Algebra of Discrete Symmetries in the Extended Poincaré Group*

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Abstract

We begin with the comprehensible review of the basics of the Lorentz, (extended) Poincare Groups and O(3,2) and O(4,1). On the basis of the Gelfand-Tsetlin-Sokolik-Silagadze research [1-3], we investigate the definitions of the discrete symmetry operators both on the classical level, and in the secondary-quantization scheme. We studied the physical content within several bases: light-front form formulation, helicity basis, angular momentum basis, on several practical examples. The conclusion is that we have ambiguities in the definitions of the the corresponding operators P, C; T, which lead to different physical consequences.

1 The Standard Definitions.

The Lorentz Group conserves the interval $ds^2 = dx^\mu dx_\mu$ in the Minkowski space with respect to (pseudo)Euclidean rotations. The Poincaré Group includes translations in the Minkowski space. The extended Poincaré Group includes discrete transformations, the unitary C, P, and the antiunitary T in the quantum field theory (QFT). The P is the space inversion: $x^0 \to x^0$, $\mathbf{x} \to -\mathbf{x}$. The T is the time reversal: $x^0 \to -x^0$, $\mathbf{x} \to \mathbf{x}$. The C is the electric charge conjugation. It is related to the PT operation: $x^0 \to -x^0$, $\mathbf{x} \to -\mathbf{x}$. The interval is also conserved under these operations. In the QFT the eigenvalues of the combined CPT are also invariants.

While [4] presented the derivation method to obtain the field operator *ab initition*, we define the field operator [5, 6] in the pseudo-Euclidean metrics as follows:

$$\Psi(x) = \frac{1}{(2\pi)^3} \sum_h \int \frac{d^3\mathbf{p}}{2E_p} \left[u_h(\mathbf{p}) a_h(\mathbf{p}) e^{-ip\cdot x} + v_h(\mathbf{p}) b_h^\dagger(\mathbf{p}) e^{+ip\cdot x} \right] . \quad (1)$$

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Hence, the Dirac equation is:

$$[i\gamma^\mu \partial_\mu - m]\,\Psi(x) = 0\,. \tag{2}$$

At least, 3 methods of its derivation exist [7, 8, 9]:

- the Dirac one (the Hamiltonian should be linear in $\partial/\partial x^i$, and be compatible with $E_p^2 - \mathbf{p}^2 c^2 = m^2 c^4$);
- the Sakurai one (based on the equation $(E_p - \sigma\cdot\mathbf{p})(E_p + \sigma\cdot\mathbf{p})\phi = m^2\phi$);
- the Ryder one (the relation between 2-spinors at rest is $\phi_R(\mathbf{0}) = \pm\phi_L(\mathbf{0})$ and boosts).

It has solutions of the positive energies and the negative energies. The latter are reinterpreted as the antiparticles. $E_p = \sqrt{\mathbf{p}^2 + m^2}$, $c = \hbar = 1$, $g^{\mu\nu} = diag\{1,-1,-1,-1\}$. The solutions in the momentum representation are: $u_h(\mathbf{p}) = column(\phi_R^h(\mathbf{p})\ \phi_L^h(\mathbf{p}))$. Next,

$$u_h = \begin{pmatrix} \exp(+\boldsymbol{\sigma}\cdot\boldsymbol{\varphi})\phi_R^h(\mathbf{0}) \\ \exp(-\boldsymbol{\sigma}\cdot\boldsymbol{\varphi})\phi_L^h(\mathbf{0}) \end{pmatrix}\,,\ v_h(\mathbf{p}) = \gamma^5 u_h(\mathbf{p})\,, \tag{3}$$

where $cosh(\varphi) = E_p/m$, $sinh(\varphi) = |\mathbf{p}|/m$, $\hat{\varphi} = \mathbf{p}/|\mathbf{p}|$, h is the polarization index. It is shown that the parity operator can be chosen as

$$P = e^{i\alpha_s}\gamma^0 R\,,\ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{4}$$

because

$$\left[i\gamma^\mu \partial'_\mu - m\right]\Psi(x^{\mu'}) = 0\,, (change\ of\ variables)\,, \tag{5}$$

where

$$\Psi(x^{\mu'}) = \mathcal{A}\Psi(x^\mu)\,, (lineality)\,. \tag{6}$$

These conditions can be satisfied by γ^0 matrix in the Weyl basis. R can be chosen $R \equiv (\theta \to \pi - \theta, \phi \to \pi + \phi, r \to r)$. For fermions it is well known that a particle and an antiparticle have opposite eigenvalues of the parity operator in this $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz Group. In the QFT we should have:

$$U_P \psi(x) U_P^\dagger = e^{i\alpha_s}\gamma^0\psi(x')\,. \tag{7}$$

So,

$$U_P a_h(\mathbf{p}) U_P^\dagger = e^{+i\alpha_s} a_h(\mathbf{p}')\,,\ U_P b_h(\mathbf{p}) U_P^\dagger = -e^{-i\alpha_s} b_h(\mathbf{p}')\,. \tag{8}$$

The operator U_P can be constructed in the usual way, see [5] and [6]. The charge operator interchange the particle and the antiparticle. For example, in the Dirac case on the classical level:

$$u_\uparrow \to -v_\downarrow,\ u_\downarrow \to +v_\uparrow, v_\uparrow \to +u_\downarrow,\ v_\downarrow \to -u_\uparrow, \tag{9}$$

Thus, we can write thanks to E. Wigner:

$$\mathcal{C}_{1/2} = e^{i\alpha_c} \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} \mathcal{K},\ \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \,. \tag{10}$$

In the QFT we should have:

$$U_C \psi(x) U_C^\dagger = e^{i\alpha_c} C \psi^\dagger(x) \,. \tag{11}$$

So,

$$U_C a_h(\mathbf{p}) U_C^\dagger = e^{+i\alpha_c} b_h(\mathbf{p}) \,,\ U_C b_h(\mathbf{p}) U_C^\dagger = e^{-i\alpha_c} a_h(\mathbf{p}) \,. \tag{12}$$

See, however, Ref. [11], where **two** possibilities for the charge conjugation operator have been proposed.

The time reversal operator is antiunitary (see Wigner and [4]). Let us remind that the operator of hermitian conjugation does not act on c-numbers on the left side of the equation (13). This fact is conected with the properties of an antiunitary operator: $\left[V^{^T} \lambda A (V^{^T})^{-1}\right]^\dagger = \left[\lambda^* V^{^T} A (V^{^T})^{-1}\right]^\dagger = \lambda \left[V^{^T} A^\dagger (V^{^T})^{-1}\right]$.

$$\left[V^{^T}_{[1/2]} \Psi(x^\mu) (V^{^T}_{[1/2]})^{-1}\right]^\dagger = S(T) \Psi^\dagger(x''^{\mu}) \ . \tag{13}$$

We can see that C and P anticommutes in the Dirac case:

$$\{C, P\}_+ = 0 \ , P^2 = 1, \ , C^2 = 1 \,, \tag{14}$$

and $(CPT) = \pm 1$. However, we present the opposite case later, where $(CPT) = \pm i$, which is related to the commutation (anticommutation) of the C and P operators.

The Table in p. 157 of Ref. [5] gives us the properties of the scalar, 4-vector, tensor, axial-vector and pseudoscalar under these transformations in the case of the "Dirac-like-parity" definitions. However, see the next Section.

2 Anomalous Representations of the Inversion Group.

The previous Section perfectly describes the CPT properties of the charged fermions. Nevertheless, the authors of [1, 2, 10] proposed another class of representations of the full Lorentz Group long ago . As it was shown recently, it may be applied to the (anti)bosons of the opposite parities, and to the (anti)fermions of undefined parities. The latter are not the eigenstates of the parity operator, but they are the eigenstates of the charge-conjugate operator. Gelfand, Tsetlin and Sokolik noted that there exist representations of the full Lorentz Group of the anomalous parity. Originally, this concept was intended to be applied to explanation of the decay of $K-$ mesons.

Briefly, the examples are: one can note that in the $(1/2, 1/2)$ representation (or for x^μ) the operators of the space inversion (t_{01}), the time reversal (t_{10})

and the combined space-time inversion (t_{11}) are commutative. They form the inversion group together with the unit element. Let us search the projective representations of the Lorentz group combined with the discrete group. As opposed to the usual case, $t_{01}t_{10} = t_{11}$, $t_{10}t_{11} = t_{01}$, $t_{01}t_{11} = t_{10}$, for instance, one can drop the condition of the commutativity, and one can form the projective rep. with $T_{10}T_{01} = -T_{11}$, or $T_{11}T_{11} = -1$, see the full Table in [1]. They noted that there are **two** normal-parity (in their notation) and **two** anomalous parity reps. for (bi)spinors. Then, they extended the concept of the anomalous parity to any rep. of the proper Lorentz Group characterized by the numbers (k_0, k_1) and $(-k_0, k_1)^*$. When

$$[T_{i'k'}, T_{i''k''}]_+ = 0\,, \tag{15}$$

this is the case of the anomalous parity (later, this was confirmed by Nigam and Foldy [12]). G. Sokolik noted that this concept is related to the concept of the 5-dim representations of the proper orthogonal group with pseudo-Euclidean metrics. For example,

$$T_{10} \sim H_{54} = \exp(i\pi I_{54}/2)\ ,\ T_{11} \sim H_{43}H_{21} = \exp(i\pi I_{43})\exp(i\pi I_{21})\,, \tag{16}$$

$$T_{01} = T_{11}T_{10}\,. \tag{17}$$

T_{10}, T_{01}, T_{11} leave invariant the extended 8-component Dirac equation only (compare with [13] and [14]):

$$\Gamma_\mu \partial^\mu \psi + m\psi = 0\,,\ \Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} \tag{18}$$

They claimed relations to the concepts (known in that time):

- Istopic Spin;
- Fusion Theory;
- the non-linear Heisenberg Theory

were mentioned. The corresponding matrix representations of the anomalous-parity representations have been presented:

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, T_{10} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, T_{11} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{19}$$

and

$$T_{01} = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, T_{10} = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}, T_{11} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{20}$$

Later Wigner [10] repeated their results at the Istanbul School lectures (1962). And Silagadze [3] rediscovered and applied this possibility in 1992. The conclusion of these papers is: we noted that both new versions of the reps. of the full Lorentz Group (commuting spinor and anticommuting boson representations) lead to the doubling of the dimensionality of the $\psi-$ function.

3 The Self/Anti-self Charge Conjugate States.

The content of this Section contains the material of [11]. The conclusions are: we have constructed another explicit example of the BWW-GTS theory. The matter of physical dynamics connected with this mathematical construct should be solved in future as depended on what gauge interactions with potential fields do we introduce [14] and what experimental setup do we choose.

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