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Electromagnetic Foundation of Dirac Theory

Michael Ibison

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Abstract The dynamics of classical charges subject to a particular variant of electromagnetic direct particle interaction are shown to derive from a homogeneous differential equation in a Clifford multivector. Under appropriate conditions the multivector can be factorized to give a Dirac Equation whose bi-spinor operands are eigenvectors of the multivector, thereby giving an electromagnetic basis for the Dirac Equation.

The Clifford multivector is an ensemble of vector and bi-vector contributions from the potential and Faraday of the auxiliary ('adjunct') fields of direct particle interaction, each member generated by a unique current. The presumption of light-speed motion of the charge generates non-linear constraints on these fields. These conditions are shown to be responsible for the otherwise enigmatic eigenvalue selection / 'wavefunction collapse' behavior characteristic of Dirac bi-spinors.

Though time-symmetric adjunct fields are intrinsic to the direct particle action paradigm, their elimination has been the main focus of previous work in this field in order to conform with Maxwell field theory. By contrast, this work presents the time-symmetric fields as the foundation of Dirac bi-spinors. Even so, accidentally we discover a novel explanation of the emergence of exclusively retarded radiation from the direct action paradigm that makes no appeal to special boundary conditions.

Keywords Quantum Theory · Dirac's Equation · Clifford Algebra · direct particle interaction · time-symmetry · Majorana spinors

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M. Ibison
Tel.: +1 512 669 6713
E-mail: ngc5548@gmail.com

1 Introduction

1.1 Historical Context

Direct Particle Interaction, henceforth DPI, is a version of electromagnetism distinct from the Maxwell theory that was first proposed by Schwarzschild [1], Tetrode [2], and Fokker [3], in which the EM fields and potentials are not independent dynamical variables, and the only electromagnetic contribution to the action comes from direct interaction between 4-currents. All electromagnetic energy and momentum is to be accounted for in the interaction between charges, so that any EM energy leaving a charge must be destined for absorption by another charge. Accordingly DPI does not admit strictly vacuum degrees of freedom, strictly on-shell photons, or radiation exactly as portrayed by field theory. Since its inception a challenge for DPI has been an explanation for the observational evidence apparently in favor of exclusively retarded radiation.[1] Though Wheeler and Feynman [4],[5] showed that radiation-like behavior, including radiation reaction, could arise within DPI if the future is sufficiently absorbing, the subsequent discovery of accelerating cosmological expansion rendered their explanation untenable because the universe is nearly transparent on the forward light-cone [6],[7] (see also the works by Pegg [8],[9]). The books by Hoyle and Narlikar [10],[11] and Davies [[6] are recommended for a comprehensive review of Direct Particle Interaction.

1.2 Relation to other work

Clifford Formalism The focus of this work is on exposing the classical foundations of Quantum Theory, employing Clifford algebra primarily as an intermediate tool, eventually departing from the Clifford formalism to obtain Dirac bi-spinors that are strictly compliant with the traditional Dirac Theory. We share with Hestenes [12],[13] (see also [14]) that the Dirac equation be founded on real (versus complex) quantities, though there are differences both in how that is implemented here, and in the outcome. As shown by Rodriguez [15],[16], the Clifford object that is the operand of the Dirac-Hestenes equation operator is not a Dirac bi-spinor, and does not share the same Fierz Identities as those of the Dirac bi-spinor that is the focus of this work (see [17]).[2]

Pilot Wave Model This work has in common with the pilot wave model of de Broglie [18],[19] and Bohm [20]-[23] in both the non-relativistic (Schrödinger) domain and its relativistic extension (for example [24],[25]) that the electron is a classical point charge following flow lines generated by a 'field'. The book by Holland [26] is recommended for a thorough exposition of the de Broglie Bohm theory. See [27]-[29] for journal-paper reviews of the Broglie Bohm theory, including its extension to quantum field theory. In common with those extensions, and of relevance to this work, the

[1] An outcome of this work is that the observational facts are compatible with a different interpretation.

[2] The Fierz Identities are bi-linear relationships between the different $n \in [0,4]$ blades in the outer product $\psi\overline{\psi}$ (i.e. between the $\left\langle \psi\overline{\psi} \right\rangle_n$) as a consequence of the reduced number of degrees of freedom in a multivector restricted to this form.

original - Schrödinger domain - model has since been re-cast by Hiley [30], [31], [24] in terms of Clifford algebra.

Though the psi-field and associated quantum potential of the pilot wave model are sufficient for the task of reproducing standard theory, the de Broglie Bohm model is silent on the origin of the field. Even so, that model is to be credited for its pioneering role in expanding the language employed to 'explain' QM to include a classical particle (in addition to the wavefunction) and by providing an example of a successful epistemological alternative to the Bohr / Copenhagen doctrine.

Random Walks There have been efforts to mimic the Schrödinger and Dirac equations with classical diffusion processes / random walks by Nagasawa [35], Nelson [36],[37], Ord [38]-[40] and others, e.g. [41]. Though Nelson in particular seems to have had some success in reproducing quantum behavior from diffusion processes, the rules governing the jump probabilities do not appear to have a strong physical motivation. It is important that all of these have in common that in order to establish a 'classical' probability distribution that matches those of QM the diffusion jump probability at $\mathbf{x}$ are not Markovian, but depend on the (probabilistic) history of visits to $\mathbf{x}$. Possibly there is a connection with the ensembles of mutually exclusive possibilities that play a prominent role in this work.

Time-Symmetric Presentations of QM Though Cramer [45]-[47] does not attempt to give an explicit electromagnetic foundation for the wavefunction, his 'Transactional Interpretation' of QM captures something of role of time-symmetric exchanges in this work that are crucial to the emergence of Dirac dynamics from an entirely classical EM framework. Cramer's casting - in the non-relativistic domain - of the Schrödinger wavefunction and its charge conjugate as 'offer' and 'accept' waves approximately correspond, respectively, with the retarded and advanced components of time-symmetric exchanges.

The theory of weak-value measurements due to Aharonov, Albert, and Vaidman [48] that grant equal status to the initial and final boundary conditions on the wavefunction has helped draw attention to the time-symmetry already present in traditional quantum theory, but which derives, according to this work, from the time-symmetry of the EM fields that underlie the wavefunction. Sutherland [49]-[51] makes a case for retro-causal influences underpinning QM, granting the final boundary condition employed to explain weak-value measurement the same status as the initial boundary condition, with the effect that the wavefunction at all intermediate times depends symmetrically on both – in all cases. With this construction he is able to give an entirely local 'ontological explanation' for entangled-state behavior such as in the Bell experiment which does not refer to a preferred frame. The claim here is not that QM is at fault predictively, or that its predictions are at odds with special relativity, but that the particle and wavefunction can be given an ontological status at all *intermediate* times consistent with special relativity.

Price, Wharton, Evans and Miller [53]-[56], have argued not only that the nonlocality intrinsic to QM is suggestive of retro-causal influences, but have suggested (correctly, from the perspective of this work) this be taken as evidence of a direct particle interaction foundation of quantum dynamics.

Barut Zanghi Paper Barut and Zanghi [57] showed how to reproduce the algebraic structure of the observables of the Dirac Theory with a classical theory of a point charge augmented with spinor degrees of freedom. The goal of that work was not to reproduce the Dirac *equation*, however. Its achievement was in constructing a classical analog that was faithful to the Dirac equation so that 'canonical quantization' reproduces the algebra of the observables of the Dirac theory. By contrast this work reproduces not only the Dirac equation '*ab initio*' from a particular variant of classical EM theory (and therefore the algebra of its observables) but also the attendant machinery of eigenvalue selection by observation, neither of which were the aim or focus of the Barut-Zanghi work.

2 Direct Particle Interaction

2.1 Action

The electromagnetic direct particle interaction is

$$I_{DPI} = -\int \mathrm{d}^4 x \int \mathrm{d}^4 x' G(x - x') \mathbb{j}(x) \circ \mathbb{j}(x') \tag{1}$$

where

$$\begin{aligned} G(x) &= \frac{1}{4\pi} \delta\left(x^2\right) \Rightarrow \partial^2 G(x) = \delta^4(x) \\ \mathbb{j}(x) &= |e| \int \mathrm{d}\lambda\, \mathbb{v}(\lambda) \delta^4(x - q(\lambda)); \quad \mathbb{v}(\lambda) = \frac{\mathrm{d}\mathbb{q}(\lambda)}{\mathrm{d}\lambda}. \end{aligned} \tag{2}$$

$\mathbb{j}$ is a Lorentz vector, $q(\lambda)$ is a Lorentz vector, $\mathbb{v}$ is a 4-vector and also a Lorentz vector when λ is a Lorentz scalar. A double strike font signifies the object is to be considered an element of $\mathrm{Cl}_{1,3}(\mathbb{R})$ rendered in $\mathrm{M}_4(\mathbb{C})$, where appropriate. (An exception introduced later is the EM multivector, which is in $\mathrm{Cl}_{1,3}(\mathbb{C})$). x and $q(\lambda)$ are also Lorentz vectors, but so-written are considered to be represented more conventionally, i.e. in $\mathbb{R}^4$ with Minkowski norm, and $x^2 = x^\mu x_\mu$ etc. Where necessary we refer to components in 3+1 D, e.g. $x = (t, \mathbf{x})$. Hence, since they all refer to the same object, $\mathbb{q}(\lambda) \cong q(\lambda) \cong \left(q^0(\lambda), \mathbf{q}(\lambda)\right)$.

Due to the structure of (1) an anti-symmetric component of $G(x)$ makes no contribution to the action. Consequently DPI effectively mandates a Green's function that is invariant under negation of any of the coordinates, and is thereby distinguished from traditional theory by its restriction to time-symmetric interactions relative to the sources.

Let the currents be broken into segments parameterized by laboratory time t: $q(\lambda) \to \{q_1(t), q_2(t), \ldots, q_N(t)\}$ each with constant sign of $\mathrm{d}q_l(t)/\mathrm{d}t$, and where $q_l^0(t) = t$. The current in (2) is then

$$\mathbb{j}(x) = \sum_{l=1}^{N} \mathbb{j}_l(x); \;\; \mathbb{j}_l(x) = e_l \mathbb{v}_l(t) \delta^3(\mathbf{x} - \mathbf{q}_l(t)); \;\; \mathbb{v}_l(t) = (1, \mathbf{v}_l(t)) \tag{3}$$

where $\mathbf{v}_l(t) = \mathrm{d}\mathbf{q}_l(t)/\mathrm{d}t$. Using (3) in (1) and denying self-action leads to

$$\begin{aligned} I_{DPI} &\rightarrow -\sum_{k,l=1;k\neq l}^{N} \int \mathrm{d}^4x \int \mathrm{d}^4x' G(x-x')\, \mathbb{j}_k(x) \circ \mathbb{j}_l(x') \\ &= -\sum_{k,l=1;k\neq l}^{N} \frac{e_k e_l}{4\pi} \int \mathrm{d}t \int \mathrm{d}t' \delta\left(s_{k,l}^2(t,t')\right) \mathbb{v}_k(t) \circ \mathbb{v}_l(t') \end{aligned} \tag{4}$$

where $s_{k,l}(t,t') = q_k(t) - q_l(t')$.

2.2 Adjunct fields

Classical Current With reference to the second part of (4) the subsequent introduction of x to denote pre-existing $\mathbb{R}^4$ spacetime is an intermediate computational tool. This applies to the current (2), which in DPI therefore has a derivative status relative to $q(t)$. To be consistent with the adjunct potential as coined by Wheeler and Feynman (see below) the subjects of (2) and (3) should be called *adjunct* currents.

Adjunct Potential The adjunct potential [4],[5] generated by the l^{th} charge is

$$\mathbb{A}_l(x) = \int \mathrm{d}^4x' G(x-x')\, \mathbb{j}_l(x') = \frac{e_l}{4\pi} \int \mathrm{d}t' \mathbb{v}_l(t') \delta\left((x - q_l(t'))^2\right) \tag{5}$$

a consequence of which is

$$\partial^2 \mathbb{A}_l(x) = \mathbb{j}_l(x)\,. \tag{6}$$

The total adjunct potential from N charges is

$$\mathbb{A}(x) = \sum_{l=1}^{N} \mathbb{A}_l(x)\,. \tag{7}$$

We will also need to refer to the potential of all but the l^{th} current

$$\mathbb{A}_{\bar{l}}(x) = \mathbb{A}(x) - \mathbb{A}_l(x)\,. \tag{8}$$

The technique of distinguishing between fields according to their origin is due to Leiter [58]. Note that x in $\mathbb{A}(x)$ should not be taken to imply a pre-existing $\mathbb{R}^4$ spacetime 'canvas'; direct particle interaction grants the adjunct potential a physically meaningful role only on the worldline of a charge.

The Lorenz gauge is mandated by the structure of (5), in particular because the Green's function $G(x,x') \rightarrow G(x-x')$ depends only on the coordinate difference $x - x'$:[3]

$$\partial\!\!\!/ \circ \mathbb{A}_l(x) = \int \mathrm{d}^4x' \partial\!\!\!/ G(x-x') \circ \mathbb{j}_l(x') = \int \mathrm{d}^4x' G(x-x') \partial\!\!\!/' \circ \mathbb{j}_l(x') = 0. \tag{9}$$

[3] $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ has the usual meaning. $\mathrm{a} \circ \mathrm{b} = (\mathrm{ab} + \mathrm{ba})/2$ is the scalar product of two Clifford vectors. Likewise $a \circ b = a^\mu b_\mu$.

Clearly (9) implies $\partial\!\!\!/ \circ \mathbb{A}(x) = 0$. Applying (5) and (8) to Eq. (4) gives

$$I_{DPI} = -\sum_{l=1}^{N} \int \mathrm{d}^4 x\, \mathbb{j}_l(x) \circ \mathbb{A}_{\bar{l}}(x) = -\int \mathrm{d}^4 x\, \mathbb{j}(x) \circ \mathbb{A}(x) - I_{self} \tag{10}$$

where

$$I_{self} = -\sum_{l=1}^{N} \int \mathrm{d}^4 x\, \mathbb{j}_l(x) \circ \mathbb{A}_l(x)\,. \tag{11}$$

Note that the $\mathrm{q}_l(t)$ are the only dynamical degrees of freedom - the action is not extremized by variation of the $\mathbb{A}_l(x)$.

Properties The adjunct potential of direct particle interaction differs from a potential of traditional field theory in that the adjunct potential:

i) Is always sourced.
ii) Necessarily satisfies the Lorenz gauge condition.
iii) Is time-symmetric relative to the source.
iv) Is physically consequential only where it originates and where it is terminated.[4][5]

Consequent to iv) is that the solutions of $\partial^2 \mathbb{A} = \mathbb{0}$ are everywhere physically inconsequential.

Adjunct Faraday The adjunct Faraday bi-vector is [6]

$$\mathbb{F}_l = \mathbb{F}_l(x) = \partial\!\!\!/ \wedge \mathbb{A}_l = \partial\!\!\!/ \mathbb{A}_l - \partial\!\!\!/ \circ \mathbb{A}_l = \partial\!\!\!/ \mathbb{A}_l. \tag{12}$$

We will need also

$$\mathbb{F}(x) = \sum_{l=1}^{N} \mathbb{F}_l(x)\,, \quad \mathbb{F}_{\bar{l}}(x) = \mathbb{F}(x) - \mathbb{F}_l(x)\,. \tag{13}$$

Taking into account (6) (using $\partial\!\!\!/^2 = \partial^2$) the 'field equations' appear to be those of the Maxwell electrodynamics in the Lorenz gauge

$$\partial\!\!\!/ \mathbb{A}_l = \mathbb{F}_l, \quad \partial\!\!\!/ \mathbb{F}_l = \mathbb{j}_l \tag{14}$$

though $\mathbb{A}_l$ and $\mathbb{F}_l$ are under-constrained by (14) because they admit an unphysical complementary function solution to $\partial^2 \mathbb{A}_l = \mathbb{0}$.

Eq. (4) is time-reparameterization invariant wherein t plays the role of a 'speed parameter' for the space-time curve $q = q(t)$ in $\mathbb{R}^4$. Accordingly the worldlines in (4)

[4] The Wheeler and Feynman adjunct potential satisfies i), ii) and iii) only. The termination requirement iv) is understood but not built in to the structure. Their adjunct potential is mathematically indistinguishable from a field-theory potential satisfying the same conditions (i.e. just i), ii) and iii)) because it is non-zero on all future and past oriented null rays passing through the worldline of the source. On that basis Hoyle and Narlikar have argued (incorrectly from the point of view of this work) that the electromagnetic direct action stress-energy is essentially no different from that of the Maxwell theory.

[5] Feynman subsequently changed his position on the role of self-action, and so by implication on the status of the adjunct potential at its source.

[6] $\mathrm{a} \wedge \mathrm{b} = (\mathrm{ab} - \mathrm{ba})/2$ is the anti-symmetric product of two vectors.

can be parameterized with any monotonic function of t. Alternatively the action can be written without any reference to t, for example as

$$I_{DPI} = -\frac{e^2}{4\pi} \sum_{k,l=1;k\neq l}^{N} \int \int \mathrm{d}q_k \circ \mathrm{d}q_l \delta\Big((q_k - q_l)^2\Big). \tag{15}$$

From the absence of **x** in the actions (4) and (15) it can be inferred that any generalized Fourier space representation of the adjunct current and potential will have a status equal to that of the **x**-space representation. Representation independence of the fields will play a role in the description of dynamics of the currents.

2.3 Time-Symmetry

The DPI action employs a Green's function that is time-symmetric. Accordingly the adjunct potential and Faraday are time-symmetric relative to their source. The physical content of DPI however is restricted to the interactions at both ends of a light-like connection. These null ray line segments extend along the forward and backward light cone from a nominally local charge. Their angular distribution and their distribution in time (forwards versus backwards) depend on the distribution of other charges in space and time. Taking into account Cosmological evolution this distribution will not generally be time-symmetric - except perhaps at the future conformal singularity. Further, potentials superpose, with the result that the total incoming response potential might in extreme cases vanish, even though it is the result of any number of other, distant charges.[7] Broadly then, though DPI is a time-symmetric theory, the manifestation of that property depends on the actual distribution of matter.

In contrast with earlier attempts to reconcile DPI with observation, in this work we allow for the possibility that the advance component of the DPI adjunct potential is not, in general, canceled at its source by the response of other charges. An outcome is that the universal system of charges can remain tightly coupled by whatever symmetric component remains, post recombination. In Section 5 the totality of DPI modes are shown to correspond to those of an elastic lattice with optical and acoustic branches. The modes of the optical branch correspond very closely to the vacuum modes of field theory, thereby explaining the emergence of retarded radiation without appeal to a thermodynamic arrow of time. The acoustic modes are subsequently shown to underpin the Dirac wavefunction. Effectively, this work resolves the difficulties attributed to DPI with a re-interpretation of the supposed deficiency as the foundation of QM.

[7] This is the foundation of the Wheeler-Feynman absorber theory, wherein the presumed complete future absorption results in complete cancellation of the advanced component of the response.

3 Light-speed charge in a given potential

3.1 Light-speed motion

We depart from classical traditional by asserting light-speed motion of the electron

$$\mathbb{v}_l^2(t) = 0 \forall l \in [1, N]. \tag{16}$$

Justifications for this assertion are:

i) The time-symmetric interaction appears to demand that the mass be dynamically determined,[8] which (16) achieves, though not uniquely so.[9]
ii) The self-energy of a classical light-speed charge is ill-defined by traditional classical theory, which ambiguity can be removed in favor of a (definite) finite energy in that limit - without affecting the predictions of classical theory at subluminal speeds [60].
iii) The eigenvalues of the velocity operator for the Dirac electron are ±1.
iv) The Dirac Equation is an outcome of this (classical) analysis.

Eq. (16) can be enforced via a semi-holonomic constraint in an action

$$I_{LS} = -\frac{1}{2}\sum_{l=1}^{N}\int \mathrm{d}t\, \mu_l(t)\, \mathbb{v}_l^2(t) \tag{17}$$

extremized by variation of $\mu_l(t)$. For I_{LS} to be a Lorentz scalar the $\mu_l(t)$ must transform as dt. Alternatively each path can be parameterized with a monotonically increasing Lorentz scalar, including an appropriately defined frame-independent time. It turns out however that the Euler equations will be such as to grant $\mu_l(t)$ the appropriate transformation property automatically.[10]

With (10) the full action is

$$I = I_{LS} + I_{DPI} = -\sum_{l=1}^{N}\int \mathrm{d}t \left(\frac{1}{2}\mu_l(t)\, \mathbb{v}_l^2(t) + e_l \mathbb{v}_l(t) \circ \mathbb{A}_{\bar{l}}(q_l(t))\right) \tag{18}$$

$\mathbb{A}_{\bar{l}}(q_l(t))$ is the adjunct potential of all but the l^{th} charge evaluated on the path of the l^{th} charge. It is the 'incoming' adjunct potential relative to the current with label l. The Euler equations are the corresponding Newton-Lorentz equations [11]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_l(t)\, \mathbb{v}_l(t)\right] = e_l \left\langle \mathbb{F}_{\bar{l}}(q_l(t))\, \mathbb{v}_l(t)\right\rangle_1 \tag{19}$$

[8] To be submitted.

[9] A property of the time-symmetric interaction is that the adjunct potential response of other nominally distant charges to the motion of the local charge is contemporaneous with the motion of the latter. The causal loop is closed with the requirement that the local electron motion in the presence of the incoming response potential is consistent with the motion that brought about that response. In the work cited here electron mass appears as an eigenvalue of the fields of that exchange.

[10] This outcome is an automatic consequence of the relationship (22) established with the potential, which is a true Lorentz vector.

[11] $\langle\rangle_1$ extracts the vector part of its Clifford operand, $\langle\rangle_2$ extracts the bi-vector part, etc.

where, using an over-dot to identify the target of $\not\partial$,

$$\left\langle \mathbb{F}_{\bar{l}}(q_l(t))\mathbb{v}_l(t)\right\rangle_1 = \left[\not\partial\left[\mathbb{v}_l(t)\circ\dot{\mathbb{A}}_{\bar{l}}(x)\right]\right]_{x=q_l(t)} - \frac{\mathrm{d}\mathbb{A}_{\bar{k}}(q_l(t))}{\mathrm{d}t} \tag{20}$$

in which terms (19) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_l(t)\mathbb{v}_l(t)+e_l\mathbb{A}_{\bar{l}}(q_l(t))\right] = e_l\left[\not\partial\left[\mathbb{v}_l(t)\circ\dot{\mathbb{A}}_{\bar{l}}(x)\right]\right]_{x=q_l(t)}. \tag{21}$$

The left–hand side is the time rate of change of the total (mechanical plus electromagnetic) 4-momentum of the local charge. The electromagnetic part of the momentum is specific to the charge 'in' the potential $\mathbb{A}_{\bar{l}}$ at $q_l(t)$.

3.2 First integral of Newton-Lorentz equation

Null Incoming Potential Suppose initially that the incoming potential is null. Then a particular solution of (21) is

$$\mu_l(t)\mathbb{v}_l(t)+e_l\mathbb{A}_{\bar{l}}(q_l(t)) = 0 \tag{22}$$

and the total momentum is zero. The time-component of (22) gives that

$$\mu_l(t) = -e_l\phi_{\bar{l}}(q_l(t)) \tag{23}$$

and therefore

$$\mathbb{v}_l(t) = \mathbb{A}_{\bar{l}}(q_l(t))/\phi_{\bar{l}}(q_l(t)) \Rightarrow \mathbf{v}_l(t) = \mathbf{A}_{\bar{l}}(q_l(t))/\phi_{\bar{l}}(q_l(t)). \tag{24}$$

Hence the null current follows the flow lines of an incoming null adjunct potential. [12]

General Case Any non-null potential can be decomposed into null components. It will turn out to be useful to decompose the incoming Faraday in an analogous way, which in combination will give rise to 4 distinguishably different null potentials. Initially we presume that the charge follows just one of those null potentials, accepting the possibility of subsequent revision to account for the presence of the other potentials. It turns out that however that it is always possible to find a decomposition in which the 4 null paths are independent of each other, provided the potentials are modes of the acoustic branch (see below). Optical branch mode potentials (Section 5.1) require separate treatment however.

To implement this strategy let an arbitrary incoming potential be decomposed into r null potentials

$$\mathbb{A}_{\bar{l}}(x) = \sum_{n=1}^{r}\mathbb{A}_{\bar{l},n}(x); \quad \mathbb{A}_{\bar{l},n}^2(x) = 0 \tag{25}$$

where for now the number of terms r in the decomposition is left undetermined. Each $\mathbb{A}_{\bar{l},n}(x)$ generates a set of flow-lines, the possible occupancy of each member of which

[12] Since only derivatives of the incoming potential appear in (21) it follows that a more general solution is $\mu_l(t)\mathbb{v}_l(t)+e_l\mathbb{A}_{\bar{l}}(q_l(t)) = e_l\mathrm{u}_{\bar{l}}$ for any constant vector $\mathrm{u}_{\bar{l}}$.

by a charge will initially be considered independently, in accord with the above. Then the solution (24) can be applied to each of these:

$$\mathbb{v}_{l,n}(t) = \mathbb{A}_{\bar{l},n}(q_{l,n}(t)) / \phi_{\bar{l},n}(q_{l,n}(t)); \quad \left[\mathbb{A}_{\bar{l},n}(q_{l,n}(t))\right]^2 = 0 \tag{26}$$

Here $q_{l,n}(t)$ is the worldline of the l^{th} charge following the flow-line of the n^{th} null potential in an r-fold decomposition of the potential of all other charges. Provided the $\mathbb{A}_{\bar{l},n}(x)$ independently satisfy the Lorenz gauge then it is straight-forward to show that

$$\mathrm{d}\phi_{\bar{l},n}(q_{l,n}(t)) / \mathrm{d}t = \left[\not{\partial} \circ \mathbb{A}_{\bar{l},n}(x)\right]_{x=q_{l,n}(t)} = 0. \tag{27}$$

It follows that the paths $q_{l,n}(t)$ that satisfy (26) are the characteristics of $\phi_{\bar{l},n}$, i.e. on which $\phi_{\bar{l},n}$ is constant. In particular

$$\phi_{\bar{l},n}(q_{l,n}(t)) = \phi_{\bar{l},n}(q_{l,n}(0)). \tag{28}$$

Connection with de Broglie Bohm Model Eq. (26) with (28) is the classical electromagnetic foundation of the de Broglie Bohm pilot-wave mechanism, consistent with which is the absence of a role in the dynamics for magnitude of the 4-potential. When however the description is subsequently extended to cover an ensemble of charges the time-component of the null potential (which in this context can be equated with the 'magnitude') will be 're-purposed' to carry information about the occupation probabilities of the flow lines.

3.3 Signs of mass and charge

The stipulation that the time component of $v_l(t)$ is equal to 1 forces the parameterization of all particles to be in the same direction along the time dimension. Informally this means that all charges proceed forwards in time, regardless of the sign of the charge. To align with convention we also arrange for the sign of the dynamic mass to be positive. Taking into account (28), the time component of the n^{th} potential in an r-fold decomposition of (22) is

$$\mu_{l,n}(t) = \mu_{l,n}(0) = -e_{l,n}\phi_{\bar{l},n}(q_{l,n}(0)). \tag{29}$$

A positive constant mass therefore requires

$$\mu_{l,n}(0) = |e| \left|\phi_{\bar{l},n}(q_{l,n}(0))\right| \Rightarrow e_{l,n} = -|e| \operatorname{sgn}\left(\phi_{\bar{l},n}(q_{l,n}(0))\right). \tag{30}$$

The sign of the charge is the negative of the sign of $\phi_{\bar{l},n}(q_{l,n}(0))$, which implies a restriction of electron / positron flow-lines to the positive / negative 'half-cycles' of a time-varying potential and a corresponding restriction on the current vector (Section 7.2).

4 Ensembles

4.1 Sum over mutually exclusive possibilities

Eq. (26) with (28) is the first order differential equation [13]

$$\frac{dq_{l,n}(t)}{dt} = \frac{1}{\phi_{\bar{l},n}(q_{l,n}(0))} A_{\bar{l},n}(q_{l,n}(t)); \quad \left[\mathbb{A}_{\bar{l},n}(q_{l,n}(t))\right]^2 = 0. \tag{31}$$

If $A_{\bar{l},n}(x)$ is given then in principle $q_{l,n}(t)$ can be found by solving (31). Taking into account (30) the 4-current $\mathbb{j}_l(x)$ in (3) must be distinguished accordingly as one of $\mathbb{j}_{l,n}(x)$ for $n \in [1,r]$

$$\mathbb{j}_{l,n}(x) = -\frac{|e|\delta^3(\mathbf{x}-\mathbf{q}_{l,n}(t))}{\left|\phi_{\bar{l},n}(q_{l,n}(0))\right|} \mathbb{A}_{\bar{l},n}(q_{l,n}(t)) = -\frac{|e|\delta^3(\mathbf{x}-\mathbf{q}_{l,n}(t))}{\left|\phi_{\bar{l},n}(q_{l,n}(0))\right|} \mathbb{A}_{\bar{l},n}(x) \tag{32}$$

where $q_{l,n}(t)$ is a solution of (31). Let us write

$$\mathbf{q}_{l,n}(t) = \boldsymbol{\delta}_{l,n}(t) + \mathbf{q}_{l,n}(0) \tag{33}$$

where $\boldsymbol{\delta}_{l,n}(0) = \mathbf{0}$, ie $\boldsymbol{\delta}_{l,n}(t)$ is the particular solution of (31) that passes through the origin at $t = 0$. We now form a statistically-weighted ensemble, summing the currents (32) over the initial conditions. Let $p_{l,n}(\mathbf{q}_{l,n}(0))$ be the weight of the n^{th} null current passing through $\mathbf{x} = \mathbf{q}_{l,n}(0)$ at $t = 0$. Then

$$\{\mathbb{j}_{l,n}(x)\} = \int d^3 q_{l,n}(0)\, p_{l,n}(\mathbf{q}_{l,n}(0))\, \mathbb{j}_{l,n}(x) \tag{34}$$

is an ensemble current, the members of which are mutually exclusive when there is just one local charge. Consider the particular weights

$$p_{l,n}(\mathbf{q}_{l,n}(0)) = \left|\phi_{\bar{l},n}(q_{l,n}(0))\right| \beta^2 / |e| = \left|\phi_{\bar{l},n}(0, \mathbf{q}_{l,n}(0))\right| \beta^2 / |e| \tag{35}$$

where β is a constant with dimensions L^{-1}. Substitution of (35) and (32) into (34) and using (33) gives

$$\{\mathbb{j}_{l,n}(x)\} = -\beta^2 \int d^3 q_{l,n}(0)\, \delta^3(\mathbf{x}-\mathbf{q}_{l,n}(t))\, \mathbb{A}_{\bar{l},n}(x) = -\beta^2 \mathbb{A}_{\bar{l},n}(x). \tag{36}$$

The ensemble current $\{\mathbb{j}_{l,n}(x)\}$ is conserved iff each of $\mathbb{A}_{\bar{l},n}(x)$ satisfy the Lorenz gauge, and vice-versa. Despite appearances, Eq. (35) is not a restriction on the weights because $\phi_{\bar{l},n}(q_{l,n}(0))$ is not *given*. A consequence of (36) is that $\phi_{\bar{l},n}(x)$ (now) satisfies a homogeneous coupled differential equation (see Section 5), for which $\phi_{\bar{l},n}(q_{l,n}(0))$ can be cast as a boundary condition - with no constraint on its functional form.

We now form an r-fold ensemble of the null ensemble currents:

$$\{\mathbb{j}_l(x)\} = \sum_{n=1}^{r} \{\mathbb{j}_{l,n}(x)\} = -\beta^2 \mathbb{A}_{\bar{l}}(x) \tag{37}$$

---

[13] Here we revert to a component representation of the Lorentz vectors to avoid discussion of functions of Clifford vectors necessitated by writing $d\mathbb{q}_{l,n}(t)/dt = \mathbb{A}_{\bar{l},n}(\mathbb{q}_{l,n}(t))/\phi_{\bar{l},n}(\mathbb{q}_{l,n}(t))$.

where we used (25). Introducing the *ensemble* adjunct potential

$$\{\mathbb{A}_l(x)\} = \int d^4x' G(x-x')\{\mathbb{j}_l(x')\} \tag{38}$$

the ensemble version of (14) is

$$\not{\partial}\{\mathbb{A}_l(x)\} = \{\mathbb{F}_l(x)\}, \quad \not{\partial}\{\mathbb{F}_l(x)\} = -\beta^2 \mathbb{A}_{\bar{l}}(x). \tag{39}$$

These ensembles simulate smooth fields satisfying differential equations. They hide the constraint that the adjunct potential is physically consequential only at the point of contact with the charge. And in the form (39) they hide the non-linear constraint that the generators of the flow lines for the current are null.

4.2 Post hoc enforcement of mutual exclusion

Nothing in the above enforces mutual exclusion; the $p_{l,n}(\mathbf{q}_{l,n}(0))$ above appear to be independent. By contrast, if it is known that there is just one particle then mutual exclusivity of flow-line occupancy requires [14]

$$p_{l,n}\left(\mathbf{q}_{l,n}(0),\mathbf{q}'_{l,n}(0)\right) = \delta^3\left(\mathbf{q}_{l,n}(0)-\mathbf{q}'_{l,n}(0)\right)p_{l,n}(\mathbf{q}_{l,n}(0)) \tag{40}$$

and suitably extended to cover higher orders of correlation. An implementation, viable at least in a single particle theory, is to compute the dynamics at first ignoring mutual exclusion, treating (34) as an ordinary integral and (37) as an ordinary sum - i.e. both in the sense of a superposition - and enforce mutual exclusion only on products of mutually exclusive possibilities. For example, squaring $\{\mathbb{j}_l(x)\}$ in (37), and supposing for simplicity that $n \in \{1,2\}$, one has

$$\{\mathbb{j}_l(x)\}^2 = \{\mathbb{j}_{l,1}(x)\}^2 + \{\mathbb{j}_{l,2}(x)\}^2 + 2\{\mathbb{j}_{l,1}(x)\}\circ\{\mathbb{j}_{l,2}(x)\} \tag{41}$$

The first two terms on the right are null, the third term vanishes because it is the product of two-mutually exclusive possibilities, and therefore $\{\mathbb{j}_l(x)\}$ is effectively null. This property extends to the $\{\mathbb{j}_{l,n}(x)\}$ given by (42): squaring (37), one has

$$\{\mathbb{j}_{l,n}(x)\}^2 = \int d^3a \int d^3b\, p_{l,n}(\mathbf{a})\, p_{l,n}(\mathbf{b})\, \mathbb{j}_{l,n}(x;\mathbf{a})\circ \mathbb{j}_{l,n}(x;\mathbf{b}) \tag{42}$$

To remove the mutually-exclusive terms one can make the replacement

$$p_{l,n}(\mathbf{a})\, p_{l,n}(\mathbf{b}) \to p_{l,n}(\mathbf{a},\mathbf{b}) = p_{l,n}(\mathbf{a})\,\delta^3(\mathbf{a}-\mathbf{b}) \tag{43}$$

whereupon (42) becomes

$$\{\mathbb{j}_{l,n}(x)\}^2 = \int d^3a\, p_{l,n}(\mathbf{a})\,[\mathbb{j}_{l,n}(x;\mathbf{a})]^2 = 0. \tag{44}$$

as required.

[14] If a and b are discrete and mutually exclusive then $p(a|b) = \delta_{a,b}$, and Bayes Theorem $p(a,b) = p(a|b)\,p(b)$ becomes $p(a,b) = \delta_{a,b}\,p(b) = \delta_{a,b}\,p(a)$.

The SU(2) representation of a null vector can be 'factorized' as an outer-product of Weyl spinors. A null Faraday, which turns out also to play a prominent role in the dynamics, can be similarly factorized. Subsequently we show that nullity is automatically preserved when the dynamics is expressed in terms of Weyl spinors rather than Lorentz vectors and bi-vectors. Further, and of relevance to the above, mutual exclusion can be then enforced by requiring that products of Weyl spinors that are factors of mutually exclusive null currents do not contribute to expectation of observables. This issue is briefly revisited in Section 8. The suggestive connection with the anti-commutators of QFT is not discussed in this document however, which is primarily focused on the single particle theory.

4.3 Back reaction

If the incoming potential is insensitive to flow-line occupancy then it is no different from the ensemble of incoming potentials, in which case

$$\mathbb{A}_{\bar{l},n}(x) = \left\{\mathbb{A}_{\bar{l},n}(x)\right\} \Rightarrow \mathbb{A}_{\bar{l}}(x) = \left\{\mathbb{A}_{\bar{l}}(x)\right\}. \tag{45}$$

Sensitivity of the potential to flow-line occupancy connotes a 'back-reaction' from the larger system and is ignored here because doing so achieves the goal of this work, which is convergence with Dirac theory in Minkowski space-time. Using (45) in (39) gives

$$\not{\partial}\{\mathbb{A}_l(x)\} = \{\mathbb{F}_l(x)\}, \quad \not{\partial}\{\mathbb{F}_l(x)\} = -\beta^2\left\{\mathbb{A}_{\bar{l}}(x)\right\} \tag{46}$$

Note that the relationship (46) is exclusively between ensembles. There is no corresponding direct relationship between particular members of the ensemble $\mathbb{j}_l(x)$ and $\mathbb{A}_{\bar{l}}(x)$.[15]

5 Normal Modes

5.1 Acoustic and optical branches

Eqs. (46) are equivalent to

$$\partial^2\{\mathbb{A}_l(x)\} = -\beta^2\left\{\mathbb{A}_{\bar{l}}(x)\right\} \tag{47}$$

subject to the constraints

$$\not{\partial}\circ\{\mathbb{A}_l(x)\} = \not{\partial}\circ\left\{\mathbb{A}_{\bar{l}}(x)\right\} = 0. \tag{48}$$

Eq. (47) is a differential difference equation. The same information can be represented in a pair of homogeneous differential equations, which can be constructed with the help of an equation 'adjoint' to (47). Suppressing arguments

$$\partial^2\left\{\mathbb{A}_{\bar{l}}\right\} = -\beta^2\sum_{k=1;k\neq l}^{N}\left\{\mathbb{A}_{\bar{k}}\right\} = -\beta^2\sum_{k=1;k\neq l}^{N}\left[\{\mathbb{A}\} - \{\mathbb{A}_k\}\right] = -\beta^2\left[(N-1)\{\mathbb{A}\} - \left\{\mathbb{A}_{\bar{l}}\right\}\right]. \tag{49}$$

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[15] Each member current is delta-valued on the worldline of the charge, whereas every incoming potential - every member of $\{\mathbb{A}(x)\}$ - is a smooth function of co-dimension 1 in $\mathbb{R}^4$ on the double light-cone of its source.

Here $\{\mathbb{A}\}$ is the total ensemble potential

$$\{\mathbb{A}\} = \sum_{l=1}^{N} \{\mathbb{A}_l\} = \{\mathbb{A}_l\} + \left\{\mathbb{A}_{\bar{l}}\right\} \tag{50}$$

using which (49) can be written just in terms of $\{\mathbb{A}_l\}$ and $\left\{\mathbb{A}_{\bar{l}}\right\}$

$$\partial^2 \left\{\mathbb{A}_{\bar{l}}\right\} = -\beta^2 \left[(N-1)\{\mathbb{A}_l\} + (N-2)\left\{\mathbb{A}_{\bar{l}}\right\}\right]. \tag{51}$$

Eqs. (47) and (51) form the coupled system

$$\begin{bmatrix} \partial^2, & \beta^2 \\ (N-1)\beta^2, & \partial^2 + (N-2)\beta^2 \end{bmatrix} \begin{bmatrix} \{\mathbb{A}_l\} \\ \left\{\mathbb{A}_{\bar{l}}\right\} \end{bmatrix} = 0. \tag{52}$$

Adding the two rows gives an equation for the total adjunct potential

$$\left[\partial^2 + \kappa^2\right]\{\mathbb{A}\} = 0 \tag{53}$$

where $\kappa = \beta\sqrt{N-1}$. Subtracting the second row from $N-1$ times the first row gives

$$\left[\partial^2 - \beta^2\right]\left\{\tilde{\mathbb{A}}_l\right\} = 0 \tag{54}$$

where

$$\left\{\tilde{\mathbb{A}}_l\right\} = \{\mathbb{A}_l\} - \left\{\mathbb{A}_{\bar{l}}\right\} / (N-1) \tag{55}$$

is an anti-symmetric combination of the potential of the local charge and the potential of all other distant charges, as it acts on the local charge. The relative weights are such that the potentials of distant charges contribute coherently. The anti-symmetry is suggestive of an analogy with the optical modes of an elastic lattice, whereas $\{\mathbb{A}\}$ represents the symmetric modes of a coupled N-particle system, analogous to the acoustic modes of an elastic lattice. Eq. (53) is a Klein-Gordon equation for the total adjunct ensemble potential $\{\mathbb{A}\}$ with mass-frequency κ. Given $N \sim 10^{80}$ say, this is of order 10^{40} times the magnitude of β in (54). If κ corresponds to a known elementary particle then β must be tiny. If for example κ is the Compton frequency of the electron with wavelength 2.4×10^{-12}m, then the wavelength associated with β is of order of the present Hubble radius, and the frequency has a period of order of the Cosmological age. At frequencies much greater than this $\left\{\tilde{\mathbb{A}}_l\right\}$ behaves like a free (vacuum) potential:[16,17]

$$\partial^2 \left\{\tilde{\mathbb{A}}_l\right\} \approx 0. \tag{56}$$

Eq. (56) is a novel demonstration of the existence of endogenous quasi-vacuum modes in a DPI theory, without recourse to special boundary conditions presumed in earlier work. Examination of the connection with retarded EM radiation is outside the scope of this report, which is focused on the origin of the Dirac equation.

16 Due to the sign of β^2 in (54) the lowest non-negative energy mode has zero energy (no time variation) and a corresponding (Hubble radius) spatial variation. Note that, however small, β forces agreement between field theory and direct particle interaction on the necessity that the (free) potential satisfies the Lorenz gauge.

17 Given light-speed motion of the source, this will occur as the acceleration approaches 10^{-10} m/s^2 from above, suggestive of a connection with anomalous dispersion of velocities in the outer arms of spiral galaxies (e.g. as characterized by MOND).

5.2 Acoustic branch with no radiation

If it is known that no radiation is present, i.e. $\left\{\tilde{\mathbb{A}}_l\right\} = 0$, then (55) gives that the incoming and locally-generated adjunct ensemble potentials are proportional,

$$\{\mathbb{A}_l\} = \left\{\mathbb{A}_{\bar{l}}\right\} / (N-1) \tag{57}$$

in which case the total potential is

$$\{\mathbb{A}\} = \{\mathbb{A}_l\} + \left\{\mathbb{A}_{\bar{l}}\right\} = N\{\mathbb{A}_l\} \tag{58}$$

and the local potential $\{\mathbb{A}_l\}$ satisfies the Klein-Gordon equation (53). It follows from (57) that under these conditions (of no radiation), the local current is proportional to its own potential as

$$\{\mathbb{j}_l\} = -\kappa^2\{\mathbb{A}_l\} \tag{59}$$

and Eq. (47) now reads

$$\left[\partial^2 + \kappa^2\right]\{\mathbb{A}_l\} = 0 \tag{60}$$

with the subsidiary condition

$$\not{\partial} \circ \{\mathbb{A}_l\} = 0. \tag{61}$$

5.3 EM multivector

The Lorenz gauge constraint can be incorporated into the dynamics via the ensemble multivector

$$\{\mathbb{Q}_l\} = \kappa\{\mathbb{A}_l\} + i\{\mathbb{F}_l\} \tag{62}$$

(where $\{\mathbb{F}_l\} = \not{\partial}\{\mathbb{A}_l\}$), in which terms (60) and (61) can be combined into the 'multivector Dirac equation'

$$\left[\not{\partial} + i\kappa\right]\{\mathbb{Q}_l\} = 0. \tag{63}$$

Hereafter we refer to any linear combination of the potential and Faraday as an 'EM multivector' (to distinguish it from any other multivector containing other non-zero blades). We note in passing that in the Majorana representation (63) can be expressed entirely in terms of real quantities. Suppressing the particle label and re-writing as (63)

$$\left[\left[\not{\partial} + i\kappa\right]/i\right]\left[\{\mathbb{Q}\}/i\right] = 0$$

then

$$\left[\not{\partial} + i\kappa\right]/i = \begin{bmatrix} \frac{\partial}{\partial x} + \kappa & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} - \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + \kappa & -\frac{\partial}{\partial y} + \frac{\partial}{\partial t} & 0 \\ 0 & -\frac{\partial}{\partial y} - \frac{\partial}{\partial t} & \frac{\partial}{\partial x} + \kappa & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} + \frac{\partial}{\partial t} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + \kappa \end{bmatrix} \tag{64}$$

and, expressed in terms of contra-variant components,

$$\{\mathbb{Q}\}/i = \left\{\begin{bmatrix} \kappa A_x - E_y & -\kappa A_z + B_y & E_z - B_x & -\kappa\phi + \kappa A_y + B_z + E_x \\ -\kappa A_z - B_y & -\kappa A_x - E_y & \kappa\phi - \kappa A_y + B_z + E_x & -E_z + B_x \\ E_z + B_x & -\kappa\phi - \kappa A_y - B_z + E_x & \kappa A_x + E_y & -\kappa A_z + B_y \\ \kappa\phi + \kappa A_y - B_z + E_x & -E_z - B_x & -\kappa A_z - B_y & -\kappa A_x + E_y \end{bmatrix}\right\}. \tag{65}$$

It is established in Section 6 that $\mathbb{Q}$ transforms like the outer product of a Dirac bi-spinor with its adjoint. By contrast, each of the individual columns in $\mathbb{Q}$ in (65) do not transform like a Dirac bi-spinor, even though in any frame each of those columns satisfies a Dirac-like equation.

6 Dirac Equation

6.1 Multivector projections

Eq. (63) is a coupled first order system in the components of the potential and Faraday. The focus of this work is on the ensemble current, which can found from solutions of (63) provided the conditions described in Section 3.2 are met. If so then each flow line of each of the null components of that current is a possible - mutually exclusive - path for the local electron.

One could form a Dirac equation of sorts by right multiplication of (63) with a constant 4-vector to project onto $\mathbb{C}^4$. But $\{\mathbb{Q}_l(x)\}$ times a constant 4-vector does not transform as a bi-spinor (see below). By contrast a Lorentz invariant 4-vector (bi-spinor) description of the dynamics can be obtained from a projection of the phase-space representation of the Clifford Multivector, because in that representation there is no constraint that the projection 4-vector be constant. The dimensionality of $\{\mathbb{Q}_l(x)\}$ mandates there are 4 such independent projections that generate 4 Dirac equations, each associated with a different conserved current.

6.2 Multivector eigenvectors (Dirac bi-spinors)

Phase-space representation We suppress the particle index l, and distinguish between real-space, phase-space, and Fourier domain functions by their arguments. Using the transform convention

$$f(k) = \int d^4x e^{ik\circ x} f(x) \Rightarrow f(x) = (2\pi)^{-4} \int d^4k e^{-ik\circ x} f(k) \tag{66}$$

let

$$f(x,k) = e^{-ik\circ x} f(k) \tag{67}$$

for any function $f(k)$ so that

$$f(x) = (2\pi)^{-4} \int d^4k f(x,k). \tag{68}$$

In these terms the multivector (62) is

$$\{\mathbb{Q}(x,k)\} = \kappa\{\mathbb{A}(x,k)\} + i\{\mathbb{F}(x,k)\} = [\kappa + \mathbb{k}]\{\mathbb{A}(x,k)\} \tag{69}$$

and (63) can be written as either of [18]

$$\left[\partial\!\!\!/ + i\kappa\right]\{\mathbb{Q}(x,k)\} = i[\kappa - \mathbb{k}]\{\mathbb{Q}(x,k)\} = 0. \tag{70}$$

The second of (70) can be written

$$\mathbb{P}_-(k)\{\mathbb{Q}(x,k)\} = 0 \tag{71}$$

where $\mathbb{P}_\sigma(k) = (\kappa + \sigma\mathbb{k})/2\kappa$, $\sigma = \pm 1$, are a complementary pair of projection matrixes each with rank 2. Consequently, $\{\mathbb{Q}(x,k)\}$ has rank 2, and can be decomposed, therefore, as the sum of two outer-products of 4-component vectors in $\mathbb{C}^4$, though the form of that decomposition is constrained by conditions on $\{\mathbb{Q}(x,k)\}$ due to the reality of the underlying potential and Faraday, and – relatedly - the symmetries of their matrix representations.

Substitution of (69) into (70) gives the Klein-Gordon type condition $k^2 = \kappa^2$. The two roots can be accommodated by reduction of the dimensionality of the k-space integrations, replacing (68) with

$$\{\mathbb{Q}(x)\} = (2\pi)^{-3}\int \mathrm{d}^3k\left(\{\mathbb{Q}_+(\mathbf{k})\}e^{ik\circ x} + \{\mathbb{Q}_-(\mathbf{k})\}e^{-ik\circ x}\right) \tag{72}$$

where

$$k = \left(k^0, \mathbf{k}\right); \quad k^0 = {}_+\sqrt{\kappa^2 + \mathbf{k}^2}. \tag{73}$$

One infers from (68) that

$$\{\mathbb{Q}(x,k)\} = 2\pi\left(\{\mathbb{Q}_+(\mathbf{k})\}e^{ik\circ x} + \{\mathbb{Q}_-(\mathbf{k})\}e^{-ik\circ x}\right)\delta\left(k^0 - {}_+\sqrt{\kappa^2 + \mathbf{k}^2}\right) \tag{74}$$

Since $\{\mathbb{Q}(x,k)\}$ has rank 2, $\{\mathbb{Q}_+(\mathbf{k})\}$ and $\{\mathbb{Q}_-(\mathbf{k})\}$ must each have at least rank 1. (Because $\{\mathbb{Q}_+(\mathbf{k})\}e^{ik\circ x}$ and $\{\mathbb{Q}_-(\mathbf{k})\}e^{-ik\circ x}$ are functionally independent from the point of view of a Fourier decomposition of solutions of (70) it cannot be the case that one of these has rank 2, and the other rank 0.) Taking $\{\mathbb{Q}(x,k)\}$ to be the more fundamental physical quantity, we now seek rank 1 representations of $\{\mathbb{Q}_+(\mathbf{k})\}$ and $\{\mathbb{Q}_-(\mathbf{k})\}$ that have sufficient degrees of freedom to satisfy symmetry constraints on $\{\mathbb{Q}(x,k)\}$.

Relativistic Covariance Corresponding to a Lorentz transformation

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu; \quad L^T L = 1 \tag{75}$$

where $L = \{L^\mu{}_\nu\}$, the transformation rule for $\partial\!\!\!/$ is

$$\partial\!\!\!/ \to \partial\!\!\!/' = \mathbb{S}\partial\!\!\!/\mathbb{S}^{-1} \tag{76}$$

for a constant matrix $\mathbb{S}(L)$. One finds

$$\partial\!\!\!/ = \gamma^\mu\frac{\partial}{\partial x^\mu} \to \gamma^\nu\frac{\partial}{\partial x'^\mu} = \gamma^\nu\frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial}{\partial x^\mu} = \gamma^\nu\left(L^{-1}\right)^\mu{}_\nu\frac{\partial}{\partial x^\mu} \tag{77}$$

[18] k is $k\!\!\!/$ of the traditional Feynman slash notation.

and therefore $\mathbb{S}(L)$ is the solution of

$$\mathbb{S}(L)\gamma^{\mu}\mathbb{S}^{-1}(L) = \gamma^{\nu}\left(L^{-1}\right)^{\mu}{}_{\nu} \tag{78}$$

which up to an overall scalar factor is [62]

$$\mathbb{S}(L) = e^{[\gamma_b,\gamma_a]\omega^{ab}}; \quad \omega^{ab} = \left(g^{ab} - L^{ab}\right)/8. \tag{79}$$

Since $\not{\partial}$ is a proto-typical vector it follows that the potential must transform likewise

$$\{\mathbb{A}(x)\} \to \{\mathbb{A}'(x')\} = \mathbb{S}\{\mathbb{A}(x)\}\mathbb{S}^{-1} \tag{80}$$

and therefore

$$\{\mathbb{F}(x)\} = \not{\partial}\{\mathbb{A}(x)\} \to \{\mathbb{F}'(x')\} = \not{\partial}'\{\mathbb{A}'(x')\} = \mathbb{S}\{\mathbb{F}(x)\}\mathbb{S}^{-1}. \tag{81}$$

Consequently

$$\{\mathbb{Q}(x)\} \to \{\mathbb{Q}'(x')\}\mathbb{S}\{\mathbb{Q}(x)\}\mathbb{S}^{-1} \tag{82}$$

and (70) is invariant under Lorentz transformations:

$$\left[\not{\partial} + i\kappa\right]\{\mathbb{Q}(x)\} \to \left[\not{\partial}' + i\kappa\right]\{\mathbb{Q}'(x')\} = \mathbb{S}\left[\not{\partial} + i\kappa\right]\{\mathbb{Q}(x)\}\mathbb{S}^{-1} = 0. \tag{83}$$

It follows from (82) that $\{\mathbb{Q}(x)\}$ transforms as an outer-product of Dirac-theory bi-spinors $\psi(x)$, the transformation rule for which (see for example [62]) is $\psi'(x') = \mathbb{S}\psi(x)$. Since $\{\mathbb{Q}(x,k)\}$ has rank two it must be decomposable in $\mathrm{M}_4(\mathbb{C})$ as

$$\{\mathbb{Q}(x,k)\} = r(x,k)\overline{s}(x,k) + u(x,k)\overline{v}(x,k) \tag{84}$$

where r, s transform as Dirac bi-spinors, $\overline{s}, \overline{v}$ transform as adjoint bi-spinors, and the overbar has the traditional meaning for a bi-spinor ψ that $\overline{\psi} = \psi^{\dagger}\gamma^{0}$.

Consistent with (84), and taking into account the discussion above, we now seek a sufficient decomposition of $\{\mathbb{Q}_{+}(\mathbf{k})\}$ and $\{\mathbb{Q}_{-}(\mathbf{k})\}$ in (72) as

$$\{\mathbb{Q}_{+}(\mathbf{k})\} = r(\mathbf{k})\overline{s}(\mathbf{k}), \quad \{\mathbb{Q}_{-}(\mathbf{k})\} = u(\mathbf{k})\overline{v}(\mathbf{k}) \tag{85}$$

whereupon (72) becomes

$$\{\mathbb{Q}(x)\} = (2\pi)^{-3}\int \mathrm{d}^3k\left(r(\mathbf{k})\overline{s}(\mathbf{k})e^{ik\circ x} + u(\mathbf{k})\overline{v}(\mathbf{k})e^{-ik\circ x}\right). \tag{86}$$

We have not used the ensemble notation for the bi-spinors because $r(\mathbf{k})$ and $\overline{s}(\mathbf{k})$ in (85) for example are outer-product vector *factors* of an ensemble - they do not each represent an ensemble of bi-spinors.

Restriction to $\mathrm{Cl}_{1,3}(\mathbb{R})$ The degrees of freedom in $r, s\ \bar{s}, \bar{v}$ must be restricted to conform with intrinsic symmetries of the gamma matrixes

$$\gamma^{\mu} = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^0 \mathrm{C} \gamma^{\mu *} \mathrm{C} \gamma^0. \tag{87}$$

The first of (87) applied to a real-space potential and Faraday yield

$$\gamma^0 \left\{ \mathbb{A}^{\dagger}(x) \right\} \gamma^0 = \{\mathbb{A}(x)\} \tag{88}$$

and

$$\gamma^0 \left\{ \mathbb{F}^{\dagger}(x) \right\} \gamma^0 = \gamma^0 \left[\partial\!\!\!/ \{\mathbb{A}(x)\} \right]^{\dagger} \gamma^0 = \gamma^0 \left\{ \mathbb{A}^{\dagger}(x) \right\} \overleftarrow{\partial\!\!\!/}^{\dagger} \gamma^0 = \{\mathbb{A}(x)\} \overleftarrow{\partial\!\!\!/} = -\{\mathbb{F}(x)\} \tag{89}$$

and therefore

$$\gamma^0 \left[i \{\mathbb{F}(x)\} \right]^{\dagger} \gamma^0 = i \{\mathbb{F}(x)\}. \tag{90}$$

Applied to (62) these give

$$\{\mathbb{Q}(x)\} = \gamma^0 \left\{ \mathbb{Q}^{\dagger}(x) \right\} \gamma^0. \tag{91}$$

A similar application of the second of (87) leads to

$$\{\mathbb{Q}(x)\} = \gamma^0 \mathrm{C} \{\mathbb{Q}^*(x)\} \mathrm{C} \gamma^0. \tag{92}$$

Recalling (66), (67) and (68), the phase-space representations of $\{\mathbb{A}\}$, $\{\mathbb{F}\}$, and $\{\mathbb{Q}\}$ must have the same symmetries, and therefore

$$\{\mathbb{Q}(x,k)\} = \gamma^0 \left\{ \mathbb{Q}^{\dagger}(x,k) \right\} \gamma^0 = \gamma^0 \mathrm{C} \{\mathbb{Q}^*(x,k)\} \mathrm{C} \gamma^0. \tag{93}$$

The first of these implies that $\mathbb{Q}(x)\gamma^0$ and $\mathbb{Q}(x,k)\gamma^0$ are Hermitian. Applied to (86) the decomposition is restricted to

$$\{\mathbb{Q}(x)\} = (2\pi)^{-3} \int \mathrm{d}^3 k \left(r(\mathbf{k}) \bar{s}(\mathbf{k}) e^{ik \circ x} + s(\mathbf{k}) \bar{r}(\mathbf{k}) e^{-ik \circ x} \right). \tag{94}$$

Denoting the charge-conjugate of a bi-spinor by $\psi^c = \gamma^0 \mathrm{C} \psi^*$, the second of (93) connotes charge conjugation invariance of the whole matrix, which requires $r^c(\mathbf{k}) = s(\mathbf{k}) \Leftrightarrow r(\mathbf{k}) = s^c(\mathbf{k})$, and therefore

$$\{\mathbb{Q}(x)\} = (2\pi)^{-3} \int \mathrm{d}^3 k \left[\psi\left(\tfrac{1}{2}\mathbf{k}\right) \overline{\psi^c}\left(\tfrac{1}{2}\mathbf{k}\right) e^{ik \circ x} + \psi^c\left(\tfrac{1}{2}\mathbf{k}\right) \bar{\psi}\left(\tfrac{1}{2}\mathbf{k}\right) e^{-ik \circ x} \right] \tag{95}$$

for some bi-spinor $\psi(\mathbf{k})$. Through a change of scale of the integration (95) can be written

$$\{\mathbb{Q}(x)\} = \pi^{-3} \int \mathrm{d}^3 k \left[\psi(x,\mathbf{k}) \overline{\psi^c}(x,\mathbf{k}) + \psi^c(x,\mathbf{k}) \bar{\psi}(x,\mathbf{k}) \right] \tag{96}$$

where

$$\psi(x,\mathbf{k}) = \psi(\mathbf{k}) e^{ik_c \circ x} \tag{97}$$

and where the wave-vector is now

$$k_c = \left(k_c^0, \mathbf{k}\right); \quad k_c^0 = {}_{+}\sqrt{\kappa_c^2 + \mathbf{k}^2}, \quad \kappa_c = \kappa/2. \tag{98}$$

The subscript c alludes to the Compton frequency, which is half the frequency of the rest-frame adjunct potential. Eq. (96) implies that $\{\mathbb{Q}_-(\mathbf{k})\} = \{\mathbb{Q}_+^c(\mathbf{k})\}$, and also that (72) could be written more efficiently as

$$\{\mathbb{Q}(x)\} = \pi^{-3}\int d^3k\left(\{\mathbb{Q}(\mathbf{k})\}e^{ik\circ x} + \{\mathbb{Q}^c(\mathbf{k})\}e^{-ik\circ x}\right) \tag{99}$$

where

$$\{\mathbb{Q}^c(\mathbf{k})\} = \gamma^0 C\{\mathbb{Q}^*(\mathbf{k})\}C\gamma^0 \tag{100}$$

where

$$\{\mathbb{Q}(\mathbf{k})\} = \psi(\mathbf{k})\overline{\psi^c}(\mathbf{k}). \tag{101}$$

6.3 Dirac Equation

Applying (68) to (74), a Fourier phase factor form of the multivector differential equation (63) is

$$\left[\partial\!\!\!/ + i\kappa\right]\{\mathbb{Q}(x,k)\} = 0. \tag{102}$$

With the substitution (99) Eq. (63) can also be expressed in the form

$$\left[\partial\!\!\!/ + i\kappa\right]\{\mathbb{Q}(\mathbf{k})\}e^{ik\circ x} = 0 \tag{103}$$

with the component form of k given in (73). This is sufficient because the charge conjugate of (103) takes care of the second term in (99). Expressed instead in terms of the eigenvector decomposition (101), Eq. (103) is

$$\left[\partial\!\!\!/ + i\kappa\right]\psi(\mathbf{k})\overline{\psi^c}(\mathbf{k})e^{ik\circ x} = 0 \Rightarrow \left[\partial\!\!\!/ + i\kappa_c\right]\psi(\mathbf{k})\overline{\psi^c}(\mathbf{k})e^{ik_c\circ x} = 0. \tag{104}$$

Using (97) it follows that a sufficient condition for the satisfaction of (63) is that each phase-space component $\psi(x,\mathbf{k})$, $\forall x,\mathbf{k}$ satisfies the Dirac equation

$$\left[\partial\!\!\!/ + i\kappa_c\right]\psi(x,\mathbf{k}) = i\left[\kappa_c + \mathbb{k}_c\right]\psi(x,\mathbf{k}) = 0. \tag{105}$$

7 Dirac Currents

7.1 Electron-positron current

Solutions $\{\mathbb{Q}_l(x)\}$ of (63) can be assembled from solutions $\psi(x,\mathbf{k})$ of (105) using (96), the bi-vector and vector parts of which are the ensemble Faraday and ensemble potential, respectively. The latter is [19]

$$\{\mathbb{A}(x)\} = \frac{1}{\pi^3\kappa}\int d^3k\left[\left\langle\psi(x,\mathbf{k})\overline{\psi^c}(x,\mathbf{k})\right\rangle_1 + \left\langle\psi^c(x,\mathbf{k})\overline{\psi}(x,\mathbf{k})\right\rangle_1\right]. \tag{106}$$

[19] The Minkowski components of a Clifford vector $\mathbb{V} = \left\langle\psi\overline{\psi^c}\right\rangle_1 = V^\mu\gamma_\mu$ are $V^\mu = \overline{\psi^c}\gamma_\mu\psi$.

The ensemble potential is proportional to an ensemble of local currents through (59). Specifically

$$\{\mathbb{j}(x)\} = -\frac{2\kappa_c}{\pi^3}\int \mathrm{d}^3k \Big[\Big\langle \psi(x,\mathbf{k})\overline{\psi^c}(x,\mathbf{k})\Big\rangle_1 + \Big\langle \psi^c(x,\mathbf{k})\overline{\psi}(x,\mathbf{k})\Big\rangle_1\Big], \tag{107}$$

where we used $\kappa = 2\kappa_c$. Let us confirm that $\{\mathbb{A}(x)\}$ and therefore $\{\mathbb{j}(x)\}$ satisfy the Lorenz gauge condition. Suppressing arguments

$$\partial\circ\{\mathbb{A}\} = \frac{1}{\pi^3\kappa}\int \mathrm{d}^3k \Big\langle \partial\!\!\!/ \Big[\Big\langle \psi\overline{\psi^c}\Big\rangle_1 + \Big\langle \psi^c\overline{\psi}\Big\rangle_1\Big]\Big\rangle_0 = \frac{1}{\pi^3\kappa}\int \mathrm{d}^3k \Big[\overline{\psi^c}\overset{\leftrightarrow}{\partial\!\!\!/}\psi + \overline{\psi}\overset{\leftrightarrow}{\partial\!\!\!/}\psi^c\Big]. \tag{108}$$

This vanishes because $\partial\!\!\!/\psi = -i\kappa_c\psi$, $\partial\!\!\!/\psi^c = -i\kappa_c\psi^c$, $\overline{\psi}\overset{\leftarrow}{\partial\!\!\!/} = i\kappa_c\overline{\psi}$, and $\overline{\psi^c}\overset{\leftarrow}{\partial\!\!\!/} = i\kappa_c\overline{\psi}^c$. Hence $\{\mathbb{j}(x)\}$ is a conserved current. Due to (35) we will refer to the $\{\mathbb{j}(x)\}$ given by (107) as the electron-positron ensemble current. The overall factor $2\kappa_c/\pi^3$ can be replaced to comply with a normalization condition on the charge.

To facilitate a physical interpretation of the current we express the ψ in terms of the eigenvectors of charge conjugation. These are Majorana bi-spinors, which will be denoted here by a change of font to $\boldsymbol{\psi}$. They can be projected out of an arbitrary ψ using

$$\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k}) = \hat{\mathbb{P}}_{\sigma_e}[\psi(x,\mathbf{k})] = \frac{1}{2}[\psi(x,\mathbf{k}) + \sigma_e\psi^c(x,\mathbf{k})]. \tag{109}$$

The inverse relations are

$$\psi(x,\mathbf{k}) = \boldsymbol{\psi}_+(x,\mathbf{k}) + \boldsymbol{\psi}_-(x,\mathbf{k}), \quad \psi^c(x,\mathbf{k}) = \boldsymbol{\psi}_+(x,\mathbf{k}) - \boldsymbol{\psi}_-(x,\mathbf{k}). \tag{110}$$

Substitution of (110) into (96) gives

$$\{\mathbb{Q}(x)\} = \frac{2}{\pi^3}\int \mathrm{d}^3k \Big[\boldsymbol{\psi}_+(x,\mathbf{k})\overline{\boldsymbol{\psi}}_+(x,\mathbf{k}) - \boldsymbol{\psi}_-(x,\mathbf{k})\overline{\boldsymbol{\psi}}_-(x,\mathbf{k})\Big] \tag{111}$$

in which terms (107) is

$$\{\mathbb{j}(x)\} = -\frac{4\kappa_c}{\pi^3}\int \mathrm{d}^3k \Big[\Big\langle \boldsymbol{\psi}_+(x,\mathbf{k})\overline{\boldsymbol{\psi}}_+(x,\mathbf{k})\Big\rangle_1 - \Big\langle \boldsymbol{\psi}_-(x,\mathbf{k})\overline{\boldsymbol{\psi}}_-(x,\mathbf{k})\Big\rangle_1\Big]. \tag{112}$$

Using the defining property (109) it can be shown that outer-products of Majorana bi-spinors have vector and bi-vector parts only - all other Clifford blades vanish - as do all bi-linear combinations of the vector and bi-vector parts. These constraints can be represented by expressing the individual outer-products of the Majorana bi-spinors above in terms of electron and positron multivectors $\mathbb{Q}_{\sigma_e}$ where, suppressing arguments

$$\{\mathbb{Q}_{\sigma_e}\} = \boldsymbol{\psi}_{\sigma_e}\overline{\boldsymbol{\psi}}_{\sigma_e} = \kappa\{\mathbb{A}_{\sigma_e}\} + i\{\mathbb{F}_{\sigma_e}\}, \tag{113}$$

and where, suppressing braces, the component forms of the potential and Faraday satisfy

$$A_{\sigma_e} = \phi_{\sigma_e}\Big(1, \hat{\mathbf{E}}_{\sigma_e}\times\hat{\mathbf{B}}_{\sigma_e}\Big), \quad \mathbf{E}_{\sigma_e}\cdot\mathbf{B}_{\sigma_e} = \mathbf{B}^2_{\sigma_e} - \mathbf{E}^2_{\sigma_e} = 0. \tag{114}$$

$(\mathbf{E}_{\sigma_e}, \mathbf{B}_{\sigma_e}, \mathbf{A}_{\sigma_e})$ are mutually orthogonal therefore. Note that the individual $\{\mathbb{Q}_{\sigma_e}\}$ do not satisfy the multi-vector Dirac equation (63). Of particular relevance here is that

(114) implies that A_{σ_e} is null, and therefore that (110) is a decomposition of the total current into 2 null currents. Their further decomposition into polarized null currents is discussed below. At fixed t the ensemble $\{\mathbb{j}(x)\}$ therefore comprises 4 null currents of each charge species passing through every $\mathbf{x}$.[20]

Upon substitution of (97) into (109) the outer-products in (111) and (112) become

$$\begin{aligned}\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k})\overline{\boldsymbol{\psi}}_{\sigma_e}(x,\mathbf{k}) = &\tfrac{1}{4}\left[\psi(\mathbf{k})\overline{\psi}(\mathbf{k})+\psi^c(\mathbf{k})\overline{\psi^c}(\mathbf{k})\right]\\ &+\tfrac{\sigma_e}{4}\left[\psi^c(\mathbf{k})\overline{\psi}(\mathbf{k})e^{-2ik_c\circ x}+\psi(\mathbf{k})\overline{\psi^c}(\mathbf{k})e^{2ik_c\circ x}\right].\end{aligned} \tag{115}$$

Hence at fixed $\mathbf{k}$ the two terms in (112) each comprise an oscillatory component offset by a constant mean. Moreover, the magnitude of the mean is the same for both species. Consequently the static terms cancel upon substitution of (115) into (112), leaving

$$\{\mathbb{j}(x)\} = -\frac{\kappa_c}{\pi^3}\int \mathrm{d}^3k\left[\left\langle \psi^c(\mathbf{k})\overline{\psi}(\mathbf{k})\right\rangle_1 e^{-2ik_c\circ x}+\left\langle \psi(\mathbf{k})\overline{\psi^c}(\mathbf{k})\right\rangle_1 e^{2ik_c\circ x}\right]. \tag{116}$$

The electron-positron current (112) is purely sinusoidal therefore, as would be expected of a solution of the Klein-Gordon equation.

The time component of each current is proportional to

$$\overline{\boldsymbol{\psi}}_{\sigma_e}(x,\mathbf{k})\gamma^0\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k}) = \overline{\psi}(\mathbf{k})\gamma^0\psi(\mathbf{k})+\sigma_e\mathrm{Re}\left\{\overline{\psi}(\mathbf{k})\gamma^0\psi^c(\mathbf{k})e^{-2ik_c\circ x}\right\} \tag{117}$$

Since

$$\overline{\boldsymbol{\psi}}_{\sigma_e}(x,\mathbf{k})\gamma^0\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k}) = \boldsymbol{\psi}^\dagger_{\sigma_e}(x,\mathbf{k})\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k}) > 0 \tag{118}$$

and

$$\overline{\psi}(\mathbf{k})\gamma^0\psi(\mathbf{k}) = \psi^\dagger(\mathbf{k})\psi(\mathbf{k}) > 0 \tag{119}$$

it follows from (117) that the sign of the charge is determined solely by the sign of the static term. The two terms in (112), $\left\langle \boldsymbol{\psi}_-(x,\mathbf{k})\overline{\boldsymbol{\psi}}_-(x,\mathbf{k})\right\rangle_1$ and $-\left\langle \boldsymbol{\psi}_+(x,\mathbf{k})\overline{\boldsymbol{\psi}}_+(x,\mathbf{k})\right\rangle_1$, are the currents of opposite signed species of equal mass. Nominally these are electrons and positrons, though which term corresponds to which species depends on the sign of the overall factor.

7.2 Traditional Dirac current

The electron and positron bi-spinors $\boldsymbol{\psi}_-$ and $\boldsymbol{\psi}_+$ independently solve the Dirac equation, and contribute independently to the electron and positron currents in (112) (the cross terms vanish). In the Majorana representation they are respectively purely real and purely imaginary - or vice-versa - up to an overall phase factor. A input to this presentation of DPI is that the individual members of the current ensemble are always null. By contrast the ensemble current is generally non-null. It will be non-null in the case of (112) due to interference between the electron and positron currents.[21] A purely electron ensemble current is constrained to move at light speed, therefore.

[20] And therefore $r = 4$ in (37).

[21] I.E.: not as a result of interference between bi-spinors.

In the Majorana representation consider now the composition

$$\psi(x,\mathbf{k}) = \boldsymbol{\psi}_{-}^{(1)}(x,\mathbf{k}) + i\boldsymbol{\psi}_{-}^{(2)}(x,\mathbf{k}). \tag{120}$$

For definiteness we can take $\boldsymbol{\psi}_{-}$ to be real, from which it is inferred the second term in (120) mimics the positron contribution to the total wavefunction. The implication of the subscript however is that it contribute to the total current with the same sign as the first term, i.e. so that

$$\{\mathbb{j}_{\mathrm{Dirac}}(x)\} = \frac{4\kappa_c}{\pi^3}\int \mathrm{d}^3k\left[\left\langle \boldsymbol{\psi}_{-}^{(1)}(x,\mathbf{k})\overline{\boldsymbol{\psi}}_{-}^{(1)}(x,\mathbf{k})\right\rangle_1 + \left\langle \boldsymbol{\psi}_{-}^{(2)}(x,\mathbf{k})\overline{\boldsymbol{\psi}}_{-}^{(2)}(x,\mathbf{k})\right\rangle_1\right] \tag{121}$$

and therefore

$$\{\mathbb{j}_{\mathrm{Dirac}}(x)\} = \frac{2\kappa_c}{\pi^3}\int \mathrm{d}^3k\left\langle \psi(x,\mathbf{k})\overline{\psi}(x,\mathbf{k})\right\rangle_1 \tag{122}$$

where we used $\langle\overline{\psi^c}\psi^c\rangle_1 = \langle\overline{\psi}\psi\rangle_1$. This is the traditional Dirac current, to be compared with (107). Evidently (122) is the particle current without regard for the sign of the charge.[22] It remains conserved because the currents of the two charge species are independently conserved. Eq. (122) can also be interpreted as a purely electron current provided the sign of ϕ in (29) is associated with the mass rather than the charge (as it is in (30)). Hence the Dirac current fixes the sign of the charge in exchange for indeterminacy in the sign of the energy, swapping their status relative to the electron-positron current.

7.3 External Coupling

Eq. (122) is a non-null ensemble current of a single charge species. Appropriately interpreted there no problem with its use as a generator of exclusively electronic flow lines from solutions of the free Dirac equation. There is a problem however with the appearance of that current in the interaction

$$L_{int} = -\int \mathrm{d}^4x\{\mathbb{j}_{\mathrm{Dirac}}(x)\}\circ\mathbb{A}_{\mathrm{ext}}(x) \tag{123}$$

where $\mathbb{A}_{\mathrm{ext}}(x)$ is a vacuum potential [23] This is because (123) couples an EM potential to a current without regard for the sign of the charge carrier, or because it couples positive and negative energy states - depending on the interpretation given to $\{\mathbb{j}_{\mathrm{Dirac}}\}$. Use of (123) is the source of well-known problems with the traditional presentation of the single particle Dirac theory. Discussion of alternative forms of the interaction within a single particle theory is outside the scope of this work.

[22] Relatedly, unlike (116), Eq. (122) is not a solution of the Klein-Gordon equation.

[23] In the context of this work, more accurately it is the anti-symmetric $\left\{\widetilde{\mathbb{A}}_l\right\}$ that solves 55.

7.4 Dynamic independence of the currents

Each charge current can be further analyzed into two null independently conserved currents distinguished by their polarization.[24] A projector with the required properties (see for example [62]) is

$$\mathbb{P}_{\sigma_p} = \frac{1}{2}\left[1 + \sigma_p \gamma_5 \mathbb{n}\right]; \quad \sigma_p = \pm 1 \tag{124}$$

where $\mathbb{n}$ is any vector satisfying $n \circ k_c = 0$ and $n^2 = -1$. Denoting the projections by $\boldsymbol{\psi}_{\sigma_p,\sigma_e} = \mathbb{P}_{\sigma_p} \boldsymbol{\psi}_{\sigma_e}$, the electron-positron ensemble current (112) associated with a general solution of (105) can be decomposed into 4 null components

$$\{\mathbb{j}(x)\} = \frac{1}{\pi^3} \sum_{\sigma_p,\sigma_e=\pm 1} \sigma_e \int \mathrm{d}^3 k \left\{\mathbb{j}_{\sigma_p,\sigma_e}(x,\mathbf{k})\right\} \tag{125}$$

where

$$\left\{\mathbb{j}_{\sigma_p,\sigma_e}(x,\mathbf{k})\right\} = -4\kappa_c \sigma_e \left\langle \boldsymbol{\psi}_{\sigma_p,\sigma_e}(x,\mathbf{k}) \overline{\boldsymbol{\psi}}_{\sigma_p,\sigma_e}(x,\mathbf{k}) \right\rangle_1 . \tag{126}$$

The absence in the current of cross terms of the form $\boldsymbol{\psi}_{\sigma_p,\sigma_e}(x,\mathbf{k}) \overline{\boldsymbol{\psi}}_{\sigma'_p,\sigma'_e}(x,\mathbf{k})$ where σ'_p, σ'_e differ from σ_p, σ_e is an outcome of the properties of the charge and polarization projectors. In this work these represent 4 potentially coinciding ensembles of possible paths of a light-speed charge following the flow lines of a null potential, as described in Section 3.2. Crucial to the applicability of the method of Section 3.2 to the general case that the incoming potential and Faraday are non-null is that these currents are dynamically independent. Specifically, this requires that the Faraday of each current does not act on any of the other 3 currents, which is that a charge following the flow lines of a potential does not experience a Lorentz force from any of the other 3 potentials. Extending (113) to

$$\left\{\mathbb{Q}_{\sigma_p,\sigma_e}\right\} := \boldsymbol{\psi}_{\sigma_p,\sigma_e} \overline{\boldsymbol{\psi}}_{\sigma_p,\sigma_e} = \kappa \left\{\mathbb{A}_{\sigma_p,\sigma_e}\right\} + i \left\{\mathbb{F}_{\sigma_p,\sigma_e}\right\} \tag{127}$$

the requirement can be expressed as

$$\left\langle \left\{\mathbb{F}_{\sigma_p,\sigma_e}\right\} \left\{\mathbb{A}_{\sigma'_p,\sigma'_e}\right\} \right\rangle_1 = 0 \tag{128}$$

except perhaps when $\sigma_p = \sigma'_p$ and $\sigma_e = \sigma'_e$. It turns out that this constraint can be satisfied, and forces a particular association between the bi-vector and vector parts of each of the 4 possible outer products in (126). A proof will be given elsewhere. Here we quote the result, which is that the condition (128) is met if

$$\left\{\mathbb{F}_{\sigma_p,\sigma_e}\right\} = \not{\partial} \left\{\mathbb{A}_{\sigma_p,\bar{\sigma}_e}\right\}. \tag{129}$$

Notice that the Faraday part of the outer product in (113) does not derive from the potential part of the same outer product. Hence the Majorana bi-spinor $\boldsymbol{\psi}_+$ is the generator of the electron potential and the positron Faraday, for example.[25]

[24] We use polarization rather than spin because the electron and positron multi-vectors alone do not generate angular momentum. This follows from the decomposition (113) consistent with the remarks in 7.1, together with the fact that spin angular momentum is a Clifford pseudo-vector. Any angular momentum, if present, must be a joint property of the electron and positron multivectors therefore.

[25] Or vice-versa.

8 Superposition, anti-commutation, and wavefunction collapse

Eq. (125) is an integral superposition of an outer-product of phase-space bi-spinors, each term corresponding to a single Fourier **k**-space component of the current. The constraint that the current is null will be satisfied if each of the $\left\{\mathbb{j}_{\sigma_p,\sigma_e}(x,\mathbf{k})\right\}$ - i.e. for each possible $\sigma_p, \sigma_e, \mathbf{k}$ over all x - is mutually exclusive. Under these conditions each term in the superposition (i.e. the integrand) is a candidate for the role of *sole* contributor to a single instance current, whose relative magnitude therefore corresponds to the probability of that being the case in any single instance.

Mutual exclusion of each contribution is a consequence of a particular decomposition (of the current vector into Majorana bi-spinors) in which the contributing terms are individually null. The Majorana bi-spinors and the Dirac equation they satisfy derive from factorization of the multivector [26]. Mutual exclusion therefore applies to solutions of the Dirac equation, represented in any function space.

From the perspective of this work the freedom to choose the function space originates in the Dirac multi-vector equation, not in the Dirac equation. The distinction is important because singular value decomposition of the multivector does not generally commute with transformation of the function space. The Fourier space bi-spinors appearing in (105) differ from the Fourier transform of real-space solutions of the real space Dirac equation, for example.

This distinction can be removed by attributing the property of mutual exclusion to the bi-spinors - rather than to the null currents they generate. This requires that the bi-spinors be treated as delta-correlated in whatever function space the current is expressed. Delta-correlation can be enforced through appropriate anti-commutation rules. In a discrete function space they can be achieved simply by striking out off-diagonal terms in the density matrix. In all cases the amplitude of the remaining diagonal terms will retain their role as the probabilities of each term being the sole contribution to the current in any single instance, in that representation.

9 Summary

The Dirac Equation is shown to derive from an equation for the Clifford multivector of the time-symmetric potential and Faraday of classical direct particle electrodynamics. The probabilistic aspect is seen to be a consequence of embedding the dynamics of a single current in an ensemble of hypothetical currents. Wavefunction collapse / representation-independent eigenvalue selection is shown to be a consequence of non-linear constraints on the solutions of a linear differential equation.

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[26] More accurately: singular value decomposition.

Conflict of interest

The authors declare that they have no conflict of interest.

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