gamma}M_m}\|VT_{z_m^j}U_m\|_{L^2} \to 0. \end{aligned}$$

The last expression above implies that the term (3.92) cannot be canceled by the remaining terms in (3.91). Therefore (3.91) and consequently the set of equations (3.87) has no solution in this case.

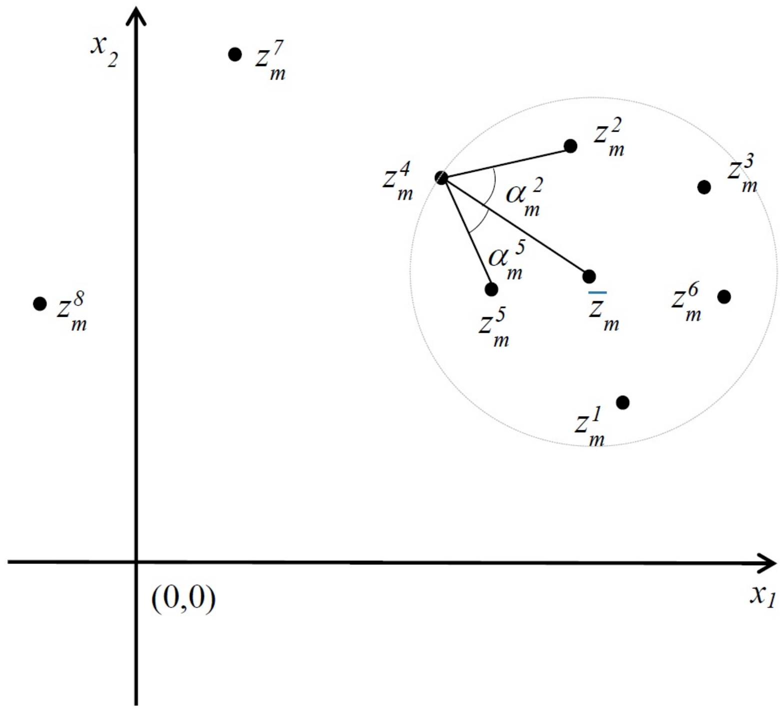

Figure 3.3: An example of grouping of the U_m's (each centered at a z_m^j) used while discussing splitting case (i). We assume that $n = 2$ and $d = 8$. The set $\{z_m^j : j = 1, \ldots, 6\}$ is a connected component whose baricenter is $\bar{z}_m$. $\bar{z}_m$ is also the center of the circle shown in the picture. Clearly z_m^4 is located at a maximal distance from $\bar{z}_m$ (i.e., $q = 4$). We consider a finite dimensional equation (3.91) with projection on to the function $\partial_{x_k} U_m(\cdot - z_m^4)$, where x_k is parallel to $\bar{z}_m - z_m^4$. We show that the terms containing U_m's centered at z_m^2 and z_m^5, which are connected to z_m^4, cannot be canceled in (3.91) by the remaining terms. This gives us the desired contradiction. The lower bound for $\cos \alpha_m^l$ used in Case (i) can be easily verified from the picture.

Case (ii): Assume that (3.89) holds. By adding the equations in (3.87) corresponding to each of the $z_m^i \in Z_m^1$, see Figure 3.4, we obtain

$$0 = \Big\langle \partial_{x_m} \widehat{U}_m, V(\widehat{U}_m + \widehat{W}_m) + \sigma(\widehat{U}_m + \widehat{W}_m)^{2p+1} - \sigma \widehat{U}_m^{2p+1} - \sigma \sum_{i=1}^{d} T_{z_m^i} U_m^{2p+1} \Big\rangle$$
$$+ C \sum_{j=1}^{d} \|V T_{z_m^j} U_m\|_{L^2}^2 + \sum_{i \neq j,\ i,j=1}^{d} O\Big(e^{-2\sqrt{E_* - \gamma}|z_m^i - z_m^j|}\Big). \tag{3.93}$$

Since the functions U_m are radially symmetric, one can check that for any $i, j \in \{1, 2, \ldots d\}$ and any direction x_k

$$\big\langle T_{z_m^i} \partial_{x_k} U_m, T_{z_m^j} U_m^{2p+1} \big\rangle + \big\langle T_{z_m^j} \partial_{x_k} U_m, T_{z_m^i} U_m^{2p+1} \big\rangle = 0.$$

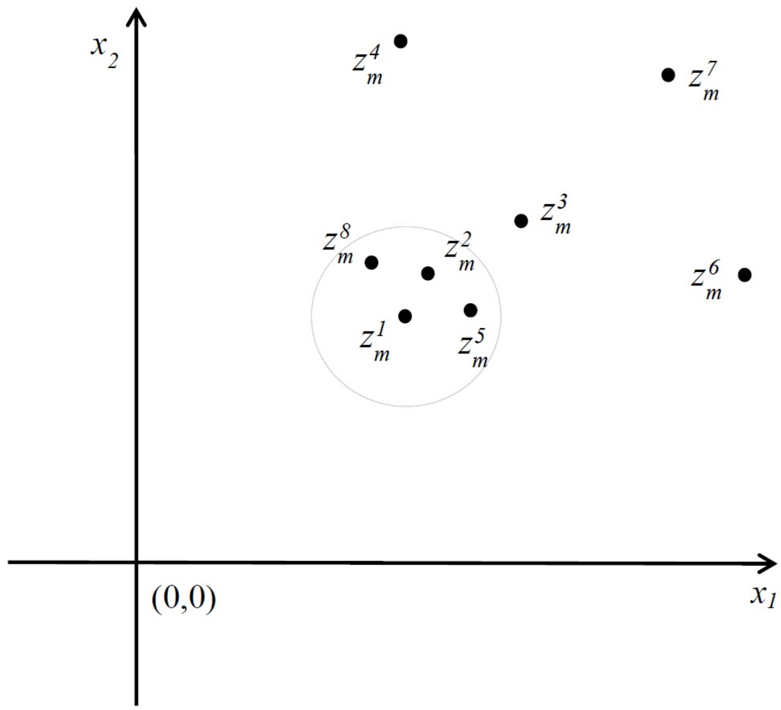

Figure 3.4: An example of grouping of the U_m's (each centered at a z_m^j) used while discussing splitting case (ii). We assume that $n = 2$ and $d = 8$. The norm $\|VT_{z_m^1} U_m\|_{L^2}$ is the maximum of the set $\{\|VT_{z_m^j} U_m\|_{L^2} \,|\, j = 1, 2, \ldots 8\}$ and z_m^2, z_m^5 and z_m^8 are near z_m^1. By adding the U_m's centered at z_m^1, z_m^2, z_m^5 and z_m^8 we get a function $\widehat{U}_m$. By treating $\widehat{U}_m$ like U_m in the case of compactness we get the desired contradiction.

In particular the above equality implies that

$$\left\langle \partial_{x_m} \widehat{U}_m, \sum_{i \in Z_m^1} T_{z_m^i} U_m^{2p+1} \right\rangle = 0. \tag{3.94}$$

Note that for some $\delta > 0$

$$\lim_{m \to \infty} \frac{|z_m^i - z_m^1|}{|z_m^1|} > \delta \quad \text{for each} \quad z_m^i \in Z_m^2.$$

Therefore

$$\lim_{m \to \infty} \frac{|z_m^i - z_m^k|}{|z_m^1|} > \delta \quad \text{for each} \quad z_m^k \in Z_m^1,\ z_m^i \in Z_m^2.$$

From the above discussion we get the estimate

$$\left\langle \partial_{x_m} \widehat{U}_m, V\widehat{W}_m + \sigma(\widehat{U}_m + \widehat{W}_m)^{2p+1} - \sigma \widehat{U}_m^{2p+1} - \sigma \sum_{l \in Z_m^2} T_{z_m^l} U_m^{2p+1} \right\rangle = O(e^{-\sqrt{E_* - \gamma}\delta |z_m^1|}). \tag{3.95}$$

The hypothesis in (3.16) implies that for some $C > 0$

$$\|VT_{z_m^1} U_m\|_{L^2}^2 > Ce^{-\sqrt{E_*-\gamma}\delta|z_m^1|}.$$

Using the above estimate, (3.94) and (3.95), we can rewrite (3.93) as

$$0 = \left\langle \partial_{x_m} \widehat{U}_m, V\widehat{U}_m \right\rangle + C\sum_{j=1}^{d} \|VT_{z_m^j} U_m\|_{L^2}^2 + \sum_{i\neq j,\ i,j=1}^{d} O\Big(e^{-2\sqrt{E_*-\gamma}|z_m^i - z_m^j|}\Big). \tag{3.96}$$

We will next show that the first term in the above equation dominates the other two terms. Comparing the first term with the third term, it follows from (3.89) that for a sufficiently small $\gamma > 0$, as $m \to \infty$ (considering a subsequence if necessary)

$$\begin{aligned} & M_m \left(\frac{-\ln\left|\left\langle \partial_{x_m} V, \widehat{U}_m^2 \right\rangle\right|}{M_m} - 2\sqrt{E_* - \gamma} \right) \to -\infty \\ \iff \quad & -\ln\left|\left\langle \partial_{x_m} V, \widehat{U}_m^2 \right\rangle\right| - 2\sqrt{E_* - \gamma} M_m \to -\infty \\ \iff \quad & \frac{e^{-2\sqrt{E_*-\gamma}M_m}}{\left|\left\langle \partial_{x_m} V, \widehat{U}_m^2 \right\rangle\right|} \to 0. \end{aligned}$$

We next compare the first term with the second term of (3.96). Using the hypothesis (3.15) we get, as in the compactness case, that

$$\lim_{m\to\infty} \frac{\left|\langle VT_{z_m^1} U_m, VT_{z_m^1} U_m \rangle\right|^2}{\left|\left\langle \widehat{U}_m^2, \partial_{x_m} V \right\rangle\right|} \le \lim_{m\to\infty} \frac{\left|\left\langle \widehat{U}_m V, \widehat{U}_m V \right\rangle\right|^2}{\left|\left\langle \widehat{U}_m^2, \partial_{x_m} V \right\rangle\right|} = 0.$$

Hence the term $\langle \partial_{x_m} \widehat{U}_m, V\widehat{U}_m \rangle$ in (3.96) cannot be canceled by the remaining two terms. Thus we get that (3.93), and consequently the set of equations (3.87), has no solution in this case.

Since either Case (i) or Case (ii) must occur, we get the contradiction that the set of equations (3.87) has no solution if splitting occurs. This completes the proof of this theorem.

□

We currently do not have a similar result for arbitrary solutions of the time-independent Schrödinger equation (1.3) with focussing nonlinearity. The reason is that the delicate estimates in the nonlinear system of finite equations with finitely many unknowns obtained after a Lyapunov-Schmidt decomposition when attempting to show that the profiles cannot approach the limit $|x| = \infty$. Currently the estimates are using the positivity of the profiles, but we hope to remove this restriction in the future.

4 Behavior of bound states at large E

In this section we analyze the limit points of zeroes of F i.e., bound states of (1.1), at the $E = \infty$ boundary. Note that Corollary 2.3 prevents branches of bound states from reaching this limit in the defocusing case $\sigma > 0$. However, in the focusing case $\sigma < 0$, we expect that all branches of bound states will end up at $E = \infty$ after maybe undergoing a finite number of bifurcations. Indeed, Theorem 2.2 guarantees it for maximal branches along which e-values of L_+ do not approach zero. For any maximal ground state branches on finite interval $E \in (E_-, E_+)$, $E_+ < \infty$ we already know from Theorem 3.3 that it has (at least) a limit point (ψ_{E_+}, E_+) which is a zero of F. In the analytic case we get that the limit point is unique and branch can be analytically continued past the bifurcation point (ψ_{E_+}, E_+), see [3]. By repeating this argument the branch extended analytically past the possible bifurcation points either forms a loop or ends up at the boundary of the Fredholm domain $H^2(\mathbb{R}^n, \mathbb{C}) \times [0, \infty)$. But Remark 2.7, respectively Theorem 3.1, exclude the $E = 0$, respectively the finite E infinite H^2 norm portion of the boundary. Therefore, with the exception of the loops, all ground state branches will reach $E = \infty$. We think that the restrictions to ground state branches and analytical maps will be both removed in the near future, hence all bound state branches will have an extension to $E = \infty$ boundary. Moreover, as we shall see in the examples of Section 6, by analyzing the bound state branches near $E = \infty$ we will be able to discover all the non-loop branches in the interior of the Fredholm domain i.e., solutions (ψ_E, E) of (1.3) with $E > 0$, together with their stability and bifurcation points.

The first result shows that the L^2 norm $\mathcal{N}(E)$, the energy $\mathcal{E}(E)$ and consequently the H^1 and H^2 norms of any $E \mapsto \psi_E$, C^1 curve of solutions of (1.3) have a very precise behavior as $E \to \infty$, see Theorem 4.1. This leads to a unique choice for a renormalization (change of variables) which renders the curve bounded in H^2 as $E \to \infty$. We then employ concentration compactness to show that such a curve, modulo the renormalization, must actually converge to a superposition of finitely many, nontrivial solutions of the limiting equation:

$$-\Delta u + u + \sigma |u|^{2p} u = 0. \tag{4.1}$$

This is the well known and studied, constant coefficient, nonlinear Schrödinger equation, see for example [4]. Its set of solutions is invariant under the Euclidian group of transformations in $\mathbb{R}^n$ and is formed by smooth, exponentially decaying functions, see Appendix. In general we can show that the limiting point of our curve of bound states must be a finite superposition of solutions of (4.1). However, since the limiting equation has a unique (up to translations in $\mathbb{R}^n$) positive solution u_∞, see [23], we can be much more precise with the ground state branches, namely we can show that, up to rotations in the complex plane, they must converge in H^2 to a superposition of u_∞ each centered at a critical point of the potential V, see Theorem 4.2.

Reciprocally, we show that solutions of our problem (1.3) in a sufficiently small H^2 neighborhood of any superposition of $\pm u_\infty$ each centered at a distinct critical point of the potential, form a unique, 2 dimensional C^1 manifold obtained by rotating a curve of real valued solutions, see Theorem 4.3. The orbital stability of these solutions is studied in the same theorem.

Remark 4.1. *Note that the two main theorems of this section determines almost all ground state branches at large E. Indeed Theorem 4.2 shows that, modulo a rotation and renormalization, any such branch must converge in H^2 to a finite superposition of positive, u_∞, solutions of* (4.1) *each centered at a critical point of V. Theorem 4.3 confirms that near such a superposition, with at most one u_∞ at each critical point of V, there is a unique set of ground states and, moreover, the set can be organized as a C^1 manifold. We are missing the case when the superposition has multiple profiles u_∞ approaching* the same *critical point. This situation occurs in related problems, see [26], and is* *allowed by Theorem 4.2 but not analyzed by Theorem 4.3. We will analyze it in a future paper, our current partial results show that multiple profiles at a minima of V cannot occur but saddle points and maxima of V allow for any number of profiles to approach it along the directions given by the e-vectors of the Hessian of V at the saddle or maximal point corresponding to negative e-values.*

In the next theorem we use the following notation. For any $z \in \mathbb{R}^n$, let T_z be the shift operator defined as follows: For any function u defined on $\mathbb{R}^n$, $T_z u(x) = u(x - z)$ for all $x \in \mathbb{R}^n$. We define $\mathcal{I} = \{1, 2, \dots n\}$.

Theorem 4.1. *Suppose that $\sigma < 0$ and the potential V is such that*

$$V \in L^\infty(\mathbb{R}^n), \qquad \lim_{|x|\to\infty} V(x) = 0 \qquad \text{and} \qquad x.\nabla V \in L^\infty(\mathbb{R}^n). \tag{4.2}$$

Consider a C^1 branch of real-valued solutions (ψ_E, E) of (1.3) on the interval $[E_1, \infty)$, for some $E_1 > E_0$. Then the following assertions hold:

(i) There exists a $0 < b < \infty$ such that

$$\lim_{E\to\infty} \frac{\|\psi_E\|_{L^{2p+2}}^{2p+2}}{E^{1-\frac{n}{2}+\frac{1}{p}}} = b, \tag{4.3}$$

$$\lim_{E\to\infty} \frac{\|\psi_E\|_{L^2}^{2}}{E^{\frac{1}{p}-\frac{n}{2}}} = \frac{-\sigma}{2}\left(\frac{2p+2-np}{p+1}\right) b, \tag{4.4}$$

$$\lim_{E\to\infty} \frac{\|\nabla\psi_E\|_{L^2}^{2}}{E^{1-\frac{n}{2}+\frac{1}{p}}} = \frac{-\sigma}{2}\left(\frac{np}{p+1}\right) b. \tag{4.5}$$

Moreover, the function u_E defined as

$$u_E(x) = E^{-\frac{1}{2p}} \psi_E(E^{-\frac{1}{2}}x) \qquad \forall x \in \mathbb{R}^n \tag{4.6}$$

is such that

$$\lim_{E\to\infty} \|u_E\|_{H^1} = -\sigma b > 0 \tag{4.7}$$

and u_E solves the equation

$$-\Delta u + E^{-1} V_{E^{-\frac{1}{2}}} u + u + \sigma |u|^{2p} u = 0. \tag{4.8}$$

Here $V_{E^{-\frac{1}{2}}}(x) = V(E^{-\frac{1}{2}}x)$ for all $x \in \mathbb{R}^n$.

(ii) For any sequence $(E_m)_{m=1}^\infty$ in $[E_1, \infty)$ with $\lim_{m\to\infty} E_m = \infty$, there exists a subsequence $(E_{m_k})_{k=1}^\infty$, a positive integer M and a subsequence $(x_k^i)_{k=1}^\infty$ in $\mathbb{R}^n$ for each $i \in \{1, 2, \dots M\}$ such that

$$\lim_{k\to\infty} \left\| u_{E_{m_k}} - \sum_{i=1}^{M} u_i(\cdot - x_k^i) \right\|_{H^2} = 0 \tag{4.9}$$

and for every $1 \le i, j \le M$ with $i \neq j$

$$\lim_{k\to\infty} |x_k^i - x_k^j| = \infty \quad \text{and} \quad \text{either} \quad \lim_{k\to\infty} \frac{x_k^i}{\sqrt{E_{m_k}}} = x^i \quad \text{or} \quad \lim_{k\to\infty} \frac{x_k^i}{\sqrt{E_{m_k}}} = \infty. \tag{4.10}$$

Here $u_{E_{m_k}}$ is the scaled version of $\psi_{E_{m_k}}$ obtained by letting $E = E_{m_k}$ in (4.6) and each u_i is a solution of (4.1).

Remark 4.2. *The above theorem applies to any C^1 branch of solutions of* (1.3) *and not just to C^1 branches of ground states. Also in this theorem p can be smaller than $1/2$ i.e., $0 < p < 2/(n-2)$.*

Proof. Consider a C^1 branch of real-valued solutions (ψ_E, E) of (1.3) on the interval $[E_1, \infty)$ for some $E_1 > E_0$. From (2.12) recall the energy functional

$$\mathcal{E}(E) = \int_{\mathbb{R}^n} |\nabla \psi_E(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} V(x)|\psi_E(x)|^2 \,\mathrm{d}x + \frac{\sigma}{p+1} \int_{\mathbb{R}^n} |\psi_E(x)|^{2p+2} \,\mathrm{d}x.$$

Recall $\mathcal{N}(E) = \|\psi_E\|^2_{L^2}$, see (3.1). Then from (2.13) we have

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}E} = -E\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}E}$$

and from (2.15) we have

$$\mathcal{E}(E) + \frac{\sigma p}{p+1}\|\psi_E\|^{2p+2}_{L^{2p+2}} = -E\mathcal{N}(E).$$

Differentiating the above equation with respect to E gives

$$\frac{\mathrm{d}}{\mathrm{d}E}\|\psi_E\|^{2p+2}_{L^{2p+2}} = \frac{p+1}{-\sigma p}\mathcal{N}(E). \tag{4.11}$$

Since ψ_E is a weak solution of (1.3), using $x.\nabla\psi_E$ as a test function and integrating by parts we get the Pohozaev type identity

$$(n-2)\|\nabla\psi_E\|^2_{L^2} + \int_{\mathbb{R}^n} (nV(x) + x.\nabla V(x))|\psi_E(x)|^2 \mathrm{d}x + \frac{n\sigma}{p+1}\|\psi_E\|^{2p+2}_{L^{2p+2}} = -En\|\psi_E\|^2_{L^2}. \tag{4.12}$$

Adding (4.12) with (2.15) multiplied by $-(n-2)$ gives

$$\int_{\mathbb{R}^n} (2V(x) + x.\nabla V(x))|\psi_E(x)|^2 \mathrm{d}x + \sigma\left(\frac{n}{p+1} + 2 - n\right)\|\psi_E\|^{2p+2}_{L^{2p+2}} = -2E\|\psi_E\|^2_{L^2},$$

which can be written equivalently as

$$2E\mathcal{N}(E) + \int_{\mathbb{R}^n} (2V(x) + x.\nabla V(x))|\psi_E(x)|^2 \mathrm{d}x = -\sigma\left(2 - \frac{np}{p+1}\right)\|\psi_E\|^{2p+2}_{L^{2p+2}}.$$

Since we have $2p < 4/(n-2)$ for $n > 2$, it follows that $2 - np/(p+1) > 0$ for any n. Bounding the integral term in the identity above via Hölder inequality we get

$$2(E-C)\mathcal{N}(E) \le -\sigma\frac{2p+2-np}{p+1}\|\psi_E\|^{2p+2}_{L^{2p+2}} \le 2(E+C)\mathcal{N}(E), \tag{4.13}$$

where $2C = 2\|V\|_{L^\infty} + \|x.\nabla V\|_{L^\infty}$. Denoting $Q(E) = \|\psi_E\|^{2p+2}_{L^{2p+2}}$ and using (4.11) in (4.13) it follows that

$$\left(\frac{1}{p} + 1 - \frac{n}{2}\right)\frac{Q(E)}{E+C} \le \frac{dQ(E)}{dE} \le \left(\frac{1}{p} + 1 - \frac{n}{2}\right)\frac{Q(E)}{E-C}. \tag{4.14}$$

Fix $E_2 > \max\{C, E_1\}$ and $E_3 > E_2$ and integrate the above expression on $[E_2, E_3]$ to get

$$\frac{Q(E_2)}{(E_2+C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}}(E_3+C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)} \le Q(E_3) \le \frac{Q(E_2)}{(E_2-C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}}(E_3-C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}$$

and therefore

$$\frac{Q(E_2)}{(E_2+C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}} \leq \frac{Q(E_3)}{E_3^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}} \leq \frac{Q(E_2)}{(E_2-C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}}.$$

This implies (4.3) since

$$\lim_{E_2\to\infty} \frac{\frac{Q(E_2)}{(E_2-C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}}}{\frac{Q(E_2)}{(E_2+C)^{\left(\frac{1}{p}+1-\frac{n}{2}\right)}}} = 1.$$

Now (4.4) follows from dividing (4.13) by $E^{1-n/2+1/p}$ and passing to the limit as $E \to \infty$, while (4.5) follows from dividing (2.15) by $E^{1-n/2+1/p}$ and passing to the limit as $E \to \infty$.

Using the fact that ψ_E solves (1.3), it is easy to verify that u_E solves (4.8). Moreover (4.6) combined with (4.3)-(4.5) gives

$$\begin{aligned}
\|u_E\|_{L^{2p+2}}^{2p+2} &= E^{-1-\frac{1}{p}+\frac{n}{2}}\|\psi_E\|_{L^{2p+2}}^{2p+2} \overset{E\to\infty}{\longrightarrow} b,\\
\|u_E\|_{L^2}^2 &= E^{-\frac{1}{p}+\frac{n}{2}}\|\psi_E\|_{L^2}^2 \overset{E\to\infty}{\longrightarrow} \frac{-\sigma}{2}\frac{2p+2-np}{p+1}b,\\
\|\nabla u_E\|_{L^2}^2 &= E^{-1-\frac{1}{p}+\frac{n}{2}}\|\nabla\psi_E\|_{L^2}^2 \overset{E\to\infty}{\longrightarrow} \frac{-\sigma}{2}\frac{np}{p+1}b.
\end{aligned} \tag{4.15}$$

Adding the last two equations we get (4.7). This completes the proof of part (i) of the theorem.

To prove part (ii) of the theorem, consider a sequence $(E_m)_{m=1}^\infty$ in $[E_1,\infty)$ with $\lim_{m\to\infty} E_m = \infty$. Define

$$u_{E_m}(x) = E_m^{-\frac{1}{2p}}\psi_{E_m}(E_m^{-\frac{1}{2}}x) \qquad \forall\, x \in \mathbb{R}^n.$$

From part (i) we know that the sequence $(u_{E_m})_{m=1}^\infty$ is bounded in H^1 and the sequence $(\|u_{E_m}\|_{L^2})_{m=1}^\infty$ converges to a nonzero constant, see (4.15). Define

$$\Psi_m = \frac{u_{E_m}}{\|u_{E_m}\|_{L^2}}.$$

Like in Section 3 (see discussion below (3.19)), we will apply concentration compactness theory to $(\Psi_m)_{m\in\mathbb{N}}$ by defining the concentration function ρ and the variable μ for the sequence $(\Psi_m)_{m\in\mathbb{N}}$. Out of three possible cases in the concentration compactness theory we will show that vanishing cannot occur and compactness and splitting imply (4.9) and (4.10) for a subsequence of u_{E_m}.

Vanishing ($\mu = 0$): In this case there is a subsequence of $(\Psi_m)_{m\in\mathbb{N}}$, again denoted by $(\Psi_m)_{m\in\mathbb{N}}$, such that along this subsequence

$$\lim_{m\to\infty}\|\Psi_m\|_{L^q} = 0 \quad \text{for all} \quad 2 < q < \frac{2n}{n-2} \qquad (2 < q < \infty \text{ if } n=2,\ 2 < q \leq \infty \text{ if } n=1). \tag{4.16}$$

Since $u_{E_m} = \|u_{E_m}\|_{L^2}\Psi_m$ and part (i) gives

$$\lim_{E_m\to\infty}\|u_{E_m}\|_{L^2} = a > 0, \tag{4.17}$$

it follows from (4.16) that there is a subsequence of $(u_{E_m})_{m\in\mathbb{N}}$, again denoted by $(u_{E_m})_{m\in\mathbb{N}}$, such that along this subsequence

$$\lim_{m\to\infty} \|u_{E_m}\|_{L^q} = 0 \quad \text{for all} \quad 2 < q < \frac{2n}{n-2} \qquad (2 < q < \infty \text{ if } n = 2,\ 2 < q \le \infty \text{ if } n = 1). \tag{4.18}$$

Note that u_{E_m} solves (4.8) (with $E = E_m$ of course). Therefore

$$u_{E_m} = (-\Delta + 1)^{-1} \left[-R_m^2 V_{R_m} u_{E_m} - \sigma |u_{E_m}|^{2p} u_{E_m} \right], \tag{4.19}$$

where $R_m = (E_m)^{-1/2}$ and $V_{R_m}(x) = V(R_m x)$ for all $x \in \mathbb{R}^n$. Clearly $\lim_{m\to\infty} R_m = 0$. Since $L^2 \hookrightarrow H^{-1}$ and $\|u_{E_m}\|_{L^2}$ is uniformly bounded, we get that for some $C > 0$

$$\lim_{m\to\infty} \|R_m^2 V_{R_m} u_{E_m}\|_{H^{-1}} \le C \lim_{m\to\infty} \|R_m^2 V_{R_m} u_{E_m}\|_{L^2} \le C \lim_{m\to\infty} R_m^2 \|V\|_{L^\infty} \|u_{E_m}\|_{L^2} = 0. \tag{4.20}$$

Fix r such that if $n > 2$, then $2 < r < 2n/(n-2)$ and $2n/(n+2) < r/(2p+1) < 2$ and if $n \le 2$, then $2 < r < \infty$ and $1 < r/(2p+1) < 2$. For $n > 2$, the fact that such an r exists follows easily from the observation that $p < 2/(n-2)$ implies $(2p+1)2n/(n+2) < 2n/(n-2)$. Then it follows that $L^{\frac{r}{2p+1}} \hookrightarrow H^{-1}$ and so for some $C > 0$

$$\lim_{m\to\infty} \|\, |u_{E_m}|^{2p} u_{E_m} \|_{H^{-1}} \le C \lim_{m\to\infty} \| |u_{E_m}|^{2p} u_{E_m} \|_{L^{\frac{r}{2p+1}}} = C \lim_{m\to\infty} \|u_{E_m}\|_{L^r}^{2p+1} = 0. \tag{4.21}$$

Here we have used (4.18) to obtain the last equality. It now follows easily from (4.20) and (4.21) and the fact that the operator $(-\Delta+1)^{-1} : H^{-1} \mapsto H^1$ is unitary that

$$\lim_{m\to\infty} \|u_{E_m}\|_{H^1} = 0$$

which contradicts (4.17). Therefore vanishing cannot happen.

Compactness ($\mu = 1$): Since $u_{E_m} = \|u_{E_m}\|_{L^2} \Psi_m$ and (4.17) holds, it follows from concentration compactness theory that if $\mu = 1$, then there exists a subsequence of $(E_m)_{m\in\mathbb{N}}$, also denoted as $(E_m)_{m\in\mathbb{N}}$, a corresponding sequence $(y_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^n$ and a function $\hat{u}_* \in H^1(\mathbb{R}^n)$ such that for each $m \in \mathbb{N}$, u_{E_m} can be decomposed as follows:

$$u_{E_m} = T_{y_m} \hat{u}_m + \hat{v}_m, \tag{4.22}$$

where $\hat{u}_m, \hat{v}_m \in H^1(\mathbb{R}^n)$ and as $m \to \infty$

(i) $\hat{u}_m \overset{H^1}{\rightharpoonup} \hat{u}_*$ and $\hat{u}_m \overset{L^q}{\to} \hat{u}_*$ for $2 \le q < 2n/(n-2)$ $(2 \le q < \infty$ if $n = 2, 2 \le q \le \infty$ if $n = 1)$,

(ii) $\hat{v}_m \overset{H^1}{\rightharpoonup} 0$ and $\hat{v}_m \overset{L^q}{\to} 0$ for $2 \le q < 2n/(n-2)$ $(2 \le q < \infty$ if $n = 2, 2 \le q \le \infty$ if $n = 1)$.

The pair (u_{E_m}, R_m) satisfies (4.8) and therefore, by translating all the functions in this equation by $-y_m$, we get

$$(-\Delta + R_m^2 T_{-y_m} V_{R_m} + 1) T_{-y_m} u_{E_m} + \sigma T_{-y_m} |u_{E_m}|^{2p} u_{E_m} = 0.$$

Here $R_m = E_m^{-1/2}$ and $V_{R_m}(x) = V(R_m x)$ for all $x \in \mathbb{R}^n$. Taking the L^2 scalar product of the above equation with any $\phi \in C_0^\infty(\mathbb{R}^n)$ gives

$$\langle \nabla T_{-y_m} u_{E_m}, \nabla\phi\rangle + R_m^2 \langle T_{-y_m} V_{R_m} T_{-y_m} u_{E_m}, \phi(x)\rangle + \langle T_{-y_m} u_{E_m}, \phi\rangle + \sigma \left\langle T_{-y_m} |u_{E_m}|^{2p} u_{E_m}, \phi\right\rangle = 0.$$

Substituting for u_{E_m} from (4.22) into the last equation and taking the limit as $m \to \infty$ we obtain, using (i) and (ii) above, that

$$\langle \nabla\hat{u}_*, \nabla\phi\rangle + \langle \hat{u}_*, \phi\rangle + \sigma \left\langle |\hat{u}_*|^{2p}\hat{u}_*, \phi\right\rangle = 0.$$

Thus $\hat{u}_*$ is a weak solution (and therefore a strong, H^2 solution) of (4.1). Redefining $\hat{v}_m$ to be $\hat{v}_m + T_{y_m}\hat{u}_m - T_{y_m}\hat{u}_*$ we obtain the following different decomposition for u_{E_m} in place of (4.22):

$$u_{E_m} = T_{y_m}\hat{u}_* + \hat{v}_m. \tag{4.23}$$

The redefined functions $\hat{v}_m$ in (4.23) also have the convergence properties mentioned in (ii) above. Since (u_{E_m}, R_m) satisfies (4.8), substituting the above decomposition for u_{E_m} into (4.8) and using the fact that $\hat{u}_*$ satisfies (4.1), we obtain

$$\hat{v}_m = -(-\Delta+1)^{-1}\left[R_m^2 V_{R_m} T_{y_m}\hat{u}_* + (R_m^2 V_{R_m} + \sigma(2p+1)|T_{y_m}\hat{u}_*|^{2p})\hat{v}_m + N(T_{y_m}\hat{u}_*, \hat{v}_m)\right]. \tag{4.24}$$

Here $N(U,v) : H^2(\mathbb{R}^n)\times H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ is the nonlinear function from (3.7),

$$N(U,v) = \sigma|U+v|^{2p}(U+v) - \sigma|U|^{2p}U - \sigma(2p+1)|U|^{2p}v.$$

From the limit $\hat{v}_m \stackrel{L^2}{\to} 0$ as $m\to\infty$ from (ii), $\lim_{m\to\infty} R_m = 0$ and $u^* \in L^\infty$ (see Proposition 7.1), it follows that

$$\lim_{m\to\infty} \|R_m^2 V_{R_m} T_{y_m}\hat{u}_* + (R_m^2 V_{R_m} + \sigma(2p+1)|T_{y_m}\hat{u}_*|^{2p})\hat{v}_m\|_{L^2} = 0$$

and from Lemma 3.1 and the limits for $\hat{v}_m$ in (ii) it follows that

$$\lim_{m\to\infty} \|N(T_{y_m}\hat{u}_*, \hat{v}_m)\|_{L^2} = 0. \tag{4.25}$$

Hence from (4.24) we get that

$$\lim_{m\to\infty} \|\hat{v}_m\|_{H^2} = 0. \tag{4.26}$$

Suppose that $(R_m y_m)_{m\in\mathbb{N}}$ has a converging subsequence, denoted again by $(R_m y_m)_{m\in\mathbb{N}}$, converging to $x^1 \in \mathbb{R}^n$. Then it follows from (4.23) and (4.26) that (4.9) and the second limit in (4.10) hold with $M=1$, $u_1 = \hat{u}_*$ and $x_k^1 = y_k$. On the other hand, if $(R_m y_m)_{m\in\mathbb{N}}$ has no bounded subsequence, then it follows from (4.23) and (4.26) that (4.9) and the third limit in (4.10) hold with $M=1$, $u_1 = \hat{u}_*$ and $x_k^1 = y_k$. In either case, the first limit in (4.10) holds vacuously. This completes the proof of part (ii) in the compactness case.

Splitting ($0<\mu<1$): Since $u_{E_m} = \|u_{E_m}\|_{L^2}\Psi_m$ and (4.17) holds, it follows from concentration compactness theory that if $0<\mu<1$, then there exists a subsequence of $(E_m)_{m\in\mathbb{N}}$, again denoted as $(E_m)_{m\in\mathbb{N}}$, on which we can decompose u_{E_m} as follows:

$$u_{E_m} = \sum_{j=1}^{d} T_{y_m^j} u_m^j + u_m^0 + \hat{v}_m, \tag{4.27}$$

where $u_m^j, v_m \in H^1(\mathbb{R}^n)$ for all $j \in \mathcal{J} = \{0,1,\dots d\}$ and there exist non-zero functions u_*^j such that as $m\to\infty$

(i) $u_m^j \overset{H^1}{\rightharpoonup} u_*^j$ and $u_m^j \overset{L^q}{\to} u_*^j$ for $2 \le q < 2n/(n-2)$ ($2 \le q < \infty$ if $n = 2$, $2 \le q \le \infty$ if $n = 1$),

(ii) $|y_m^j| \to \infty$ and $|y_m^j - y_m^k| \to \infty$ for $j, k \in \{1, 2, \ldots d\}$ and $j \neq k$,

(iii) $\hat{v}_m \overset{H^1}{\rightharpoonup} 0$ and $\hat{v}_m \overset{L^q}{\to} 0$ for $2 < q < 2n/(n-2)$ ($2 < q < \infty$ if $n = 2$, $2 < q \le \infty$ if $n = 1$).

Concentration compactness theory does not imply that $u_*^0 \neq 0$, but we assume it only to simplify our presentation. All the arguments in this proof remain valid if $u_*^0 = 0$. Letting $u = u_{E_m}$ and $E = E_m$ in (4.8) and taking the L^2 scalar product of the resulting equation with $\phi \in C_0^\infty(\mathbb{R}^n)$ we get

$$\big\langle \nabla u_{E_m}, \nabla \phi \big\rangle + \big\langle (R_m^2 V_{R_m} + 1) u_{E_m}, \phi \big\rangle + \sigma \big\langle |u_{E_m}|^{2p} u_{E_m}, \phi \big\rangle = 0.$$

Here $R_m = E_m^{-1/2}$ and $V_{R_m}(x) = V(R_m x)$ for all $x \in \mathbb{R}^n$. Substituting for u_{E_m} from (4.27) into the above equation and taking the limit as $m \to \infty$ gives

$$\big\langle \nabla u_*^0, \nabla \phi \big\rangle + \big\langle u_*^0, \phi \big\rangle + \sigma \big\langle |u_*^0|^{2p} u_*^0, \phi \big\rangle = 0.$$

Therefore u_*^0 is a weak solution (and hence a strong, H^2 solution) of (4.1). Similarly for any $j \in \{1, 2, \ldots d\}$, taking the L^2 scalar product of the equation

$$(-\Delta + 1 + R_m^2 T_{-y_m^j} V_{R_m}) T_{-y_m^j} u_{E_m} + \sigma T_{-y_m^j} |u_{E_m}|^{2p} u_{E_m} = 0$$

(which is nothing but a translated version of (4.8) with $E = E_m$ and $u = u_{E_m}$) with $\phi \in C_0^\infty(\mathbb{R}^n)$ and taking the limit as $m \to \infty$ we can conclude that u_*^j is a weak solution (and hence a strong, H^2 solution) to (4.1). Hence for each $j \in \mathcal{J}$ the function u_*^j is a solution of (4.1). Redefining $\hat{v}_m$ to be $\hat{v}_m + u_m^0 + \sum_{j=1}^d T_{y_m^j} u_m^j - u_*^0 - \sum_{j=1}^d T_{y_m^j} u_*^j$ we obtain the following different decomposition for u_{E_m}:

$$u_{E_m} = \sum_{j=1}^d T_{y_m^j} u_*^j + u_*^0 + \hat{v}_m. \tag{4.28}$$

The redefined $\hat{v}_m$ satisfy the limits in (iii) above and from Proposition 7.2 in the Appendix we get that there exist $C > 0$ and $\gamma \in (0, 1)$ such that

$$|u_*^j(x)| < C e^{-\sqrt{1-\gamma}|x|} \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, j \in \mathcal{J}. \tag{4.29}$$

Since (u_{E_m}, R_m) satisfies (4.8), substituting the decomposition in (4.28) for u_{E_m} into (4.8) and using the fact that each u_*^j satisfies (4.1), we get

$$\begin{aligned} \hat{v}_m = -(-\Delta + 1)^{-1} \Bigg[& R_m^2 V_{R_m} \sum_{j=0}^d T_{y_m^j} u_*^j + \Big(R_m^2 V_{R_m} + \sigma(2p+1) \Big| \sum_{j=0}^d T_{y_m^j} u_*^j \Big|^{2p} \Big) \hat{v}_m \\ & -\sigma \sum_{j=0}^d T_{y_m^j} |u_*^j|^{2p} u_*^j + \sigma \Big| \sum_{j=0}^d T_{y_m^j} u_*^j \Big|^{2p} \sum_{j=0}^d T_{y_m^j} u_*^j + N\Big(\sum_{j=0}^d T_{y_m^j} u_*^j, \hat{v}_m \Big) \Bigg]. \end{aligned} \tag{4.30}$$

Here $y_m^0 = 0$ and $N(U, v) : H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ is the nonlinear function defined in (3.7). We will now show that the term in the bracket in the above expression (to which $-(-\Delta + 1)^{-1}$ is applied) converges to 0 in $L^2(\mathbb{R}^n)$ as $m \to \infty$. Indeed, since $u_*^j \in L^2(\mathbb{R}^n)$ for all $j \in \mathcal{J}$, $V \in L^\infty(\mathbb{R}^n)$, $\sup_{m\in\mathbb{N}} \|v_m\|_{L^2} < \infty$ and $\lim_{m\to\infty} R_m = 0$, we have

$$\lim_{m\to\infty} \Big\| R_m^2 V_{R_m} \sum_{j=0}^{d} T_{y_m^j} u_*^j + R_m^2 V_{R_m} \hat{v}_m \Big\|_{L^2} = 0. \tag{4.31}$$

From (4.29) and the limits for y_m^j in (ii) we get that

$$\lim_{m\to\infty} \Bigg\| \Big| \sum_{j=0}^{d} T_{y_m^j} u_*^j \Big|^{2p} \sum_{j=0}^{d} T_{y_m^j} u_*^j - \sum_{j=0}^{d} T_{y_m^j} |u_*^j|^{2p} u_*^j \Bigg\|_{L^2} = 0. \tag{4.32}$$

From (4.29) we have $u_*^j \in L^q(\mathbb{R}^n)$ for any $q \in [1, \infty]$. This, along with the limits for $\hat{v}_m$ in (iii), implies using Hölder's inequality that

$$\lim_{m\to\infty} \Bigg\| \Big| \sum_{j=0}^{d} T_{y_m^j} u_*^j \Big|^{2p} \hat{v}_m \Bigg\|_{L^2} = 0. \tag{4.33}$$

Since $\|u_{E_m}\|_{H^2}$ is bounded in uniformly in m and each $u_*^j \in H^2(\mathbb{R}^n)$, it follows that $\|\hat{v}_m\|_{H^2}$ is bounded uniformly in m. Hence from Lemma 3.1 and (iii) it follows that

$$\lim_{m\to\infty} \Big\| N\Big(\sum_{j=0}^{d} T_{y_m^j} u_*^j, \hat{v}_m \Big) \Big\|_{L^2} = 0. \tag{4.34}$$

From (4.30)-(4.34) we get that

$$\lim_{m\to\infty} \|\hat{v}_m\|_{H^2} = 0. \tag{4.35}$$

For each $j \in \mathcal{J}$, if $(R_m y_m^j)_{m\in\mathbb{N}}$ has a converging subsequence, then denote it (again) as $(R_m y_m^j)_{m\in\mathbb{N}}$ and its limit point as x^j. It now follows from (4.28), (4.35) and the limits for y_m^j in (ii) that (4.9) and (4.10) hold with $M = d+1$, $u_i = u_*^{i-1}$ and $x_k^i = y_k^{i-1}$ for $i \in \{1, 2, \dots M\}$. This completes the proof of part (ii) in the splitting case.

Theorem 4.1 is now completely proven. $\square$

Recall that $\mathcal{I} = \{1, 2, \dots n\}$. For any $z \in \mathbb{R}^n$ and any function u defined on $\mathbb{R}^n$, $T_z u(x) = u(x-z)$ for all $x \in \mathbb{R}^n$. In the next theorem, under additional hypothesis on p and the potential V, we strengthen the results in part (ii) of Theorem 4.1 as follows: If (ψ_E, E) is a branch of ground states on $[E_1, \infty)$, then each u_i in (4.9) is the unique radially symmetric positive solution u_∞ of (4.1), the third limit in (4.10) does not occur and each x^i in (4.10) is a critical point of the potential V.

Theorem 4.2. *Let $\sigma < 0$ and $p \geq 1/2$. Assume that the potential V satisfies the conditions in (4.2) and $\nabla V \in W^{1,\infty}(\mathbb{R}^n)$. Furthermore, let the radial derivative $\partial_r V(x)$ of V and a set of its tangential derivatives $\{\partial_{s_i} V(x), i = 1, 2, \dots n-1\}$, where $s_1, s_2, \dots s_{n-1}$ are a set of mutually orthogonal directions which are also orthogonal to the radial direction, satisfy the following hypotheses: There exists a set $\mathcal{M}$ of measure zero such that*

$$\limsup_{x\in\mathbb{R}^n\setminus\mathcal{M},\ |x|\to\infty} \frac{V^2(x)}{|\partial_r V(x)|} < \infty \tag{4.36}$$

and there exist constants $\mathcal{C}, \beta, \rho > 0$ such that for each $x \in \mathbb{R}^n \setminus \mathcal{M}$ with $|x| > \rho$

$$\mathcal{C}|\partial_r V(x)| > e^{-\beta|x|}, \tag{4.37}$$

$$\mathcal{C}|\partial_r V(x)| > \Big(\sum_{i=1}^{n-1} |\partial_{s_i} V(x)|^2\Big)^{\frac{1}{2}}. \tag{4.38}$$

Under the above assumptions, consider a C^1 curve (ψ_E, E) of real-valued positive ground states of (1.3), see Definition 2.1, on the interval $[E_1, \infty)$ for some $E_1 > 0$. Then for any sequence $(E_m)_{m=1}^{\infty}$ in $[E_1, \infty)$ with $\lim_{m\to\infty} E_m = \infty$, there exists a subsequence $(E_{m_k})_{k=1}^{\infty}$, a positive integer M and a subsequence $(x_k^i)_{k=1}^{\infty}$ in $\mathbb{R}^n$ for each $i \in \{1, 2, \dots M\}$ such that

$$\lim_{k\to\infty} \Big\| u_{E_{m_k}} - \sum_{i=1}^{M} u_\infty(\cdot - x_k^i) \Big\|_{H^2} = 0 \tag{4.39}$$

and for every $1 \le i, j \le M$ with $i \ne j$

$$\lim_{k\to\infty} |x_k^i - x_k^j| = \infty \quad \textit{and} \quad \lim_{k\to\infty} \frac{x_k^i}{\sqrt{E_{m_k}}} = x^i. \tag{4.40}$$

Here $u_{E_{m_k}}$ is the scaled version of $\psi_{E_{m_k}}$ obtained by letting $E = E_{m_k}$ in (4.6), u_∞ is the unique radially symmetric positive solution of (4.1). Furthermore, if the potential V is continuously differentiable in a neighborhood of x^i, then x^i must be a critical point of V.

Remark 4.3. *The hypotheses in (4.36)-(4.38) in the above theorem impose restrictions on the behavior of the potential near $x = \infty$ alone. In particular, near infinity the potential should not have any critical points and must either strictly decrease or strictly increase to zero in the radial direction. The radial derivative of the potential multiplied by a sufficiently large constant must, in magnitude, dominate the tangential ones. These assumptions hold for physical potentials which model a field that is homogeneous and isotropic at infinity. The conclusions of the above theorem implies that for large E the ground states on a C^1 branch of real-valued positive ground states are essentially a sum of finitely many humps. Each of these humps is a re-scaling (via (4.6)) of the radially symmetric profile u_∞, and each of them concentrates at a critical point of the potential V. In particular, these humps cannot drift towards infinity. The question of whether such C^1 branches of ground states actually exist is addressed in Theorem 4.3.*

Proof. Consider a C^1 branch of real-valued positive ground states (ψ_E, E) of (1.3) on the interval $[E_1, \infty)$ for some $E_1 > 0$. Let $(E_m)_{m=1}^{\infty}$ be a sequence with $\lim_{m\to\infty} E_m = \infty$. Define

$$u_{E_m}(x) = E_m^{-\frac{1}{2p}} \psi_{E_m}(E_m^{-\frac{1}{2}} x) \qquad \forall x \in \mathbb{R}^n.$$

From part (i) of Theorem 4.1 we know that the sequence $(u_{E_m})_{m=1}^{\infty}$ is bounded in H^1 and the sequence $(\|u_{E_m}\|_{L^2})_{m=1}^{\infty}$ converges to a nonzero constant, see (4.15). Define

$$\Psi_m = \frac{u_{E_m}}{\|u_{E_m}\|_{L^2}}.$$

Like in the proof of part (ii) of Theorem 4.1, we will apply concentration compactness theory to $(\Psi_m)_{m\in\mathbb{N}}$. We have already established in that proof that vanishing ($\mu = 0$) cannot occur. We will show that if either compactness ($\mu = 1$) or splitting ($0 < \mu < 1$) occurs, then (4.39) and (4.40) hold for a subsequence of u_{E_m}.

Compactness ($\mu = 1$): In the proof of part (ii) of Theorem 4.1 we have established that along a subsequence of $(E_m)_{m\in\mathbb{N}}$, again denoted as $(E_m)_{m\in\mathbb{N}}$, u_{E_m} can be written as follows:

$$u_{E_m} = T_{y_m}\hat{u}_* + \hat{v}_m. \tag{4.41}$$

Here $\hat{u}_*$ is a strong, H^2 solution of (4.1) and

$$\lim_{m\to\infty} \|\hat{v}_m\|_{H^2} = 0. \tag{4.42}$$

Since (ψ_E, E) is a ground state branch, we have $u_{E_m} > 0$ for all m. It now follows using (4.41) and (4.42) that for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^n)$

$$0 \le \liminf_{m\to\infty}\langle u_{E_m}, T_{y_m}\phi\rangle = \langle \hat{u}_*, \phi\rangle + \liminf_{m\to\infty}\langle \hat{v}_m, T_{y_m}\phi\rangle = \langle \hat{u}_*, \phi\rangle,$$

which implies that $\hat{u}_*$ is a nonnegative function. Since every nonnegative [23] solution of (4.1) must be a translation of the function u_∞, we have $\hat{u}_* = T_{y_*}u_\infty$ for some $y_* \in \mathbb{R}^n$. Redefining y_m to be $y_m + y_*$ we get

$$u_{E_m} = T_{y_m}u_\infty + \hat{v}_m. \tag{4.43}$$

To complete the proof of this theorem in the compactness case, we will show that the sequence $(R_m y_m)_{m\in\mathbb{N}}$ remains bounded and one of its subsequences, again denoted as $(R_m y_m)_{m\in\mathbb{N}}$, converges to some $x^1 \in \mathbb{R}^n$. It will then follow from (4.43) and (4.42) that (4.39) and (4.40) hold with $M = 1$ and $x_k^1 = y_k$. Finally we will show that if the potential V is continuously differentiable in a neighborhood of x^1, then x^1 must be a critical point of V.

To show that $(R_m y_m)_{m\in\mathbb{N}}$ is a bounded sequence, we will assume the contrary, i.e. there exists a subsequence of $(y_m)_{m\in\mathbb{N}}$, also denoted by $(y_m)_{m\in\mathbb{N}}$, such that

$$\lim_{m\to\infty} R_m|y_m| = \infty. \tag{4.44}$$

Define the linear operator $L_+(u, R) : H^2(\mathbb{R}^n) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n)$ as follows:

$$L_+(u, R)[v] = (-\Delta + R^2 V_R + 1)v + \sigma(2p + 1)|u|^{2p}v. \tag{4.45}$$

Note that the definition of L_+ here is different from the definition in Section 3 due to the scaling (4.6) used to obtain u_{E_m} from ψ_{E_m}. Since for the decomposition of u_{E_m} in (4.43) we have $\lim_{m\to\infty} y_m = \infty$ by assumption (4.44) and (4.42) holds, for reasons similar to those expressed in the case of bifurcation at finite E (see discussion in the paragraph below (3.33)), we wish to decompose u_{E_m} with respect to the kernel of $L_+(T_{y_m}u_\infty, R_m)$. Since the kernel of $-\Delta + 1 + \sigma(2p + 1)T_{y_m}|u_\infty|^{2p}$ approaches the kernel of $L_+(T_{y_m}u_\infty, R_m)$ for large m, we will work with the former. The kernel of $-\Delta + 1 + \sigma(2p + 1)T_{y_m}|u_\infty|^{2p}$ is span $\{T_{y_m}\partial_{x_k}u_\infty | k = 1, 2, \ldots n\}$, which is the set of infinitesimal generators of translations of $T_{y_m}u_\infty$ in $\mathbb{R}^n$. Hence the projection onto the kernel can be absorbed into translations of the profile u_∞. To be precise, we apply Lemma 3.2 to (4.43) to obtain the following decomposition for sufficiently large m:

$$u_{E_m} = T_{y_m+s_m}u_\infty + v_m = T_{z_m}u_\infty + v_m, \tag{4.46}$$

where $z_m = y_m + s_m$ and v_m satisfies

$$\langle v_m, \partial_{x_k} T_{y_m+s_m}u_\infty\rangle = 0 \quad \text{for all} \quad k = 1, 2, \dots n.$$

Moreover, as $m \to \infty$ we have $R_m \to 0$, $|s_m| \to 0$ and $R_m|z_m| \to \infty$. Furthermore, since $v_m = \hat{v}_m + T_{y_m}u_\infty - T_{y_m+s_m}u_\infty$ and (4.42) holds and $T_{y_m}u_\infty - T_{y_m+s_m}u_\infty \overset{H^2}{\to} 0$ (because $s_m \to 0$), we have

$$\lim_{m\to\infty} \|v_m\|_{H^2} = 0.$$

We assume, with no loss of generality, that the decomposition of u_{E_m} in (4.46) holds for all m.

Since u_{E_m} is a solution for (4.8), using (4.46) we obtain

$$(-\Delta + R_m^2 V_{R_m} + 1)(T_{z_m}u_\infty + v_m) + \sigma|T_{z_m}u_\infty + v_m|^{2p}(T_{z_m}u_\infty + v_m) = 0. \tag{4.47}$$

We will analyze the above equation using a Lyapunov-Schmidt decomposition. Our approach here will be identical to the one used in Section 3 and is summarized next. First we obtain a set of equations - an infinite dimensional equation and a set of finite dimensional equations - which is equivalent to (4.47). We show the existence of a unique solution v_m, as a function of z_m, to the infinite dimensional equation via contraction mapping. Using this solution we study the set of finite dimensional equations to obtain the contradiction that $u_{E_m} = T_{y_m}u_\infty + \hat{v}_m$ does not solve (4.8) if (4.44) holds. Hence (4.44) cannot hold.

Recall the linear operator $L_+(u, R)$ from (4.45) and the nonlinear function $N(U, v) : H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ from (3.7):

$$N(U, v) = \sigma|U + v|^{2p}(U + v) - \sigma|U|^{2p}U - \sigma(2p + 1)|U|^{2p}v.$$

Using L_+ and N, we can rewrite (4.47) as

$$L_+(T_{z_m}u_\infty, R_m)[v_m] + R_m^2 V_{R_m} T_{z_m}u_\infty + N(T_{z_m}u_\infty, v_m) = 0. \tag{4.48}$$

Consider the following projection operator $P_m^\perp$ which projects onto the orthogonal complement of the subspace

$$\ker(L_+(T_{z_m}u_\infty, R_m) - R_m^2 V_{R_m}) = \operatorname{span}\left\{T_{z_m}\partial_{x_k}u_\infty \middle| k \in \mathcal{I}\right\}$$

in $L^2(\mathbb{R}^n)$:

$$P_m^\perp\phi = \phi - \sum_{k=1}^n \langle\phi, T_{z_m}\partial_{x_k}u_\infty\rangle \frac{T_{z_m}\partial_{x_k}u_\infty}{\|T_{z_m}\partial_{x_k}u_\infty\|_{L^2}^2} \qquad \forall\phi \in L^2(\mathbb{R}^n).$$

Let $P_m^\perp L^2(\mathbb{R}^n)$ denote the range of $P_m^\perp$. Clearly (4.48) is equivalent to the set of equations

$$P_m^\perp L_+(T_{z_m}u_\infty, R_m)[v_m] + P_m^\perp R_m^2 V_{R_m} T_{z_m}u_\infty + P_m^\perp N(T_{z_m}u_\infty, v_m) = 0, \tag{4.49}$$

which is an infinite dimensional equation and

$$\langle T_{z_m}\partial_{x_k}u_\infty, R_m^2 V_{R_m} T_{z_m}u_\infty + R_m^2 V_{R_m} v_m + N(T_{z_m}u_\infty, v_m)\rangle = 0 \qquad \forall k \in \mathcal{I}, \tag{4.50}$$

which is a set of finite dimensional equations. Clearly the operator

$$P_m^\perp(-\Delta + 1 + \sigma|T_{z_m}u_\infty|^{2p})P_m^\perp : H^2(\mathbb{R}^n) \cap P_m^\perp L^2(\mathbb{R}^n) \mapsto P_m^\perp L^2(\mathbb{R}^n),$$

has a bounded inverse and the norm of the inverse operator is independent of m. The operators $P_m^\perp L_+(T_{z_m}u_\infty, R_m)P_m^\perp$ and $P_m^\perp(-\Delta+1+\sigma|T_{z_m}u_\infty|^{2p})P_m^\perp$ differ by $P_m^\perp R_m^2 V(R_m x)P_m^\perp$, and $\|P_m^\perp R_m^2 V(R_m x)P_m^\perp\|_{H^2\mapsto L^2} \to 0$ as $m \to \infty$. Hence for sufficiently large m, $P_m^\perp L_+(T_{z_m}u_\infty, R_m)P_m^\perp$ is boundedly invertible with a bound independent of m, i.e.

$$\|[P_m^\perp L_+(T_{z_m}u_\infty, R_m)P_m^\perp]^{-1}\|_{L^2\mapsto H^2} \le K.$$

Since $P_m^\perp v_m = v_m$, (4.49) can be written as

$$v_m = [P_m^\perp L_+(T_{z_m}u_\infty, R_m)P_m^\perp]^{-1}P_m^\perp(R_m^2 V_{R_m} T_{z_m}u_\infty + N(T_{z_m}u_\infty, v_m)). \tag{4.51}$$

Fix $\delta > 0$ and $M \in \mathbb{N}$ such for all $m > M$

$$\|v_m\|_{H^2} \le \delta \quad \text{and} \quad KC(\|u_\infty\|_{H^2})2\delta \le \frac{1}{2}. \tag{4.52}$$

Here $C(\|u_\infty\|_{H^2})$ is the constant in (3.14) with $L = \|u_\infty\|_{H^2}$. It now follows via the contraction mapping principle that there exists a unique solution v_m to (4.51) for all $m > M$ such that

$$\|v_m\|_{H^2} \le 2KR_m^2\|V_{R_m}T_{z_m}u_\infty\|_{L^2}. \tag{4.53}$$

We will assume, with no loss of generality, that a v_m satisfying (4.53) exists for all m.

Using the solution v_m of (4.49), regarded as a function of z_m, we will now consider the set of finite dimensional equations (4.50) to be equations in the unknown z_m and show that under our assumption (4.44), these equations have no solution.

Let x_m denote the radial direction at z_m. For each $m \in \mathbb{N}$, consider the following finite dimensional equation obtained as a linear combination of the equations in (4.50):

$$\langle T_{z_m}\partial_{x_m}u_\infty, R_m^2 V_{R_m} T_{z_m}u_\infty\rangle + \langle T_{z_m}\partial_{x_m}u_\infty, R_m^2 V_{R_m} v_m + N(T_{z_m}u_\infty, v_m)\rangle = 0. \tag{4.54}$$

We will show that, under our hypothesis on the behavior of the potential $V(x)$ as $|x| \to \infty$, the dominant term in (4.54) as $R_m z_m \to \infty$ is $\langle T_{z_m}\partial_{x_m}u_\infty, R_m^2 V_{R_m} T_{z_m}u_\infty\rangle$ and by dividing the left hand side of (4.54) with it and taking the limit as $m \to \infty$, we get the contradiction that $1 = 0$.

Fix $\theta \in (0, \pi/2)$ such that $2\mathcal{C}\tan\theta < 1$, where $\mathcal{C}$ is the constant in the hypothesis (4.38). Let B_m be the closed ball in $\mathbb{R}^n$ with center z_m and radius $|z_m|\sin\theta$. Then for the first term in (4.54)

$$-2\langle T_{z_m}\partial_{x_m}u_\infty, R_m^2 V_{R_m} T_{z_m}u_\infty\rangle = R_m^3 \int\limits_{B_m\cup\mathbb{R}^n\setminus B_m} u_\infty^2(x - z_m)\partial_{x_m}V(R_m x)\mathrm{d}x.$$

Since there exist constants $C > 0$ and $\gamma \in (0, 1/2)$ such that $|u_\infty(x)| < Ce^{-\sqrt{1-\gamma}|x|}$ and $|\partial_{x_k} u_\infty(x)| < Ce^{-\sqrt{1-\gamma}|x|}$ for all $x \in \mathbb{R}^n$, it follows that for sufficiently large m

$$\begin{aligned}\left| R_m^3 \int\limits_{\mathbb{R}^n \setminus B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \right| &\le CR_m^3 \|\nabla V\|_{L_\infty} \int\limits_{\mathbb{R}^n \setminus B_m} e^{-2\sqrt{1-\gamma}|x-z_m|}\mathrm{d}x \\ &\le CR_m^3 e^{-2\sqrt{1-2\gamma}|z_m|\sin\theta}. \end{aligned} \tag{4.55}$$

For each $x \in B_m$, let $\{\theta_i(x) \big| i \in \mathcal{I}\}$ with $|\theta_i(x)| < \theta$ be the set of angles using which the derivative of the potential along the direction x_m can be written as a linear combination of the derivatives along the radial and tangential directions at x, i.e.

$$\partial_{x_m} V(R_m x) = \partial_r V(R_m x)\cos\theta_n(x) + \sum_{i=1}^{n-1} \partial_{s_i} V(R_m x)\sin\theta_i(x),$$

where $\partial_r V$ and $\partial_{s_i} V$ are the radial and tangential derivatives introduced in the theorem. The function $\partial_{x_m} V(R_m x)$ has the same sign a.e. in B_m for large m. Indeed, since $\lim_{m\to\infty} R_m z_m = \infty$, for almost every $x \in B_m$ we can use the hypothesis in (4.37) and (4.38) to get

$$\begin{aligned} |\partial_{x_m} V(R_m x)| &= \left|\partial_r V(R_m x)\cos\theta_n(x) + \sum_{i=1}^{n-1} \partial_{s_i} V(R_m x)\sin\theta_i(x)\right| \\ &\ge \cos\theta\, |\partial_r V(R_m x)| - \Big(\sum_{i=1}^{n-1} |\partial_{s_i} V(R_m x)|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n-1} \sin^2\theta_i(x)\Big)^{\frac{1}{2}} \\ &\ge \cos\theta\, |\partial_r V(R_m x)| - \mathcal{C}\sin\theta\, |\partial_r V(R_m x)| \qquad (4.56) \\ \implies \quad |\partial_{x_m} V(R_m x)| &> Ce^{-\beta R_m |x|}. \end{aligned}$$

Using the above estimate it follows that for m large

$$\left| R_m^3 \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \right| \ge CR_m^3 \int\limits_{B_m} u_\infty^2(x - z_m)e^{-\beta R_m |x|}\mathrm{d}x \tag{4.57}$$

$$\ge CR_m^3 e^{-2\beta R_m |z_m|}, \tag{4.58}$$

which along with (4.55) gives

$$\lim_{m\to\infty} \frac{\left| \int\limits_{\mathbb{R}^n \setminus B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \right|}{\left| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \right|} = 0. \tag{4.59}$$

For the remaining terms in (4.54) we have the following estimates using (3.14) and (4.53):

$$\begin{aligned} &|\langle T_{z_m}\partial_{x_m} u_\infty, R_m^2 V_{R_m} v_m + N(T_{z_m} u_\infty, v_m)\rangle| \\ &\le R_m^2 \|\partial_{x_m} V_{R_m} T_{z_m} u_\infty\|_{L^2} \|v_m\|_{L^2} + R_m^2 \|V_{R_m} T_{z_m} u_\infty\|_{L^2} \|v_m\|_{H^1} + \|T_{z_m}\partial_{x_m} u_\infty\|_{L^2} \|N(T_{z_m} u_\infty, v_m)\|_{L^2} \\ &\le CR_m^4 \big(\|\partial_{x_m} V_{R_m} T_{z_m} u_\infty\|_{L^2}^2 + \|V_{R_m} T_{z_m} u_\infty\|_{L^2}^2 \big). \end{aligned} \tag{4.60}$$

Now as in (4.55), using the decay estimates for $|u_\infty|$ and $|\partial_{x_i} u_\infty|$, we get

$$\Big| \int\limits_{\mathbb{R}^n \setminus B_m} u_\infty^2(x - z_m) V^2(R_m x)\mathrm{d}x \Big| < Ce^{-2\sqrt{1-2\gamma}|z_m|\sin\theta}, \tag{4.61}$$

$$\Big| \int\limits_{\mathbb{R}^n \setminus B_m} u_\infty^2(x - z_m)(R_m \partial_{x_m} V(R_m x))^2\mathrm{d}x \Big| < Ce^{-2\sqrt{1-2\gamma}|z_m|\sin\theta}. \tag{4.62}$$

Since $\partial_{x_m} V(R_m x)$ does not change sign in B_m (excluding a set of measure 0), using (4.36) along with (4.56) gives us the following estimates:

$$\begin{aligned}
&\lim_{m\to\infty} \frac{R_m \Big| \int\limits_{B_m} u_\infty^2(x - z_m) V^2(R_m x)\mathrm{d}x \Big|}{\Big| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \Big|} \\
&= \lim_{m\to\infty} \frac{\Big| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)[R_m \frac{V^2(R_m x)}{\partial_{x_m} V(R_m x)}]\mathrm{d}x \Big|}{\Big| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \Big|} = 0
\end{aligned} \tag{4.63}$$

and

$$\begin{aligned}
&\lim_{m\to\infty} \frac{R_m^3 \Big| \int\limits_{B_m} u_\infty^2(x - z_m)(\partial_{x_m} V(R_m x))^2\mathrm{d}x \Big|}{\Big| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \Big|} \\
&= \lim_{m\to\infty} \frac{\Big| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)[R_m^3 \partial_{x_m} V(R_m x)]\mathrm{d}x \Big|}{\Big| \int\limits_{B_m} u_\infty^2(x - z_m)\partial_{x_m} V(R_m x)\mathrm{d}x \Big|} = 0.
\end{aligned} \tag{4.64}$$

In deriving the last limit we have used the assumption $\nabla V \in L^\infty(\mathbb{R}^n)$. Using the estimates in (4.55), (4.59), (4.61), (4.62), (4.63) and (4.64) we see that the first term in (4.54)

$$\langle T_{z_m} \partial_{x_m} u_\infty, R_m^2 V_{R_m} T_{z_m} u_\infty \rangle$$

cannot be canceled by the remaining terms in (4.54) when m is large. Therefore it follows that there are no solutions to (4.47) such that the sequence $(R_m z_m)_{m\in\mathbb{N}}$, and consequently $(R_m y_m)_{m\in\mathbb{N}}$, is unbounded. Hence (4.44) cannot hold and so $(R_m y_m)_{m\in\mathbb{N}}$ must be a bounded sequence.

Consider a subsequence of $(R_m y_m)_{m\in\mathbb{N}}$, again denoted as $(R_m y_m)_{m\in\mathbb{N}}$, converging to $x^1 \in \mathbb{R}^n$. From (4.46) and (4.42) it follows that (4.39) and (4.40) holds with $M = 1$ and $x_k^1 = y_k$. Suppose that V is continuously differentiable at x^1. We will now complete the proof of the theorem in the compactness case by showing x^1 is a critical point of the potential. Indeed, for each $k \in \mathcal{I}$, (4.50) can be written using (3.14) and (4.53) (via calculations similar to those used to derive (4.60)) as

$$\int_{\mathbb{R}^n} \partial_{x_k} V(R_m x + R_m z_m) u_\infty^2(x)\mathrm{d}x = O\big(R_m \|V_{R_m} T_{z_m} u_\infty\|_{L^2}^2 + R_m \|V_{R_m} T_{z_m} \partial_{x_1} u_\infty\|_{L^2}^2\big).$$

For every $x \in \mathbb{R}^n$, we have $\partial_{x_k} V(R_m x + R_m z_m) u_\infty^2(x) \to \partial_{x_k} V(x^1) u_\infty^2(x)$ as $m \to \infty$ and $|\partial_{x_1} V(R_m x + R_m z_m) u_\infty^2(x)| \leq \|\nabla V\|_{L^\infty} u_\infty^2(x)$ for all m. Since $\|\nabla V\|_{L^\infty} u_\infty^2 \in L^1(\mathbb{R}^n)$, it follows using the Lebesgue dominated convergence theorem that as $m \to \infty$ the left hand side of the above equation converges to $\partial_{x_k} V(x^1) \|u_\infty\|_{L^2}^2$. Clearly the right hand side of the above equation converges to 0. Hence we have

$$\partial_{x_k} V(x^1) \|u_\infty\|_{L^2}^2 = 0 \qquad \forall\, k \in \mathcal{I},$$

i.e., x^1 is a critical point of the potential.

Splitting ($0 < \mu < 1$): In the proof of part (ii) of Theorem 4.1 we have established in the splitting case that along a subsequence of $(E_m)_{m\in\mathbb{N}}$, again denoted as $(E_m)_{m\in\mathbb{N}}$, u_{E_m} can be written as follows:

$$u_{E_m} = \sum_{j=1}^{d} T_{y_m^j} u_*^j + u_*^0 + \hat{v}_m\,. \tag{4.65}$$

Here u_*^j is a strong solution of (4.1) for each $j \in \mathcal{J} = \{0, 1, \dots d\}$,

$$\lim_{m\to\infty} \|\hat{v}_m\|_{H^2} = 0, \tag{4.66}$$

and as $m \to \infty$

$$|y_m^j| \to \infty \quad \text{and} \quad |y_m^j - y_m^k| \to \infty \qquad \forall\, j,k \in \{1,2,\dots d\},\ j \neq k. \tag{4.67}$$

Concentration compactness theory does not imply that $u_*^0 \neq 0$, but we assume it only to simplify our presentation. All the arguments in this section remain valid if $u_*^0 = 0$. Since (ψ_E, E) is a ground state branch, we have $u_{E_m} > 0$ for all m. It now follows using (4.65) and (4.66) that for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^n)$ and any $j \in \mathcal{J}$, using the notation $y_m^0 = 0$, we have

$$0 \leq \liminf_{m\to\infty} \langle u_{E_m}, T_{y_m^j}\phi\rangle = \langle u_*^j, \phi\rangle + \liminf_{m\to\infty} \langle \hat{v}_m, T_{y_m}\phi\rangle = \langle u_*^j, \phi\rangle\,,$$

which implies that $\hat{u}_*^j$ is a nonnegative function. Since every nonnegative solution of (4.1) must be a translation of the function u_∞, see [23], we have $u_*^j = T_{y_*^j} u_\infty$ for some $y_*^j \in \mathbb{R}^n$. Redefining y_m^j to be $y_m^j + y_*^j$ for each $j \in \mathcal{J}$ we get the following different decomposition for u_{E_m}:

$$u_{E_m} = \sum_{j=1}^{d} T_{y_m^j} u_\infty + T_{y_m^0} u_\infty + \hat{v}_m. \tag{4.68}$$

Here $y_m^0 \in \mathbb{R}^n$ is a constant and the y_m^js satisfy (4.67).

To complete the proof of this theorem in the splitting case, we will show that the sequence $(R_m y_m^j)_{m\in\mathbb{N}}$ remains bounded for all $j \in \mathcal{J}$ and one of its subsequences, again denoted as $(R_m y_m^j)_{m\in\mathbb{N}}$, converges to some $x^j \in \mathbb{R}^n$. It will then follow from (4.68) and (4.66), that (4.39) and (4.40) hold with $M = d+1$ and $x_k^1 = y_k$. Finally we will show that if the potential V is continuously differentiable in a neighborhood of x^j, then x^j must be a critical point of V.

To show that $(R_m y_m^j)_{m\in\mathbb{N}}$ is a bounded sequence for each $j \in \mathcal{J}$, we will assume the contrary i.e., we suppose that there exists a $0 < d_0 < d$ such that $(R_m y_m^j)_{m\in\mathbb{N}}$ is a bounded sequence for each $j \in \mathcal{J}_0 = \{0, 1, \dots d_0\}$ and for each $j \in \mathcal{J} \setminus \mathcal{J}_0$ the sequence $(y_m^j)_{m\in\mathbb{N}}$ has a subsequence, also denoted by $(y_m^j)_{m\in\mathbb{N}}$, such that

$$\lim_{m\to\infty} R_m|y_m^j| = \infty \qquad \forall\, j \in \mathcal{J} \setminus \mathcal{J}_0. \tag{4.69}$$

Recall the linear operator $L_+(u, R) : H^2(\mathbb{R}^n) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n)$ from (4.45):

$$L_+(u, R)[v] = (-\Delta + R^2 V_R + 1)v + \sigma(2p + 1)|u|^{2p} v.$$

Since for the decomposition of u_{E_m} in (4.68) we have $\lim_{m\to\infty} y_m^j = \infty$ and $\lim_{m\to\infty} |y_m^j - y_m^k| = \infty$ for $j, k \in \{1, 2, \dots d\}$ and (4.66) holds, for reasons similar to those expressed in the case of bifurcation at finite E (see discussion in the paragraph below (3.72)), we wish to decompose u_{E_m} with respect to the kernel of $L_+(\sum_{j=0}^{\infty} T_{y_m} u_\infty, R_m)$. From the theory of Schrödinger operators with potentials separated by a large distance, the kernel of $L_+(\sum_{j=0}^{\infty} T_{y_m} u_\infty, R_m)$ approaches the union of the kernels of $-\Delta + 1 + \sigma(2p+1)T_{y_m^j}|u_\infty|^{2p}$, $j \in \mathcal{J}$, for large m. We will therefore work with the latter. The kernel of $-\Delta + 1 + \sigma(2p+1)T_{y_m^j}|u_\infty|^{2p}$ is span $\left\{T_{y_m^j}\partial_{x_k}u_\infty \,|\, k = 1, 2, \dots n\right\}$, which is the set of infinitesimal generators of translations of $T_{y_m^j}u_\infty$ in $\mathbb{R}^n$. Hence the projection onto this kernel can be absorbed into translations of the profile u_∞. To be precise, we apply Proposition 7.6 (also see Remark 7.2) with $\psi = 0$, $\phi_i = 0$ for all i, $u_j = u_\infty$ and $y_j = y_m^j$ for $j \in \{0, 1, \dots d\}$ to u_{E_m} in (4.68) to conclude, using the limits $\hat{v}_m \stackrel{H^2}{\to} 0$ and $|y_m^j - y_m^k| \to \infty$ as $m \to \infty$, that for sufficiently large m:

$$u_{E_m} = \sum_{j=0}^{d} T_{y_m^j + s_m^j} u_\infty + v_m = \sum_{j=0}^{d} T_{z_m^j} u_\infty + v_m, \tag{4.70}$$

where $z_m^j = y_m^j + s_m^j$ and v_m satisfies

$$\left\langle v_m, T_{z_m^j}\partial_{x_k}u_\infty\right\rangle = 0 \qquad \forall k \in \mathcal{I}, \quad \forall j \in \mathcal{J}.$$

Moreover, as $m \to \infty$ we have $R_m \to 0$, $|s_m^j| \to 0$ for $j \in \mathcal{J}$, $R_m|z_m^j|$ remains bounded for $j \in \mathcal{J}_0$ and $R_m|z_m^j| \to \infty$ for $j \in \mathcal{J} \setminus \mathcal{J}_0$. Furthermore, since $v_m = \hat{v}_m + \sum_{j=0}^{d} T_{y_m^j} u_\infty - \sum_{j=0}^{d} T_{y_m^j + s_m^j} u_\infty$ and (4.66) holds and $\sum_{j=0}^{d} T_{y_m^j} u_\infty - \sum_{j=0}^{d} T_{y_m^j + s_m^j} u_\infty \stackrel{H^2}{\to} 0$ (because $s_m^j \to 0$), we have

$$\lim_{m\to\infty} \|v_m\|_{H^2} = 0.$$

We assume, with no loss of generality, that the decomposition of u_{E_m} in (4.70) holds for all m.

The arguments presented next to establish that the assumption in (4.69) cannot hold are a complex extension of those used to show that $(R_m z_m)_{m\in\mathbb{N}}$ is bounded in the compactness case. Define

$$\tilde{u}_m = \sum_{j=0}^{d} T_{z_m^j} u_\infty, \qquad \tilde{u}_m^0 = \sum_{j=0}^{d_0} T_{z_m^j} u_\infty, \qquad \tilde{u}_m^1 = \sum_{j=d_0+1}^{d} T_{z_m^j} u_\infty. \tag{4.71}$$

Since u_{E_m} solves (4.8), we substitute its decomposition in (4.70) into (4.8) to get

$$(-\Delta + 1 + R_m^2 V_{R_m})(\tilde{u}_m + v_m) + \sigma\, |\tilde{u}_m + v_m|^{2p}\, (\tilde{u}_m + v_m)(x) = 0. \tag{4.72}$$

As in the compactness case, we will analyze the above equation using the Lyapunov-Schmidt decomposition. We first derive an equivalent family of equations: an infinite dimensional equation and a set of finite dimensional equations. We then solve the infinite dimensional equation for v_m as a function of the z_m^js and derive some useful estimates for its decay. Using this solution and its decay estimates we study the set of finite dimensional equations to derive the contradiction that if (4.69) holds, then (4.70) does not solve (4.8). Therefore $\{R_m z_m^j\}_{m\in\mathbb{N}}$ must be a bounded sequence for all $j \in \mathcal{J}$.

Recall the linear operator $L_+(u,R) : H^2(\mathbb{R}^n) \times \mathbb{R} \mapsto L^2(\mathbb{R}^n)$ from (4.45) and the nonlinear operator $N(U,v) : H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ from (3.7),

$$N(U,v) = \sigma|U+v|^{2p}(U+v) - \sigma|U|^{2p}U - \sigma(2p+1)|U|^{2p}v,$$

which satisfies the bound in (3.14). Using these operators we can rewrite (4.72) as

$$L_+(\tilde{u}_m, R_m)[v_m] + N(\tilde{u}_m, v_m) + R_m^2 V_{R_m} \sum_{j=0}^{d} T_{z_m^j} u_\infty + \sigma \left|\tilde{u}_m\right|^{2p} \tilde{u}_m - \sigma \sum_{j=0}^{d} T_{z_m^j} |u_\infty|^{2p} u_\infty = 0. \quad (4.73)$$

Define the operator $P_m^\perp$ on $L^2(\mathbb{R}^n)$ to be the orthogonal projection onto the subspace

$$\left\{ T_{z_m^j} \partial_{x_k} u_\infty \middle| k \in \mathcal{I}, j \in \mathcal{J} \right\}^\perp.$$

Note that the definitions of $P_m^\perp$ here is different from the compactness case. For simplicity of notation, let us rewrite (4.73) as

$$F(v_m, z_m) = 0,$$

where z_m is the set $\{z_m^0, z_m^1, \ldots z_m^d\}$. This equation is equivalent to the following set of equations:

$$P_m^\perp F(v_m, z_m) = 0, \quad (4.74)$$

which is an infinite dimensional equation to be solved for v_m as a function of z_m, and

$$\langle F(v_m, z_m), \partial_{x_k} u_\infty(x - z_m^j)\rangle = 0, \quad j \in \mathcal{J}, \quad k \in \mathcal{I}, \quad (4.75)$$

which is a set of finite dimensional equations to be solved for z_m using the solution v_m of (4.74).

We will first solve the infinite dimensional equation (4.74). When m is sufficiently large, R_m is sufficiently small and $|z_m^j|$ and $|z_m^j - z_m^k|$ are sufficiently large for all $j,k \in \mathcal{J}$. We can write $L_+(\tilde{u}_m, R_m) = \widetilde{L}_m + \widetilde{W}_m$, where

$$\widetilde{L}_m = -\Delta + 1 + \sigma(2p+1) \sum_{j=0}^{d} \left|T_{z_m^j} u_\infty\right|^{2p}, \qquad \widetilde{W}_m = R_m^2 V_{R_m} + \sigma(2p+1)|\tilde{u}_m|^{2p} - \sigma(2p+1) \sum_{j=0}^{d} \left|T_{z_m^j} u_\infty\right|^{2p}.$$

Clearly $\|\widetilde{W}_m\|_{H^2 \mapsto L^2} \to 0$ as $m \to \infty$. Applying Proposition 7.5 in the Appendix to $\widetilde{L}_m$, with $R = R_m$, $L_{R_m} = \widetilde{L}_m$, $E(R_m) = 1$, $s_k(R_m) = z_m^k$ and $V_k = \sigma(2p+1)|u_\infty|^{2p}$, and then applying the spectral perturbation theory to $\widetilde{L}_m + \widetilde{W}_m$ by regarding $\widetilde{W}_m$ as a perturbation, it follows that the operator $P_m^\perp L_+(\tilde{u}_m, R_m) P_m^\perp : H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ has a bounded inverse and the norm of the inverse operator can be bounded uniformly in m, for large m. Hence for sufficiently large m, using (4.73) and $P_m^{\perp} v_m = v_m$, we can rewrite the infinite dimensional equation (4.74) as

$$v_m = -(P_m^{\perp} L_+(\tilde{u}_m, R_m) P_m^{\perp})^{-1} \Big(N(\tilde{u}_m, v_m) + R_m^2 V_{R_m} \tilde{u}_m + \sigma \left|\tilde{u}_m\right|^{2p} \tilde{u}_m - \sigma \sum_{k=0}^{d} T_{z_m^k} |u_\infty|^{2p} u_\infty \Big). \tag{4.76}$$

Like in the case of compactness, using (3.14), the smallness of $\|v_m\|_{H^2}$ and the estimates

$$|u_\infty(x)| < Ce^{-\sqrt{1-\gamma}|x|} \qquad \forall\, x \in \mathbb{R}^n, \tag{4.77}$$

for some $C > 0$ and $\gamma \in (0, 1)$, the existence of a unique solution v_m to (4.76) can be established by applying the contraction mapping principle.

While analyzing the finite dimensional equations (4.75) it will be useful to have estimates for the decay of $\|v_m\|_{H^2}$ explicitly in terms of z_m^js. Indeed, like in the compactness case (see (4.53)), we can easily derive that for some $C > 0$

$$\|v_m\|_{H^2} \le C R_m^2 \sum_{j=0}^{d} \|V_{R_m} T_{z_m^j} u_\infty\|_{L^2}. \tag{4.78}$$

However this estimate is not useful since $\|V_{R_m} T_{z_m^j} u_\infty\|_{L^2}$ need not decay to 0 as $z_m^j \to \infty$ for $j \in \mathcal{J}_0$. This is a consequence of $(R_m z_m^j)_{m\in\mathbb{N}}$ being bounded for $j \in \mathcal{J}_0$. Hence we will express v_m as

$$v_m = P_m^{\perp} v_m^0 + v_m^1.$$

Here v_m^0 will be associated with z_m^j's for $j \in \mathcal{J}_0$ and v_m^1 will be associated with z_m^j's for $j \in \mathcal{J} \setminus \mathcal{J}_0$. For v_m^0 we will derive certain exponential in space decay estimates (see (4.86)) and for the v_m^1 we will derive an estimate like (4.78) (see (4.88)). Using these estimates we will analyze the equations (4.75) to show that our assumption (4.69) leads to a contradiction.

We first derive the estimate (4.86). Let $P_0^{\perp}$ be the projection in $L^2(\mathbb{R}^n)$ onto its subspace

$$S_m = \Big\{ T_{z_m^j} \partial_{x_k} u_\infty \,\big|\, j \in \mathcal{J}_0,\, k \in \mathcal{I} \Big\}^{\perp}. \tag{4.79}$$

Recall $\tilde{u}_m^0$ from (4.71). Consider the equation

$$P_0^{\perp} \Big[(-\Delta + 1 + R_m^2 V_{R_m}) \big(\tilde{u}_m^0 + v_m^0 \big) + \sigma |\tilde{u}_m^0 + v_m^0|^{2p} \big(\tilde{u}_m^0 + v_m^0 \big) \Big] = 0. \tag{4.80}$$

We wish to find a $v_m^0 \in H^2(\mathbb{R}^n)$ that solves this equation. For sufficiently large m, the linear operator $F_L : S_m \cap H^2(\mathbb{R}^n) \mapsto S_m \cap L^2(\mathbb{R}^n)$ defined as

$$F_L(v) = -P_0^{\perp} \Big[(-\Delta + 1 + R_m^2 V_{R_m}) v + \sigma (2p+1) |\tilde{u}_m^0|^{2p} v \Big]$$

is invertible with the norm of its inverse operator bounded uniformly in m. This can be established using Proposition 7.5 in the Appendix and the spectral perturbation theory (see, for instance, the discussion above (4.76)). Hence we can write (4.80) as

$$v_m^0 = F_L^{-1} P_0^\perp \Big(R_m^2 V_{R_m} \tilde{u}_m^0 + \sigma |\tilde{u}_m^0|^{2p} \tilde{u}_m^0 - \sigma \sum_{j=0}^{d_0} T_{z_m^j} |u_\infty|^{2p} u_\infty + N(\tilde{u}_m^0, v_m^0) \Big) \tag{4.81}$$
$$\equiv F_N(v_m^0).$$

For m sufficiently large and δ sufficiently small, the map F_N is a contraction on the space $S_m \cap H^2(\mathbb{R}^n) \cap B_\delta$, where B_δ is a closed ball of radius δ around the origin in $H^2(\mathbb{R}^n)$. Hence for each m large, a $v_m^0 \in S_m \cap H^2(\mathbb{R}^n)$ solving (4.80) exists via the contraction mapping theorem and

$$\|v_m^0\|_{H^2} = O(R_m^2) + \sum_{k \neq j,\ j,k=0}^{d_0} O(\exp^{-\sqrt{1-\gamma}|z_m^j - z_m^k|}). \tag{4.82}$$

Our assumption $p \geq 1/2$, along with the constraint $0 < p < 2/(n-2)$, implies that $n \leq 5$. We will now show that $\lim_{m\to\infty} \|v_m^0\|_{W^{2,4}} = 0$, which will imply that v_m^0 is a continuous function and

$$\lim_{m\to\infty} \|v_m^0\|_{L^\infty} = 0. \tag{4.83}$$

From (4.81) we have

$$v_m^0 = -P_0^\perp (-\Delta + 1)^{-1} P_0^\perp \Big[R_m^2 V_{R_m} \tilde{u}_m^0 + (R_m^2 V_{R_m} + \sigma(2p+1)|\tilde{u}_m^0|^{2p}) v_m^0$$
$$+ \sigma |\tilde{u}_m^0|^{2p} \tilde{u}_m^0 - \sigma \sum_{j=0}^{d_0} T_{z_m^j} |u_\infty|^{2p} u_\infty + N(\tilde{u}_m^0, v_m^0) \Big].$$

Since $(-\Delta+1)^{-1} : L^4 \mapsto W^{2,4}$ is a bounded operator, it follows from the above expression that to establish (4.83), it suffices to show that $\lim_{m\to\infty} \|R_m^2 V_{R_m} \tilde{u}_m^0 + (R_m^2 V_{R_m} + \sigma(2p+1)|\tilde{u}_m^0|^{2p}) v_m^0\|_{L^4} = 0$, $\lim_{m\to\infty} \||\tilde{u}_m^0|^{2p} \tilde{u}_m^0 - \sigma \sum_{j=0}^{d_0} T_{z_m^j} |u_\infty|^{2p} u_\infty\|_{L^4} = 0$ and $\lim_{m\to\infty} \|N(\tilde{u}_m^0, v_m^0)\|_{L^4} = 0$. The first limit follows easily using $\lim_{m\to\infty} R_m = 0$, (4.82) which implies $\lim_{m\to\infty} \|v_m^0\|_{L^4} = 0$ and the fact $u_\infty \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ which implies that $\|\tilde{u}_m^0\|_{L^4} < C$ for all m. The second limit follows easily from the estimate in (4.77) and the fact that $|z_m^j - z_m^k| \to \infty$ as $m \to \infty$. We only need to prove the third limit. As shown in Lemma 3.1, there exists $C > 0$ such that

$$|N(U(x), v(x))| \leq C \left[|U(x)|^{2p-1} |v(x)|^2 + |v(x)|^{2p+1} \right] \qquad \forall\, x \in \mathbb{R}^n.$$

If $U \in L^\infty(\mathbb{R})^n$, then

$$\|N(U,v)\|_{L^4} \leq C \|U\|_{L^\infty}^{2p-1} \|v\|_{L^8}^2 + C \|v\|_{L^{8p+4}}^{2p+1}.$$

Since $H^2(\mathbb{R}^n) \hookrightarrow L^8(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n) \hookrightarrow L^{8p+4}(\mathbb{R}^n)$ for $n \leq 5$ and $0 < p < 2/(n-2)$, it follows that

$$\|N(U,v)\|_{L^4} \leq C \|U\|_{L^\infty}^{2p-1} \|v\|_{H^2}^2 + C \|v\|_{H^2}^{2p+1}.$$

From this estimate, (4.82) and $\tilde{u}_m^0 \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we get $\lim_{m\to\infty} \|N(\tilde{u}_m^0, v_m^0)\|_{L^4} = 0$ as desired.

From (4.81) we get that there exist $\beta_i^j \in \mathbb{R}$, for each $i \in \mathcal{I}$ and $j \in \mathcal{J}_0$, such that

$$(-\Delta + 1 + R_m^2 V_{R_m} + \sigma(2p+1)|\tilde{u}_m^0|^{2p}) v_m^0 + R_m^2 V_{R_m} \tilde{u}_m^0$$

$$+\sigma|\tilde{u}_m^0|^{2p}\tilde{u}_m^0 - \sigma\sum_{j=0}^{d_0} T_{z_m^j}|u_\infty|^{2p}u_\infty + N(\tilde{u}_m^0, v_m^0) = \sum_{j=0}^{d_0}\sum_{i=1}^{n}\beta_i^j T_{z_m^j}\partial_{x_i}u_\infty.$$

Each of the β_i^js depend on m. By taking the L^2 scalar product of the above equation with the functions $T_{z_m^j}\partial_{x_i}u_\infty$, and using the estimate for $\|v_m^0\|_{H^2}$ in (4.82) and the estimate for N in (3.14), we can show that

$$\beta_i^j = O(R_m^2) + \sum_{k\neq r,\ r,k=0}^{d_0} O(\exp^{-\sqrt{1-\gamma}|z_m^r - z_m^k|}).$$

Define $W_m = \tilde{u}_m^0 + v_m^0$ and consider the set

$$K_+ = \Big\{x \in \mathbb{R}^n \Big| W_m(x) > \max_{i\in\mathcal{I},\ j\in\mathcal{J}_0} |\partial_{x_i}u_\infty(x - z_m^j)|\Big\}.$$

On K_+ we have

$$\Delta W_m(x) \geq \Bigg(1 + \sigma|W_m(x)|^{2p} - R_m^2|V_{R_m}(x)| - \sum_{j=0}^{d_0}\sum_{i=1}^{n}|\beta_i^j|\Bigg)W_m(x).$$

Fix $0<\gamma<1$. Fix $M\in\mathbb{N}$ such that for all $m>M$

$$\sum_{j=0}^{d_0}\sum_{i=1}^{n}|\beta_i^j| < \frac{\gamma}{4}, \qquad R_m^2\|V\|_{L^\infty} < \frac{\gamma}{4}, \qquad \|v_m^0\|_{L^\infty} < \frac{\gamma}{8}.$$

Define

$$\mathcal{B} = \bigcap_{j=0}^{d_0}\mathbb{R}^n \setminus B(z_m^j, L),$$

where $B(z_m^j, L)$ denotes an open ball in $\mathbb{R}^n$ with center z_m^j and radius L and $L>0$ is chosen sufficiently large such that $|W_m(x)|^{2p} \leq \gamma/4$ for all $x\in\mathcal{B}$ and $m>M$. This is possible due to the exponential decay of u_∞. Therefore

$$\Delta W_m(x) \geq (1-\gamma)W_m(x) \qquad \forall x \in \mathcal{B}\cap K_+. \tag{4.84}$$

Consider the functions $\Phi_j(x) = C_0 e^{-\sqrt{1-\gamma}|x-z_m^j|}$ for $j\in\mathcal{J}_0$ with $C_0>0$ being a constant (to be chosen later). These functions satisfy

$$\Delta\Phi_j(x) \leq (1-\gamma)\Phi_j(x) \qquad \forall x\in\mathbb{R}^n\setminus 0. \tag{4.85}$$

Define $\Phi = \sum_{j=0}^{d_0}\Phi_j$. It follows from (4.84) and (4.85) that

$$\Delta(W_m - \Phi)(x) \geq (1-\gamma)(W_m-\Phi)(x) \qquad \forall x\in K_+\cap\mathcal{B}.$$

Hence there exists no positive maxima for the function $W_m - \Phi$ in the set $K_+\cap\mathcal{B}$. Choose C_0 sufficiently large so that

$$\Phi(x) \geq W_m(x) \quad \forall x\in\partial\mathcal{B}, \qquad \Phi(x) \geq \max_{j\in\mathcal{J}_0,\ i\in\mathcal{I}}|\partial_{x_i}u_\infty(x-z_m^j)| \qquad \forall x\in\mathbb{R}^n,$$

where $\partial\mathcal{B}$ denotes the boundary of $\mathcal{B}$. Also, on the boundary ∂K_+ of K_+ we have

$$W_m(x) = \max_{j\in\mathcal{J}_0,\ i\in\mathcal{I}} |\partial_{x_i} u_\infty(x - z_m^j)|.$$

Moreover, since $v_m^0 \in W^{2,4}$ and $n \leq 5$, we have $v_m^0(x) \to 0$ as $|x| \to \infty$ which implies that $W_m(x) \to 0$ as $|x| \to \infty$. It follows from the above discussion that

$$\Phi(x) \geq W_m(x) \qquad \forall x \in K_+ \cap \mathcal{B}.$$

A similar argument applied to the function $-W_m$ on the set

$$K_- = \Big\{x \in \mathbb{R}^n \Big| \ -W_m(x) > \max_{j\in\mathcal{J}_0,\ i\in\mathcal{I}} |\partial_{x_i} u_\infty(x - z_m^j)|\Big\}$$

gives

$$\Phi(x) \geq -W_m(x) \quad \forall x \in K_- \cap \mathcal{B}.$$

Therefore, we can conclude that for each $x \in \mathcal{B}$ and some $C > 0$

$$\begin{aligned} &|\tilde{u}_m^0(x) + v_m^0(x)| \leq \max_{j\in\mathcal{J}_0,\ i\in\mathcal{I}} \{|\partial_{x_i} u_\infty(x - z_m^j)|, \Phi(x)\} \\ \Longrightarrow\ &|v_m^0(x)| \leq C \sum_{j=0}^{d_0} e^{-\sqrt{1-\gamma}|x - z_m^j|}. \end{aligned} \tag{4.86}$$

For $x \in \mathbb{R}^n \setminus \mathcal{B}$, we have $|v_m^0(x)| \leq \|v_m^0\|_{L^\infty}$ which tends to 0 as $m \to \infty$.

Next we derive estimates for the H^2 norm of $v_m^1 = v_m - P_m^\perp v_m^0$. This function is the part of v_m associated with the z_m^js for which the sequence $(R_m z_m^j)_{m\in\mathbb{N}}$ is unbounded. Since $v_m^0 \in S_m$ (S_m is defined in (4.79)) we have

$$P_m^\perp v_m^0 = v_m^0 - \sum_{k=1}^{n} \sum_{j=d_0+1}^{d} \left\langle T_{z_m^j} \partial_{x_k} u_\infty, v_m^0 \right\rangle T_{z_m^j} \partial_{x_k} u_\infty = v_m^0 - P_z v_m^0.$$

Here P_z is the projection in L^2 onto its subspace spanned by

$$\big\{T_{z_m^j} \partial_{x_k} u_\infty,\ j \in \mathcal{J} \setminus \mathcal{J}_0,\ k \in \mathcal{I}\big\}.$$

To keep the notation simple, we do not indicate explicitly the dependence of P_z on m. Recall $\tilde{u}_m^1$ from (4.71). Then (4.72) can be written as

$$\begin{aligned} 0 =&(-\Delta + 1 + R_m^2 V_{R_m})(\tilde{u}_m^1 + W_m + v_m^1 - P_z v_m^0) \\ &+ \sigma|\tilde{u}_m^1 + W_m + v_m^1 - P_z v_m^0|^{2p}(\tilde{u}_m^1 + W_m + v_m^1 - P_z v_m^0). \end{aligned}$$

This equation is equivalent to

$$
\begin{aligned}
0 =&(-\Delta + 1 + R_m^2 V_{R_m}) v_m^1 - R_m^2 V_{R_m} \sum_{k=1}^{n} \sum_{j=d_0+1}^{d} \langle T_{z_m^j} \partial_{x_k} u_\infty, v_m^0 \rangle T_{z_m^j} \partial_{x_k} u_\infty \\
&+ \sigma \left| \tilde{u}_m^1 + W_m + v_m^1 - P_z v_m^0 \right|^{2p} \left(\tilde{u}_m^1 + W_m + v_m^1 - P_z v_m^0 \right) - \sigma \sum_{j=d_0+1}^{d} T_{z_m^j} |u_\infty|^{2p} u_\infty \\
&- \sigma |W_m|^{2p} W_m + \sum_{j=0}^{d_0} \sum_{i=1}^{n} \beta_i^j T_{z_m^j} \partial_{x_i} u_\infty + R_m^2 V_{R_m} \tilde{u}_m^1 \\
&+ \sum_{k=1}^{n} \sum_{j=d_0+1}^{d} (2p+1) \sigma \langle T_{z_m^j} \partial_{x_k} u_\infty, v_m^0 \rangle T_{z_m^j} |u_\infty|^{2p} \partial_{x_k} u_\infty
\end{aligned}
$$

which, using the operators $L_+ \left(\tilde{u}_m^1 + W_m - P_z v_m^0, R_m \right)$ and $N \left(\tilde{u}_m^1 + W_m - P_z v_m^0, v_m^1 \right)$, can be rewritten as

$$
\begin{aligned}
0 =&L_+ \left(R_m, \tilde{u}_m^1 + W_m - P_z v_m^0 \right) v_m^1 + N \left(\tilde{u}_m^1 + W_m - P_z v_m^0, v_m^1 \right) + \sum_{j=0}^{d_0} \sum_{i=1}^{n} \beta_i^j T_{z_m^j} \partial_{x_i} u_\infty \\
&+ \sigma \left| \tilde{u}_m^1 + W_m - P_z v_m^0 \right|^{2p} \left(\tilde{u}_m^1 + W_m - P_z v_m^0 \right) - \sigma \sum_{j=d_0+1}^{d} T_{z_m^j} |u_\infty|^{2p} u_\infty - \sigma |W_m|^{2p} W_m \\
&+ R_m^2 V_{R_m} \tilde{u}_m^1 - R_m^2 V_{R_m} \sum_{k=1}^{n} \sum_{j=d_0+1}^{d} \langle T_{z_m^j} \partial_{x_k} u_\infty, v_m^0 \rangle T_{z_m^j} \partial_{x_k} u_\infty \\
&+ \sum_{k=1}^{n} \sum_{j=d_0+1}^{d} (2p+1) \sigma \langle T_{z_m^j} \partial_{x_k} u_\infty, v_m^0 \rangle T_{z_m^j} |u_\infty|^{2p} \partial_{x_k} u_\infty. \qquad (4.87)
\end{aligned}
$$

The argument based on Proposition 7.5 and spectral perturbation theory (see discussion above (4.76)) used to show that $P_m^\perp L_+(\tilde{u}_m, R_m) P_m^\perp$ is invertible, with a uniform in m bound for its inverse, can be used to show that the same is true for $P_m^\perp L_+ \left(\tilde{u}_m^1 + W_m - P_z v_m^0, R_m \right) P_m^\perp$. Indeed, let $L_+ \left(\tilde{u}_m^1 + W_m - P_z v_m^0, R_m \right) = \widetilde{L}_m + \widetilde{W}_m$, where $\widetilde{L}_m$ is as defined above (4.76) and

$$
\widetilde{W}_m = R_m^2 V_{R_m} + \sigma(2p+1) \Big(|\tilde{u}_m + v_m^0 - P_z v_m^0|^{2p} - \sum_{j=0}^{d} \left| T_{z_m^j} u_\infty \right|^{2p} \Big).
$$

Using the fact that $\|v_m^0\|_{H^2} \to 0$ and $|y_m^j - y_m^k| \to \infty$ as $m \to \infty$, we get $\|\widetilde{W}_m\|_{H^2 \mapsto L^2} \to 0$ as $m \to \infty$. Applying the argument above (4.76) to $\widetilde{L}_m$ and $\widetilde{W}_m$ defined here, we get $L_m = P_m^\perp L_+ \left(\tilde{u}_m^1 + W_m - P_z v_m^0, R_m \right) P_m^\perp$ is invertible for large m and the norm of its inverse operator can be uniformly bounded. Therefore applying $P_m^\perp$ to (4.87) and rewriting it as $v_m = L_+^{-1}$[remaining terms] and using $P_m^\perp \sum_{j=0}^{d_0} \sum_{i=1}^{n} \beta_i^j T_{z_m^j} \partial_{x_i} u_\infty = 0$, we can deduce via contraction mapping principle that

$$
\|v_m^1\|_{H^2} \leq C R_m^2 \|V_{R_m} \tilde{u}_m^1\|_{L^2} + \sum_{j=d_0+1}^{d} \sum_{k=0,\, j \neq k}^{d} O(e^{-\sqrt{1-\gamma}|z_m^j - z_m^k|}). \qquad (4.88)
$$

In summary, we have

$$v_m = P_m^{\perp} v_m^0 + v_m^1 = v_m^0 - P_z v_m^0 + v_m^1$$

with $v_m^0(x) \le \sum_{k=0}^{d_0} Ce^{-\sqrt{1-\gamma}|x-z_m^k|}$ for all $x \in \mathbb{R}^n$, which using the definition of P_z, gives

$$\|P_z v_m^0\|_{H^2} + \|v_m^1\|_{H^2} \le CR_m^2 \|V_{R_m} \tilde{u}_m^1\|_{L^2} + \sum_{j=d_0+1}^{d} \sum_{k=0,\, j\neq k}^{d} O(e^{-\sqrt{1-\gamma}|z_m^j - z_m^k|}).$$

Next we will study the set of finite dimensional equations in (4.75) obtained by projecting (4.73) on the functions $T_{z_m^j} \partial_{x_k} u_\infty$. Explicitly, the set of equations are

$$\Big\langle T_{z_m^j} \partial_{x_k} u_\infty, R_m^2 V_{R_m} \sum_{i=0}^{d} T_{z_m^i} u_\infty + N(\tilde{u}_m, v_m) \Big\rangle + \Big\langle T_{z_m^j} \partial_{x_k} u_\infty, \sigma |\tilde{u}_m|^{2p} \tilde{u}_m - \sigma \sum_{i=0}^{d} T_{z_m^i} |u_\infty|^{2p} u_\infty \Big\rangle$$

$$+ \Big\langle T_{z_m^j} \partial_{x_k} u_\infty \big(R_m^2 V_{R_m} + (2p+1)\sigma \big[\tilde{u}_m^{2p} - T_{z_m^j} u_\infty^{2p} \big] \big), v_m \Big\rangle = 0 \tag{4.89}$$

for $j \in \mathcal{J}$ and $k \in \mathcal{I}$. We will show that the set of equations in (4.89) has no solution under our assumption (4.69). Our analysis will focus on two types of terms in (4.89) - terms involving interaction between $T_{z_m^j} u_\infty$ and $T_{z_m^k} u_\infty$ for $j \neq k$ and $j, k \in \mathcal{J} \setminus \mathcal{J}_0$ (such terms arise from the second inner product in (4.89)) and terms involving interaction between the various $T_{z_m^i} u_\infty$ and the potential V with $i \in \mathcal{J} \setminus \mathcal{J}_0$. We will consider two cases, one of which must occur. In the first case we assume that the former terms, which decay slower than $e^{-\sqrt{1+\gamma}|z_m^j - z_m^k|}$ dominate the latter terms. Then, by combining the equations in (4.89) appropriately we construct a new equation in which the former terms dominate all the other terms. This will imply that there is no solution to (4.89). In the second case, we will combine the equations in (4.89) appropriately to obtain a new equation in which the former terms (at least the important ones) are eliminated. Under the assumption that the terms involving the potential dominate terms decaying like $e^{-2\sqrt{1-\gamma}|z_m^j - z_m^k|}$, we can apply the argument from the proof of the compactness case to conclude that (4.89) has no solution. We will now discuss the two cases rigorously. In the proof below, we assume for simplicity that the set $\mathcal{J} \setminus \mathcal{J}_0$ has at least two elements. If it has only one element, then the proof of contradiction is similar to the proof in the compactness case, see remark below (4.104).

Let $M_m = \min\{|z_m^j - z_m^k| \,|\, j, k \in \mathcal{J} \setminus \mathcal{J}_0, j \neq k\}$. We assume without of loss of generality, by considering subsequences if necessary, that each of the bounded sequences $(M_m / |z_m^j - z_m^k|)_{m \in \mathbb{N}}$ is converging and that $\|V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}$ is the maximum of the set $\{\|V_{R_m} T_{z_m^j} u_\infty\|_{L^2}, j \in \mathcal{J} \setminus \mathcal{J}_0\}$ for each m and also that each of the sequences $|z_m^{d_0+1} - z_m^k| / |z_m^{d_0+1}|$ converges either to a finite limit or to ∞. Partition $\{z_m^0, z_m^1, \ldots z_m^d\}$ into sets Z_m^1 and Z_m^2 such that for each $z_m^k \in Z_m^1$ and no $z_m^k \in Z_m^2$ we have

$$\lim_{m \to \infty} \frac{|z_m^k - z_m^{d_0+1}|}{|z_m^{d_0+1}|} = 0. \tag{4.90}$$

Define

$$\hat{u}_m = \sum_{z_m^k \in Z_m^1} T_{z_m^k} u_\infty \quad \text{and} \quad \hat{w}_m = \sum_{z_m^k \in Z_m^2} T_{z_m^k} u_\infty.$$

In Case (i) we suppose that

$$\limsup_{m\to\infty} \frac{-\ln \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2}}{M_m} > 1 \qquad \forall j \in \mathcal{J} \setminus \mathcal{J}_0 \tag{4.91}$$

holds and in Case (ii) we suppose that

$$\liminf_{m\to\infty} \frac{-\ln \left| R_m^2 \left\langle V_{R_m}, \partial_{x_m} \hat{u}_m^2 \right\rangle \right|}{M_m} < 2, \tag{4.92}$$

where x_m is the direction along the vector $z_m^{d_0+1}$, holds. One of the above two conditions must hold. Indeed, if (4.91) does not hold for any $j \in \mathcal{J}\setminus\mathcal{J}_0$ then, since we have assumed that $\|V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}$ is the maximum of the set $\{\|V_{R_m} T_{z_m^j} u_\infty\|_{L^2} \big| j \in \mathcal{J} \setminus \mathcal{J}_0\}$ for each m, it follows that

$$\limsup_{m\to\infty} \frac{-\ln \|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}}{M_m} \leq 1.$$

Using this and the estimate

$$\lim_{m\to\infty} \frac{\ln \left| R_m^2 \left\langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \right\rangle \right|}{\ln \|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}^2} = 0, \tag{4.93}$$

which is established below, it follows that

$$\begin{aligned}
&\liminf_{m\to\infty} \frac{-\ln \left| R_m^2 \left\langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \right\rangle \right|}{M_m} \\
&= \liminf_{m\to\infty} \frac{-\ln \|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}^2}{M_m} \frac{-\ln \left| R_m^2 \left\langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \right\rangle \right|}{-\ln \|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}^2} \\
&= 0,
\end{aligned}$$

which implies that if (4.91) does not hold, then (4.92) holds, i.e. either Case (i) or Case (ii) must occur. We will now establish (4.93) by showing that

$$\lim_{m\to\infty} \frac{\|R_m^2 V_{R_m} \hat{u}_m\|_{L^2}^2}{|R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle|} = 0. \tag{4.94}$$

This limit and $\hat{u}_m > T_{z_m^{d_0+1}} u_\infty$ imply that $(\|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}^2)/(|R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle|) \to 0$ as $m \to \infty$ which, along with $\lim_{m\to\infty} \|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2} = 0$, implies (4.93). To show (4.94), we fix $\theta \in (0, \pi/2)$ such that $2\mathcal{C} \tan\theta < 1$, where $\mathcal{C}$ is the constant in the hypothesis (4.38). Let B_m be the closed ball in $\mathbb{R}^n$ with center $z_m^{d_0+1}$ and radius $|z_m^{d_0+1}| \sin\theta$. By adopting the approach used in the compactness case, see (4.55)-(4.59) and (4.61)-(4.64) and the associated discussion, we get using the exponential decay of u_∞, (4.90), $\lim_{m\to\infty} |R_m z_m^{d_0+1}| = \infty$ and the hypothesis in (4.36)-(4.38) that

$$\lim_{m\to\infty} \frac{\|R_m^2 V_{R_m}\hat{u}_m\|_{L^2}^2}{|R_m^2\langle \partial_{x_m}\hat{u}_m, V_{R_m}\hat{u}_m\rangle|} \le C \lim_{m\to\infty} \frac{R_m\left|\sum_{z_m^k\in Z_m^1}\int\limits_{B_m} u_\infty^2(x-z_m^k)V^2(R_m x)\mathrm{d}x\right|}{\left|\sum_{z_m^k\in Z_m^1}\int\limits_{B_m} u_\infty^2(x-z_m^k)\partial_{x_m}V(R_m x)\mathrm{d}x\right|}$$

$$= C \lim_{m\to\infty} \frac{R_m\left|\sum_{z_m^k\in Z_m^1}\int\limits_{B_m} u_\infty^2(x-z_m^k)\partial_{x_m}V(R_m x)\frac{V^2(R_m x)}{\partial_{x_m}V(R_m x)}\mathrm{d}x\right|}{\left|\sum_{z_m^k\in Z_m^1}\int\limits_{B_m} u_\infty^2(x-z_m^k)\partial_{x_m}V(R_m x)\mathrm{d}x\right|}$$

$$= 0.$$

Hence (4.94), and therefore (4.93), holds.

We will now consider the two cases. We say that the sequences $(z_m^j)_{m\in\mathbb{N}}$ and $(z_m^k)_{m\in\mathbb{N}}$ are connected if and only if

$$\lim_{m\to\infty} \frac{M_m}{|z_m^j - z_m^k|} = 1.$$

Case (i): Assume that (4.91) holds. We say that the sequences $(z_m^{a_1})_{m\in\mathbb{N}}, (z_m^{a_2})_{m\in\mathbb{N}}, \dots (z_m^{a_r})_{m\in\mathbb{N}}$, where each $a_i \in \mathcal{J}\setminus\mathcal{J}_0$, form a connected component if two conditions are satisfied: (a) For any $i,j\in\{a_1,a_2,\dots a_r\}$ with $i\ne j$, there exists a chain $(b_1,b_2,\dots b_t)$, where each $b_k\in\{a_1,a_2,\dots a_r\}$, $b_1=i$ and $b_t=j$, such that the sequences $(z_m^{b_k})_{m\in\mathbb{N}}$ and $(z_m^{b_{k+1}})_{m\in\mathbb{N}}$ are connected for each $k\in\{1,2,\dots r-1\}$ and (b) For any $i\notin\{a_1,a_2,\dots a_r\}$, there exists no $j\in\{a_1,a_2,\dots a_r\}$ such that $(z_m^i)_{m\in\mathbb{N}}$ is connected to $(z_m^j)_{m\in\mathbb{N}}$.

Consider a connected component with at least two sequences, say $(z_m^{d_0+1})_{m\in\mathbb{N}}, (z_m^{d_0+2})_{m\in\mathbb{N}}, \dots (z_m^{d_0+r})_{m\in\mathbb{N}}$. For each m, the baricenter of these z_m^js is

$$\bar{z}_m = \frac{z_m^{d_0+1}+z_m^{d_0+2}+\dots+z_m^{d_0+r}}{r}.$$

By going to a subsequence if necessary, suppose that $q\in\{d_0+1,d_0+2,\dots d_0+r\}$ is such that $|z_m^q-\bar{z}_m|\ge|z_m^j-\bar{z}_m|$ for all $j\in\{d_0+1,d_0+2,\dots d_0+r\}$ and all m. For each z_m^l connected to z_m^q (clearly $l>d_0$) it follows that

$$\cos\alpha_m^l \ge \frac{|z_m^l-z_m^q|}{2|\bar{z}_m-z_m^q|},$$

where α_m^l is the angle between the vectors $z_m^l-z_m^q$ and $\bar{z}_m-z_m^q$. Also, we have

$$\begin{aligned}\limsup_{m\to\infty}\frac{|\bar{z}_m-z_m^q|}{M_m} &= \limsup_{m\to\infty}\frac{|z_m^{d_0+1}-z_m^q+\dots+z_m^{q-1}-z_m^q+z_m^{q+1}-z_m^q+\dots+z_m^{d_0+r}-z_m^q|}{rM_m}\\ &\le \limsup_{m\to\infty}\sum_{i=d_0+1}^{d_0+r}\frac{|z_m^i-z_m^q|}{rM_m}\le\frac{(r-1)^2}{r}\le\frac{(d-1)^2}{d},\end{aligned}$$

which implies that for each z_m^l connected to z_m^q

$$\liminf_{m\to\infty}\frac{|z_m^l-z_m^q|}{2|\bar{z}_m-z_m^q|}\ge\liminf_{m\to\infty}\frac{r|z_m^l-z_m^q|}{2(r-1)\sum_{i=d_0+1}^{d_0+r-1}|z_m^i-z_m^{i+1}|}\ge\frac{r}{2(r-1)^2}\ge\frac{d}{2(d-1)^2}.$$

By letting $j = q$ in (4.89), we get that for some for a fixed $\epsilon > 0$

$$0 = \Big\langle T_{z_m^q}\partial_{x_k} u_\infty, \sigma(2p+1) T_{z_m^q}[u_\infty^{2p}] \sum_{\substack{l: z_m^l, z_m^q \text{ is} \\ \text{connected}}} T_{z_m^l} u_\infty \Big\rangle$$

$$+ C R_m^2 \sum_{j=d_0+1}^{d} \| V_{R_m} T_{z_m^j} u_\infty \|_{L^2} + O\Big(\sum_{i\neq j,\ i,j=d_0+1}^{d} e^{-(1+\epsilon)\sqrt{1-\gamma}|z_m^i - z_m^j|} \Big). \tag{4.95}$$

Here we have used the hypothesis in (4.37), which implies that $|V(x)| > Ce^{-\beta|x|}$ for almost all large $|x|$, to dominate some exponentially decaying terms (such as the product $T_{z_m^q} u_\infty T_{z_m^j} u_\infty$ for $j \in \mathcal{J}_0$). By choosing $x_k \parallel \bar{z}_m - z_m^q$ in (4.95) and by denoting the direction parallel to $(z_m^l - z_m^q)$ by x_{lq}, we get for m large

$$\begin{aligned}
&\Big| \Big\langle T_{z_m^q}\partial_{x_k} u_\infty, \sigma(2p+1) T_{z_m^q}[u_\infty^{2p}] \sum_{\substack{l: z_m^l, z_m^q \text{ is} \\ \text{connected}}} T_{z_m^l} u_\infty \Big\rangle \Big| \\
&= \Big| \sigma(2p+1) \sum_{\substack{l: z_m^l, z_m^q \text{ is} \\ \text{connected}}} \cos\alpha_m^l \big\langle T_{z_m^q}\partial_{x_{lq}} u_\infty, T_{z_m^q}[u_\infty^{2p}] T_{z_m^l} u_\infty \big\rangle \Big| \\
&\geq \frac{C(\gamma) d}{2(d-1)^2} \exp^{-\sqrt{1+\gamma} M_m},
\end{aligned} \tag{4.96}$$

for any $\gamma > 0$. While deriving the above expression, we have used two estimates. The first is the easily verifiable estimate that if $x_i \perp (z_m^l - z_m^q)$, then

$$\big\langle T_{z_m^q}\partial_{x_i} u_\infty, \sigma(2p+1) T_{z_m^q}[u_\infty^{2p}] T_{z_m^l} u_\infty \big\rangle = 0.$$

This is used to obtain the equality in the second line of (4.96). The second estimate is if $z_m^l - z_m^q$ is along the direction of x_i and m is large, then for every $\gamma > 0$, there exists $C(\gamma) > 0$ such that

$$-\big\langle T_{z_m^q}\partial_{x_i}(u_\infty)^{2p+1}, T_{z_m^l} u_\infty \big\rangle \geq C(\gamma) \exp^{-\sqrt{1+\gamma}|z_m^l - z_m^q|}. \tag{4.97}$$

This is used to derive the inequality in the third line of (4.96). The estimate in (4.97) can be established as follows. Clearly $z_m^l - z_m^q$ has all coordinates 0, except the i^{th} coordinate which is positive. We denote it by z_m^δ. Let $y = (x_1, x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_n)$ and ∂_i denote the derivative in the x_i-direction. For any $y \in \mathbb{R}^{n-1}$, consider the integral

$$\int_{\mathbb{R}} \partial_i(u_\infty)^{2p+1}(x - z_m^q) u_\infty(x - z_m^l)\mathrm{d}x_i = \int_{\mathbb{R}} \partial_i(u_\infty)^{2p+1}(x_i, y) u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i.$$

From the radial symmetry of u_∞ and the monotone decay of u_∞ along the radial direction, it follows that for any $a > 0$,

$$\partial_i(u_\infty)^{2p+1}(a, y) = -\partial_i(u_\infty)^{2p+1}(-a, y) \leq 0, \qquad u_\infty(a - z_m^\delta, y) \geq u_\infty(-a - z_m^\delta, y) > 0.$$

From these estimates it is easy to see that for any $y \in \mathbb{R}^{n-1}$

$$-\int_{\mathbb{R}} \partial_{x_i}(u_\infty)^{2p+1}(x_i, y) u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i > -\int_{A_m} \partial_{x_i}(u_\infty)^{2p+1}(x_i, y) u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i > 0. \tag{4.98}$$

Here $A_m = \{a \big| |a \pm z_m^\delta| < 1\}$. For each $\epsilon \in (0,1)$, there exists $C_1(\epsilon), C_2(\epsilon) > 0$ such that for $|x|$ large

$$C_1(\epsilon)e^{-\sqrt{1+\epsilon}|x|} \le |u_\infty(x)| \le C_2(\epsilon)e^{-\sqrt{1-\epsilon}|x|}, \qquad |\partial_r u_\infty(x)| \ge C_1(\epsilon)e^{-\sqrt{1+\epsilon}|x|}.$$

Here ∂_r is the radial derivative. Also, since $z_m^\delta \to \infty$ as $m \to \infty$, it follows that there exists $C_3(\epsilon) > 0$ for m large such that if $x_i \in A_m$ and $|y| < 1$, then

$$|\partial_i u_\infty(x)| \ge C_3(\epsilon)e^{-\sqrt{1+\epsilon}|x|}.$$

Using the above estimates for u_∞ and its derivative, we get that for m large and $|y| < 1$

$$\begin{aligned} -\int_{A_m} \partial_{x_i}(u_\infty)^{2p+1}(x_i,y)u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i &> -\frac{1}{2}\int_{-1+z_m^\delta}^{1+z_m^\delta} \partial_{x_i}(u_\infty)^{2p+1}(x_i,y)u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i \\ &> C_4(\epsilon)e^{-\sqrt{1+2\epsilon}|z_m^\delta|}, \end{aligned}$$

for some $C_4(\epsilon) > 0$. This, along with (4.98), implies that for m large

$$\begin{aligned} -\big\langle T_{z_m^q}\partial_{x_i}(u_\infty)^{2p+1}, T_{z_m^l}u_\infty\big\rangle &= -\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \partial_{x_i}(u_\infty)^{2p+1}(x_i,y)u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i\,\mathrm{d}y \\ &\ge -\frac{1}{2}\int_{|y|<1}\int_{-1+z_m^\delta}^{1+z_m^\delta} \partial_{x_i}(u_\infty)^{2p+1}(x_i,y)u_\infty(x_i - z_m^\delta, y)\mathrm{d}x_i\mathrm{d}y \\ &\ge C_4(\epsilon)e^{-\sqrt{1+2\epsilon}|z_m^\delta|}, \end{aligned}$$

for some $C_5(\epsilon) > 0$, i.e. (4.97) holds.

It follows from (4.91) that as $m \to \infty$, for a sufficiently small $\gamma > 0$ and each $j \in \mathcal{J} \setminus \mathcal{J}_0$, we have (at least on a subsequence of m) that

$$\begin{aligned} &M_m\left[\sqrt{1+\gamma} + \frac{\ln \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2}}{M_m}\right] \to -\infty \\ \iff\ & \sqrt{1+\gamma}M_m + \ln \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2} \to -\infty, \\ \iff\ & e^{\sqrt{1+\gamma}M_m}\|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2} \to 0. \end{aligned}$$

The last expression above implies that the term considered in (4.96) cannot be canceled in (4.95). Therefore (4.95) and consequently the set of equations (4.89) has no solution in this case.

Case (ii): Assume that (4.92) holds. By adding the equations in (4.89) corresponding to each of the $z_m^i \in Z_m^1$, we obtain

$$\begin{aligned} 0 =& \Big\langle \partial_{x_m}\hat{u}_m, R_m^2 V_{R_m}(\hat{u}_m + \hat{w}_m) + \sigma(\hat{u}_m + \hat{w}_m)^{2p+1} - \sigma\hat{u}_m^{2p+1} - \sigma\sum_{i=0}^{d} T_{z_m^i}u_\infty^{2p+1}\Big\rangle \\ &+ O\Big(\sum_{j=d_0+1}^{d} \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2}^2\Big) + \sum_{i=d_0+1}^{d}\sum_{\substack{k=0 \\ i\neq k}}^{d} O\left(e^{-2\sqrt{1-\gamma}|z_m^i - z_m^k|}\right). \end{aligned} \tag{4.99}$$

Since the functions u_∞ are radially symmetric, one can check that for any $i, j \in \mathcal{J} \setminus \mathcal{J}_0$ and any direction x_k

$$\left\langle T_{z_m^i} \partial_{x_k} u_\infty, T_{z_m^j} u_\infty^{2p+1} \right\rangle + \left\langle T_{z_m^j} \partial_{x_k} u_\infty, T_{z_m^i} u_\infty^{2p+1} \right\rangle = 0.$$

In particular the above inequality implies that

$$\left\langle \partial_{x_m} \hat{u}_m, \sum_{i \in Z_1} T_{z_m^i} u_\infty^{2p+1} \right\rangle = 0. \tag{4.100}$$

Note that for some $\delta > 0$

$$\lim_{m\to\infty} \frac{|z_m^i - z_m^{d_0+1}|}{|z_m^{d_0+1}|} > \delta \quad \text{for each} \quad z_m^i \in Z_2.$$

Therefore

$$\lim_{m\to\infty} \frac{|z_m^i - z_m^k|}{|z_m^{d_0+1}|} > \delta \quad \text{for each} \quad z_m^k \in Z_1, z_m^i \in Z_2.$$

From the above discussion we get the estimate

$$\left\langle \partial_{x_m} \hat{u}_m, R_m^2 V_{R_m} \hat{w}_m + \sigma(\hat{u}_m + \hat{w}_m)^{2p+1} - \sigma \hat{u}_m^{2p+1} - \sigma \sum_{l \in Z_2} T_{z_m^l} u_\infty^{2p+1} \right\rangle = O\left(e^{-\sqrt{1-\gamma}\delta |z_m^{d_0+1}|}\right). \tag{4.101}$$

The hypothesis in (4.37) implies that for some $C > 0$

$$\|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}^2 > Ce^{-\sqrt{1-\gamma}\delta|z_m^{d_0+1}|} \tag{4.102}$$

Using (4.100), (4.101) and (4.102) we can rewrite (4.99) as

$$0 = \langle \partial_{x_m} \hat{u}_m, R_m^2 V_{R_m} \hat{u}_m \rangle + O(\|R_m^2 V_{R_m} T_{z_m^{d_0+1}} u_\infty\|_{L^2}^2) + \sum_{i=d_0+1}^{d} \sum_{\substack{k=0 \\ i \neq k}}^{d} O\left(e^{-2\sqrt{1-\gamma}|z_m^i - z_m^k|}\right). \tag{4.103}$$

We will next show that the first term in the above equation dominates the other two terms. Comparing the first term with the third term, it follows from (4.92) that for a sufficiently small $\gamma > 0$, as $m \to \infty$ (considering subsequences if necessary)

$$\begin{aligned}
& M_m \left(\frac{-\ln \left| R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle \right|}{M_m} - 2\sqrt{1-\gamma} \right) \to -\infty \\
\iff \quad & -\ln \left| R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle \right| - 2\sqrt{1-\gamma} M_m \to -\infty \\
\iff \quad & \frac{e^{-2\sqrt{1-\gamma}M_m}}{|R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle|} \to 0.
\end{aligned}$$

We next compare the first term with the second term of (4.103). Using the hypothesis (4.36) we get, as in the compactness case, that

$$\lim_{m\to\infty} \frac{\left| R_m^4 \left\langle T_{z_m^{d_0+1}} u_\infty V_{R_m}, T_{z_m^{d_0+1}} u_\infty V_{R_m} \right\rangle \right|}{|R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle|} \le \lim_{m\to\infty} \frac{\left| R_m^4 \langle \hat{u}_m V(R_m x), \hat{u}_m V(R_m x) \rangle \right|}{|R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle|} = 0. \tag{4.104}$$

Hence the term $|R_m^2 \langle \partial_{x_m} \hat{u}_m, V_{R_m} \hat{u}_m \rangle|$ in (4.103) cannot be canceled by the remaining two terms. Thus we get that (4.99), and consequently the set of equations (4.89), has no solution in this case.

We remark that if the set $\mathcal{J} \setminus \mathcal{J}_0$ has only one element, then we need not consider the two cases. Indeed, using (4.37) the first term in (4.103) can be shown to be larger than $CR_m^3 e^{-2\beta R_m |z_m^d|}$ (see (4.58) in compactness case). Therefore it dominates the third term which will be $O(e^{-2\sqrt{1-2\gamma}|z_m^d|})$. The first term also dominates the second term (see (4.104)). Hence (4.99), and consequently the set of equations (4.89), has no solution.

Since either Case (i) or Case (ii) must occur, we get the contradiction that the set of equations (4.89) has no solution if our assumption (4.44) is true. Hence it must be false and $(R_m y_m^j)_{m\in\mathbb{N}}$ is a bounded sequence for all $j \in \mathcal{J}$. Next we will show that each accumulation point of the sequence $(R_m z_m^j)_{m\in\mathbb{N}}$ at which V is continuously differentiable must be a critical point of the potential. This, according to the discussion below (4.68), will complete the proof of this theorem.

Suppose that, by considering subsequences if required, that for each $j \in \mathcal{J}$, the sequence $(R_m z_m^j)_{m\in\mathbb{N}}$ is convergent. To show that if the potential is continuously differentiable at a limit of these sequences, then that limit must be a critical point of the potential, it is sufficient to assume that there exists (by relabeling the sequences if needed) a $\bar{d}$ such that $\lim_{m\to\infty} R_m z_m^j = x^0$ for each $j \in \{0, 1, \ldots \bar{d}\}$ and $\lim_{m\to\infty} |x^0 - R_m z_m^j| > 0$ for each $j \in \{\bar{d}+1, \bar{d}+2, \ldots d\}$ and prove that x^0 is a critical point of the potential whenever V is continuously differentiable at x^0. Such a proof follows.

We write u_{E_m} as

$$u_{E_m} = \sum_{j=0}^{d} T_{z_m^j} u_\infty + v_m = \tilde{u}_m + v_m = \bar{u}_m^0 + \bar{u}_m^1 + v_m,$$

where

$$\tilde{u}_m = \sum_{j=0}^{d} T_{z_m^j} u_\infty, \qquad \bar{u}_m^0 = \sum_{j=\bar{d}+1}^{d} T_{z_m^j} u_\infty, \qquad \bar{u}_m^1 = \sum_{j=0}^{\bar{d}} T_{z_m^j} u_\infty.$$

Earlier, using the decomposition $\tilde{u}_m = \tilde{u}_m^0 + \tilde{u}_m^1$, we obtained a decomposition of v_m in terms of the pair of functions (v_m^0, v_m^1). Similarly, using the decomposition $\tilde{u}_m = \bar{u}_m^0 + \bar{u}_m^1$, we can obtain another estimate for v_m in terms of a different pair of functions $(\bar{v}_m^0, \bar{v}_m^1)$. Then, defining $P_{\bar{z}}$ analogous to P_z, we have $v_m = \bar{v}_m^0 - P_{\bar{z}} \bar{v}_m^0 + \bar{v}_m^1$, where $\bar{v}_m^0(x) \le \sum_{k=\bar{d}+1}^{d} Ce^{-\sqrt{1-\gamma}|x - z_m^k|}$ and

$$\|P_{\bar{z}} \bar{v}_m^0\|_{H^2} + \|\bar{v}_m^1\|_{H^2} \le CR_m^2 \left\|V_{R_m} \bar{u}_m^1\right\|_{L^2} + \sum_{j=0}^{\bar{d}} \sum_{\substack{k=0 \\ j\neq k}}^{d} O(e^{-\sqrt{1-\gamma}|z_m^j - z_m^k|}).$$

We will now consider two distinct cases to show that x^0 must be a critical point of the potential whenever V is continuously differentiable at x^0. Since these cases are similar to Case (i) and Case (ii) discussed above, we will keep our presentation concise and avoid detailed explanations. Define $D_m = \min\{|z_m^j - z_m^k| | j, k = 0, 1, \ldots \bar{d}, j \neq k\}$ and assume that (by considering subsequences if necessary) that each of the bounded sequences $D_m / |z_m^j - z_m^k|$ is converging. Our proof is by contradiction. Hence we assume that V is continuously differentiable at x^0, but x^0 is not a critical point of the potential and

$$|\partial_{x_1} V(x^0)| > 0. \tag{4.105}$$

In Case (a) we assume that

$$1 < \limsup_{m\to\infty} \frac{-\ln \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2}}{D_m} \quad \forall j \in \{0,1,\dots \bar{d}\} \tag{4.106}$$

holds and in Case (b) we assume that

$$\liminf_{m\to\infty} \frac{-\ln \left|R_m^2 \left\langle V_{R_m}, \partial_{x_1}(\bar{u}_m^1)^2\right\rangle\right|}{D_m} < 2 \tag{4.107}$$

holds. One of these two cases must occur. Indeed, if (4.106) does not hold, then for some $r \in \{0,1,\dots \bar{d}\}$ we have

$$\limsup_{m\to\infty} \frac{-\ln \|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2}}{D_m} \le 1. \tag{4.108}$$

Since $\lim_{m\to\infty} \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2} = 0$ for all $j \in \{0,1,\dots \bar{d}\}$, we can assume with no loss of generality that $\|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2} \ge \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2}$ for all $j \in \{0,1,\dots \bar{d}\}$. Using (4.108) and the estimate

$$\lim_{m\to\infty} \frac{\ln |R_m^2 \langle V_{R_m}, \partial_{x_1}(\bar{u}_m^1)^2\rangle|}{\ln \|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2}^2} = 0, \tag{4.109}$$

which is established below, it follows that

$$\begin{aligned} &\liminf_{m\to\infty} \frac{-\ln \left|R_m^2 \left\langle V_{R_m}, \partial_{x_1}(\bar{u}_m^1)^2\right\rangle\right|}{D_m} \\ =& \liminf_{m\to\infty} \frac{-\ln \|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2}^2}{D_m} \frac{-\ln \left|R_m^2 \left\langle V_{R_m}, \partial_{x_1}(\bar{u}_m^1)^2\right\rangle\right|}{-\ln \|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2}^2} \\ =& \, 0, \end{aligned}$$

which implies that if (4.106) does not hold, then (4.107) holds, i.e. either Case (a) or Case (b) must occur. We will now establish (4.109) by showing that

$$\lim_{m\to\infty} \frac{\|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2}^2}{|R_m^2 \langle \partial_{x_1}(\bar{u}_m^1)^2, V_{R_m}\rangle|} = 0. \tag{4.110}$$

This limit and $\lim_{m\to\infty} \|R_m^2 V_{R_m} T_{z_m^r} u_\infty\|_{L^2} = 0$ imply (4.109). Note that

$$\begin{aligned} R_m^2 \langle \partial_{x_1}(\bar{u}_m^1)^2, V_{R_m}\rangle =& \, R_m^3 \sum_{j=0}^{\bar{d}} \int_{\mathbb{R}^n} u_\infty^2(x) \partial_{x_1} V(R_m x + R_m z_m^j) \mathrm{d}x \\ &+ R_m^3 \sum_{j=0}^{\bar{d}} \sum_{k\neq j,\, k=0}^{\bar{d}} \int_{\mathbb{R}^n} u_\infty(x - z_m^j) u_\infty(x - z_m^k) \partial_{x_1} V(R_m x) \mathrm{d}x. \end{aligned} \tag{4.111}$$

Since $|z_m^j - z_m^k| \to \infty$ as $m \to \infty$ for all $j,k \in \{0,1,\dots \bar{d}\}$ with $j \neq k$, $\nabla V \in L^\infty(\mathbb{R}^n)$ and $|u_\infty(x)| < Ce^{-\sqrt{1-\gamma}|x|}$ for all $x \in \mathbb{R}^n$ and for some $C > 0$ and $\gamma \in (0,1/2)$, it follows that the integrals in the second term on the right hand side of the above equation converge to 0 as $m \to \infty$. Next consider the integrals in the first term on the right hand side of the above equation. For every

$x \in \mathbb{R}^n$ and $j \in \{0,1,\dots \bar{d}\}$, we have $\partial_{x_1} V(R_m x + R_m z_m^j) u_\infty^2(x) \to \partial_{x_1} V(x^0) u_\infty^2(x)$ as $m \to \infty$ and $|\partial_{x_1} V(R_m x + R_m z_m^j) u_\infty^2(x)| \le \|\nabla V\|_{L^\infty} u_\infty^2(x)$ for all m. Since $\|\nabla V\|_{L^\infty} u_\infty^2 \in L^1(\mathbb{R}^n)$, it follows using the Lebesgue dominated convergence theorem that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} u_\infty^2(x) \partial_{x_1} V(R_m x + R_m z_m^j) \mathrm{d}x = \partial_{x_1} V(x^0) \|u_\infty\|_{L^2}^2 .$$

It therefore follows from (4.111) that

$$\lim_{m\to\infty} \frac{\langle \partial_{x_1} (\bar{u}_m^1)^2, V_{R_m} \rangle}{R_m} = (\bar{d}+1) \partial_{x_1} V(x^0) \|u_\infty\|_{L^2}^2 . \tag{4.112}$$

This along with $R_m \|V_{R_m} T_{z_m^r} u_\infty\|_{L^2}^2 \to 0$ as $m \to \infty$ implies (4.110), which in turn implies (4.109). We will now consider the two cases.

Case (a): Suppose that (4.106) holds. Let $\bar{z}_m$ be the baricenter of the set $\{z_m^0, z_m^1, \dots z_m^{\bar{d}}\}$. For some $q \in \{0,1,\dots,\bar{d}\}$, let z_m^q be located at a maximal distance from $\bar{z}_m$. For each $l \in \{0,1,\dots,\bar{d}\}$ with $l \neq q$ we have

$$\cos \alpha_m^l \ge \frac{|z_m^l - z_m^q|}{2|\bar{z}_m - z_m^q|},$$

where α_m^l is the angle between the vectors $z_m^l - z_m^q$ and $\bar{z}_m - z_m^q$. It can be shown (as in Case (i) earlier) that

$$\liminf_{m\to\infty} \frac{|z_m^l - z_m^q|}{2|\bar{z}_m - z_m^q|} \ge \frac{d+1}{2d^2}.$$

Let $j = q$ in (4.89). Then

$$\begin{aligned} 0 =& \Big\langle T_{z_m^q} \partial_{x_k} u_\infty, \sigma(2p+1) T_{z_m^q} u_\infty^{2p} \sum_{l\neq q,\ l=0}^{\bar{d}} T_{z_m^l} u_\infty \Big\rangle \\ &+ C R_m^2 \sum_{j=0}^{\bar{d}} \Big\| V_{R_m} T_{z_m^j} u_\infty \Big\|_{L^2} + O\Big(\sum_{i=0}^{\bar{d}} \sum_{i\neq j,\, j=0}^{d} e^{-(1+\epsilon)\sqrt{1-\gamma}|z_m^i - z_m^j|} \Big) \end{aligned} \tag{4.113}$$

for some fixed $\epsilon > 0$. By choosing $x_k \parallel \bar{z}_m - z_m^q$ in (4.113) we get (by adopting the approach used in Case (i)) that for m large

$$\Big| \Big\langle T_{z_m^q} \partial_{x_k} u_\infty, \sigma(2p+1) T_{z_m^q} u_\infty^{2p} \sum_{l\neq q,\ l=0}^{\bar{d}} T_{z_m^l} u_\infty \Big\rangle \Big| \ge \frac{d+1}{2d^2} C(\gamma) \exp^{-\sqrt{1+\gamma} D_m} . \tag{4.114}$$

It now follows from (4.106) that as $m \to \infty$, for a sufficiently small $\gamma > 0$ and each $j \in \{0,1,\dots \bar{d}\}$, we have (at least on a subsequence of m) that

$$e^{\sqrt{1+\gamma} D_m} \|R_m^2 V_{R_m} T_{z_m^j} u_\infty\|_{L^2} \to 0.$$

This implies that the term on the left hand side of (4.114) cannot be canceled in (4.113). But since $u_{E_m} = \tilde{u}_m + v_m$ is assumed to be a solution to the equations in (4.89) and so (4.113), we can conclude that (4.106) cannot hold.

Case (b): Suppose that (4.107) holds. By adding all the equations in (4.89) for which $x_k = x_1$ and $j \in \{0, 1, \dots \bar{d}\}$, we obtain

$$
\begin{aligned}
0 =& \Big\langle \partial_{x_1} \bar{u}_m^1, R_m^2 V_{R_m}(\bar{u}_m^0 + \bar{u}_m^1) + \sigma(\bar{u}_m^0 + \bar{u}_m^1)^{2p+1} - \sigma(\bar{u}_m^1)^{2p+1} - \sigma \sum_{j=0}^{d} T_{z_m^j} u_\infty^{2p+1} \Big\rangle \\
&+ O\Big(\sum_{j=0}^{\bar{d}} \Big\| R_m^2 V_{R_m} T_{z_m^j} u_\infty \Big\|_{L^2}^2 \Big) \\
&+ O\Big(\sum_{i=0}^{\bar{d}} \sum_{j=\bar{d}+1}^{d} e^{-\sqrt{1-\gamma}|z_m^i - z_m^j|} + \sum_{i=0}^{\bar{d}} \sum_{j\neq i,\ j=0}^{d} e^{-2\sqrt{1-\gamma}|z_m^i - z_m^j|} \Big).
\end{aligned} \tag{4.115}
$$

As in Case (ii) above, we have

$$
\Big\langle \partial_{x_1} \bar{u}_m^1, \sum_{j=0}^{\bar{d}} T_{z_m^j} u_\infty^{2p+1} \Big\rangle = 0
$$

and it can be verified that

$$
\begin{aligned}
&\Big\langle \partial_{x_1} \bar{u}_m^1, R_m^2 V_{R_m} \bar{u}_m^0 + (\bar{u}_m^0 + \bar{u}_m^1)^{2p+1} - (\bar{u}_m^1)^{2p+1} - \sum_{l=\bar{d}+1}^{d} T_{z_m^l} u_m^{2p+1} \Big\rangle \\
&= O\Big(\sum_{i=0}^{\bar{d}} \sum_{j=\bar{d}+1}^{d} e^{-\sqrt{1-\gamma}|z_m^i - z_m^j|} \Big).
\end{aligned}
$$

We can now write (4.115) as follows:

$$
\begin{aligned}
0 =& -\frac{1}{2} \int_{\mathbb{R}^n} (\bar{u}_m^1(x))^2 \partial_{x_1} V(R_m x) dx + O\Big(R_m \sum_{j=0}^{\bar{d}} \Big\| V_{R_m} T_{z_m^j} u_\infty \Big\|_{L^2}^2 \Big) \\
&+ \frac{1}{R_m^3}\, O\Big(\sum_{i=0}^{\bar{d}} \sum_{j=\bar{d}+1}^{d} e^{-\sqrt{1-\gamma}|z_m^i - z_m^j|} \Big) + \frac{1}{R_m^3}\, O\Big(\sum_{i=0}^{\bar{d}} \sum_{j\neq i,\ j=0}^{d} e^{-2\sqrt{1-\gamma}|z_m^i - z_m^j|} \Big).
\end{aligned} \tag{4.116}
$$

As $m \to \infty$, since (4.107) holds, it follows that the first term in (4.116) is much larger than the last term. Moreover this first term converges to $(\bar{d}+1)\partial_{x_1} V(x^0) \|u_\infty\|_{L^2}^2/2$, see (4.112), and is bounded away from zero (since $|\partial_{x_1} V(x^0)| > 0$) and the second and third terms tend to zero as $m \to \infty$. Therefore we obtain the contradiction that $u_{E_m} = \tilde{u}_m + v_m$ cannot be a solution to (4.89). Thus (4.107) cannot hold.

If V is continuously differentiable at x^0 and (4.105) holds, then both Case (a) and Case (b) give the contradiction that $u_{E_m} = \tilde{u}_m + v_m$ does not solve (4.89). Since one of these cases must occur it follows that (4.105) cannot hold and x^0 is a critical point of the potential.

Theorem 4.2 is now completely proven. □

Again, the reason we cannot obtain the above result for arbitrary branches of bound states is the same as in finite E case, see the discussion at the end of Section 3.

The next theorem proves the existence of C^1 branches of ground states considered in the previous theorem. We show that given a finite set of m critical points of the potential, there exists a unique C^1 branch of ground states consisting of m humps. Each of these humps is a re-scaling of u_∞ using (4.6) and localizes at one of the m critical points.

Theorem 4.3. *Let $\sigma < 0$. For each $E > E_0$, consider the nonlinear equation*

$$-\Delta u(x) + E^{-1}V(E^{-\frac{1}{2}}x)u(x) + u(x) + \sigma|u|^{2p}u(x) = 0 \qquad \forall\, x \in \mathbb{R}^n, \tag{4.117}$$

where the potential $V : \mathbb{R}^n \to \mathbb{R}$ satisfies Hypothesis (H1) and (H2). Let $\{x_1, x_2, \dots x_m\}$ be a finite subset of all the critical points of V at which V is twice differentiable and has a nonsingular Hessian. Fix either $\mu_i = 1$ or $\mu_i = -1$ for each $i \in \{1, 2, \dots m\}$. Then there exists an $\varepsilon > 0$ and an $E_\varepsilon > 0$ such that for each $E > E_\varepsilon$, (4.117) has a unique real-valued solution $\mathcal{U}_E \in H^2(\mathbb{R}^n)$ that satisfies the following estimate:

$$\|\mathcal{U}_E - \sum_{i=1}^{m} \mu_i u_\infty(\cdot - x_i\sqrt{E})\|_{H^2(\mathbb{R}^n)} < \varepsilon. \tag{4.118}$$

Here u_∞ is the unique, radially symmetric and positive solution to (4.1). Moreover, the map $E \mapsto \mathcal{U}_E$ is a C^1 curve and along this curve

$$\lim_{E\to\infty} \|\mathcal{U}_E - \sum_{i=1}^{m} \mu_i u_\infty(\cdot - x_i\sqrt{E})\|_{H^2(\mathbb{R}^n)} = 0. \tag{4.119}$$

In addition, any solution (U, E) of (4.117) satisfying $E > E_\varepsilon$ and $\inf_{\theta\in(0,2\pi]} \|U - e^{i\theta}\mathcal{U}_E\|_{H^2} < \varepsilon$ is of the form $e^{i\theta}\mathcal{U}_E$ for some $\theta \in (0, 2\pi]$.

Furthermore when $p \geq 1/2$, for large E, one can count the number of negative eigenvalues of the linearized operator $L_+(\mathcal{U}_E, E) : H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$:

$$L_+(\mathcal{U}_E, E)[v] = (-\Delta + E^{-1}V_{E^{-\frac{1}{2}}} + 1)v + \sigma(2p+1)|\mathcal{U}_E|^{2p}v,$$

where $V_{E^{-\frac{1}{2}}}(x) = V(E^{-\frac{1}{2}}x)$ for all $x \in \mathbb{R}^n$:

$$\#neg(L_+(\mathcal{U}_E, E)) = m + \sum_{i=1}^{m} n_i, \tag{4.120}$$

where n_i is the number of negative eigenvalues of the Hessian of V at x_i.

Consequently, if the nonlinearity is super-critical, $p > 2/n$, then all branches are orbitally unstable, while for critical and subcritical nonlinearities, $p \leq 2/n$, the branches emerging from a single profile u_∞ centered at a local minima of the potential are the only stable ones.

Proof. Using Lemma 7.6 and Remarks 7.2 and 7.3 it can be shown that the following two claims are equivalent:

(1) there exists $\varepsilon > 0$ and an $E_\varepsilon > 0$ such that for each $E > E_\varepsilon$, (4.117) has a unique real-valued solution $\mathcal{U}_E$ satisfying (4.118),

(2) there exists $\varepsilon > 0$ and an $E_\varepsilon > \|V\|_{L^\infty}$ such that for each $E > E_\varepsilon$, (4.117) has a unique solution $\mathcal{U}_E$ of the form

$$\mathcal{U}_E = \sum_{i=1}^{m} \mu_i T_{x_i\sqrt{E}+s_i} u_{i,E} + v_E\,, \tag{4.121}$$

where v_E, $u_{i,E}$ and s_i satisfy the following: $v_E \in H^2(\mathbb{R}^n)$ with $\|v_E\|_{H^2(\mathbb{R}^n)} < \varepsilon$ and for each $i \in \{1, 2, \dots m\}$, $|s_i| < \varepsilon$ and $u_{i,E}$ is the unique radially symmetric positive function in $H^2(\mathbb{R}^n)$ satisfying

$$-\Delta u_{i,E} + u_{i,E} + \frac{V(x_i)}{E} u_{i,E} + \sigma |u_{i,E}|^{2p} u_{i,E} = 0\,, \tag{4.122}$$

and furthermore the following orthogonality relations hold:

$$\langle v_E, T_{x_i\sqrt{E}+s_i} \partial_{x_k} u_{i,E} \rangle = 0 \qquad \forall\, k \in \{1, 2, \dots n\}, \quad \forall\, i \in \{1, 2, \dots m\}. \tag{4.123}$$

(To simplify the notation, the dependence of s_i on E is not explicitly shown.)

To show that claim (1) implies claim (2), note that: (a) the x_i's are all distinct and so for $i \neq j$, $\sqrt{E}|x_i - x_j| \to \infty$ as $E \to \infty$, (b) each $u_{i,E}$ is a scaled version of u_∞ i.e.,

$$u_i(x) = \left(1 + \frac{V(x_i)}{E}\right)^{\frac{1}{2p}} u_\infty\left(\sqrt{1 + \frac{V(x_i)}{E}}\, x\right) \qquad \forall\, x \in \mathbb{R}^n, \tag{4.124}$$

and so

$$\lim_{E\to\infty} \sum_{i=1}^{m} \|u_{i,E} - u_\infty\|_{H^2} = 0,$$

and hence we can replace $T_{x_i\sqrt{E}} u_\infty$ with $T_{x_i\sqrt{E}} u_{i,E}$ in (4.118) to get a new estimate for $\mathcal{U}_E$ instead of (4.118), which will be valid for large E and (c) Lemma 7.6 can be applied with $u = \mathcal{U}_E$ (satisfying the new estimate), $y_i = \sqrt{E}x_i$, $u_i = u_{i,E}$ and $\psi = \phi_i = 0$ to get the decomposition in (4.122) (which is similar to the decomposition in (7.24)). Showing that claim (2) implies claim (1) is simpler. We omit the details.

We will complete the proof of this theorem below by establishing claim (2). For this we will assume that a function $\mathcal{U}_E$ that can be decomposed as in (4.121) and satisfying (4.123) solves (4.117) and then establish the existence of s_is and v_E, with appropriate properties, whenever E is sufficiently large. Since we are interested in finding s_is in a small neighborhood of the origin in $\mathbb{R}^n$, while deriving all our estimates below we will assume that

$$\sup_{i=1,2,\dots m} |s_i| \leq 1\,. \tag{4.125}$$

In the rest of this proof, to simplify our notation, the dependence of $u_{i,E}$ on E will not be shown explicitly and $u_{i,E}$ will instead be written as u_i. We will use the notation $s = [s_1, s_2, \dots s_m]^\top$.

Substituting $\mathcal{U}_E$ from (4.121) into (4.117) and using the fact that u_i solves (4.122), we get that

$$\begin{aligned} &- \Delta v_R + R^2 \sum_{i=1}^{m} (V_R - V_i) \mu_i T_{x_i/R+s_i} u_i + R^2 V_R v_R + v_R \\ &+ \sigma \left| \sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i + v_R \right|^{2p} \left(\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i + v_R \right) - \sigma \sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} |u_i|^{2p} u_i = 0\,. \end{aligned} \tag{4.126}$$

Here we have introduced the notation

$$V_i = V(x_i), \qquad R = \frac{1}{\sqrt{E}}, \qquad V_R(x) = V(Rx), \qquad v_R(x) = v_E(x) \qquad \forall\, x \in \mathbb{R}^n.$$

Recall the nonlinear map $N(U, v)$ introduced in (3.7):

$$N(U, v) = \sigma|U + v|^{2p}(U + v) - \sigma|U|^{2p}U - \sigma(2p + 1)|U|^{2p}v,$$

which satisfies the bound in (3.8) and (3.14). Like in the proof of Theorem 4.2 (see (4.48) and (4.73)), we will rewrite (4.126) using the linear operator $L_+(u, R)$, which is denoted differently in this proof for convenience, and the nonlinear operator N. Define the linear operator $L_\infty^{R,s} : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as follows: for all $v \in H^2(\mathbb{R}^n)$,

$$L_\infty^{R,s} v = \, - \Delta v + R^2 V_R v + v + \sigma(2p + 1)\Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big|^{2p} v\,.$$

Using $L_\infty^{R,s}$ and N, (4.126) can be rewritten as

$$\begin{aligned} &L_\infty^{R,s} v_R + R^2 \sum_{i=1}^{m} (V_R - V_i)\mu_i T_{x_i/R+s_i} u_i + N\Big(\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i, v_R\Big) \\ &+ \sigma\Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big|^{2p}\Big(\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big) - \sigma \sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} |u_i|^{2p} u_i \,=\, 0. \end{aligned} \tag{4.127}$$

We can write $L_\infty^{R,s} = L_+^{R,s} + L_\delta^{R,s}$, where $L_+^{R,s}, L_\delta^{R,s} : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are defined as follows: for all $v \in H^2(\mathbb{R}^n)$

$$L_+^{R,s} v = \, - \Delta v + v + \sigma(2p + 1) \sum_{i=1}^{m} |T_{x_i/R+s_i} u_\infty|^{2p} v\,,$$

$$L_\delta^{R,s} v \,=\, R^2 V_R v + \sigma(2p + 1)\Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big|^{2p} v - \sigma(2p + 1) \sum_{i=1}^{m} |T_{x_i/R+s_i} u_\infty|^{2p} v\,.$$

Using (4.124) we get $\|L_\delta^{R,s}\|_{H^2 \to L^2} \to 0$ as $R \to 0$, uniformly in s satisfying (4.125). For each $i \in \{1, 2, \dots m\}$, we have

$$\ker\big[\, - \Delta + 1 + \sigma(2p + 1)|T_{x_i/R+s_i} u_\infty|^{2p}\big] \,=\, \operatorname{span}\{T_{x_i/R+s_i} \partial_{x_k} u_\infty \,\big|\, k \,=\, 1, 2, \dots m\}.$$

Applying Proposition 7.5 in the Appendix to $L_+^{R,s}$, with $E(R)$, $s_k(R)$ and V_k in the proposition being 1, $x_k/R + s_k$ and $\sigma(2p+1)|u_\infty|^{2p}$, respectively, and then applying the spectral perturbation theory to $L_+^{R,s} + L_\delta^{R,s}$ by regarding $L_\delta^{R,s}$ as a perturbation, it follows that there exists $C, R_1 > 0$ such that $\widetilde{P}_{R,s}^\perp L_\infty^{R,s} \widetilde{P}_{R,s}^\perp$ is invertible and

$$\|(\widetilde{P}_{R,s}^\perp L_\infty^{R,s} \widetilde{P}_{R,s}^\perp)^{-1}\|_{L^2 \mapsto H^2} < C \tag{4.128}$$

for all $R \in (0, R_1)$ and all s satisfying (4.125). Here $\widetilde{P}_{R,s}$ is the orthogonal projection operator onto

$$\operatorname{span}\{T_{x_i/R+s_i} \partial_{x_j} u_\infty \,\big|\, i \,=\, 1, 2, \dots m,\ j \,=\, 1, 2, \dots n\}.$$

Let P (by suppressing its dependence on R and s) be the orthogonal projection operator onto

$$\operatorname{span}\{T_{x_i/R+s_i}\partial_{x_k}u_i \,\big|\, i = 1,2,\dots m,\ k = 1,2,\dots m\}.$$

Using (4.124), it follows that $\|\widetilde{P}_{R,s} - P\|_{L^2\mapsto L^2} \to 0$ and $\|\widetilde{P}_{R,s} - P\|_{H^2\mapsto H^2} \to 0$ as $R \to 0$. Hence from (4.128) we get that there exists $R_2 \in (0, R_1)$ such that $P^\perp L_\infty^{R,s} P^\perp : P^\perp L^2 \cap H^2 \to P^\perp L^2$ is boundedly invertible for all $R \in (0, R_2)$ and s satisfying (4.125) and there exists $K > 0$ such that

$$\sup_{\substack{R\in(0,R_2),\\ s \text{ satisfying } (4.125)}} \|(P^\perp L_\infty^{R,s} P^\perp)^{-1}\|_{P^\perp L_2\to P^\perp L^2\cap H^2} < K. \tag{4.129}$$

Recall from (4.123) that $P^\perp v_R = v_R$. Define

$$D(R,s) = \sigma\Big|\sum_{i=1}^m \mu_i T_{x_i/R+s_i}u_i\Big|^{2p}\Big(\sum_{i=1}^m \mu_i T_{x_i/R+s_i}u_i\Big) - \sigma\sum_{i=1}^m \mu_i T_{x_i/R+s_i}|u_i|^{2p}u_i\,.$$

It is easy to see that (4.127) is equivalent to the following set of equations:

$$P^\perp L_\infty^{R,s} P^\perp v_R + P^\perp R^2 \sum_{i=1}^m (V_R - V_i)\mu_i T_{x_i/R+s_i}u_i + P^\perp N\Big(\sum_{i=1}^m \mu_i T_{x_i/R+s_i}u_i, P^\perp v_R\Big) + P^\perp D(R,s) = 0 \tag{4.130}$$

and for $k = 1,2,\dots n$ and $i = 1,2,\dots m$

$$\Big\langle L_\infty^{R,s} v_R + R^2\sum_{j=1}^m (V_R - V_j)\mu_j T_{x_j/R+s_j}u_j + N\Big(\sum_{j=1}^m \mu_j T_{x_j/R+s_j}u_j, v_R\Big), T_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle + \Big\langle D(R,s), T_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle = 0. \tag{4.131}$$

A tedious but elementary calculation gives that for some $\gamma \in (0,1)$,

$$\|D(R,s)\|_{L^2} \ \text{ is }\ O(e^{-(1-\gamma)\min\{|x_i-x_j|/R\,|\,i,j=1,2,\dots m,\ i\neq j\}}).$$

In particular, $\|D(R,s)\|_{L^2}$ is $O(R^m)$ for any integer $m > 0$, uniformly in s satisfying (4.125).

We will complete the proof of claim (2) in two steps. In the first step we will solve (4.130) for v_R, for each s satisfying (4.125) and all R sufficiently small. We will see that the dependence of v_R on R and s is C^1 and $v_R \to 0$ as $R \to 0$, uniformly in s satisfying (4.125). In the second step, using some estimates for v_R derived in the first step, we will solve (4.131) for s, for all R sufficiently small. We will see that the dependence of s on R is C^1 and $s \to 0$ as $R \to 0$. This will complete the proof of claim 2 and also establish the claim in the theorem that the map $E \mapsto \mathcal{U}_E$ is C^1 and satisfies (4.119).

We now proceed with the first step. So as to explicitly indicate the dependence of v_R on s, below we use the notations $v_{R,s}$ instead of v_R. But we will continue to use the notation $P^\perp$ by suppressing its dependence on R and s. We can now rewrite (4.130) as

$$\begin{aligned} v_{R,s} &= (P^\perp L_\infty^{R,s} P^\perp)^{-1}\Bigg[P^\perp R^2 \sum_{i=1}^m (V_R - V_i)\mu_i T_{x_i/R+s_i} u_i + P^\perp N\bigg(\sum_{i=1}^m \mu_i T_{x_i/R+s_i} u_i, P^\perp v_{R,s}\bigg) \\ &\qquad\qquad + P^\perp D(R,s)\Bigg] \\ &= v_{R,s}^0 + (P^\perp L_\infty^{R,s} P^\perp)^{-1} P^\perp N\bigg(\sum_{i=1}^m \mu_i T_{x_i/R+s_i} u_i, P^\perp v_{R,s}\bigg). \end{aligned} \tag{4.132}$$

Here

$$v_{R,s}^0 = (P^\perp L_\infty^{R,s} P^\perp)^{-1}\big[P^\perp R^2 \sum_{i=1}^m (V_R - V_i)\mu_i T_{x_i/R+s_i} u_i + P^\perp D(R,s)\big].$$

Since $V \in L^\infty(\mathbb{R}^n)$, V is twice differentiable at $x = x_i$ and x_i is a critical point for V for every $i \in \{1, 2, \dots m\}$, there exists $c_1 > 0$ such that $|V(x) - V(x_i)| \le c_1 |x - x_i|^2$ for all $x \in \mathbb{R}^n$ and every i. Therefore if $R \in (0, R_2)$ and (4.125) holds, then

$$\begin{aligned} \sum_{i=1}^m \|(V_R - V_i) T_{x_i/R+s_i} u_i\|_{L^2}^2 &= \sum_{i=1}^m \int_{\mathbb{R}^n} [(V(Rx) - V(x_i)) u_i(x - x_i/R - s_i)]^2 \mathrm{d}x \\ &= \sum_{i=1}^m \int_{\mathbb{R}^n} [(V(Rx + x_i + Rs_i) - V(x_i)) u_i(x)]^2 \mathrm{d}x \\ &\le c_2 R^4 \end{aligned} \tag{4.133}$$

for some constant $c_2 > 0$. The above estimate, the fact that $\|D(R,s)\|_{L^2}$ is $O(R^m)$ and (4.129) imply that $\|v_{R,s}^0\|_{H^2}$ is $O(R^4)$. We will now apply the contraction mapping principle to (4.132). From Lemma 3.1 we get that there exists a constant $C > 0$ (independent of s_i, since (4.125) holds) such that

$$\bigg\| N\bigg(\sum_{i=1}^m \mu_i T_{x_i/R+s_i} u_i, P^\perp v_1\bigg) - N\bigg(\sum_{i=1}^m \mu_i T_{x_i/R+s_i} u_i, P^\perp v_2\bigg)\bigg\|_{L^2} \le C\|v_1 - v_2\|_{H^2}^{\min\{2,2p+1\}} \tag{4.134}$$

for all $v_1, v_2 \in H^2(\mathbb{R}^n)$ with $\|v_1\|_{H^2}, \|v_2\|_{H^2} < 1$ and all $R \in (0, R_2)$. Fix $r \in (0,1)$ and $R_3 \in (0, R_2)$ such that

$$\|v_{R,s}^0\|_{H^2} < \frac{r}{2} \qquad \forall\ R \in (0, R_3) \quad \text{and} \quad 4KCr^{\min\{1,2p\}} < 1.$$

Let $B(0,r)$ denote the closed ball of radius r centered at the origin in $H^2(\mathbb{R}^n)$. Then for each $R \in (0, R_3)$ and s satisfying (4.125), the map

$$\mathcal{N}_{R,s}(v) = v_{R,s}^0 + (P^\perp L_\infty^R P^\perp)^{-1} P^\perp N\bigg(\sum_{i=1}^m \mu_i T_{x_i/R+s_i} u_i, P^\perp v\bigg)$$

is a contraction on $B(0,r)$. It now follows from the contraction mapping principle that for each $R \in (0, R_3)$ and s satisfying (4.125), there exists a unique solution $v_{R,s} \in B(0,r)$, with $P^\perp v_{R,s} = v_{R,s}$, that solves (4.132). Since the dependence of $\mathcal{N}_{R,s}$ on R and s is C^1 and $\mathcal{N}_{R,s}$ are uniform contractions on the set $\{(R,s)\,|\,R \in (0,R_3), s \text{ satisfies (4.125)}\}$, it follows that the dependence of $v_{R,s}$ on R and s is also C^1. Furthermore, for each $R \in (0,R_3)$ and s satisfying (4.125), $\mathcal{N}_{R,s}$ is a contraction on $B(2\|v^0_{R,s}\|,0)$ as well, meaning that $\|v_{R,s}\|_{H^2}$ is also $O(R^4)$. This completes step 1. We now proceed with step 2.

Differentiating (4.122) with respect to x_k we get that for each $i \in \{1,2,\dots m\}$

$$-\Delta\partial_{x_k}u_i + \partial_{x_k}u_i + R^2V_i\partial_{x_k}u_i + \sigma(2p+1)|u_i|^{2p}\partial_{x_k}u_i = 0.$$

Using the above expression and the definition of $L^{R,s}_\infty$, it is easy to see that (4.131) can be rewritten as follows: for $k = 1,2,\dots n$ and $i = 1,2,\dots m$

$$\begin{aligned}
&\Big\langle R^2(V_R - V_i)T_{x_i/R+s_i}u_i, T_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle + \Big\langle R^2\sum_{j=1,\ j\neq i}^{m}(V_R - V_j)\mu_jT_{x_j/R+s_j}u_j,\, \mu_iT_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle\\
&\quad + \Big\langle R^2(V_R - V_i)v_{R,s} + N\Big(\sum_{j=1}^{m}\mu_jT_{x_j/R+s_j}u_j, v_{R,s}\Big) + D(R,s),\, \mu_iT_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle\\
&\quad - \Big\langle \sigma(2p+1)|T_{x_i/R+s_i}u_i|^{2p}v_{R,s} - \sigma(2p+1)\Big|\sum_{j=1}^{m}\mu_jT_{x_j/R+s_j}u_j\Big|^{2p}v_{R,s},\, \mu_iT_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle = 0.
\end{aligned} \tag{4.135}$$

We will now simplify the first term in (4.135). Denote the Hessian of V at x (if it exists) by $\nabla^2V(x)$ and its k^{th} row by $\nabla^2_kV(x)$. Using the fact that each x_i is a critical point of V at which V is twice differentiable, we get that

$$\begin{aligned}
\Big\langle R^2(V_R - V_i)T_{x_i/R+s_i}u_i, T_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle &= \frac{R^3}{2}\int_{\mathbb{R}^n}\partial V_{x_k}(Rx + x_i + Rs_i)\,u_i^2(x)\,\mathrm{d}x\\
&= \frac{R^4}{2}\int_{\mathbb{R}^n}\nabla^2_kV(x_i).(x+s_i)\,u_i^2(x)\,\mathrm{d}x + R^4g(R,s_i)\\
&= \frac{R^4}{2}(\nabla^2_kV(x_i).\,s_i)\|u_i\|^2_{L^2} + R^4g(R,s_i).
\end{aligned} \tag{4.136}$$

Here, using Taylor's theorem and the exponential decay of u_i, we have $\lim_{R\to0}g(R,s_i) = 0$ uniformly in s_i satisfying (4.125). Next we will show that the rest of the terms in (4.135) (except the first one) are $o(R^4)$. Using the fact that the functions u_i and $\partial_{x_k}u_i$ are in L^∞ and decay exponentially, it can be shown via elementary estimates that

$$\Bigg\|\Bigg(T_{x_i/R+s_i}|u_i|^{2p} - \Big|\sum_{j=1}^{m}\mu_jT_{x_j/R+s_j}u_j\Big|^{2p}\Bigg)T_{x_i/R+s_i}\partial_{x_k}u_i\Bigg\|_{L^2}$$

is $O(R^m)$, uniformly in s satisfying (4.125), for any integer m. Hence the same is true for

$$\Bigg|\Big\langle |T_{x_i/R+s_i}u_i|^{2p}v_{R,s} - \Big|\sum_{j=1}^{m}\mu_jT_{x_j/R+s_j}u_j\Big|^{2p}v_{R,s}, T_{x_i/R+s_i}\partial_{x_k}u_i\Big\rangle\Bigg|.$$

Finally, using (4.136), the above estimate and the estimates $\|v_{R,s}\|_{H^2}$ is $O(R^4)$, $\|D(R,s)\|_{L^2}$ is $O(R^m)$ for any integer m and $\|N\big(\sum_{j=1}^m \mu_j T_{x_j/R+s_j} u_j, v_{R,s}\big)\|_{L^2}$ is $o(R^4)$ (which follows from (4.134)), all three valid uniformly in s satisfying (4.125), we get that (4.135) can be written as

$$(\nabla_k^2 V(x_i).\, s_i)\|u_i\|_{L^2}^2 + \mathcal{G}_R^{ik}(s) = 0 \qquad \forall\, k \in \{1, 2, \dots n\}, \quad \forall\, i \in \{1, 2, \dots m\}, \tag{4.137}$$

where $|\mathcal{G}_R^{ik}(s)|$ is $o(1)$ in R, uniformly in s satisfying (4.125). Furthermore, since $v_{R,s}$ is differentiable with respect to s, we can show that $\mathcal{G}_R^{ik}(s)$ is differentiable with respect to s and its derivative is also $o(1)$ in R, uniformly in s satisfying (4.125). For each i, denote $\mathcal{G}_R^i = [\mathcal{G}_R^{i1}, \mathcal{G}_R^{i2}, \dots \mathcal{G}_R^{in}]^\top$ and let

$$\mathcal{G}_R = \left[\frac{[\nabla^2 V(x_1)]^{-1}\mathcal{G}_R^1}{\|u_1\|_{L^2}^2}, \frac{[\nabla^2 V(x_2)]^{-1}\mathcal{G}_R^2}{\|u_2\|_{L^2}^2}, \dots \frac{[\nabla^2 V(x_m)]^{-1}\mathcal{G}_R^m}{\|u_m\|_{L^2}^2} \right]^\top.$$

Then we can rewrite (4.137) as

$$\|u_i\|_{L^2}^2 \nabla^2 V(x_i)\, s_i + \mathcal{G}_R^i(s) = 0 \qquad \forall\, i \in \{1, 2, \dots m\},$$

which in turn can be rewritten as follows:

$$s = \mathcal{G}_R(s), \tag{4.138}$$

where $\mathcal{G}_R(s)$ and its derivative with respect to s are both $o(1)$ in R, uniformly in s satisfying (4.125). This means that there exists $R_4 \in (0, R_3)$ such that for each $R \in (0, R_4)$, the map $\mathcal{G}_R$ is a contraction on the closed unit ball in $\mathbb{R}^{n\times m}$, with a contraction constant independent of R. Hence for each $R \in (0, R_4)$ there exists a unique $s \in \mathbb{R}^n$, with $|s| \le 1$, that solves (4.138). Furthermore, since the dependence of $\mathcal{G}_R$ on R is C^1, $\mathcal{G}_R$ are uniform contractions for $R \in (0, R_4)$ and $\mathcal{G}_R$ is $o(1)$ in R, uniformly in s satisfying (4.125), it follows that the dependence of s on R is C^1 and s is also $o(1)$ in R. This completes the proof of step 2.

We have now shown that there exists $\varepsilon > 0$ and an $E_\varepsilon > 0$ such that for each $E > E_\varepsilon$, (4.117) has a unique real-valued solution $\mathcal{U}_E$ satisfying (4.118) and the map $E \mapsto \mathcal{U}_E$ is a C^1 curve satisfying (4.119). Next we will count the number of negative eigenvalues of the linearized operator $L_+(\mathcal{U}_E, E)$ along this curve. This will complete the proof of this theorem. We use the notation $\mathcal{U}_R = \mathcal{U}_E$ and $v_R = v_E$ and write $L_+(\mathcal{U}_E, E)$ in terms of R i.e., for all $v \in H^2(\mathbb{R}^n)$

$$L_+(\mathcal{U}_R, R)[v] = -\Delta v + R^2 V_R v + v + \sigma(2p+1)|\mathcal{U}_R|^{2p} v.$$

First we will discuss the spectral properties of $L_+(\mathcal{U}_R, R)$. From (4.124) we have $\lim_{R\to 0} \|u_i - u_\infty\|_{H^2} = 0$. Using this and the fact that $\|v_R\|_{H^2}$ is $O(R^4)$, it can be verified that $L_+(\mathcal{U}_R, R)$ can be expressed as $\widetilde{L}_R + \widetilde{W}_R$, where

$$\widetilde{L}_R[v] = -\Delta v + v + \sigma(2p+1)\sum_{i=1}^m |T_{x_i/R+s_i} u_\infty|^{2p} v \qquad \forall\, v \in H^2(\mathbb{R}^n)$$

and $\lim_{R\to 0} \|\widetilde{W}_R\|_{H^2 \mapsto L^2} = 0$. The first eigenvalue γ of $(-\Delta + 1) + \sigma(2p+1)|u_\infty|^{2p}$ is simple and negative and its second eigenvalue is zero with multiplicity n. Applying the spectral theory for operators with potentials separated by large distances (see proof of Proposition 7.5 in the Appendix) and Proposition 7.5 to $\widetilde{L}_R$, with $E(R)$, V_k and s_k in the proposition being 1, $\sigma(2p+1)|u_\infty|^{2p}$ and $x_k/R+s_k$, respectively, and then applying the spectral perturbation theory to $\widetilde{L}_R+\widetilde{W}_R$ by regarding $\widetilde{W}_R$ as a perturbation, it follows that for R small there exist exactly m eigenvalues $\gamma_1^R, \gamma_2^R, \ldots \gamma_m^R$ of $L_+(\mathcal{U}_R, R)$ which converge to $\gamma < 0$ as $R \to 0$ and exactly mn eigenvalues $\lambda_1^R, \lambda_2^R, \ldots \lambda_{mn}^R$ of $L_+(\mathcal{U}_R, R)$ with corresponding eigenfunctions $\phi_1^R, \phi_2^R, \ldots \phi_{mn}^R$ (we can suppose that these functions form an orthonormal set in L^2 as $L_+(\mathcal{U}_R, R)$ is self-adjoint) such that for each $k \in \{1, 2, \ldots mn\}$

$$\lim_{R\to 0} \lambda_k^R = 0, \qquad L_+(\mathcal{U}_R, R)\phi_k^R = \lambda_k^R \phi_k^R, \qquad \phi_k^R = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^k(R) T_{x_i/R+s_i}\partial_{x_j} u_i + \Phi_k^R,$$

where $\langle \Phi_k^R, T_{x_i/R+s_i}\partial_{x_j} u_i\rangle = 0$ and $a_{ij}^k(R)$ is bounded for all i, j, k and R small. All other eigenvalues of $L_+(\mathcal{U}_R, R)$ remain positive for all sufficiently small R. Furthermore, $\widetilde{P}_R^{\perp} L_+(\mathcal{U}_R, R)\widetilde{P}_R^{\perp}$ is invertible for all R sufficiently small and there exists $C > 0$ independent of R small such that

$$\|(\widetilde{P}_R^{\perp} L_+(\mathcal{U}_R, R)\widetilde{P}_R^{\perp})^{-1}\|_{L^2 \mapsto H^2} < C. \tag{4.139}$$

Here $\widetilde{P}_R$ is the orthogonal projection operator onto

$$\operatorname{span}\{T_{x_i/R+s_i}\partial_{x_j} u_\infty \,\big|\, i = 1, 2, \ldots m,\ j = 1, 2, \ldots n\}.$$

To complete the proof of this theorem we need to identify the signs of the eigenvalues λ_k^R for R small. This is done below.

Let P_R be the orthogonal projection operator onto

$$\operatorname{span}\{T_{x_i/R+s_i}\partial_{x_j} u_i \,\big|\, i = 1, 2, \ldots m,\ j = 1, 2, \ldots n\}.$$

Using (4.124), it follows that $\|\widetilde{P}_R - P_R\|_{L^2\mapsto L^2} \to 0$ and $\|\widetilde{P}_R - P_R\|_{H^2\mapsto H^2} \to 0$ as $R \to 0$. Hence it follows from (4.139) that $P_R^{\perp} L_+(\mathcal{U}_R, R) P_R^{\perp}$ is invertible for all R sufficiently small and there exists $C > 0$ independent of R small such that

$$\|(P_R^{\perp} L_+(\mathcal{U}_R, R) P_R^{\perp})^{-1}\|_{L^2 \mapsto H^2} < C. \tag{4.140}$$

Define the operator $L_R, W_R : H^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ as follows: For each $v \in H^2(\mathbb{R}^n)$,

$$L_R v = \ -\Delta v + v + \sigma(2p+1)\Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big|^{2p} v,$$

$$W_R v = R^2 V_R v + \sigma(2p+1)\Big(\Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i + v_R\Big|^{2p} - \Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big|^{2p}\Big) v.$$

Then clearly

$$L_+(\mathcal{U}_R, R)v = L_R v + W_R v.$$

Using the approach used to obtain (3.14) in Section 3 we get

$$\Big\|\Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i + v_R\Big|^{2p} - \Big|\sum_{i=1}^{m} \mu_i T_{x_i/R+s_i} u_i\Big|^{2p}\Big\|_{L^2} = O(\|v_R\|_{H^2}) \tag{4.141}$$

and since $\|v_R\|_{H^2}$ is $O(R^4)$ it follows that $\|W_R\|_{H^2\mapsto L^2}$ is $O(R^2)$. Using L_R and W_R, (4.130) can be rewritten as

$$\begin{aligned} &P_R^\perp L_R P_R^\perp v_R + P_R^\perp R^2 \sum_{i=1}^m (V_R - V_i)\mu_i T_{x_i/R+s_i} u_i \\ = \; &-P_R^\perp W_R P_R^\perp v_R - P_R^\perp N\Big(\sum_{i=1}^m \mu_i T_{x_i/R+s_i} u_i, P_R^\perp v_R\Big) - P_R^\perp D(R,s). \end{aligned} \tag{4.142}$$

Using the estimate $\|v_R\|_{H^2}$ is $O(R^4)$ and (3.14) it follows that the right hand side of the above equation is $O(R^6)$. From (4.140) and $\lim_{R\to 0}\|W_R\|_{H^2\mapsto L^2} = 0$, we get that $P_R^\perp L_R P_R^\perp$ is invertible for all R sufficiently small and there exists a $C > 0$ independent of R small such that $\|(P_R^\perp L_R P_R^\perp)^{-1}\|_{L^2\mapsto H^2} < C$. Therefore, for R small, using (4.142) we can write $v_R = v_R^0 + v_R^1$ so that v_R^0 is the unique solution of

$$P_R^\perp L_R v_R^0 = -P_R^\perp R^2 \sum_{i=1}^m (V_R - V_i)\mu_i T_{x_i/R+s_i} u_i \tag{4.143}$$

and $\|v_R^1\|_{H^2}$ is $O(R^6)$.

Apply $P_R^\perp$ to $\lambda_k \phi_k^R = L_R\phi_k^R + W_R\phi_k^R$ to obtain

$$\begin{aligned} &\big(P_R^\perp L_R P_R^\perp - \lambda_k^R + P_R^\perp W_R P_R^\perp\big)\Phi_k^R = P_R^\perp \sum_{i=1}^m\sum_{j=1}^n a_{ij}^k(R)(R^2V_i - W_R)T_{x_i/R+s_i}\partial_{x_j} u_i \\ &- P_R^\perp \sigma(2p+1)\sum_{i=1}^m\sum_{j=1}^n a_{ij}^k(R)\Big(\Big|\sum_{t=1}^m \mu_t T_{x_t/R+s_t}u_t\Big|^{2p} - |T_{x_i/R+s_i}u_i|^{2p}\Big)T_{x_i/R+s_i}\partial_{x_j}u_i\,. \end{aligned}$$

The operator (from $H^2 \to L^2$) on the left hand side of the above equation is invertible for R small with uniform in R bound for the inverse. This follows from our earlier discussion on the invertibility of $P_R^\perp L_R P_R^\perp$ and the limits $\lambda_k^R \to 0$ and $\|W_R\|_{H^2\mapsto L^2} \to 0$ as $R \to 0$. Using (4.133) and (4.141) it follows that the term on the right hand side of the above equation is $O(R^4)$ in L^2. Hence we get that $\|\Phi_k^R\|_{H^2}$ is $O(R^4)$. Taking the innerproduct of $L_+(\mathcal{U}_R, R)\phi_k^R = \lambda_k^R\phi_k^R$ with $T_{x_i/R+s_i}\partial_{x_j}u_i$ we get

$$\begin{aligned} &a_{ij}^k(R)\lambda_k^R\|\partial_{x_j}u_i\|_{L^2}^2 = \Big\langle \sum_{t=1}^m\sum_{q=1}^n a_{tq}^k(R)(W_R - R^2V_t)T_{x_t/R+s_t}\partial_{x_q}u_t, T_{x_i/R+s_i}\partial_{x_j}u_i\Big\rangle + o(R^4) \\ &= \Big\langle \sum_{q=1}^n a_{iq}^k(R)(W_R - R^2V_i)T_{x_i/R+s_i}\partial_{x_q}u_i, T_{x_i/R+s_i}\partial_{x_j}u_i\Big\rangle + o(R^4) \\ &= \Big\langle \sum_{q=1}^n a_{iq}^k(R)(W_R - R^2V_R)T_{x_i/R+s_i}\partial_{x_q}u_i, T_{x_i/R+s_i}\partial_{x_j}u_i\Big\rangle \\ &\qquad + \Big\langle \sum_{q=1}^n a_{iq}^k(R)R^2[V_R - V_i])T_{x_i/R+s_i}\partial_{x_q}u_i, T_{x_i/R+s_i}\partial_{x_j}u_i\Big\rangle + o(R^4). \end{aligned} \tag{4.144}$$

We will now derive estimates for the terms containing $W_R - R^2 V_R$ in the above expression. Consider the sets

$$S_+^R = \Big\{ x \in \mathbb{R}^n \Big| \sum_{j=1}^m \mu_j u_j(x - x_j/R - s_j) - |v_R(x)| > 0 \Big\},$$

$$S_-^R = \Big\{ x \in \mathbb{R}^n \Big| \sum_{j=1}^m \mu_j u_j(x - x_j/R - s_j) + |v_R(x)| < 0 \Big\}$$

and define

$$\begin{aligned}\mathbf{H}^R = &\Big(\Big| \sum_{t=1}^m \mu_t T_{x_t/R+s_t} u_t + v_R \Big|^{2p} - \Big| \sum_{t=1}^m \mu_t T_{x_t/R+s_t} u_t \Big|^{2p} \Big) \\ &- 2p \ \mathrm{sign}\Big(\sum_{t=1}^m \mu_t T_{x_t/R+s_t} u_t \Big) \Big| \sum_{t=1}^m \mu_t T_{(x_t/R+s_t)} u_t \Big|^{2p-1} v_R.\end{aligned}$$

Adapting the approach used to derive (3.14), it can be shown that $\|\mathbf{H}^R\|_{L^2(S_+^R \cup S_-^R)}$ is $O(\|v_R\|_{H^2}^{\min\{2p,2\}})$ if $p > 1/2$ and 0 if $p = 1/2$. So $\|\mathbf{H}^R\|_{L^2(S_+^R \cup S_-^R)}$ is $o(R^4)$. Since $\|v_R\|_{H^2}$ is $O(R^4)$, it follows for $n = 1, 2, 3$ that $v_R \overset{L^\infty}{\to} 0$ as $R \to 0$. For $n \geq 4$, the same conclusion can be arrived at by first using regularity to show that v_R converges to 0 in $W^{2,q}$ for any $q \geq 2$ (see Appendix). Using this and the exponential decays of u_i and its derivative, we have $\|\partial_{x_q} T_{x_i/R+s_i} u_i\|_{L^2(\mathbb{R}^n \setminus \{S_+^R \cup S_-^R\})}$ is $O(R)$ and so

$$\left| \int_{\mathbb{R}^n \setminus \{S_+^R \cup S_-^R\}} \mathbf{H}^R(x) \partial_{x_q} u_i(x - x_i/R - s_i) \mathrm{d}x \right| \leq C \|\partial_{x_q} T_{x_i/R+s_i} u_i\|_{L^2(\mathbb{R}^n \setminus \{S_+^R \cup S_-^R\})} \|v_R\|_{L^2} = o(R^4). \tag{4.145}$$

Clearly there exist coefficients $c_{tq}(R)$ which are $O(R^4)$ such that (4.143) can be written as

$$L_R v_R^0 = -R^2 \sum_{t=1}^m (V_R - V_t) \mu_t T_{x_t/R+s_t} u_t + \sum_{t=1}^m \sum_{r=1}^n c_{tr}(R) T_{x_t/R+s_t} \partial_{x_r} u_t.$$

Taking the derivative of the above equation along the x_q direction we get

$$\begin{aligned}&L_R \partial_{x_q} v_R^0 + \sigma(2p+1)2p \ \mathrm{sign}\Big(\sum_{t=1}^m \mu_t T_{x_t/R+s_t} u_t \Big) \Big| \sum_{t=1}^m \mu_t T_{x_t/R+s_t} u_t \Big|^{2p-1} \Big(\sum_{t=1}^m \mu_t T_{x_t/R+s_t} \partial_{x_q} u_t \Big) v_R^0 \\ = &- R^2 \partial_{x_q} V_R \sum_{t=1}^m \mu_t T_{x_t/R+s_t} u_t - R^2 \sum_{t=1}^m (V_R - V_t) \mu_t T_{x_t/R+s_t} \partial_{x_q} u_t + \sum_{t=1}^m \sum_{r=1}^n c_{tr}(R) T_{x_t/R+s_t} \partial^2_{x_q x_r} u_t.\end{aligned} \tag{4.146}$$

Taking the innerproduct of the above equation with $T_{x_i/R+s_i} \partial_{x_j} u_i$ and $v_R = v_R^0 + v_R^1$ with $\|v_R^1\|_{H^2}$

is $O(R^6)$ we get that

$$
\begin{aligned}
&\Big\langle \sigma(2p+1)2p\ \mathrm{sign}\Big(\sum_{t=1}^{m}\mu_t T_{x_t/R+s_t}u_t\Big)\Big|\sum_{t=1}^{m}\mu_t T_{x_t/R+s_t}u_t\Big|^{2p-1} v_R T_{x_i/R+s_i}\partial_{x_q}u_i, T_{x_i/R+s_i}\partial_{x_j}u_i\Big\rangle \\
&\qquad\qquad\qquad + \big\langle R^2[V_R - V_i]T_{x_i/R+s_i}\partial_{x_q}u_i, T_{x_i/R+s_i}\partial_{x_j}u_i\big\rangle \\
&= -R^3\int_{\mathbb{R}^n}\partial_{x_q}V(Rx)u_i(x-x_i/R-s_i)\partial_{x_j}u_i(x-x_i/R-s_i)\mathrm{d}x + o(R^4) \\
&= \frac{R^4}{2}\partial^2_{x_q x_j}V(x_i)\|u_i\|^2_{L^2} + o(R^4). \qquad (4.147)
\end{aligned}
$$

Recall that

$$W_R - R^2V_R = \sigma(2p+1)\Big(\Big|\sum_{t=1}^{m}\mu_t T_{x_t/R+s_t}u_t + v_R\Big|^{2p} - \Big|\sum_{t=1}^{m}\mu_t T_{x_t/R+s_t}u_t\Big|^{2p}\Big).$$

Using $\|\mathbf{H}^R\|_{L^2(S_+^R\cup S_-^R)}$ is $o(R^4)$ and (4.145), it follows from (4.144) and (4.147) that

$$a^k_{ij}(R)\lambda^R_k\|\partial_{x_j}u_i\|^2 = \frac{R^4}{2}\sum_{q=1}^{n}a^k_{iq}\partial^2_{x_q x_j}V(x_i)\|u_i\|^2_{L^2} + o(R^4).$$

Recall that $\lim_{R\to 0}\|u_i - u_\infty\|_{H^2} = 0$. Using this, the above equation yields

$$a^k_{ij}(R)\lambda^R_k\|\partial_{x_j}u_\infty\|^2 = \frac{R^4}{2}\sum_{q=1}^{n}a^k_{iq}\partial^2_{x_q x_j}V(x_i)\|u_\infty\|^2_{L^2} + o(R^4). \qquad (4.148)$$

Define

$$A_k(R) = \begin{bmatrix} A^1_k(R) & A^2_k(R) & \dots & A^m_k(R)\end{bmatrix},$$

where

$$A^i_k(R) = \begin{bmatrix} a^k_{i1}(R) & a^k_{i2}(R) & \dots & \dots & a^k_{in}(R)\end{bmatrix}.$$

Let $\nabla^2\mathbf{V}$ be the block diagonal matrix with diagonal entries $\nabla^2V(x_1), \nabla^2V(x_2), \dots \nabla^2V(x_m)$. Here $\nabla^2V(x_i)$ is the Hessian of V at x_i. Multiplying (4.148) by $a^l_{ij}(R)$ and summing the result over i and j gives

$$\lambda^R_k\|\partial_{x_j}u_\infty\|^2_{L^2}A_k(R)A_l^\top(R) = \frac{R^4}{2}\|u_\infty\|^2_{L^2}A_k(R)\nabla^2\mathbf{V}A_l^\top(R) + o(R^4). \qquad (4.149)$$

Let $\mathbf{A}_R$ be the $nm\times nm$ matrix whose k^{th} row is $\|\partial_{x_j}u_\infty\|_{L^2}A_k(R)$ and Λ_R be the $nm\times nm$ diagonal matrix with diagonal entries $\lambda^R_1, \lambda^R_2, \dots \lambda^R_{nm}$. Then it follows from (4.149) that

$$\Lambda_R\mathbf{A}_R\mathbf{A}_R^\top = \frac{R^4}{2}\frac{\|u_\infty\|^2_{L^2}}{\|\partial_{x_j}u_\infty\|_{L^2}}\mathbf{A}_R\nabla^2\mathbf{V}\,\mathbf{A}_R^\top + o(R^4). \qquad (4.150)$$

Since the ϕ^R_ks form an orthonormal set in L^2 and each $\|\Phi^R_k\|_{L^2}$ is $O(R^4)$, it follows that $\lim_{R\to 0}\mathbf{A}_R\mathbf{A}_R^\top - I_{2m\times 2m} = 0$. Let $\mathbf{E}_R$ be the unitary matrix obtained by applying the Gram-Schmidt orthonormalization process to the rows of $\mathbf{A}_R$. Then $\lim_{R\to 0}\mathbf{A}_R - \mathbf{E}_R = 0$ and (4.150) can written as

$$\Lambda_R = \frac{R^4}{2}\frac{\|u_\infty\|^2_{L^2}}{\|\partial_{x_j}u_\infty\|_{L^2}}\mathbf{E}_R\nabla^2\mathbf{V}\,\mathbf{E}_R^\top + o(R^4). \qquad (4.151)$$

Hence the sign of the eigenvalues $\{\lambda_1, \lambda_2, \dots \lambda_{nm}\}$ are determined by the sign of the eigenvalues of the Hessian at the critical points $\{x_1, x_2, \dots x_m\}$.

Finally we discuss the orbital stability with respect to the dynamical system (1.1) of the branches (ψ_E, E) of solutions of (1.3) obtained from $(\mathcal{U}_E, E)$ via the renormalization (4.6):

$$\psi_E(x) = E^{2p}\mathcal{U}_E(x\sqrt{E}).$$

The linearized operators $L_+(\psi_E, E)$ and $L_+(\mathcal{U}_E, E)$ have the same spectra. Consequently, if the number of profiles $m \geq 2$ formula (4.120) shows that L_+ has at least two negative eigenvalues. Since along this branches L_- is nonnegative, by the result in [8], we get that they are orbitally unstable. If there is a single profile $m = 1$ centered at a local maxima or saddle point the branch is orbitally unstable by the same argument. If the profile at a local minima we have that L_+ has exactly one negative eigenvalue. The result in [9] now implies that the orbital stability is determined by $\frac{d}{dE}\mathcal{N}(E)$. From the formula (4.4) we deduce that the slope is positive for subcritical nonlinearity $p < 2/n$ and hence these branches are orbitally stable. But the slope is negative for $p > 2/n$ and this branches are then orbitally unstable. One can actually show that for $p = 2/n$ the slope remains positive and hence these branches are also stable.

This completes the proof of the theorem. □

5 Symmetry Breaking Bifurcation

In this section we show how our results on the maximal extension of branches of bound states and their behavior at the $E = \infty$ limit, see Theorems 4.1 and 4.3, can be put together to show that, in the presence of symmetry, there is a symmetry breaking bifurcation point along the ground state branch emerging from $(0, E_0)$ where $-E_0$ is the lowest eigenvalue of $-\Delta + V$, see Proposition 2.2. We start by showing that if the potential in the Schrödinger equation (1.3) is invariant under a subgroup of Euclidian symmetries, then this branch and its maximal extension must be invariant under the same subgroup.

Proposition 5.1. *Consider the time independent Schrödinger equation* (1.3) *with potential V satisfying (H1)-(H3) and*

$$V(Rx) = V(x) \text{ a.e. } x \in \mathbb{R}^n, \qquad \forall\ R \in G \preceq E(n),$$

where G is a subgroup of the Euclidian group of isometries on $\mathbb{R}^n$, denoted here by $E(n)$. Then we have:

$$\psi_E(Rx) = \psi_E(x) \text{ a.e. } x \in \mathbb{R}^n, \qquad \forall\ R \in G \preceq E(n) \text{ and } \forall E \in \mathcal{I}$$

where $\{(\psi_E, E) \mid E \in \mathcal{I}\}$ is the maximal extension, see Corollary 2.2, of the real valued curve of solutions of (1.3) *given in Proposition 2.2, which bifurcates from the lowest eigenvalue $-E_0$ of $-\Delta + V$. Moreover, for all $E \in \mathcal{I}$ there is $\epsilon,\ \delta > 0$ such that any solution (ψ, E_*) of* (1.3) *satisfying $|E_* - E| < \delta$ and $\min_{0 \leq \theta < 2\pi} \|\psi - e^{i\theta}\psi_E\|_{H^2} < \epsilon$, where (ψ_E, E) is on the above maximal curve, is also invariant under the group G.*

Proof. Since the Laplacian is invariant under Euclidian transformations, and V is invariant under the action of G we deduce that, for all $R \in G$, $(\psi(Rx), E)$ is a solution of (1.3) provided $(\psi(x), E)$ is a solution. Consider now

$$\mathcal{G} = \{E \in \mathcal{I} \mid \psi_E \text{ is invariant under the action of } G\} \subseteq \mathcal{I}.$$

We will show that $\mathcal{G}$ is non-empty, open and closed in $\mathcal{I}$ and, since the later is an interval, we conclude that $\mathcal{G} = \mathcal{I}$. Then the last part of the theorem follows from part (iii) in Corollary 2.2.

We first prove $\mathcal{G} \neq \emptyset$. We know from Corollary 2.2 that a small subinterval $\mathcal{I}_\delta = (E_0, E_0 + \delta)$ if $\sigma < 0$ or $\mathcal{I}_\delta = (E_0 - \delta, E_0)$ if $\sigma > 0$ of the maximal interval $\mathcal{I}$ is obtained via the local bifurcation result in Proposition 2.2. Using the fact that the Sobolev norms are invariant under Euclidian transformations, with the notations in that proposition, for each $E \in \mathcal{I}_\delta$, and all $R \in G$ we have:

$$\|\psi_E(Rx)\|_{H^2} = \|\psi_E(x)\|_{H^2} \leq \epsilon.$$

Since $\psi_E(Rx)$ is also a real valued solution of (1.3), by part (i) of the proposition we infer that $\psi_E(Rx) = \pm\psi_E(x)$. Now the eigenvector ψ_0 solves (1.2) and, consequently is invariant under the action of G. Indeed $\psi_0(Rx)$ also solves the same equation, and, since $-E_0$ is a simple eigenvalue we have $\psi_0(Rx) = a\psi_0(x)$ for some $a \in \mathbb{C}$. But both have the same L^2 norm and are real valued, hence $a = \pm 1$. Finally they are both positive, hence $a = 1$. Consequently, by the invariance of the L^2 scalar product under Euclidian transformations we have:

$$\langle \psi_0(x), \psi_E(x)\rangle = \langle \psi_0(Rx), \psi_E(Rx)\rangle = \langle \psi_0(x), \psi_E(Rx)\rangle.$$

Hence, both $\psi_E(x)$ and $\psi_E(Rx)$ have the same $\psi_0(x)$ projection. By part (i) of Proposition 2.2 we have $\psi_E(Rx) = \psi_E(x)$. Since this is valid for all $R \in G$ and all $E \in \mathcal{I}_\delta$ we deduce $\emptyset \neq \mathcal{I}_\delta \subseteq \mathcal{G}$.

We now show that $\mathcal{G}$ is open in $\mathcal{I}$. Fix $E_1 \in \mathcal{G}$. From Corollary 2.2 $L_+(\psi_{E_1}, E_1)$ is non-singular, therefore we can apply Theorem 2.1 at (ψ_{E_1}, E_1). With its notations, for each $E \in (E_1 - \delta, E_1 + \delta)$ we have

$$\|\psi_E(Rx) - \psi_{E_1}(x)\|_{H^2} = \|\psi_E(Rx) - \psi_{E_1}(Rx)\|_{H^2} = \|\psi_E(x) - \psi_{E_1}(x)\|_{H^2} \leq \epsilon.$$

By the uniqueness part of Theorem 2.1 we must have $\psi_E(Rx) = \psi_E(x)$ for all $E \in (E_1 - \delta, E_1 + \delta)$, hence $\mathcal{G}$ is open in $\mathcal{I}$.

Finally, we show that $\mathcal{G}$ is closed in $\mathcal{I}$. Consider $\{E_k\}_{k\in\mathbb{N}} \subset \mathcal{G}$ such that $\lim_{k\to\infty} E_k = E \in \mathcal{I}$. By triangle inequality and the invariance of H^2 norm under Euclidian transformations we have, for any $R \in G$:

$$\|\psi_E(Rx) - \psi_E(x)\|_{H^2} \leq \|\psi_E(Rx) - \psi_{E_k}(Rx)\|_{H^2} + \|\psi_{E_k}(Rx) - \psi_E(x)\|_{H^2} = 2\|\psi_{E_k}(x) - \psi_E(x)\|_{H^2}.$$

By passing to the limit $k \to \infty$ and using the continuity in H^2 of the maximal branch we get $\lim_{k\to\infty} \|\psi_{E_k}(x) - \psi_E(x)\|_{H^2}$. Therefore $\psi_E(Rx) = \psi_E(x)$ for all $R \in G$ and $\mathcal{G}$ is closed in $\mathcal{I}$.

The proof is now complete. □

Note that this proposition combined with Theorem 3.2 shows that, in the case of defocusing nonlinearity ($\sigma > 0$), all ground states of (1.3) are symmetric. However, in the case of focusing nonlinearity ($\sigma < 0$), symmetric ground states are generally unstable for large values of the parameter E, and the ground state branch emerging from $(0, E_0)$ undergoes bifurcations:

Theorem 5.1. *Consider the time independent Schrödinger equation* (1.3) *with potential V satisfying (H1)-(H3),* (4.36)*-*(4.38)*,*

$$V(Rx) = V(x) \ a.e. \ x \in \mathbb{R}^n, \qquad \forall\ R \in G \preceq E(n),$$

and assume that the critical points of V invariant under the action of G are non-degenerate and are not local minima. Then any C^1 curve of ground states $E \mapsto \psi_E,\ E \in (E_\epsilon, \infty)$ with no singularity points and invariant under the action of G must be orbitally unstable. In particular the maximal extension of the ground state branch bifurcating from $(0, E_0)$ has a bifurcation point (singularity) at a finite $E = E_+,\ E_0 < E_+ < \infty$.

Proof. From Theorem 4.2 we deduce that,

$$u_E(x) = \frac{1}{E^{2p}} \psi_E\left(\frac{x}{\sqrt{E}}\right)$$

has on a sequences $\{E_k\}_{k\in\mathbb{N}} \subset \mathbb{R},\ \lim_{k\to 0} E_k = \infty$ the property:

$$\lim_{k\to\infty} \|u_{E_k}(x) - \sum_{j=1}^{d} u_\infty(x - x_j\sqrt{E_k})\|_{H^2} = 0,$$

for some integer $d \in \mathbb{N}$ and a subset of critical points of the potential V, $\mathcal{J} = \{x_j,\ j = \overline{1, d}\}$, where u_∞ is the unique, positive, radially symmetric solution of (4.1). Using the invariance of ψ_E under the action of G we get for all $E \in (E_\epsilon, \infty)$ and all $R \in G$:

$$u_E(Rx) = \frac{1}{E^{2p}} \psi_E\left(\frac{Rx}{\sqrt{E}}\right) = \frac{1}{E^{2p}} \psi_E\left(\frac{x}{\sqrt{E}}\right) = u_E(x) \ a.e.$$

Consequently

$$\begin{aligned}
&\|\sum_{j=1}^{d} u_\infty(Rx - Rx_j\sqrt{E_k}) - \sum_{j=1}^{d} u_\infty(x - x_j\sqrt{E_k})\|_{H^2} \\
\le&\|\sum_{j=1}^{d} u_\infty(Rx - Rx_j\sqrt{E_k}) - u_{E_k}(Rx)\|_{H^2} + \|u_{E_k}(Rx) - \sum_{j=1}^{d} u_\infty(x - x_j\sqrt{E_k})\|_{H^2} \\
=&2\|u_{E_k}(x) - \sum_{j=1}^{d} u_\infty(x - x_j\sqrt{E_k})\|_{H^2}
\end{aligned}$$

and, for large k, we deduce that

$$\sum_{j=1}^{d} u_\infty(Rx - Rx_j\sqrt{E_k}) = \sum_{j=1}^{d} u_\infty(x - x_j\sqrt{E_k}), \qquad \forall R \in G.$$

In particular, if $x_j \in \mathcal{J}$ then $Rx_j \in \mathcal{J}$ for all $R \in G$.

In conclusion card$\mathcal{J} \ge 2$ or $\mathcal{J}$ has a single critical point of the potential which is invariant under the action of G. In the latter case the critical point is either a saddle or a local maxima, by hypotheses. In both cases, Theorem 4.3 implies that $L_+(\psi_E, E)$ has at least two negative eigenvalues and the branch is orbitally unstable.

Finally we prove the last part by contradiction. Assume that Case (b) in Corollary 2.2 does not hold. Then by the same corollary the positive ground state curve (ψ_E, E) bifurcating from $(0, E_0)$ forms a C^1 curve in $E \in (E_0, \infty)$. By the proof above $L_+(\psi_E, E),\ E \in (E_0, \infty)$ has at least two negative eigenvalue in contradiction to Remark 2.3.

Consequently Case (b) in Corollary 2.2 occurs showing that the maximal interval for this curve is (E_0, E_+), $E_0 < E_+ < \infty$ and an eigenvalue of $L_+(\psi_E, E)$ approaches zero as $E \to E_+$. Now, if we assume the stronger hypothesis on the potential (3.15)-(3.17), then there is actually a limit point (ψ_{E_+}, E_+) of this curve which is still a ground state and at which $L_+(\psi_{E_+}, E_+)$ has non-trivial but finite dimensional kernel, see Corollary 3.1.

The theorem is now completely proven. □

The result generalizes to higher dimensions the one in [18]. We emphasize that this generalization is not at all trivial and relies on the delicate arguments of Theorem 4.2. For example, if we consider a potential symmetric under reflections with respect to a hyperplane, it could be possible that the ground states on the branch bifurcating from $(0, E_0)$ approach one profile u_∞ sliding along the hyperplane towards infinity. More precisely, in Theorem 4.1 $M = 1$ and x_k^1 is on the hyperplane with $\lim_{k\to\infty} \frac{x_k^1}{\sqrt{E_{m_k}}} = \infty$. In this case, L_+ will have only one negative eigenvalue and the contradiction cannot be obtained. However, our local bifurcation from $|x| = \infty$ argument in Theorem 4.2 excludes this phenomenon and leads to the symmetry breaking bifurcation above.

There are other results identifying such symmetry breaking bifurcations, see for example [16, 19], but they rely on perturbative techniques and require special potentials, such as double wells with very large separations, to show that the bifurcation happens at small amplitudes.

Our result is comparable to the ones obtained via variational techniques, see for example [1, 10], but has the advantage of actually showing, besides the change in shape of the ground states, the existence of a singularity point which can then be analyzed with local bifurcation techniques. Note that if one of the eigenvector in the kernel of $L_+(\psi_{E_+}, E_+)$ is not invariant under the action of G, then some branches emerging from this point are no longer symmetric, see [7]. Moreover, our Theorem 4.3, shows that all ground states emerging from one single profile centered at a local minima of a potential are orbitally stable while they might not be global minimizers of the energy with respect to fixed mass constraint. For example, such global minima will not exist when nonlinearity is L^2 supercritical i.e., $p > 2/n$. Finally, we believe that our results can be extended to other equations including the Hartree equation, see Section 6.4, while the results in [1] can only be adapted to our problem with $p > 2/n$ by changing the minimization problem to functionals unrelated to the dynamics of the Hamiltonian structure of the equation, see for example [31]. In this case they will carry no information about the stability of the minimizer, see the discussion in [15].

In the following section we show that our results can be put together to obtain the global picture of all ground state branches besides showing the existence of bifurcation points along symmetric branches.

6 Conclusions and Examples

To summarize our paper, we have completely determined the ground states of the Schrödinger equation (1.1) for defocusing nonlinearity $\sigma > 0$, see Theorem 3.2 and Remark 3.1. We also analyzed the behavior of all bound states i.e., solutions of the time independent Schrödinger equation (1.3) at both small and large H^1 (or equivalently H^2) norms. More precisely Proposition 2.1 shows that there are no nontrivial (ψ_E, E) bound states with $-E$ away from the spectrum of $-\Delta + V$ and $\|\psi_E\|_{H^1}$ sufficiently small, while for a simple eigenvalue $-E_*$ of $-\Delta + V$ the nearby, small bound states form a C^1 two dimensional manifold obtained by rotating a curve $E \mapsto \psi_E,\ |E - E_*| < \delta$ of real valued solutions, see Proposition 2.2. We have also shown that the bound states group themselves in C^1 manifolds, see Theorem 2.1, and there are no such manifolds formed from very large bound states, see Theorem 3.1, except when $E \mapsto \infty$. In the latter case, up to a re-normalization, the bound states converge to a finite superposition of solutions of a limiting equation, see Theorem 4.1. Again, we can be more precise when it comes to ground states manifolds, which, modulo rotations, converge to finitely many translations of the unique, positive and radially symmetric solution of the limiting equation, with each translation centering the radially symmetric solution at a critical point of the potential, see Theorem 4.2. Moreover, in a neighborhood of any such finite superposition of positive, radially symmetric solutions of the limiting equation, each translated to a different critical point of the potential, the bound states form a unique C^1 manifold obtained by rotating a curve $E \mapsto \psi_E,\ E \in (E_*, \infty)$ of positive solutions, hence ground states, see Theorem 4.3. We now show, on a few examples, how to combine these results with basic tools in global bifurcation theory in order to find information about all bound states of a given Schrödinger equation, far beyond the symmetry breaking phenomenon described in the previous section.

6.1 Defocusing Nonlinearity

Consider the Schrödinger equation (1.1) with $\sigma > 0$ and assume hypothesis (H1), (H2) hold. If the spectral condition (H3) is satisfied then our Theorem 3.2 states that all ground states form a unique, two dimensional, C^1 manifold generated by rotating a real valued curve which starts at $(\psi_{E_0} \equiv 0, E_0)$ and is defined on the whole interval $(0, E_0)$ i.e., it approaches the boundary $E = 0$, where the linearization fails to be Fredholm. Note that ours is a much stronger result compared to the ones presented in [14]. It is true that we are dealing with a simpler nonlinearity however we are hoping to extend our technique to a much larger class of nonlinearities, hence fully generalizing the results in [14]. Moreover, if (H3) does not hold i.e., $-\Delta + V \geq 0$, then Proposition 2.3 part (ii) implies there are no ground states of (1.3) for $E > 0$. For $E \leq 0$, see [2] for non-existence of ground states or any other bound states provided they are required to decay sufficiently fast (but polynomially in $|x|$) as $|x| \to \infty$.

Proposition 2.3 also implies that there are no excited states (solutions of (1.3) which are not ground states) that can be rotated to a real valued one if $-E_0$ is the only negative e-value of $-\Delta + V$. Now let us assume that a second negative eigenvalue $0 > -E_1 > -E_0$ exists and is simple. By Proposition 2.2 there is a curve $E \mapsto \psi_E,\ E \in (E_1 - \delta, E_1)$ of real valued excited states. This curve can undergo bifurcations with interesting implications about its dynamical stability, see [19] for the particular case of a double well potential with large separation. However, if the wells are not well separated for the bifurcation to occur at small amplitude or, if we want to work with more general potentials we run into the problem of not knowing the structure of solutions of (1.3) at the $E = 0$ boundary of the Fredholm domain. Indeed, assume that case (b-) in part (ii) of our maximal extension Theorem 2.2 does not hold. Then it follows from the same theorem that the curve can be uniquely continued (no bifurcations) on the interval $(0, E_1)$. We would need to know what kind of limit points can the curve approach as $E \searrow 0$ and the number of negative eigenvalues of the linearized operator near these limits to determine if our assumption is contradicted and bifurcation points emerge at $E_* > 0$, see the previous section for a similar but rigorous argument. The problem is that at $E = 0$ the linearization of our map $DF_\psi(\psi, 0)$ ceases to be Fredholm as zero is at the edge of the continuous spectrum. Hence, any curve approaching the $E = 0$ boundary bifurcates from the edge of the continuous spectrum of the linearization at its limit. Such bifurcations are notoriously difficult to handle, and, while some progress has been made in the case of periodic potentials, see [5], the problem remains open for our type of potentials.

Moving now to analyzing all excited states that bifurcate from the trivial solution at a simple eigenvalue of $-\Delta + V$, see Proposition 2.2, global bifurcation theory implies that a connected component of the emerging curve ends up at $E = 0$, or converges back to $(0, E_*)$, where $-E_* < 0$ is a another eigenvalue of $-\Delta + V$, or blows up in H^2 norm, see [13, 29, 3]. With our techniques we can exclude the latter case with one exception. Indeed, let's assume that p is an integer, hence the nonlinearity is real analytic. If we start with the branch bifurcating from $(0, E_+)$ via proposition 2.2 and extend it on its maximal interval $(E_, E_+)$ via Theorem 2.2, we have the following dichotomy: either $E_- = 0$ and we are done or $E_- > 0$ and there exists a sequence $E_n \searrow E_-$ such that $L_+(\psi_{E_n}, E_n)$ has an eigenvalue λ_n with $\lim_{n\to\infty} \lambda_n = 0$. But ψ_{E_n} is bounded in H^2, see Theorem 3.1, and, by using the compactness results in Corollary 7.1 with a possible passing to a subsequence we get a limit point in $H^2 \times \mathbb{R}$, (ψ_{E_-}, E_-) which, by continuity, is still a solution of (1.3) and 0 is an eigenvalue of $L_+(\psi_{E_-}, E_-)$. Now the structure zeroes of the analytic F at a singular point, see [3], implies that the whole curve has the limit point (ψ_{E_-}, E_-) as $E \searrow E_-$ and it can be analytically continued past the singularity with L_+ also becoming non-degenerate past the singularity. It is possible that E is now increasing along the continuation of the initial curve i.e., (ψ_{E_-}, E_-) is a turning point. In this case we again consider the maximal interval $(E_, E'_+)$ on which the curve can be extended without any singularity. Note that by Corollary 2.3 we must have $E'_+ \leq E_0$. On such a maximal curve it is possible that the limit point is $(0, E'_+)$ in which case $-E'_+$ must be an eigenvalue of $-\Delta + V$, and we are done. By repeating the above argument we either get to $E_- = 0$ or to a limit point $(0, E_*)$ after finitely many steps, or we have a countable number of bifurcations. In the latter case we project the analytically continued curve on the $(E, \mathcal{N} = \|\cdot\|^2_{L^2})$ plane. If we get something like in the Figure 6.1 i.e., infinite many loops around which the curve turns counterclockwise and the sum of the encompassed areas is infinite, then it is possible that the curve blows up in H^2 norm before reaching $E = 0$. However, if the sum of the area is finite then we are guaranteed to have $E \to 0$. Indeed, in this case the L^{2p+2} norm remains bounded and the argument in Theorem 3.1 can be reproduced as follows. First note that when the projection of the curve on the $(E, \mathcal{N}(E) = \|\psi_E\|^2_{L^2})$ plane forms a loop the self intersection point is not a bifurcation point, instead the actual solutions projected at this point differ in L^{2p+2} norm to the power $2p+2$ by the factor $\frac{p+1}{-\sigma p}$ multiplying the area encompassed by the loop, see (3.4). If the motion around the loop is counterclockwise this norm jumps up when the curve returns to the same projected point while if the motion around the loop is clockwise the norm jumps down. In between these loops we can estimate the change in L^{2p+2} norm by breaking the curve into C^1 parts parameterized

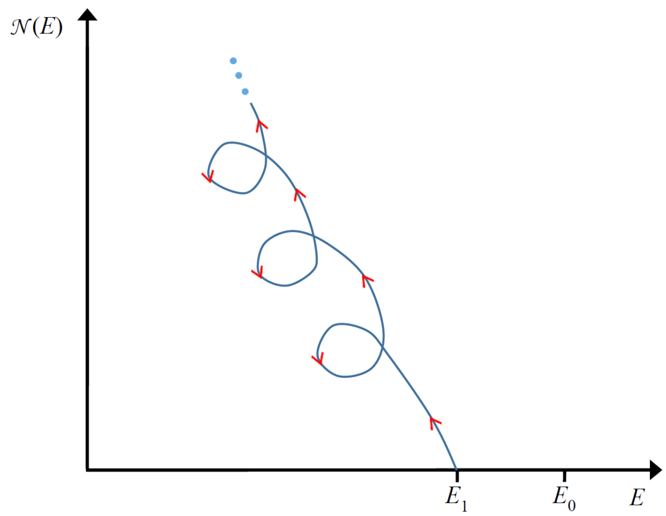

Figure 6.1: The curve shown above is the projection of a branch of excited bound states (E, ψ_E) of (1.3) with $\sigma > 0$ to the $(E, \mathcal{N} = \| \cdot \|_{L^2}^2)$ plane. If the sum of the areas encompassed by the counterclockwise loops of this curve is finite, then the curve approaches $E = 0$. If not, it is possible that $0 < E_- \leq E$ for all E on the curve. Note that $E \leq E_0$ for any E on the curve by Proposition 2.3

by E, and using the differential equation (3.4). If the up jumps add up to a finite number and there is $0 < E_- \leq E$ for all E along the curve (note that we also have $E \leq E_0$) the end result is a uniform bound of the L^{2p+2} norm along the entire curve. As in the proof of Theorem 3.1 we then get uniform bounds of the L^2 norms and, via regularity theory, uniform bounds for the H^2 norm. This means we have infinitely many bifurcation points in a bounded set in $H^2 \times (0, \infty)$ which is bounded away from zero in E. By the compactness result Corollary 7.1 these bifurcation points will have an accumulation point which is excluded by the structure of zeroes of F at singularity points, see [3]. Hence our assumption that there exists $E_- > 0$ such that $E_- \leq E$ for all E along the curve is false and the curve approaches the $E = 0$ boundary.

With this result and provided we know the limit points of solutions of (1.3) at $E = 0$ we can fully analyze the excited states. We will show how to do it for attractive nonlinearity because, in this case, the curves "end up" at $E = \infty$ where our results in Section 4 have already determined the limit points at least for ground state branches.

6.2 One Well Potentials with Focusing Nonlinearity

Consider the time independent Schrödinger equation (1.3) with $\sigma < 0$ and assume the potential V satisfies hypothesis (H1), (H2) and, in addition it has only one critical point $x_0 \in \mathbb{R}^n$ which is a non-degenerate minima i.e., V is twice differentiable at x_0 with the gradient equal to zero and the Hessian strictly positive definite. From Theorem 4.3 we know that here is a unique curve of ground states $E \mapsto \psi_E,\ E > E_\varepsilon$ which via the renormalization (4.6) converges in H^2, as $E \to \infty$,

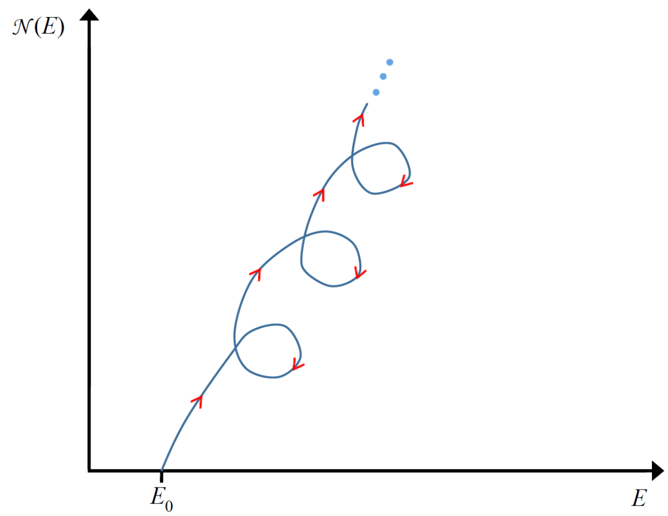

Figure 6.2: The curve shown above is the projection of a branch of ground states (E, ψ_E) of (1.3) with $\sigma < 0$ to the $(E, \mathcal{N} = \|\cdot\|^2_{L^2})$ plane. If the sum of the areas encompassed by the clockwise loops in this curve is finite, then the curve approaches $E = \infty$. If not, it is possible that $E \leq E_+ < \infty$ for all E on the curve. Note that $E \geq E_0$ for any E on the curve by Proposition 2.3

to the unique, positive, radially symmetric solution of (4.1) centered at x_0, $u_\infty(x - x_0)$. If we add the spectral hypothesis (H3) we also have the unique curve of ground states emerging from $(0, E_0)$, see Proposition 2.2. Our *conjecture* is that these two curves can be smoothly extended until they connect to form a unique C^1 curve, which, together with its rotations give all the ground states for this problem. Indeed the conjecture has been proven in the particular case of one dimension, $n = 1$, for an even potential that is strictly increasing for $x > 0$, see [12]. With the results developed in this paper we can prove the conjecture in any dimension and without any symmetry assumptions except for two obstacles: infinitely many bifurcations creating a similar phenomenon with the one described in Figure 6.1, see now Figure 6.2, and canceling bifurcations, see Figure 6.3.

Indeed, let's start at the curve of positive ground states emerging from $(0, E_0)$. Use Theorem 2.2 to extend it on its maximal interval (E_0, E_+). If $E_+ = \infty$ then Theorem 4.2 implies that the curve, modulo the renormalization (4.6) converges in H^2 to $u_\infty(x - x_0)$, i.e. the two curves are one and the same and the conjecture is proven (we will discuss the fact that there are no other ground states except the rotations of this curve later). If $E_+ < \infty$ then Corollary 3.1 implies that the curve has a limit point (ψ_{E_+}, E_+) which is still a ground state, provided V satisfies the additional hypotheses (3.15)-(3.17). Assuming now that p is an integer, hence F is real analytic, we can use the structure of zeroes of an analytic map at a singular point to uniquely extend the curve past the singularity at E_+. As in the previous subsection, it is possible that the extended curve moves to the left i.e., (ψ_{E_+}, E_+) is a turning point. In this case we again consider the maximal interval $(E_, E_+)$ on which the curve can be extended without any singularity. Note that by Corollary 2.3 we must have $E_- \geq E_0$. On such a maximal curve it is not possible that the limit point is $(0, E_-)$ because there are no eigenvalues of $-\Delta + V$ smaller than $-E_0$ and if $E_- = E_0$ then we get a contradiction with the fact that the only positive ground states near $(0, E_0)$ are the ones on the initial (E_0, E_+)

curve, see Proposition 2.2. By repeating the above argument we either get to $E_+ = \infty$ after finitely many steps or we have an infinite but countable number of bifurcations. In the latter case we project the analytically continued curve on the $(E, \mathcal{N}(E) = \|\psi_E\|_{L^2}^2)$ plane. If we get something like in the Figure 6.2 i.e., infinite many loops around which the curve turns clockwise and the sum of the encompassed areas is infinite, then it is possible that the curve blows up in H^2 norm before reaching $E = \infty$. However, if there are only finitely many such loops or the areas encompassed by them adds up to a finite number then, as in the previous subsection we first get uniform bounds of the L^{2p+2} norms along the entire curve, then, using the arguments in Theorem 3.1 we get uniform bounds for the L^2 norms followed by uniform bounds in H^1 and H^2. If we now assume that there exists $E_+ < \infty$ such that for all E on the curve $E \leq E_+$ then the curve lies in a bounded set of $H^2 \times \mathbb{R}$ (recall that $E \geq E_0$ by Corollary 2.3) and, by Theorem 3.3, the infinitely many bifurcation points along the curve will have an accumulation point. This is in contradiction with the structure of zeroes of F at singularity points, see [3]. Hence, there is a sequence of ground states along this curve (ψ_{E_k}, E_k) with $\lim_{k\to\infty} E_k = \infty$. In between these E_k points we can estimate the change in L^{2p+2} norm by breaking the curve into C^1 parts parameterized by E, and using the differential inequalities (4.14). If the up jumps in L^{2p+2} norm (at the endpoints of the loops in Figure 6.2) add up to a finite number we get the same results as for a single C^1 segment of the curve parameterized by E, see Theorem 4.1, namely that there is a number $b > 0$ such that:

$$\lim_{k\to\infty} \frac{\|\psi_{E_k}\|_{L^{2p+2}}^{2p+2}}{E_k^{1-\frac{n}{2}+\frac{1}{p}}} = b,$$

from which, as in the proof of Theorem 4.1, it follows that:

$$\begin{aligned} \lim_{k\to\infty} \frac{\|\psi_{E_k}\|_{L^2}^2}{E_k^{\frac{1}{p}-\frac{n}{2}}} &= \frac{-\sigma}{2}\left(\frac{2p+2-np}{p+1}\right) b, \\ \lim_{k\to\infty} \frac{\|\nabla\psi_{E_k}\|_{L^2}^2}{E_k^{1-\frac{n}{2}+\frac{1}{p}}} &= \frac{-\sigma}{2}\left(\frac{np}{p+1}\right) b. \end{aligned}$$

Hence, after the renormalization (4.6) the sequence converges to a superposition of solutions of (4.1). But, since we already assumed that V satisfies the hypotheses (3.15)-(3.17) which are stronger than (4.36)-(4.38), we deduce that the solutions of (4.1) must be centered at critical points of the potential, therefore they are centered at x_0. By Theorem 4.3 they are on the unique curve emerging from $u_\infty(x - x_0)$, therefore the two curves are connected.

We have finished proving a weaker form of our conjecture, namely that the small amplitude, positive, ground states for this problem form a curve which can be uniquely continued to $E = \infty$, where it converges, modulo the renormalization (4.6) to the unique, positive, radially symmetric solution of (4.1) centered at the minimum point x_0. We now turn to the question of existence of other ground states outside the above curve. Suppose there is a positive ground state (ψ_{E_*}, E_*) not on the above curve. Then from Proposition 2.3 we have $E_* \geq E_0$ and, depending on the kernel of $L_+(\psi_{E_*}, E_*)$ we have two cases. If it is trivial, then Theorem 2.1 implies that nearby ground states from a two dimensional, C^1 manifold obtained by rotating a curve of real valued (and positive) solutions of (1.3). By applying the above argument to this curve as opposed to the one emerging from $(0, E_0)$ we obtain that its unique analytic continuation either forms a closed loop or reaches

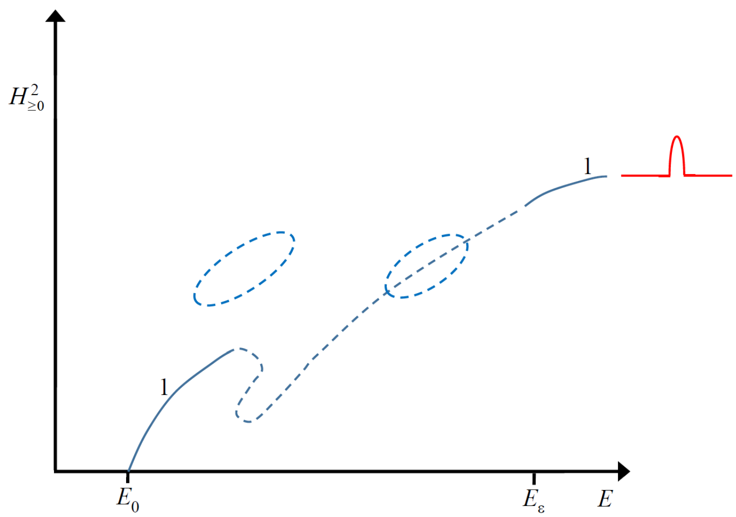

Figure 6.3: A sketch of ground state branches in the case of one well potentials. The number on the branches indicate the number of negative eigenvalues of $L_+(\psi_E, E)$ along it. The positive ground state branch near $E = E_0$ can be continued to $E = E_\infty$ where it connects to the one profile branch bifurcating from the minima of the potential. We conjecture that modulo rotations, it contains all the ground states of this problem. However, at present, we cannot exclude the occurrence of loops. We also conjecture that the branch connecting $E = E_0$ to $E = E_\infty$ has no singularities along it. But at present we cannot exclude the represented canceling bifurcations.

$E = \infty$ where, modulo renormalization (4.1), it converges in H^2 to $u_\infty(x - x_0)$. By the uniqueness of the curve emerging from $u_\infty(x-x_0)$, (ψ_{E_*}, E_*) must, in this situation, be on the curve connecting $(0, E_0)$ with $u_\infty(x - x_0)$ which contradicts the choice of (ψ_{E_*}, E_*). It remains that (ψ_{E_*}, E_*) and nearby solutions of (1.3) are part of a curve that forms a loop and its rotations. Now, if the kernel of $L_+(\psi_{E_*}, E_*)$ is non-trivial then, by the structure of zeroes of an analytic F near a singularity we have that either (ψ_{E_*}, E_*) is an isolated zero i.e., solution of (1.3), in which case we can say it forms a degenerate loop, or there are finitely many, finite dimensional, analytical manifolds of zeroes of F passing through (ψ_{E_*}, E_*). Now, if there is a sequence of points on these manifolds accumulating to (ψ_{E_*}, E_*) at which the kernel of L_+ is trivial, then there is a C^1 curve $E \mapsto \psi_E,\ E$ sufficiently close to E_* which ends at E_*. Choosing one point on this curve we are back to previous case and the first maximal extension ends up at (ψ_{E_*}, E_*). By repeating the above argument, we again get that (ψ_{E_*}, E_*) must be on a loop.

In conclusion, for a single well potential satisfying (H1)-(H3) and (3.15)-(3.17), the ground state emerging from $(0, E_0)$ form, modulo rotations, a curve that can be continued analytically for all $E > 0$ and at the $E = \infty$ limit it converges in H^2, modulo renormalization (4.6) to the unique, positive, radially symmetric solution of (4.1) centered at x_0. Along this curve there can be only canceling bifurcations i.e., the curve experiences "snaking" or if curves of other solutions of (1.3) emerge at a bifurcation point they must form a closed loop, see Figure 6.3. All other ground states if they exist must form closed loops modulo rotations. If we assume that (H3) does not hold then the same argument, starting now with the curve emerging from $u_\infty(x - x_0)$ at $E = \infty$, shows that it ends at $E = 0$. However, we can no longer say that there are no bifurcations along this curve generating new branches of solutions that also end at $E = 0$, and we can no longer exclude other branches, not connected to this one that instead of forming loops they end up at $E = 0$. All these are due to the fact that we do not yet know the limit points at the $E = 0$ boundary.

6.3 Double Well Potentials with Focusing Nonlinearity

As in the previous subsection we consider the time independent Schrödinger equation (1.3) with $\sigma < 0$ and assume the potential V satisfies hypothesis (H1), (H2) and, in addition it has exactly three critical points, two minima and a maxima or saddle, all non-degenerate. Such potentials can be obtained by adding two one well potentials described in the previous subsection each centered at a different point. For simplicity we will start with (symmetric) potentials that are invariant under the reflection with respect to a hyperplane. Without loss of generality we can assume that the hyperplane is $\{x_1 = 0\}$ in which case we have $V(-x_1, x_2, \ldots, x_n) = V(x_1, x_2, \ldots, x_n)$. From Theorem 4.3 we know that there are seven curves of non-trivial ground states $E \mapsto \psi_E,\ E > E_\infty$ which via the renormalization (4.6) converge in H^2, as $E \to \infty$, to a superposition of positive, radially symmetric solution of (4.1) each centered at one critical point. By restricting the argument in the proof of Theorem 4.3 to the Banach space of H^2 functions which are even in x_1 we deduce that three of these curves are formed by symmetric (even in x_1) ground states, namely the ones converging to u_∞ centered at the maxima/saddle point, or centered at the two minima, or centered at all three critical points. The other four are asymmetric and converge to a superposition of u'_∞s with one centered at a minima but none at the other minima.

If we add the spectral hypothesis (H3) we also have the unique curve of ground states emerging from $(0, E_0)$ which is also formed by symmetric (even in x_1) functions, see Proposition 2.2. We can now redo the argument in the previous subsection but restricted to symmetric functions to show that this curve can be extended to $E = \infty$, therefore connect to one of the three symmetric curves described above. Note that the global bifurcation results we used in the previous subsection are valid in any Banach spaces, hence in H^2_{even}, but, in order to verify their hypotheses we need the compactness result in Theorem 3.3 and real analyticity of F defined in (2.1), hence we assume that the potential satisfies (3.15)-(3.17) and that p is an integer. Now, numerical simulations suggest that in most cases the branch starting at $(0, E_0)$ connects to the one converging, as $E \to \infty$, towards one profile (u_∞) at each minima, see Figure 6.4. We will discuss the exceptions later and remark that the next argument can be adapted to any of the three possible connections.

If the branch starting at $(0, E_0)$ connects to the profiles at each minima the number of negative eigenvalues of L_+ jumps from one to two, therefore there is at least one bifurcation point along it where the second eigenvalue of L_+ crosses zero. This is a particular case of Theorem 5.1 when the Euclidian subgroup of symmetries is the one generated by reflections with respect to the hyperplane $\{x_1 = 0\}$. But now we can go further and analyze this bifurcation, see for example [19] or, for the one dimensional version, [18]. At this point another branch of asymmetric ground states emerges along which L_+ has one negative eigenvalue and, depending on the direction along the branch, the ground states localize more and more in one of the two wells. By comparing with branches at $E = \infty$, where this branch has to end, we see see that it will have each endpoint at the profile localized at one of the minima, see Figure 6.5. Then we look at the curve of positive ground states emerging from one profile at the maxima/saddle. By the same global analytic argument enhanced

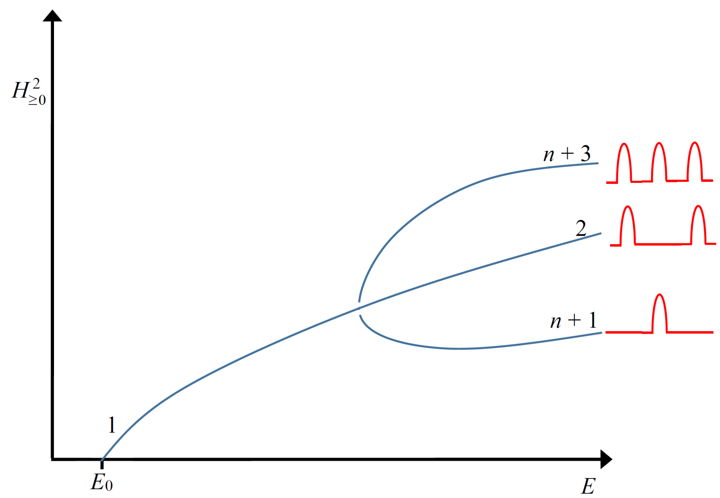

Figure 6.4: A sketch of symmetric ground states in the case of double well potentials. The numbers on the branches indicate the number of negative eigenvalues of $L_+(\psi_E, E)$ along it. The only assumption made is that the branch near $E = E_0$ connects to the one near $E = \infty$ which has one profile at each minima. The rest of the connections including all the bifurcations along them and the asymmetric branches in Figure 6.5 are determined by our arguments.

by the fact that along it $E \geq E_0$, see Corollary 2.3, and the fact that it cannot blow up in H^2 norm at a finite E, we deduce that this branch ends up back at $E = \infty$. Here it could connect to one of the negative branches obtained by rotating the curves described above by $e^{i\pi}$, however, since the branch started with positive ground states, this will require that it reaches a trivial ground state at some $E_* \geq E_0$. By Proposition 2.1 $-E_* \leq -E_0$ must be an eigenvalue of $-\Delta + V$, hence $E_* = E_0$ which contradicts the fact that the branch emerging from $(0, E_0)$ connects to the one converging, as $E \to \infty$, towards one profile at each minima. Therefore, by the uniqueness of the branch emerging at $E = \infty$ from one profile at the maxima, this branch must connect to the one converging, as $E \to \infty$, towards one profile at each critical point, see Figure 6.4. Therefore, along this branch, the number of negative eigenvalues of L_+ jumps by two, the turning point accounts for one while the second one leads to a bifurcation from which another branch of asymmetric ground states emerges and has each of its endpoints at $E = \infty$ with two profiles, one localized at one of the minima the other one at the maxima/saddle, see Figure 6.5. The branches and their connections in Figure 6.5 are fully explained and, modulo rotations, they should contain all ground states for this problem except maybe loops, as in the case of the previous subsection.

We now discuss the exceptions, when the branch starting at $(0, E_0)$ does not connect to the one converging, as $E \to \infty$, towards one profile (u_∞) at each minima. One dimensional $(n = 1)$ simulations show that this branch connects to the one converging, as $E \to \infty$, towards one profile at the maxima (there is no saddle point in one dimensions) in the "small separation" case i.e. when all three critical points are very close to each other. However, the branch emerging at $E = \infty$ from two profiles, each localized at one of the minima, no longer connects to the one with three profiles, each localized at one critical point, but instead it connects to the one converging, as $E \to \infty$, towards

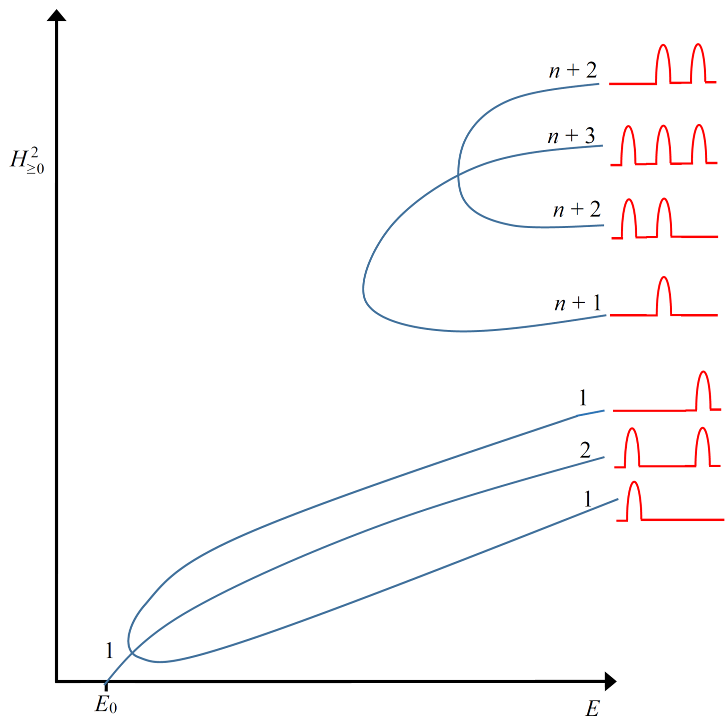

Figure 6.5: A sketch of all ground states in the case of double well potentials. The numbers on the branches indicate the number of negative eigenvalues of $L_+(\psi_E, E)$ along it. The only assumption made is that the branch near $E = E_0$ connects to the one near $E = \infty$ which has one profile at each minima. The rest of the connections including all the bifurcations along them and the asymmetric branches are determined by our arguments.

two profiles *each* approaching the maxima, see Figure 6.6. As discussed at the beginning of Section

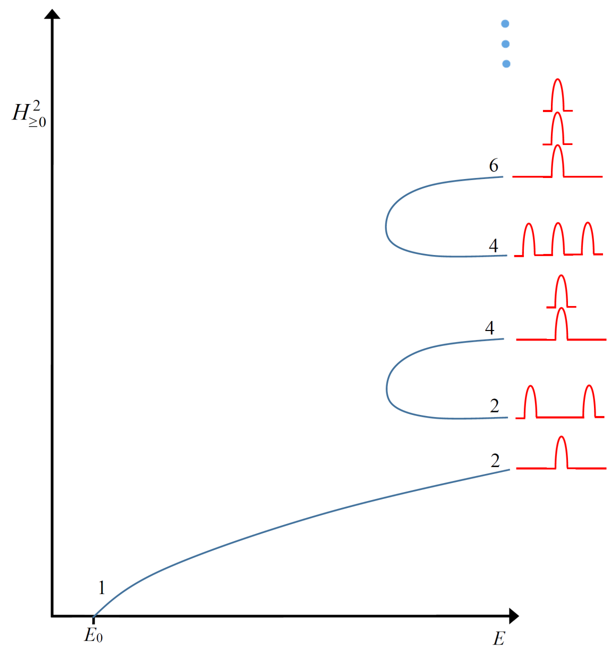

Figure 6.6: In the case of double well potentials in *one space dimension* with small separation between the wells, numerical study indicates that the symmetric branches of positive ground states connect as shown above. The branch from $E = E_0$ connects to the one that localizes (as $E \to \infty$) to one profile at the maxima. The branch that localizes (as $E \to \infty$) to a profile at both minima connects to the one that localizes to two profiles at the maxima. The branch that localizes (as $E \to \infty$) to a profile at each critical point connects to the one that localizes to three profiles at the maxima. This is different from Figure 6.4 and the consequences are explained in Section 6.3.

4 we have not yet analyzed these types of critical points. In the one dimensional case we do it in [17] and explain why this connection happen and identify all the other ground state branches and their bifurcation points. We also explain what happens as we increase the the distance between the critical points: the bifurcation diagram is stable until the two curves above touch tangentially, then the one starting at $(0, E_0)$ switches and connects to the one converging, as $E \to \infty$, towards one profile at each minima, while the one with an $E = \infty$ endpoint given by one profile at the maxima now connects to the one ending with two profiles at the maxima, see Figure 6.7. By further increasing the distance between the wells the bifurcation diagram remains stable until this curve touches tangentially the one that connects one $E = \infty$ endpoint given by three profiles each at a critical point with the other $E = \infty$ endpoint given by three profiles *each* approaching the maxima. After that the $E = \infty$ endpoint given by one profile connect to the $E = \infty$ endpoint given by three profiles each at a critical point, and, in this "large separation" of wells in the potential, the bifurcation diagram settles into the one in Figure 6.4, which we discussed above. Of course, the figure should be completed by adding $E = \infty$ endpoints given by two or more profiles each approaching the maxima which will now be connected to $E = \infty$ endpoints having the same number

of profiles at the maxima and two additional profiles each at a minima.

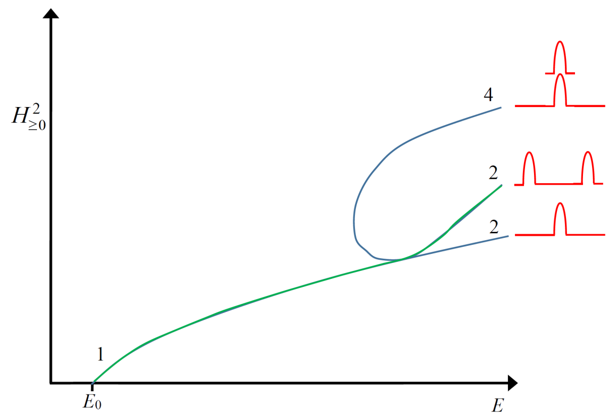

Figure 6.7: The connection of positive symmetric ground state branches depicted in Figure 6.6 remains stable as the distance between the wells is increased until the branch from E_0 becomes tangential at some point to the branch that localizes (as $E \to \infty$) to two profiles. As the distance between the wells is increased further, the branch from E_0 connects to the branch that localizes (as $E \to \infty$) to a profile at both minima and the branch that localizes (as $E \to \infty$) to one profile at the maxima connects to the one that localizes to two profiles at the maxima.

Finally, we discuss the case of non-symmetric potentials. The number and shape of the $E = \infty$ endpoints do not change but we can no longer separate them into symmetric and non-symmetric. Hence, in principle, the branch starting at $(0, E_0)$ can now connect to any of the seven branches at $E = \infty$ (or more if we add the ones which have multiple profiles approaching the maximum). Instead of analyzing each of these cases we could consider a continuous deformation connecting the current potential to a symmetric one. Starting now from the symmetric potential, the bifurcation diagram in Figure 6.5 remains unchanged due to the stability of (non-degenerate) bifurcations under perturbations. There are two ways in which a bifurcation and the connections it determines can disappear. One is when it is moved to $E = \infty$ but this will imply that we change the number of critical points of the potential. The second is when the bifurcation becomes degenerate i.e., the curves intersect at it tangentially then separate. This phenomenon was described in the previous paragraph albeit without loss of symmetry when the distance between wells is increased. In a future paper we will show that when the depth of one well is decreased, the bifurcation along the branch starting at $(0, E_0)$, see Figure 6.5, eventually degenerates and this branch ends up connecting to $E = \infty$ endpoint with a single profile at the deeper minima while the endpoint with two profiles each at a minima will now connect with the one with one profile at the higher local minima.

6.4 Conclusions

Besides describing all ground states for defocusing nonlinearity, we are on the verge of doing the same for focusing nonlinearity where their number and bifurcation points depends on the critical points of the potential. To finish the latter we still need to overcome three obstacles. The first one is illustrated in Figure 6.2 and consists of infinitely many bifurcations which, when the extended curve of bound states is projected onto the $(E, \mathcal{N} = \|\cdot\|_{L^2}^2)$ plane, lead to infinitely many loops around which the curve turns clockwise and encompasses areas which add up to infinity. In this case we do not know yet how to control the L^{2p+2} norm of the bound states on the curve and show, as in Theorem 3.1, that the curve cannot blow up in H^2 norm if the parameter E is in a compact subset of the interval $(0, \infty)$. The second obstacle is the analysis of $E = \infty$ limit points with profiles approaching the same critical point of the potential. Note that the partial results we already have show that they cannot occur at minima, hence the results presented in the examples above are essentially correct. The third obstacle only occurs when hypothesis (H3) does not hold or we are dealing with excited states of the defocusing problem. In this case, as seen in Subsections 6.1 and 6.2, branches can end up at $E = 0$ and we will need a complete analysis of the limit points at this boundary to fully describe all solutions of our problem. As for excited states in the focusing case, we have not yet shown that they form relatively compact sets when bounded in H^2 with parameter E in a compact subset of $(0, \infty)$. This result which we proved for ground states in Theorem 3.3 is essential in continuing manifolds of solutions past their possible singular (bifurcation) points, see the arguments in the subsections above. In fact, the proof of Theorem 3.3 goes through for excited states except for the case of splitting when profiles, while separating from each other, still stick together sufficiently close to have strong interacting terms. It is here where we used positivity of the profiles, valid for ground states, to get a contradiction. Nevertheless, we hope to surmount this technical detail and the obstacles discussed above in a future paper.

In conclusion, this paper shows that the algorithm described in [15], for identifying all zeroes of maps F between Banach spaces is doable. Namely

1. Identify the Fredholm domain of F i.e., the open set (which can be then broken in connected components) on which the Frechet derivative is a Fredholm operator. In our case it is $(\psi, E) \in H^2(\mathbb{R}^n, \mathbb{C}) \times (0, \infty)$.

2. Find the limit points of zeroes of F at the boundary of the Fredholm domain. In our case, for focusing nonlinearity Corollary 2.3 and Theorem 3.1 combine to show that there are no limit points at the $\{E = 0\}$ or $\{E > 0, \ \|\cdot\|_{H^2} = \infty\}$ boundaries. The limit points at $E = \infty$ boundary are a subset of the set described in Theorem 4.1 and are precisely identified for ground states in Theorems 4.2 and 4.3. For defocusing nonlinearity we are still missing the limit points at the $E = 0$ boundary.

3. Use global bifurcation techniques to continue branches of zeroes of F inside the Fredholm domain from their limit point at the boundary to their other limit point on the boundary. By comparing the discrete spectrum of the Fréchet derivative of F at the two limit points identify the bifurcations along these branches and use local bifurcation techniques to determine the other branches emerging at these bifurcation points. In the examples above we used the analytic global bifurcation theory as it only requires relative compactness of the zeroes of F in bounded domains away from the boundary of the Fredholm domain, which we proved in Theorem 3.3. Degree based global bifurcation results are also very useful but we only applied them to the defocusing case, see Subsection 6.1, because they require stronger compactness results.

Note that the algorithm should, and it actually did in our examples, identify all zeroes in the

Fredholm domain except the ones that form closed manifolds which do not reach its boundary. It adds to the classical global bifurcation theory the step where all possible limit points at the boundary of the Fredholm domain are identified. We think that this is possible in many other problems besides the nonlinear Schrödinger one. In fact we only used its Hamiltonian structure (energy, mass and the differential equation (3.2)) and Pohozaev identity to control the relevant norms of zeroes as they approach the boundary, find the appropriate renormalization which will keep these norms bounded and then show that the limit points are actual solutions of a simpler equation, see Theorem 4.1 and 3.1. All these tools are available to a large class of dispersive wave equations, see for example [15], except maybe the Pohozaev identity. The problem then reduces to identifying similar identities in the other problems or showing, as we strongly believe, that they are not necessary for finding the limit points and implementing the above algorithm.

7 Appendix

In this appendix we first recall regularity and decay properties of solutions of nonlinear Schrödinger equations. Then we prove certain compactness results for the set of solutions of families of such equations and spectral properties of their linearized operators which we use throughout this paper.

Slight adaptations of the regularity argument in [4, Section 8.1] leads to:

Proposition 7.1. *Let ψ_E be a H^1 weak solution of*

$$(-\Delta + V + E)\psi + \sigma|\psi|^{2p}\psi = f \tag{7.1}$$

where $E \in \mathbb{R}$, V is real-valued and satisfies Hypotheses (H1)-(H2), $\sigma \in \mathbb{R}$, $0 < p < 2/(n-2)$ and $f \in L^q$, for all $2 \leq q < \infty$. Then ψ_E is in $W^{2,q}(\mathbb{R}^n)$ for all $2 \leq q < \infty$, in particular it is in $L^\infty(\mathbb{R}^n)$.

As we will mostly be interested in families of equations (7.1) indexed by E, the same argument also shows:

Remark 7.1. *If (ψ_E, E) is a bounded set in $H^1 \times \mathbb{R}$ of solutions of* (7.1)*, then the set is also bounded in $W^{2,q} \times \mathbb{R}$ for all $q \geq 2$, in particular is bounded in $L^\infty \times \mathbb{R}$.*

Proposition 7.2. *Under the hypothesis of Proposition 7.1 with $E > 0$ and $f \equiv 0$, we have that for any $\gamma \in (0, E)$ there exists a $C(\gamma) > 0$ such that for all $x \in \mathbb{R}^n$ we have:*

$$\begin{aligned} |\psi_E(x)| &\leq C(\gamma)e^{-\sqrt{E-\gamma}|x|} \\ |\nabla\psi_E(x)| &\leq C(\gamma)e^{-\sqrt{E-\gamma}|x|} \end{aligned}$$

When applied to families of Schrödinger equations indexed by E the above result leads to uniform decay on bounded sets for the *defocusing nonlinearity.* Consequently the first compactness tool in [4, Section 1.7] implies relative compactness on bounded sets, see also [13]:

Corollary 7.1. *Assume that there exist constants $E_-, E_+, M > 0$ and a sequence $\{\psi_{E_k}, E_k\}_{k\in\mathbb{N}}$ of solutions for* (7.1) *with $\sigma > 0$ and $f \equiv 0$ such that*

$$E_k \in [E_-, E_+], \qquad \|\psi_{E_k}\|_{H^1} \leq M.$$

Then for any $0 < \gamma < E_-$ there exists a constant $C(\gamma)$ independent of k such that

$$|\psi_{E_k}(x)| \le C(\gamma) e^{-\sqrt{E_- - \gamma}|x|} \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, k \in \mathbb{N}.$$

Consequently, there is a subsequence of $(\psi_{E_k}, E_k)_{k\in\mathbb{N}}$, denoted again by $(\psi_{E_k}, E_k)_{k\in\mathbb{N}}$, and a solution $(\psi_{E_*}, E_*) \in H^2(\mathbb{R}^n) \times [E_-, E_+]$ of (7.1) such that

$$\lim_{k\to\infty} E_k = E_*, \qquad \lim_{k\to\infty} \|\psi_{E_k} - \psi_{E_*}\|_{H^2} = 0.$$

For attractive nonlinearity we have the much weaker result which already requires strong convergence in L^q spaces:

Proposition 7.3. *Assume that there exists $E_* \in (0,\infty)$, $\psi_* \in H^1$, $2 \le q < \frac{2n}{n-2}$, $\tilde\psi \in L^q(\mathbb{R}^n)$ and a sequence $\{\psi_{E_k}, E_k\}_{k\in\mathbb{N}}$, $E_k \in (0,\infty)$ of solutions of (7.1) with $\sigma < 0$ such that*

$$\lim_{k\to\infty} E_k = E_*, \tag{7.2}$$

$$\psi_{E_k} \overset{H^1}{\rightharpoonup} \psi_*\ , \tag{7.3}$$

$$\psi_{E_k} \overset{L^q}{\to} \tilde\psi \quad \textit{for some} \quad 2 \le q < \frac{2n}{n-2}\ (2 \le q < \infty \ \ \textit{if}\ \ n = 2,\ 2 \le q \le \infty \ \ \textit{if}\ \ n = 1), \tag{7.4}$$

then $\psi_* = \tilde\psi \in H^2(\mathbb{R}^n)$ and $\psi_{E_k} \overset{H^2}{\to} \psi_*$. Moreover (ψ_*, E_*) is a solution of (7.1).

Proof. Note that by the standard Sobolev embedding theorem, weak convergence in H^1 implies weak convergence to the same limit in L^q with q given in (7.4). Since norm convergence implies weak convergence we get $\psi_* = \tilde\psi$ by uniqueness of the weak limit.

We are now going to employ the regularity result in Remark 7.1. Since the sequence $\{\psi_{E_k}\}_{k\in\mathbb{N}}$ is weakly convergent in $H^1(\mathbb{R}^n)$, it is norm bounded in H^1. From (7.2) and $0 < E_* < \infty$ we also have that, $\{E_k\}_{k\in\mathbb{N}}$ is bounded and bounded away from zero. Hence, by Remark 7.1 we have that the sequence $\{\psi_{E_k}\}_{k\in\mathbb{N}}$ is bounded in *each* $W^{2,q}, 2 \le q < \infty$. In particular it is bounded in $L^2 \bigcap L^\infty$. Now, by the convergence (7.4) and interpolation we get that the sequence $\{\psi_{E_k}\}_{k\in\mathbb{N}}$ converges to ψ_* in *all* $L^q,\ q > 2$ spaces.

Finally, by rewriting (7.1) we get:

$$\psi_{E_k} = (-\Delta + E_k)^{-1}\left[-V\psi_{E_k} - \sigma|\psi_{E_k}|^{2p}\psi_{E_k} + f\right]. \tag{7.5}$$

Now, from the convergence above, we have

$$|\psi_{E_k}(x)|^{2p}\psi_{E_k}(x) \overset{L^2}{\to} |\psi_*(x)|^{2p}\psi_*(x).$$

Since $V \in L^\infty$ we definitely have $V(x)\psi_{E_k}(x)$ converges in $L^2(\mathbb{R}^n)$ provided $q = 2$ in hypothesis (7.4). If $q > 2$ we use the following compactness argument. Fix $\epsilon > 0$ and choose $L > 0$ sufficiently large such that

$$\|V(x)\|_{L^\infty(\{|x|\ge L\})} < \frac{\epsilon}{4M}.$$

Then, it follows from (7.4) that

$$\lim_{m\to\infty} \|\psi_{E_m} - \psi_*\|_{L^2(\{|x|<L\})} = 0,$$

and hence there exists m_0 such that

$$\|\psi_{E_m} - \psi_*\|_{L^2(\{|x|<L\})} < \frac{\epsilon}{2\|V\|_{L^\infty(\mathbb{R}^n)}} \qquad \text{for} \quad m \geq m_0.$$

Consequently for each $m > m_0$

$$\begin{aligned}\|V\psi_{E_m} - V\psi_*\|_{L^2(\mathbb{R}^n)} &\leq \|V\|_{L^\infty(\mathbb{R}^n)}\|\psi_{E_m} - \psi_*\|_{L^2(\{|x|<L\})} + \|V\|_{L^\infty(\{|x|\geq L\})}\|\psi_{E_m} - \psi_*\|_{L^2(\mathbb{R})} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4M}2M \leq \epsilon,\end{aligned}$$

which proves that $V\psi_{E_k} \xrightarrow{L^2} V\psi_*$. Now, as $E_k \to E_*$, both terms in the square bracket on the right side of (7.5) converge in $L^2(\mathbb{R}^n)$ and $(-\Delta+E_k)^{-1}$ converges to $(-\Delta+E_*)^{-1}$ in the space of bounded linear operators from $L^2(\mathbb{R}^n)$ to $H^2(\mathbb{R}^n)$. Hence we can conclude that the sequence $\{\psi_{E_k}\}_{k\in\mathbb{N}}$ converges in $H^2(\mathbb{R}^n)$. Since $H^2(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for $2 \leq q < 2n/(n-4)$, it follows from (7.4) that the sequence $\{\psi_{E_k}\}_{k\in\mathbb{N}}$ converges to ψ_* in $H^2(\mathbb{R}^n)$. Moreover, by passing to the limit in (7.5), we get

$$\psi_* = (-\Delta + E_*)^{-1}\left[-V(x)\psi_*(x) - \sigma|\psi_*(x)|^{2p}\psi_* + f\right].$$

Therefore (ψ_*, E_*) is a solution of the equation (7.1). $\square$

The above result will be combined with the following concentration compactness one, see [4, Section 1.7]. For functions $\psi \in L^2(\mathbb{R}^n)$ and numbers $r > 0$, define the concentration function:

$$\rho(\psi, r) = \sup_{y\in\mathbb{R}^n} \int_{|x-y|\leq r} ||\psi(x)|^2 dx.$$

For bounded sequences in H^1 we have:

Proposition 7.4. *Let $\{\psi_k\}_{k\in\mathbb{N}} \subset H^1(\mathbb{R}^n)$ such that there exist $a,\ M > 0$ with the properties:*

$$\begin{aligned}\|\psi_k\|^2_{L^2} &= a, \qquad \forall k \in \mathbb{N} \\ \|\psi_k\|_{H^1} &\leq M, \qquad \forall k \in \mathbb{N}\end{aligned}$$

Let

$$\mu = \lim_{r\to\infty} \liminf_{k\to\infty} \rho(\psi_k, r) \leq a.$$

Then one of the following holds.

(i) If $\mu = a$ then there is a subsequence re-denoted by $\{\psi_k\}_{k\in\mathbb{N}}$ and a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that

$$\psi_k(x - y_k) \xrightarrow{L^q} \psi_*, \qquad \forall 2 \leq q < \frac{2n}{n-2}\ (2 \leq q < \infty \ \text{if}\ n = 2,\ 2 \leq q \leq \infty \ \text{if}\ n = 1).$$

(ii) If $\mu = 0$ then there is a subsequence re-denoted by $\{\psi_k\}_{k\in\mathbb{N}}$ such that

$$\psi_k \stackrel{L^q}{\to} 0, \qquad \forall 2 < q < \frac{2n}{n-2} \ (2 < q < \infty \text{ if } n = 2,\ 2 < q \leq \infty \text{ if } n = 1).$$

(iii) If $0 < \mu < a$ then there is a subsequence re-denoted by $\{\psi_k\}_{k\in\mathbb{N}}$ and the sequences $\{v_k\}_{k\in\mathbb{N}}$, $\{w_k\}_{k\in\mathbb{N}}$, $\{z_k\}_{k\in\mathbb{N}}$ all bounded in H^1 such that

$$\begin{aligned}
\psi_k &= v_k + w_k + z_k \quad \text{and} \quad |v_k| + |w_k| \leq |\psi_k|,\\
\lim_{k\to\infty} \|v_k\|_{L^2}^2 &= \mu, \text{ and } v_k(x - y_k) \stackrel{L^q}{\to} v_*,\ \forall 2 \leq q < \frac{2n}{n-2} \ (2 \leq q \leq \infty \text{ if } n = 1), \quad (7.6)\\
\lim_{k\to\infty} \|w_k\|_{L^2}^2 &= a - \mu \quad \text{and} \quad \lim_{k\to\infty} \operatorname{dist}(\operatorname{supp} w_k,\ \operatorname{supp} v_k) = \infty,\\
\lim_{k\to\infty} \|z_k\|_{L^q} &= 0 \qquad \forall 2 \leq q < \frac{2n}{n-2} \ (2 \leq q \leq \infty \text{ if } n = 1),
\end{aligned}$$

The proof follows from the one in [4, Section 1.7] except for the splitting part (iii). The latter requires a slight adjustment in the choice of the cutoff function to insure large separation for the supports of v_k and w_k as $k \to \infty$. The fact that v_k can be chosen to satisfy the convergence properties (7.6) is very important in our analysis and follows from the following argument. With the notations in [4], if $\lim_{r\to\infty} \liminf_{k\to\infty} \rho(v_k, r) = \mu$ then we can apply part (i) to the sequence $\{v_k\}_{k\in\mathbb{N}}$ instead of $\{\psi_k\}_{k\in\mathbb{N}}$, see [4, Section 1.7] to get the convergence on a subsequence. If $\lim_{r\to\infty} \liminf_{k\to\infty} \rho(v_k, r) < \mu$ then one can show that $\lim_{r\to\infty} \liminf_{k\to\infty} \rho(w_k, r) = \mu$. Of course, this leads to a contradiction if $a - \mu < \mu$, hence $\{v_k\}_{k\in\mathbb{N}}$ must be in case (i), or leads to $\{w_k\}_{k\in\mathbb{N}}$ being in case (i) if $a - \mu = \mu$. If the latter holds we switch the roles of v_k and w_k. Finally, if $a - \mu > \mu$ we repeat the splitting argument for $\{w_k\}_{k\in\mathbb{N}}$ instead of $\{\psi_k\}_{k\in\mathbb{N}}$ which now has L^2 norm convergent to $a - \mu < a$ but the same "concentration" number μ. After no more then m steps where $m \geq \frac{a}{\mu}$ the algorithm stops and a split with one function in case (i) is produced. Note that a similar result is obtained in [24].

We now present with full proof a result on uniform bounds for the resolvent of families of linear Schrödinger operators with large separation potential. This is applied to obtain and control Lyapunov-Schmidt type decompositions when splitting, see (iii) above, occurs for sequences of ground states.

Proposition 7.5. *Let $V_1, V_2, \ldots V_m : \mathbb{R}^n \mapsto \mathbb{R}$ such that for each k, $V_k \in L^\infty(\mathbb{R}^n)$ and $\lim_{|x|\to\infty} V_k(x) = 0$. Let $s_1, s_2, \ldots s_m$ be $\mathbb{R}^n$-valued functions of $R \in (0,1)$ such that $\lim_{R\to 0} |s_i(R) - s_j(R)| = \infty$ whenever $i \neq j$ and E be a $\mathbb{R}$-valued function of $R \in (0,1)$ such that $\lim_{R\to 0} E(R) = E_* \in (0,\infty)$. Consider the family of operators*

$$\begin{aligned}
L_R &: H^2 \mapsto L^2\\
L_R &= -\Delta + E(R) + \sum_{k=1}^{m} T_{s_k(R)} V_k, \qquad R \in (0,1),
\end{aligned} \tag{7.7}$$

where we used the notation $T_y V(x) = V(x-y)$. Denote by $P_0^\perp(R)$ the projection operator in L^2 on to the orthogonal complement of the set of all eigenvectors φ that satisfy $(-\Delta + E(R) + T_{s_k(R)} V_k)\varphi = 0$ for some k. Then there exists $0 < r < 1$ and $C > 0$ such that for each $R \in (0,r)$, $P_0^\perp(R) L_R P_0^\perp(R)$ is invertible and

$$\|(P_0^\perp(R) L_R P_0^\perp(R))^{-1}\|_{H^2 \mapsto L^2} < C$$

Proof. Note that L_R is a self-adjoint operator on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ and

$$K_1 = \sup_{R\in(0,1)} \|L_R\|_{\mathcal{L}(H^2,L^2)} < \infty.$$

Since each V_k is a relatively compact operator with respect to $(-\Delta + E_*)$, it follows from Weyl's theorem that the essential spectrum of $(-\Delta + E_* + V_k)$ for all k and of L_R is $[1,\infty)$. Let the increasing sequence of numbers $\mu_1, \mu_2, \dots \mu_p$, with $\mu_{q+1} = \mu_{q+2} = \dots \mu_p = 0$ for some $0 \le q \le p$, be the collection of all the non-positive discrete eigenvalues of the operators $(-\Delta + E_* + V_k)$, $1 \le k \le m$.

Using the spectral theory for operators with potentials separated by large distances in [Morgan and Simon, 1980] and the spectral perturbation theory (to handle the fact that $E(R)$ may not be a constant) we get that as $R \to 0$, the discrete eigenvalues of L_R approach precisely the union of the discrete eigenvalues of the family of operators $\{(-\Delta + E_* + V_k)\,\big|\,1 \le k \le m\}$. Furthermore, the spectral projection operator $\widetilde{P}_\mu(R)$ associated with the set of all the eigenvalues of L_R which converge to some μ in the above union, converges in $\mathcal{L}(L^2)$ to $P_\mu(R)$ which is the projection operator on to the span of the set $\{\mathrm{Range}(P_{k,\mu}(R))\,\big|\,1 \le k \le m\}$. Here $P_{k,\mu}(R)$ is the spectral projection operator corresponding to μ for the operator $(-\Delta + E_* + T_{s_k(R)}V_k)$. From all this we can conclude that there exist $r_1 \in (0,1)$ such that for each $R \in (0, r_1)$, $\sigma(L_R)$ is the union of $\{\mu_1^R, \mu_2^R, \dots \mu_p^R\}$ and $\sigma^c(L_R)$, where $\mu_1^R, \mu_2^R, \dots \mu_p^R$ are the p smallest eigenvalues of L_R counting multiplicities and $\sigma^c(L_R) \subset (a,\infty)$ for some $a > 0$, and

$$\lim_{R\to 0} \mu_j^R = \mu_j \qquad \forall\, j \in \{1,2,\dots p\}, \qquad \lim_{R\to 0} \|P_0^\perp(R) - \widetilde{P}_0^\perp(R)\|_{\mathcal{L}(L^2)} = 0. \tag{7.8}$$

For each $j \in \{1,2,\dots p\}$, let φ_j^R be the eigenvector associated with the eigenvalue μ_j^R of L_R.

Using [30, Theorem XII.5] and the fact that L_R is a self-adjoint operator (which implies that the projection operator in that theorem is orthogonal), it follows from the first limit in (7.8) that there exists $r_2 \in (0, r_1)$ such that $\widetilde{P}_0^\perp(R) L_R \widetilde{P}_0^\perp(R) : \widetilde{P}_0^\perp(R)L^2 \cap H^2 \to \widetilde{P}_0^\perp(R)L^2$ is a boundedly invertible operator for all $R \in (0, r_2)$. We claim that r_2 can be chosen such that $\|(\widetilde{P}_0^\perp(R) L_R \widetilde{P}_0^\perp(R))^{-1}\|$ has a uniform bound on $(0, r_2)$. Suppose that this is not true. Then there exists a sequence $(R_k)_{k=1}^\infty$ in $(0, r_2)$ converging to 0 and a sequence of functions $(u_k)_{k=1}^\infty$ with $u_k \in \widetilde{P}_0^\perp(R_k)L^2 \cap H^2$ and $\|u_k\|_{H^2} = 1$ such that

$$\lim_{k\to\infty} \widetilde{P}_0^\perp(R_k) L_{R_k} \widetilde{P}_0^\perp(R_k) u_k = 0, \qquad \langle u_k, \varphi_j^{R_k}\rangle = 0 \qquad \forall\, k \ge 1, \qquad \forall\, j \in \{1,2,\dots q\}.$$

Hence $u_k \in \widetilde{P}^\perp(R_k)L^2 \cap H^2$ for each k, where $\widetilde{P}^\perp(R_k)$ is the projection operator in L^2 onto the orthogonal complement of the set

$$\mathrm{span}\{\varphi_1^{R_k}, \varphi_2^{R_k}, \dots \varphi_p^{R_k}\}.$$

Note that $\liminf \|u_k\|_{L^2} > 0$. Indeed, if this were not true, then using $\lim_{k\to\infty} L_{R_k} u_k = 0$ and $\lim_{k\to\infty} E(R_k) = E_* \in (0,\infty)$ it follows that along a subsequence $\lim_{k\to\infty}(-\Delta + E(R_k))u_k = \lim_{k\to\infty} \sum_{j=1}^m T_{s_j(R)} V_j u_k = 0$, which contradicts the fact that $\|u_k\|_{H^2} = 1$ for all k. It now follows from the operator form of the min-max principle [Theorem XIII.1, Reed and Simon] that there exists an eigenvalue $\mu_{p+1}^{R_k}$ for L_{R_k}, with eigenvector $\varphi_{p+1}^{R_k}$ different from $\varphi_j^{R_k}$ for $1 \le j \le p$, such that

$\limsup_{R_k\to 0}\mu_{p+1}^{R_k}\le 0$. This contradicts our conclusion about the spectrum of L_{R_k} (see discussion above (7.8)). Hence we can fix $r_2\in(0,r_1)$ such that $(\widetilde{P}_0^{\perp}(R)L_R\widetilde{P}_0^{\perp}(R))^{-1}$ has a uniform bound on $(0,r_2)$. Define

$$K_2=\sup_{R\in(0,r_2)}\|(\widetilde{P}_0^{\perp}(R)L_R\widetilde{P}_0^{\perp}(R))^{-1}\|_{L^2\mapsto H^2}.$$

Define the operator $\Delta P(R)=P_0^{\perp}(R)-\widetilde{P}_0^{\perp}(R)$ which maps L^2 to L^2 and H^2 to H^2. The second expression in (7.8) implies easily that the ranges of $P_0(R)$ and $\widetilde{P}_0(R)$ have the same dimension (for small R) and also that

$$\lim_{R\to 0}\|P_0(R)-\widetilde{P}_0(R)\|_{\mathcal{L}(H^2)}=0. \tag{7.9}$$

To establish (7.9) it is sufficient to show that given $\varepsilon>0$, there exists $r_\varepsilon\in(0,r_2)$ such that for each $R\in(0,r_\varepsilon)$ and every eigenvector $\psi\in\mathrm{Range}(P_{k,0}(R))$ satisfying $\|\psi\|_{L^2}=1$, for some k, there exists a $\varphi\in\mathrm{Range}(\widetilde{P}_0(R))$ such that $\|\psi-\varphi\|_{H^2}<\varepsilon$. Hence given $\varepsilon\in(0,1)$, fix $r_\varepsilon\in(0,r_2)$ such that for each $R\in(0,r_\varepsilon)$

$$|E(R)-E_*|<\frac{\varepsilon}{4+\varepsilon}. \tag{7.10}$$

$$\sup_{1\le k\le m}\ \sup_{\substack{\psi\in\mathrm{Range}(P_{k,0}(R))\\ \|\psi\|_{L^2}=1}}\left\|\sum_{j=1,j\neq k}^{m}(T_{s_j(R)}V_j)\psi\right\|_{L^2}<\frac{\varepsilon}{4}, \tag{7.11}$$

$$\|P_0(R)-\widetilde{P}_0(R)\|_{\mathcal{L}(L^2)}\left(1+\sum_{j=1}^{m}\|V_j\|_{L^\infty}\right)<\frac{\varepsilon}{4}, \tag{7.12}$$

$$\sup\big\{|\mu_j^R|\,\big|\,1\le j\le p,\lim_{R\to 0}\mu_j^R=0\big\}<\frac{\varepsilon}{4+\varepsilon}. \tag{7.13}$$

Let $R\in(0,r_\varepsilon)$. Consider a $\psi\in\mathrm{Range}(P_{k,0}(R))$ satisfying $\|\psi\|_{L^2}=1$. From (7.12) it follows that there exists a $\varphi\in\mathrm{Range}(\widetilde{P}_0(R))$ such that

$$\|\varphi-\psi\|_{L^2}\left(1+\sum_{j=1}^{m}\|V_j\|_{L^\infty}\right)<\frac{\varepsilon}{4} \tag{7.14}$$

and $\varphi=\sum_{j=1}^{t}\alpha_j\varphi_j^R$, where $\{\varphi_1^R,\varphi_2^R,\dots\varphi_t^R\}$ is an orthonormal family of eigenvectors of L_R corresponding to its eigenvalues that converge to 0. Hence $\|\varphi\|_{L^2}<(4+\varepsilon)/4$ and we have

$$(-\Delta+E_*+T_{s_k(R)}V_k)\psi=0,\qquad \Big(-\Delta+E(R)+\sum_{j=1}^{m}T_{s_j(R)}V_j\Big)\varphi=\sum_{j=1}^{t}\mu_j^R\alpha_j\varphi_j^R.$$

From these equations, using (7.10), (7.11), (7.13) and (7.14), we get

$$\|(-\Delta+E_*)(\psi-\varphi)\|_{L^2}=\left\|(E(R)-E_*)\varphi-\sum_{j=1}^{m}V_j^{s_j(R)}(\varphi-\psi)+\sum_{j=1,j\neq k}^{m}V_j^{s_j(R)}\psi-\sum_{j=1}^{t}\mu_j\alpha_j\varphi_j^R\right\|_{L^2}\le\varepsilon.$$

Since the choice $\varepsilon\in(0,1)$ is arbitrary, (7.9) follows.

From the second limit in (7.8) and (7.9) it follows that there exists $r \in (0, r_2)$ such that

$$\sup\{\|\Delta P(R)\|_{\mathcal{L}(L^2,L^2)}, \|\Delta P(R)\|_{\mathcal{L}(H^2,H^2)}\} < \inf\left\{\frac{1}{4K_1K_2}, \frac{1}{3}\right\} \qquad \forall\, R \in (0,r). \tag{7.15}$$

Consider the operator $P_0^{\perp}(R)L_RP_0^{\perp}(R) : P_0^{\perp}(R)L^2 \cap H^2 \to P_0^{\perp}(R)L^2$. For any $u \in H^2$, we have

$$P_0^{\perp}(R)L_RP_0^{\perp}(R)u = P_0^{\perp}(R)L_R\Delta P(R)u + \Delta P(R)L_R\widetilde{P}_0^{\perp}(R)u + \widetilde{P}_0^{\perp}(R)L_R\widetilde{P}_0^{\perp}(R)u. \tag{7.16}$$

It follows from (7.16), using the definitions of K_1 and K_2 and (7.15), that for each $R \in (0,r)$,

$$\begin{aligned}\|P_0^{\perp}(R)L_RP_0^{\perp}(R)u\|_{L^2} &\geq \|\widetilde{P}_0^{\perp}(R)L_R\widetilde{P}_0^{\perp}(R)u\|_{L^2} - \|P_0^{\perp}(R)L_R\Delta P(R)u\|_{L^2} - \|\Delta P(R)L_R\widetilde{P}_0^{\perp}(R)u\|_{L^2}\\ &\geq \frac{1}{2K_2}\|u\|_{H^2}. \end{aligned}\tag{7.17}$$

The range of $P_0^{\perp}(R)L_RP_0^{\perp}(R)$ is dense in $P_0^{\perp}(R)L^2$ for each $R \in (0,r)$. Indeed, suppose that for some $v \in L^2$ with $\|P_0^{\perp}(R)v\|_{L^2} = 1$,

$$\left\langle P_0^{\perp}(R)L_RP_0^{\perp}(R)u, P_0^{\perp}(R)v\right\rangle = 0 \qquad \forall\, u \in H^2(\mathbb{R}^n). \tag{7.18}$$

Fix $u \in H^2$ such that $\widetilde{P}_0^{\perp}(R)L_R\widetilde{P}_0^{\perp}(R)u = \widetilde{P}_0^{\perp}P_0^{\perp}(R)v$. Then, using (7.15) we get that

$$\|u\|_{H^2} \leq K_2\|\widetilde{P}_0^{\perp}P_0^{\perp}(R)v\|_{L^2} \leq K_2\|P_0^{\perp}(R)v - \Delta P(R)P_0^{\perp}(R)v\|_{L^2} \leq 4K_2/3.$$

Furthermore, using (7.15) and (7.16), we have

$$\begin{aligned}&\left|\left\langle P_0^{\perp}(R)L_RP_0^{\perp}(R)u, P_0^{\perp}(R)v\right\rangle\right|\\ &\geq \left|\left\langle \widetilde{P}_0^{\perp}(R)L_R\widetilde{P}_0^{\perp}(R)u, P_0^{\perp}(R)v\right\rangle\right| - \left|\left\langle P_0^{\perp}(R)L_R\Delta P(R)u, P_0^{\perp}(R)v\right\rangle\right| - \left|\left\langle \Delta P(R)L_R\widetilde{P}_0^{\perp}(R)u, P_0^{\perp}(R)v\right\rangle\right|\\ &> \left|\left\langle \widetilde{P}_0^{\perp}(R)P_0^{\perp}(R)v, P_0^{\perp}(R)v\right\rangle\right| - \frac{2}{3} > 0\end{aligned}$$

which contradicts (7.18), thereby establishing that the range of $P_0^{\perp}(R)L_RP_0^{\perp}(R)$ is dense in $P_0^{\perp}(R)L^2$ for each $R \in (0,r)$. This, along with (7.17) and the fact that $P_0^{\perp}(R)L_RP_0^{\perp}(R) \in \mathcal{L}(H^2, L^2)$, implies that $P_0^{\perp}(R)L_RP_0^{\perp}(R)$ is invertible for each $R \in (0,r)$ with $\|(P_0^{\perp}(R)L_RP_0^{\perp}(R))^{-1}\| < 2K_2$. □

Finally, we present with full proof the existence and properties of a Lyapunov-Schmidt type decompositions when splitting, see (iii) in Proposition 7.4, occurs for sequences of ground states.

Proposition 7.6. *Consider the two sets of real valued C^1 functions $\{u_1, u_2, \dots u_m\}$ and $\{\phi_1, \phi_2, \dots \phi_d\}$ in $H^2(\mathbb{R}^n)$ which satisfy*

$$|\partial_{x_k}u_j(x)| \leq Ce^{-\alpha|x|} \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, j \in \{1,2,\dots m\}, \qquad \forall\, k \in \{1,2,\dots n\}, \tag{7.19}$$

$$|\phi_i(x)| \leq Ce^{-\alpha|x|} \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, i \in \{1,2,\dots d\}, \tag{7.20}$$

for some $C, \alpha > 0$ and also satisfy

$$\langle \partial_{x_i}u_j, \partial_{x_k}u_j\rangle = 0 \qquad \forall\, j \in \{1,2,\dots m\}, \qquad \forall\, i,k \in \{1,2,\dots n\} \;\; \textit{with} \;\; i \neq k. \tag{7.21}$$

and

$$\langle \phi_r, \phi_q \rangle = 0 \qquad \forall\, r, q \in \{1, 2, \dots d\} \quad \textit{with} \quad r \neq q\,. \tag{7.22}$$

Let $\psi \in H^2(\mathbb{R}^n)$ be given. With the notation $T_y u(x) = u(x-y)$ we have that there exist constants $L, \varepsilon > 0$ and unique C^1 functions $\mathcal{S}$ and $\mathcal{A}$ defined on the set of all $y = [y_1, y_2, \dots y_m]^\top$ in $\mathbb{R}^{nm}$ and all $u \in H^1(\mathbb{R}^n)$ which satisfy the conditions

$$|y_j - y_k| > L \quad \forall j, k \in \{1, 2, \dots m\} \quad \textit{with} \quad j \neq k\,, \qquad \|u - \psi - \sum_{j=1}^{m} T_{y_j} u_j\|_{L^2} < \varepsilon\,,$$

$$|y_j| > L \qquad \forall\, j \in \{1, 2, \dots m\}\,, \tag{7.23}$$

such that using $s = \mathcal{S}(u, y) \in \mathbb{R}^{nm}$, written as $[s_1, s_2, \dots s_m]^\top$, and $a = \mathcal{A}(u, y) \in \mathbb{R}^d$, written as $[a_1, a_2, \dots a_d]^\top$, u can be decomposed as follows:

$$u = \psi + \sum_{j=1}^{m} T_{y_j + s_j} u_j + v + \sum_{i=1}^{d} a_i \phi_i\,, \tag{7.24}$$

where v satisfies

$$\langle v, T_{y_j + s_j} \partial_{x_k} u_j \rangle = 0 \qquad \forall\, k \in \{1, 2, \dots n\}\,, \qquad \forall\, j \in \{1, 2, \dots m\}\,,$$

$$\langle v, \phi_i \rangle = 0 \qquad \forall\, i \in \{1, 2, \dots d\}\,.$$

In addition, if we let $A(y) = \inf\{|y_j|, |y_j - y_k| \,\big|\, j, k \in \{1, 2, \dots m\}, j \neq k\}$ and $B(u, y) = \|u - \psi - \sum_{j=1}^{m} T_{y_j} u_j\|_{L^2}$, then

$$\lim_{A(y) \to \infty,\ B(u,y) \to 0} \mathcal{S}(u, y) = 0\,, \qquad \lim_{A(y) \to \infty,\ B(u,y) \to 0} \mathcal{A}(u, y) = 0\,. \tag{7.25}$$

Proof. Define the map $\mathbf{F} : L^2(\mathbb{R}^n) \times \mathbb{R}^{nm} \times \mathbb{R}^{nm} \times \mathbb{R}^d \mapsto \mathbb{R}^{nm} \times \mathbb{R}^d$ as follows:

$$\mathbf{F}(u, y, s, a) = [\mathbf{F}_1(u, y, s, a), \mathbf{F}_2(u, y, s, a), \dots \mathbf{F}_m(u, y, s, a), \mathbf{G}(u, y, s, a)]^\top\,,$$

where for each $j \in \{1, 2, \dots m\}$

$$\mathbf{F}_j(u, y, s, a) = [F_{j1}(u, y, s, a), F_{j2}(u, y, s, a), \dots F_{jn}(u, y, s, a)]^\top$$

with

$$F_{jk}(u, y, s, a) = \left\langle \frac{T_{y_j + s_j} \partial_{x_k} u_j}{\|\partial_{x_k} u_j\|_{L^2}^2}, u - \psi - \sum_{t=1}^{m} T_{y_t + s_t} u_t - \sum_{i=1}^{d} a_i \phi_i \right\rangle_{L^2} \qquad \forall\, k \in \{1, 2, \dots n\}$$

and

$$\mathbf{G}(u, y, s, a) = [G_1(u, y, s, a), G_2(u, y, s, a), \dots G_d(u, y, s, a)]^\top$$

with

$$G_r(u, y, s, a) = \left\langle \frac{\phi_r}{\|\phi_r\|_{L^2}^2}, u - \psi - \sum_{t=1}^{m} T_{y_t + s_t} u_t - \sum_{i=1}^{d} a_i \phi_i \right\rangle_{L^2} \qquad \forall\, r \in \{1, 2, \dots d\}.$$

We will establish the proposition by solving

$$\mathbf{F}(u, y, s, a) = 0 \tag{7.26}$$

for $(s, a) \in \mathbb{R}^{nm+d}$, given u and y. Note that an (s, a) that solves (7.26) for a given u and y implies the decomposition in (7.24). The function $\mathbf{F}$ is C^1 on $L^2(\mathbb{R}^n) \times \mathbb{R}^{nm} \times \mathbb{R}^{nm} \times \mathbb{R}^d$. Its *Fréchet* derivative with respect to (s, a), denoted as $D_{(s,a)}\mathbf{F}(u, y, s, a)$, is a $\mathbb{R}^{nm+d} \times \mathbb{R}^{nm+d}$ matrix whose elements are as follows: For $j, l \in \{1, 2, \dots m\}$ and $i, k \in \{1, 2, \dots n\}$, the $((j-1)n+k, (l-1)n+i)^{th}$ element is

$$\frac{\partial F_{jk}}{\partial s_{li}} = \left\langle \frac{T_{y_j+s_j}\partial_{x_k} u_j}{\|\partial_{x_k} u_j\|^2_{L^2}}, T_{y_l+s_l}\partial_{x_i} u_l \right\rangle - \delta_{jl}\left\langle \frac{T_{y_j+s_j}\partial^2_{x_k x_i} u_j}{\|\partial_{x_k} u_j\|^2_{L^2}}, u - \psi - \sum_{t=1}^{m} T_{y_t+s_t} u_t - \sum_{t=1}^{d} a_t \phi_t \right\rangle.$$

Here s_{li} is the i^{th} element of s_l. For $j \in \{1, 2, \dots m\}$, $k \in \{1, 2, \dots n\}$ and $q \in \{1, 2, \dots d\}$, the $((j-1)n+k, nm+r)^{th}$ element of $D_{(s,a)}\mathbf{F}(u, y, s, a)$ is

$$\frac{\partial F_{jk}}{\partial a_q} = \left\langle \frac{T_{y_j+s_j}\partial_{x_k} u_j}{\|\partial_{x_k} u_j\|^2_{L^2}}, \phi_q \right\rangle.$$

For $l \in \{1, 2, \dots m\}$, $i \in \{1, 2, \dots n\}$ and $r \in \{1, 2, \dots d\}$, the $(nm+r, (l-1)n+i)^{th}$ element is

$$\frac{\partial G_r}{\partial s_{li}} = \left\langle \frac{\phi_r}{\|\phi_r\|^2_{L^2}}, T_{y_l+s_l}\partial_{x_i} u_l \right\rangle.$$

Finally for $r, q \in \{1, 2, \dots d\}$,

$$\frac{\partial G_r}{\partial a_q} = \delta_{rq}.$$

It is easy to verify using the above expression and the decay estimates for $\partial_{x_k} u_j$ and ϕ_i that if u and y satisfy the conditions in the proposition for some $L > 1$ and ε and $|s_j| < L/4$ for each $j \in \{1, 2, \dots d\}$, then there exists a constant $C(n, \alpha) > 0$ such that

$$\left|\frac{\partial F_{jk}}{\partial s_{li}}\right| \le C(n,\alpha) L^n e^{-\alpha L/2} \qquad \text{if} \quad j \neq l,$$

$$\left|\frac{\partial F_{jk}}{\partial s_{li}}\right| \le \delta_{ik} + \big(\varepsilon + |a| \sum_{t=1}^{d} \|\phi_t\|_{L^2} + \sum_{t=1}^{m} \|T_{y_t+s_t} u_t - T_{y_t} u_t\|_{L^2}\big) \frac{\|u_j\|_{H^2}}{\|\partial_{x_k} u_j\|^2_{L^2}} \qquad \text{if} \quad j = l,$$

$$\left|\frac{\partial F_{jk}}{\partial a_q}\right| \le C(n,\alpha) L^n e^{-\alpha L/2}, \qquad \left|\frac{\partial G_r}{\partial s_{li}}\right| \le C(n,\alpha) L^n e^{-\alpha L/2}.$$

For any y, define $u_y = \psi + \sum_{j=1}^{m} T_{y_j} u_j$. Choose a $\delta > 0$. Fix $L, \varepsilon, \eta > 0$ and $\gamma \in (0, \delta)$ such that the following estimates hold: for any u and y that satisfy the conditions in the proposition, s that satisfies $\sum_{j=1}^{m} |s_j| < \gamma$, a that satisfies $|a| < \gamma$ and $\tilde{y}$ that satisfies $|\tilde{y} - y| < \eta$,

$$|\mathbf{F}(u, \tilde{y}, 0, 0)| < \frac{\gamma}{4}, \qquad \|[D_{(s,a)}\mathbf{F}(u_y, y, 0, 0)]^{-1}\| < 2, \tag{7.27}$$

$$\|D_{(s,a)}\mathbf{F}(u_y, y, 0, 0) - D_{(s,a)}\mathbf{F}(u, \tilde{y}, s, a)\| < \frac{1}{4}. \tag{7.28}$$

The existence of constants $L, \varepsilon, \eta, \gamma$ follows from the definition of $\mathbf{F}$ and the estimates for the elements of $D_{(s,a)}\mathbf{F}$. Given u and y that satisfy the conditions in the proposition, we now apply the implicit function theorem to (7.26) at its solution $(u_y, y, 0, 0)$ and conclude, using the estimates in (7.27) and (7.28), that there exists a unique s and a satisfying $\sum_{j=1}^m |s_j| < \gamma$ and $|a| < \gamma$ such that $\mathbf{F}(u, y, s, a) = 0$. The dependence of s and a on (u, y) is C^1 since $\mathbf{F}$ is C^1. We define $\mathcal{S}(u, y) = s$ and $\mathcal{A}(u, y) = a$. The estimate in (7.25) follows from the fact that we can choose $\delta > 0$ to be arbitrarily small, in which case the corresponding L will be sufficiently large and $\varepsilon, \eta, \gamma$ will be sufficiently small. The uniqueness of s and a satisfying $\sum_{j=1}^m |s_j| < \gamma$ and $|a| < \gamma$ (as mentioned above), along with the requirement that $\mathcal{S}$ and $\mathcal{A}$ be C^1 and satisfy the estimate in (7.25), implies the uniqueness of $\mathcal{S}$ and $\mathcal{A}$ we have constructed. □

We remark that in the above proof, by applying the implicit function theorem at $(u_y, y, 0, 0)$, we can conclude that there exists $L, \varepsilon_y > 0$ (ε_y depends on y) such that if u and y satisfy the conditions in the statement of the proposition with $\varepsilon = \varepsilon_y$, then there is a unique s and a unique a for which $\mathbf{F}(u, y, s, a) = 0$. But to establish the existence of an ε independent of y, we need to use the estimates in (7.27) and (7.28) which are used (in some form) in the proof of the implicit function theorem to construct closed balls for applying the contraction mapping principle.

Remark 7.2. *When $\psi = 0$ and $\phi_r = 0$ for all $r \in \{1, 2, \dots d\}$, the conclusions of the above proposition continue to hold (with $\mathcal{A} = 0$ of course) with A defined as $A(y) = \inf\{|y_j - y_k| \,\big|\, j, k \in \{1, 2, \dots m\}, j \neq k\}$ and without the restriction* (7.23)*. In particular one of the y_js can be near 0.*

Remark 7.3. *It is evident from the proof of Proposition 7.6 that the constants $L, \varepsilon > 0$ can be chosen such that they are independent of the functions $\{u_1, u_2, \dots u_m\}$ and the convergence of the maps $\mathcal{S}$ and $\mathcal{A}$ in* (7.25) *is uniform in the same functions, as long as these functions satisfy the decay estimates* (7.19)*, the conditions* (7.21) *and the inequalities*

$$\|\partial_{x_k} u_j\|_{L^2} > c_1, \qquad \|u_j\|_{H^2} < c_2 \qquad \forall\, j \in \{1, 2, \dots m\}, \qquad \forall\, k\{1, 2, \dots n\},$$

for some fixed $c_1, c_2, C, \alpha > 0$. Furthermore, the dependence of $\mathcal{S}$ and $\mathcal{A}$ on $\{u_1, u_2, \dots u_m\}$ is C^1. Indeed, consider the function $\mathbf{F}(u_1, u_2, \dots u_m, u, y, s, a)$ defined in the proof of Proposition 7.6, where this function is denoted as $\mathbf{F}(u, y, s, s)$ (in Proposition 7.6 $\{u_1, u_2, \dots u_m\}$ are fixed). We can apply the implicit function theorem to this function, by repeating the steps in the proof of Proposition 7.6, and use the fact that $\mathbf{F}$ is C^1 to conclude that the dependence of $\mathcal{S}$ and $\mathcal{A}$ on $(u_1, u_2, \dots u_d, u, y)$ is C^1.

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