Singular continuous phase for Schrödinger operators over circle diffeomorphisms with a singularity

Saša Kocić*

Dept. of Math., University of Mississippi, P. O. Box 1848, University, MS 38677-1848

October 31, 2020

Abstract

We consider a class of Schrödinger operators — referred to as Schrödinger operators over circle maps — that generalize one-frequency quasiperiodic Schrödinger operators, with a base dynamics given by an orientation-preserving homeomorphism of a circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, instead of a circle rotation. In particular, we consider Schrödinger operators over circle diffeomorphisms with a single singular point where the derivative has a jump discontinuity (circle maps with a break) or vanishes (critical circle maps). We show that in a two-parameter region — determined by the geometry of dynamical partitions and α — the spectrum of Schrödinger operators over every sufficiently smooth such map, is purely singular continuous, for every α-Hölder-continuous potential V. For $\alpha = 1$, the region extends beyond the corresponding region for the Almost Mathieu operator. As a corollary, we obtain that for every sufficiently smooth such map, with an invariant measure μ and with rotation number in a set $\mathcal{S}$ depending on the class of the considered maps, and μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator has a purely continuous spectrum, for every Hölder-continuous potential V. For circle maps with a break, this set includes some Diophantine numbers with a Diophantine exponent δ, for any $\delta > 1$.

1 Introduction

We consider a class of Schrödinger operators $H = H(T, V, x)$ on a space of square-summable sequences $\ell^2(\mathbb{Z})$, defined by

$$(Hu)_n := u_{n-1} + u_{n+1} + V(T^n x)u_n, \qquad u \in \ell^2(\mathbb{Z}), \tag{1.1}$$

*Email: skocic@olemiss.edu

where $V : \mathbb{T}^1 \to \mathbb{R}$ is a potential function, $T : \mathbb{T}^1 \to \mathbb{T}^1$ is an orientation-preserving homeomorphism of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, and $x \in \mathbb{T}^1$. For an overview of recent results on spectral theory of Schröedinger operators over dynamically defined potentials the reader is directed, e.g., to [4] (see also [13]).

When the rotation number ρ of T is irrational, this class of operators is a natural generalization of the one-frequency quasiperiodic Schrödinger operators for which $T = R_\rho$, where $R_\rho : x \mapsto x + \rho \mod 1$ is the rigid rotation. In this case, T is topologically semi-conjugate to the rotation, i.e., there is a continuous map $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that $T \circ \varphi = \varphi \circ R_\rho$. Hence, if ρ is irrational, $T^n \circ \varphi = \varphi \circ R_\rho^n$, for every $n \in \mathbb{N}$, and we have $H(T, V, x) = H(R_\rho, V \circ \varphi, y)$, where $x = \varphi(y)$, $y \in \mathbb{T}^1$.

In some cases, the spectral properties of $H(T, V, x)$ can be deduced directly from the spectral properties of the corresponding Schrödinger operator over R_ρ, using this identity. In particular, if T is an analytic circle diffeomorphism with rotation number satisfying Yoccoz's $\mathcal{H}$ arithmetic condition [25], it follows from the theory of Herman [10] and Yoccoz [25] that φ is analytic, and the spectral properties of $H(T, V, x)$, with V analytic [12] follow directly from Avila's global theory of one-frequency quasiperiodic Schrödinger operators over rotations [1]. Although for circle diffeomorphisms T with Liouville rotation numbers the conjugacy to the corresponding rotation can even be singular, certain spectral properties of $H(T, V, x)$, with potentials of the same regularity, are still analogous to those of the one-frequency quasiperiodic Schrödinger operators over rotations with the same rotation numbers.

In this paper, we initiate the study of Schrödinger operators over more general circle maps. We are interested in rigidity properties of these systems, i.e., properties of these systems that are the same in a large class. We are interested in the spectral phase diagram of Schrödinger operators over circle maps and, in particular, the singular continuous phase. Such a phase diagram emerges in one of most studied examples — the almost Mathieu family — which corresponds to $T = R_\rho$ and $V(x) = \lambda \cos(2\pi x)$. It was conjectured by Jitomirskaya [11] (Problem 8 therein), and proved by Avila, You and Zhou [2], that the almost Mathieu operator has a purely singular continuous spectrum in the region $0 < L(E) < \beta$ and that $L(E) = \beta$ is the boundary between continuous and pure point spectrum, for almost all $x \in \mathbb{T}^1$, where $L(E)$ is the Lyapunov exponent and

$$\beta = \beta(\rho) := \limsup_{n\to\infty} \frac{\ln k_{n+1}}{q_n}, \tag{1.2}$$

with k_n and $\frac{p_n}{q_n}$, $n \in \mathbb{N}$, being the partial quotients and rational convergents of $\rho \in (0,1)\backslash\mathbb{Q}$ (see section 2.2). It was shown in [12] that, in the same region, the spectrum is singular continuous for Schrödinger operators $H(T, V, x)$ with Lipschitz continuous potentials V over C^{1+BV}-smooth circle diffeomorphisms T, for almost all $x \in \mathbb{T}^1$, suggesting that $L(E) = \beta$ could be the boundary between continuous and pure point spectrum, in this case as well. A natural question to ask is if the latter holds for Schrödinger operators over general circle maps, for sufficiently regular potentials. The main result of this paper provides a negative answer to that question.

Here, we focus on spectral properties of Schrödinger operators over circle diffeomorphisms with a singularity, i.e., smooth circle diffeomorphisms with a single singular point where the derivative vanishes (critical circle maps) or has a jump discontinuity (circle maps with a break). Over the last couple of decades, these maps played a central role in the rigidity theory of circle maps — an extension of Herman's theory on the linearization of circle diffeomorphisms [9, 14, 15, 18].

In the case of circle maps with a break at x_{br}, the type of singularity is characterized by the *size of the break*

$$c := \sqrt{\frac{T'_-(x_{\mathrm{br}})}{T'_+(x_{\mathrm{br}})}} \neq 1. \tag{1.3}$$

In the case of critical circle maps, we assume that in some open neighborhood of the critical point x_c, the derivative of the map is of the order $|x - x_c|^{\gamma-1}$, i.e., $T'(x) = \Theta(|x - x_c|^{\gamma-1})$, for some $\gamma > 1$, and the type of singularity is characterized by the *order of the critical point* γ. We call a diffeomorphism with such a critical point, or with a break point, a diffeomorphism with a singularity. A diffeomorphism with a singularity is said to be C^r-smooth if it is C^r-smooth outside the singularity point.

We begin with a few more definitions. A number $\rho \in \mathbb{R}\backslash\mathbb{Q}$ is called *Diophantine* of class $D(\delta)$, for some $\delta \geq 0$, if there exists $\mathcal{C} > 0$ such that $|\rho - p/q| > \mathcal{C}/q^{2+\delta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The set of all Diophantine numbers is denoted by $D := \cup_{\delta\geq 0} D(\delta)$ and the complement of this set in $\mathbb{R}\backslash\mathbb{Q}$ is the set of Liouville numbers. If $\rho \in D(\delta) \cap (0,1)$, then $\limsup_{n\to\infty} \frac{\ln k_{n+1}}{\ln q_n} \leq \delta$ and, thus, $\beta(\rho) = 0$. We call a Liouville number $\rho \in (0,1)$ *exponentially Liouville* if $\beta(\rho) > 0$ and *super Liouville* if $\beta(\rho) = \infty$. The set of all super Liouville numbers will be denoted by $\mathcal{S}_{\mathcal{L}}$.

Let

$$\beta_e = \beta_e(\rho) := \limsup_{n\to\infty} \frac{k_{2n+1}}{q_{2n}}, \quad \text{and} \quad \beta_o = \beta_o(\rho) := \limsup_{n\to\infty} \frac{k_{2n}}{q_{2n-1}}. \tag{1.4}$$

The following theorem, which is a corollary of the main results of this paper holds for $r > 2$ and $\mathcal{S} = \mathcal{S}_{\mathrm{br}} \cup \mathcal{S}_{\mathcal{L}}$, in the case of circle maps with break, and for $r \geq 3$ and $\mathcal{S} = \mathcal{S}_{\mathcal{L}}$, in the case of critical circle maps. Here, $\mathcal{S}_{\mathrm{br}}$ is the set of $\rho \in (0,1)\backslash\mathbb{Q}$ such that $\beta_{\mathrm{br}} = \infty$, where $\beta_{\mathrm{br}} = \beta_{\mathrm{br}}(\rho) := \beta_e$ if the size of the break $c < 1$, and $\beta_{\mathrm{br}} = \beta_{\mathrm{br}}(\rho) := \beta_o$ if the size of the break $c > 1$. Since the rotation number ρ of T is irrational, T is uniquely ergodic [7]. We will denote by μ the unique invariant probability measure of T.

Theorem 1.1 *For every C^r-smooth circle diffeomorphism with a singularity T, with rotation number $\rho \in \mathcal{S}$ and the invariant measure μ, and μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator $H(T, V, x)$ has a purely continuous spectrum, for every Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$.*

Remark 1 For C^{1+BV}-smooth circle diffeomorphisms and a set $\mathcal{S} = \mathcal{S}_{\mathcal{L}}$, an analogous claim was proved in [12]. A map is said to be C^{1+BV}-smooth if it is C^1-smooth with the logarithm of the derivative of bounded variation.

Remark 2 The set $\mathcal{S} = \mathcal{S}_{\mathrm{br}} \cup \mathcal{S}_{\mathcal{L}}$ of rotation numbers for which the theorem holds in the case of circle maps with a break contains not only Liouville numbers but also some Diophantine numbers of class $D(\delta)$, for any $\delta > 1$.

Ergodic Schrödinger operators are intimately related to a family of cocycles — dynamical systems associated with each eigen-equation $Hu = Eu$. In the case of Schrödinger operators over circle maps with irrational rotation numbers, the cocycle is given by

$$(T, A) : (x, y) \mapsto (Tx, A(x, E)y), \tag{1.5}$$

where $A \in \mathrm{SL}(2, \mathbb{R})$, $x \in \mathbb{T}^1$, $y \in \mathbb{R}^2$. If $u = (u_n)_{n\in\mathbb{Z}}$ is a sequence satisfying $Hu = Eu$, then

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n(x, E) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \text{where} \quad A_n(x, E) := \begin{pmatrix} E - V(T^n x) & -1 \\ 1 & 0 \end{pmatrix} \tag{1.6}$$

is the transfer matrix. Thus,

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = P_n(x, E) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}, \tag{1.7}$$

where $P_n(x, E) := \prod_{i=n-1}^{0} A_i(x, E) = A_{n-1}(x, E) \dots A_0(x, E)$.

We define the Lyapunov exponent

$$L(E) := \lim_{n\to\infty} \int L_n(x, E)\, d\mu, \quad \text{where} \quad L_n(x, E) := \frac{1}{n} \ln \|P_n(x, E)\|. \tag{1.8}$$

Due to submultiplicativity of $P_n(x, E)$, $L(E)$ exists. Since T is ergodic, by Kingman's ergodic theorem, for almost every x,

$$L(E) = L(x, E) := \lim_{n\to\infty} \frac{1}{n} \ln \|P_n(x, E)\|. \tag{1.9}$$

Different components of the spectrum of an operator $H(T, V, x)$ are denoted by σ_{ac} (absolutely continuous), σ_{sc} (singular continuous) and σ_{pp} (pure point). We also denote by $S_{pp}(x)$ the set of eigenvalues of $H(T, V, x)$, with $\sigma_{pp}(x) = \overline{S_{pp}(x)}$. Finally, we set $\mathcal{H} = \ell^2(\mathbb{Z})$, $\mathcal{H}_{sc}(x)$ the corresponding singular continuous subspace, and $P_A(x)$ the operator of spectral projection on a Borel set A, corresponding to $H(T, V, x)$.

For circle maps with a break, we have the following claim.

Theorem 1.2 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be a $C^{2+\varepsilon}$-smooth ($\varepsilon > 0$) circle diffeomorphism with a break of size $c \neq 1$, a rotation number $\rho \in (0,1)\backslash\mathbb{Q}$, and an invariant measure μ. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$, $\alpha \in (0,1]$, we have*

(i) $S_{pp}(x) \cap \{E : 0 \leq L(E) < \alpha \max\{\frac{1}{2}\beta_{\rm br}|\ln c|, 2\beta\}\} = \emptyset$,

(ii) $P_{\{E:0<L(E)<\alpha \max\{\frac{1}{2}\beta_{\rm br}|\ln c|, 2\beta\}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

For critical circle maps, we have the following claim.

Theorem 1.3 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be any C^r-smooth critical circle map, $r \geq 3$, with a rotation number $\rho \in (0,1)\backslash\mathbb{Q}$, and an invariant measure μ. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$, $\alpha \in (0,1]$, we have*

(i) $S_{pp}(x) \cap \{E : 0 \leq L(E) < 2\alpha\beta\} = \emptyset$,

(ii) $P_{\{E:0<L(E)<2\alpha\beta\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Remark 3 The regions in the $(\beta, L(E))$ plane with purely singular continuous spectrum in Theorem 1.2 and Theorem 1.3 extend beyond the corresponding region in Theorem 1.5 of [12] for circle diffeomorphisms and, for $\alpha = 1$, beyond the corresponding region for the almost Mathieu family (Theorem 1.1 of [2]).

Theorem 1.2 and Theorem 1.3 can be stated in a unified way, and the main result of this paper can be formulated as follows. Let

$$\delta_{\max} := \limsup_{n\to\infty} \frac{|\ln \ell_n|}{q_n}, \tag{1.10}$$

where $\ell_n = \min_{I\in\mathcal{P}_{n+1}, I\subset\Delta_0^{(n-1)}} |\tau_n(I)|$ is the length of the smallest renormalized interval of partition $\mathcal{P}_{n+1}$ inside the fundamental interval $\Delta_0^{(n-1)}$ of partition $\mathcal{P}_n$ (see section 2.2).

Theorem 1.4 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be any C^r-smooth, $r \geq 3$, circle diffeomorphism with a singularity, with an irrational rotation number $\rho \in (0,1)$, and an invariant measure μ. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$, $\alpha \in (0,1]$, we have*

(i) $S_{pp}(x) \cap \{E : 0 \leq L(E) < \alpha\delta_{\max}\} = \emptyset$,

(ii) $P_{\{E:0<L(E)<\alpha\delta_{\max}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Remark 4 This theorem can be extended to circle diffeomorphisms with finitely many critical or break points.

Remark 5 It seems reasonable to expect that for Schrödinger operators over sufficiently smooth circle maps, in a large class of maps including circle diffeomorphisms with singularities, for μ-almost all $x \in \mathbb{T}^1$, and sufficiently regular potentials, the boundary between the continuous and pure point spectrum is given by $L(E) = \delta_{\max}$, i.e., that the spectrum is pure point with exponentially decaying eigenfunctions for $L(E) > \delta_{\max}$.

The proofs of these theorems use tools of both spectral theory of Schrödinger operators and one-dimensional circle dynamics. In the next section, we state a sharp version of Gordon's theorem, and introduce dynamical partitions of a circle and renormalizations of circle maps that play an important role in our analysis. In section 3, we define two sets of full invariant measure for circle maps with a break, and prove Theorem 1.2. In section 4, we define a set of full invariant measure for critical circle maps, and prove Theorem 1.3. In section 5, we give a proof of Theorem 1.4.

2 Preliminaries

2.1 A criterion for the absence of eigenvalues

In this section, we state a sharp version [12] of a theorem of Gordon [8] that has been used to prove absence of point spectra of one-dimensional operators since the pioneering work of Avron and Simon [3]. Such a sharp version was used in [2] to establish the singular continuous phase for the almost Mathieu operator.

Consider a Schrödinger operator H on $\ell^2(\mathbb{Z})$ given by the action on $u \in \ell^2(\mathbb{Z})$, as

$$(Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n. \tag{2.1}$$

As in (1.6), we can define the transfer matrix $A_n(E)$ and, as in (1.7), the n-step transfer-matrix $P_n(E) = \prod_{i=n-1}^{0} A_i(E)$. Let also $P_{-n}(E) = \prod_{i=-n}^{-1} (A_i(E))^{-1}$. Let

$$\Lambda(E) := \limsup_{|n|\to\infty} \frac{\ln \|P_n(E)\|}{n}. \tag{2.2}$$

Clearly, for bounded V, $\Lambda(E) < \infty$, for every E.

Theorem 2.1 ([12]) *Assume that there exists $\beta > 0$, and an increasing sequence of positive integers q_n diverging to infinity, such that the sequence $\{V(n)\}_{n\in Z}$ in* (2.1) *satisfies*

$$\max_{0\le j<q_n} |V(j) - V(j \pm q_n)| \le e^{-\beta q_n}. \tag{2.3}$$

If $\beta > \Lambda(E)$, then E is not an eigenvalue of operator (2.1).

Proof. We give the proof of Theorem 2.1 for completeness of the presentation. Since E is fixed, we will suppress it from the notation. Taking into account that $P_{-q_n} = \prod_{i=0}^{q_n-1} A_{i-q_n}^{-1}$ $= (\prod_{i=q_n-1}^{0} A_{i-q_n})^{-1} = (P_{q_n}^{(-q_n)})^{-1}$, and $P_{2q_n} = \prod_{i=2q_n-1}^{0} A_i = (\prod_{i=q_n-1}^{0} A_{i+q_n})P_{q_n} = P_{q_n}^{(q_n)} P_{q_n}$, where $P_n^{(k)} := \prod_{i=n-1}^{0} A_{i+k}$, and applying the telescoping identity[1], to $P_{q_n}^{-1}$ and $(P_{q_n}^{(-q_n)})^{-1}$, and P_{q_n} and $P_{q_n}^{(q_n)}$, respectively, we obtain that, for any $\epsilon > 0$, and sufficiently large n, we have

$$\|P_{-q_n} - P_{q_n}^{-1}\| < e^{(\Lambda-\beta+\epsilon)q_n}, \tag{2.4}$$

$$\|P_{2q_n} v - P_{q_n}^2 v\| < e^{(\Lambda-\beta+\epsilon)q_n} \|P_{q_n} v\|. \tag{2.5}$$

Assume there is a decaying u such that $Hu = Eu$. Let $v = (u_0, u_{-1})^T$ and assume $\|v\| = 1$. Then, for sufficiently large n we have $\max\{\|P_{q_n} v\|, \|P_{-q_n} v\|, \|P_{2q_n} v\|\} < 1/2$. Since, by the characteristic equation, $P_{q_n} - \operatorname{Tr} P_{q_n} \mathrm{I} + P_{q_n}^{-1} = 0$, using (A.2) (assuming $\epsilon < \beta - \Lambda$) and applying the characteristic equation to v, we obtain $|\operatorname{Tr} P_{q_n}| < 1$, for n large enough. Applying another form of the characteristic equation, $P_{q_n}^2 - \operatorname{Tr} P_{q_n} P_{q_n} + \mathrm{I} = 0$, again to v and using (2.5), we obtain, for large enough n, $\|P_{2q_n} v\| > 1/2$, which leads to a contradiction. **QED**

Consider the Schrödinger operator (2.1) with $V_n = V(T^n x)$ where $V : \mathbb{T}^1 \to \mathbb{R}$ is a bounded real-valued function on the circle and T is an orientation-preserving homeomorphism of a circle with an irrational rotation number ρ. Let the Lyapunov exponent $L(E)$ be defined as in (1.8). We then have

Theorem 2.2 *Assume that for some $x \in \mathbb{T}^1$, $C > 0$ and $\bar{\beta} > 0$, there is a sequence of positive integers $q_n \to \infty$ such that*

$$\sup_{0 \le i < q_n} |V_{i \pm q_n}(x) - V_i(x)| < C e^{-\bar{\beta} q_n}. \tag{2.6}$$

If $L(E) < \bar{\beta}$, then E is not an eigenvalue of the Schrödinger operator $H(T, V, x)$.

Proof. In order to apply Theorem 2.1, it suffices to prove $\limsup_{|n|\to\infty} \frac{\ln \|P_n(E)\|}{n} \le L(E)$. This follows from a result of Furman [6]. **QED**

For a sequence $q_n \to \infty$, let

$$\hat{\beta} = \hat{\beta}(x) := \liminf_{n\to\infty} \frac{\ln(\sup_{0 \le i < q_n} |x_i - x_{i \pm q_n}|)^{-1}}{q_n}, \tag{2.7}$$

where $x_i = T^i x$.

Let $\sigma_{pp}, P_A, \mathcal{H}, \mathcal{H}_{sc}$ be as in Theorem 1.2 and Theorem 1.3.

---

[1] $\hat{P}_n - \widetilde{P}_n = \sum_{i=0}^{n-1} \hat{A}_{n-1} \dots \hat{A}_{i+1} (\hat{A}_i - \widetilde{A}_i) \widetilde{A}_{i-1} \dots \widetilde{A}_0$

Theorem 2.3 *Let $V : \mathbb{T}^1 \to \mathbb{R}$ be a α-Hölder continuous real-valued function on the circle, with $\alpha \in (0,1)$. Then, we have*

(i) $S_{pp}(x) \cap \{E : 0 \leq L(E) < \alpha\hat{\beta}\} = \emptyset$,

(ii) $P_{\{E:0<L<\alpha\hat{\beta}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Proof. It suffices to prove part (i) of the claim, i.e., to exclude the point spectrum. Part (ii) of the claim then follows from Kotani's theory [19, 20, 21], x-independence of the absolutely continuous spectrum [22], and the minimality of T, since the set $\{E : L(E) > 0\}$ does not support any absolutely continuous spectrum.

If $L < \alpha\hat{\beta}$, then $v_i = V(T^i x)$ satisfy the assumption (2.6) of Theorem 2.2 for any $\bar{\beta}$ satisfying $L < \bar{\beta} < \alpha\hat{\beta}$. The claim follows. **QED**

In order to prove Theorem 1.2, and Theorem 1.3, we need appropriate bounds on $\hat{\beta}(x)$.

2.2 Dynamical partitions of a circle and renormalization

The quantity $\hat{\beta}(x)$ involves the information about the geometry of the dynamical partitions of a circle. These partitions are obtained by using the continued fraction expansion of the rotation number $\rho \in (0,1)$ of the circle map T. Every irrational $\rho \in (0,1)$ can be written uniquely as

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}} =: [k_1, k_2, k_3, \dots], \tag{2.8}$$

with an infinite sequence of partial quotients $k_n \in \mathbb{N}$. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$, obtained by the finite truncations of the continued fraction expansion (2.8). It is well-known that p_n/q_n form a sequence of best rational approximations of an irrational ρ, i.e., there are no rational numbers, with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the dynamical partitions of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with an irrational rotation number ρ, we start with an arbitrary point $x_0 \in \mathbb{T}^1$, and consider the semi-orbit $x_i = T^i x_0$, with $i \in \mathbb{N}$. The subsequence $(x_{q_n})_{n\in\mathbb{N}}$, indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, is called the sequence of dynamical convergents. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents $(x_{q_n})_{n\in\mathbb{N}}$, for the rigid rotation R_ρ has the property that its subsequence with n odd approaches x_0 from the left and the subsequence with n even approaches x_0 from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$. We also define $\Delta_i^{(n)} = T^i(\Delta_0^{(n)})$. Certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterations of a map T, cover the whole circle without intersecting each other except possibly at the end points, and form the n-th dynamical partition of the circle

$$\mathcal{P}_n := \{T^i(\Delta_0^{(n-1)}) : 0 \leq i < q_n\} \cup \{T^i(\Delta_0^{(n)}) : 0 \leq i < q_{n-1}\}. \tag{2.9}$$

Intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ are called the fundamental intervals of $\mathcal{P}_n$. These partitions are nested, in the sense that intervals of partition $\mathcal{P}_{n+1}$ are obtained by dividing intervals of partition $\mathcal{P}_n$ into finitely many intervals.

The n-th renormalization of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with rotation number ρ, with respect to partition-defining point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$, obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. If τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}\,. \tag{2.10}$$

If we identify x_0 with zero, then τ_n is just multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on $\mathbb{T}^1$.

In the following, we will use the singularity point (i.e., the break point x_{br}, in the case of circle maps with a break, or the critical point x_c, in the case of critical circle maps) as the partition-defining point x_0.

3 Schrödinger operators over circle maps with a break

3.1 Renormalizations of circle maps with a break

A C^r-smooth circle diffeomorphism (map) with a break is a map $T : \mathbb{T}^1 \to \mathbb{T}^1$, for which there exists $x_{\mathrm{br}} \in \mathbb{T}^1$ such that $T \in C^r([x_{\mathrm{br}}, x_{\mathrm{br}} + 1])$; $T'(x)$ is bounded from below by a positive constant on $[x_{\mathrm{br}}, x_{\mathrm{br}} + 1]$; the one-sided derivatives of T at x_{br} are such that the size of the break,

$$c := \sqrt{\frac{T'_-(x_{\mathrm{br}})}{T'_+(x_{\mathrm{br}})}} \neq 1. \tag{3.1}$$

The following properties of renormalizations of $C^{2+\varepsilon}$-smooth circle maps with a break, with $\varepsilon \in (0, 1)$, will be crucial to prove Theorem 1.2.

Let $V := \mathrm{Var}_{x \in \mathbb{T}^1} \ln T'(x) < \infty$.

(A) $|\ln(T^{q_n})'(x)| \leq V$, for all $x \in \mathbb{T}^1$ (at points where the derivative has breaks, both left and right derivatives are considered);

(B) There exists $K_1 > 0$ such that $\|f_n\|_{C_2} \leq K_1$, for all $n \in \mathbb{N}$;

(C) There exists $K_2 > 0$ such that $f_n'(x) \geq K_2$, for $x \in [-1, 0]$, for all $n \in \mathbb{N}$;

(D) There exists $K_3 > 0$ such that, for sufficiently large even n, if $c < 1$, and odd n, if $c > 1$, $f_n''(x) \leq -K_3$, for $x \in [-1, 0]$;

(E) There exists $K_4 > 0$ such that, for sufficiently large even n, if $c > 1$, and odd n, if $c < 1$, $f_n''(x) \geq K_4$, for $x \in [-1, 0]$.

Estimate (A), that we will refer to as Denjoy's lemma, has been proven in [17, 23]. Estimates (B), (C) and (D) have been proven in [14].

From the estimates proved in [14], we also have the following. Let $a_n = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}$ and $c_n = c^{(-1)^n}$.

Proposition 3.1 *There exists $\lambda \in (0,1)$ such that $f_n'(-1) - c_n^{-1} = \mathcal{O}(a_n + \lambda^n)$ and $f_n'(0) - c_n = \mathcal{O}(a_n + \lambda^n)$.*

We will also formulate and use the following lemma that is a generalization of a lemma by Yoccoz [5]. Yoccoz's lemma applies to C^3-smooth negative Schwarzian derivative diffeomorphisms (see section 4.1), and does not apply to renormalizations of circle maps with a break, which approach fractional linear transformations. In the following lemma, negative Schwarzian derivative condition is replaced by conditions (ii) and (iii). We give a proof of this lemma in the appendix. Let $k \in \mathbb{N}$ and let $\Delta_1, \Delta_2, \dots, \Delta_{k+1}$ be consecutive closed intervals on an interval or a circle.

Lemma 3.2 *Let $I = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k$ and let $f : I \to \Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{k+1}$ be a $C^{2+\varepsilon}$-smooth diffeomorphism, $\varepsilon \in (0,1)$, satisfying $f(\Delta_i) = \Delta_{i+1}$. Assume that there exist constants $K, K', K'' > 0$ such that*

(i) $\|f\|_{C^2} \leq K$;

(ii) *the set $B_{K'} := \{z \in I : f(z) - z \leq K'\}$ is either an open interval or empty;*

(iii) *$f''(z) \geq K''$, for every $z \in B_{K'}$.*

If $|\Delta_1|, |\Delta_k| \geq \sigma|I|$, for some $\sigma > 0$, then there exists a constant $C > 1$, such that

$$C^{-1}\frac{1}{\min\{i, k+1-i\}^2} \leq \frac{|\Delta_i|}{|I|} \leq C\frac{1}{\min\{i, k+1-i\}^2}. \tag{3.2}$$

3.2 Concave renormalization graphs and set E of full measure

In this section, we construct a set of full invariant measure for which we have appropriate control on the distances of dynamical convergents, i.e, control of the quantity $\hat{\beta}$ in (2.7), in the case of circle maps with a break. The crucial facts behind these constructions are that the graphs of the renormalizations f_n of circle maps with a break, for sufficiently large n, alternate between being convex and concave and that, in the concave case, the lengths of the intervals of the next level partition $\mathcal{P}_{n+1}$, inside a fundamental interval $\Delta_0^{(n-1)}$ of dynamical partition $\mathcal{P}_n$, grow exponentially near the end points of this interval, as the distance from these points increases as follows from Proposition 3.1.

Let $(\sigma_n)_{n\in\mathbb{N}}$, be an increasing subsequence of $2\mathbb{N}$, if $c<1$, or an increasing subsequence of $2\mathbb{N}-1$, if $c>1$, such that the corresponding sequence $(k_{\sigma_n+1})_{n\in\mathbb{N}}$ of partial quotients diverges to infinity. In this section, we assume that such a subsequence exists.

The following proposition provides estimates on the derivatives of the concave renormalizations near the end points of the renormalization interval $[-1,0]$.

Proposition 3.3 *For every $\epsilon>0$, and sufficiently large $n\in\mathbb{N}$, $|f'_{\sigma_n}(x)-c_{\sigma_n}^{-1}|\le\epsilon$, for $x\in[-1,-1+\Theta(\epsilon)]$, and $|f'_{\sigma_n}(x)-c_{\sigma_n}|\le\epsilon$, for $x\in[-\Theta(\epsilon),0]$.*

Proof. It follows directly from Proposition 3.1, since a_{σ_n} decreases exponentially in k_{σ_n+1}. **QED**

Using this proposition, we can obtain estimates on the number of iterates of renormalizations in constant size intervals near the end points, and the size of the smallest interval of partition $\mathcal{P}_{\sigma_n+1}$ inside $\mathcal{P}_{\sigma_n}$.

Proposition 3.4 *For every $\epsilon>0$, if*

$$\begin{aligned} N_1 &= \mathrm{card}\{\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})\subset[-1,-1+\epsilon)|i=0,\dots,k_{\sigma_n+1}-1\},\\ N_2 &= \mathrm{card}\{\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})\subset(-\epsilon,0]|i=0,\dots,k_{\sigma_n+1}-1\}, \end{aligned} \tag{3.3}$$

then $N_1=\frac{1}{2}k_{\sigma_n+1}+\mathcal{O}(\epsilon)k_{\sigma_n+1}$ and $N_2=\frac{1}{2}k_{\sigma_n+1}+\mathcal{O}(\epsilon)k_{\sigma_n+1}$.

Proof. To be specific, let us assume that $N_1>N_2$; the proof in the opposite case is similar. From Proposition 3.3, we have

$$|\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}})|\le\Theta((c+\Theta(\epsilon))^{N_1}), \tag{3.4}$$

and

$$|\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+(k_{\sigma_n+1}-1)q_{\sigma_n}})|\ge\Theta((c-\Theta(\epsilon))^{N_2}). \tag{3.5}$$

Since, by the Denjoy estimate (A), $|\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}})|=\Theta(|\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+(k_{\sigma_n+1}-1)q_{\sigma_n}})|)$, we have $N_1-N_2=\mathcal{O}(\epsilon)k_{\sigma_n+1}$. Since $N_1+N_2=k_{\sigma_n+1}$, and the number of intervals $\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})\not\subset$

$[-1,-1+\epsilon)\cup(-\epsilon,0]$, for $i=0,\ldots,k_{\sigma_n+1}-1$, is bounded by a constant, the claim follows. **QED**

Corollary 3.5 *For every $\epsilon>0$, and sufficiently large $n\in\mathbb{N}$,*

$$\Theta((c_{\sigma_n}-\epsilon)^{\frac{1}{2}(1+\Theta(\epsilon))k_{\sigma_n+1}}) \leq \min_{0\leq i\leq k_{\sigma_n+1}-1} |\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})| \leq \Theta((c_{\sigma_n}+\epsilon)^{\frac{1}{2}(1-\Theta(\epsilon))k_{\sigma_n+1}}). \tag{3.6}$$

Let $\epsilon>0$. Let $\eta_n\in(0,1/2)$, $n\in\mathbb{N}$. For $n\in\mathbb{N}$, let

$$\mathcal{I}_{n,0} := \left\{ I\in\mathcal{P}_{\sigma_n+1} \mid I\subset \Delta_0^{(\sigma_n-1)}\backslash\Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)|\leq (c_{\sigma_n}+\epsilon)^{\eta_n k_{\sigma_n+1}} \right\}. \tag{3.7}$$

Let

$$E_{n,0} := \bigcup_{I\in\mathcal{I}_{n,0}} I, \quad \text{and} \quad E_{n,i} := T^i(E_{n,0}), \quad \text{for} \quad i=1,\ldots,q_{\sigma_n}-1. \tag{3.8}$$

We define

$$E_n := \bigcup_{i=0}^{q_{\sigma_n}-1} E_{n,i}, \tag{3.9}$$

and

$$E := \limsup_{n\to\infty} E_n = \bigcap_{n\geq 1}\bigcup_{j\geq n} E_j. \tag{3.10}$$

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence such that the series $\sum_{n=1}^{\infty}\ln(2\eta_n)$ diverges. It suffices to take $\eta_n=\eta\in(0,1/2)$. In particular, $\eta_n k_{\sigma_n+1}\to\infty$, as $n\to\infty$.

Proposition 3.6 *For sufficiently large $n\in\mathbb{N}$, $\mu(E_n)\geq 1-2\eta_n$.*

Proof. For sufficiently large n, the number of the elements I of partition $\mathcal{P}_{\sigma_n+1}$ inside of $\Delta_0^{(\sigma_n-1)}\backslash\Delta_0^{(\sigma_n+1)}$, that do not belong to $E_{n,0}$ is smaller than $2\eta_n k_{\sigma_n+1}-2$. Otherwise, there exists $C_0>0$ such that, for $\epsilon>0$ not larger than that of Proposition 3.3, a $\Theta(\epsilon)$-neighborhood of at least one of the end points of $[-1,0]$ contains at least $\eta_n k_{\sigma_n+1}-C_0-1$ rescaled intervals $\tau_{\sigma_n}(I)$ with $I\in\mathcal{P}_{\sigma_n+1}$ and $I\subset\Delta_0^{(\sigma_n-1)}\backslash\Delta_0^{(\sigma_n+1)}$, but $I\notin\mathcal{I}_{n,0}$. Here, we have also used the fact that the number of such intervals I with $\tau_{\sigma_n}(I)\cap(-1+\Theta(\epsilon),-\Theta(\epsilon))\neq\emptyset$ is less than $2C_0$, for some constant $C_0>0$. Since the length of these rescaled intervals increases exponentially in these ϵ-neighborhoods near the end points -1 and 0, as one moves away from the end points, with rate at least $c_{\sigma_n}^{-1}-\epsilon$, the length of the largest of them would be at least

$$(c_{\sigma_n}+\epsilon)^{\eta_n k_{\sigma_n+1}}(c_{\sigma_n}^{-1}-\epsilon)^{\eta_n k_{\sigma_n+1}-C_0-1} = \frac{(1+(c_{\sigma_n}^{-1}-c_{\sigma_n})\epsilon-\epsilon^2)^{\eta_n k_{\sigma_n+1}}}{(c_{\sigma_n}^{-1}-\epsilon)^{C_0+1}} > \Theta(\epsilon), \tag{3.11}$$

for sufficiently large n, if $\epsilon > 0$ has been chosen small enough, which leads to a contradiction.

Since the partition $\mathcal{P}_{\sigma_n}$ consists of q_{σ_n} "large" intervals $\Delta_i^{(\sigma_n-1)} = T^i(\Delta_0^{(\sigma_n-1)})$, for $i = 0, \dots, q_{\sigma_n} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n-1)})$ and q_{σ_n-1} "small" intervals $\Delta_i^{(\sigma_n)} = T^i(\Delta_0^{(\sigma_n)})$, for $i = 0, \dots, q_{\sigma_n-1} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n)})$, and since the interval $\Delta_0^{(\sigma_n-1)}$ consists of the union of k_{σ_n+1} disjoint (except at the end points) intervals $\Delta_{q_{\sigma_n-1}+iq_{\sigma_n}}^{(\sigma_n)} \in \mathcal{P}_{\sigma_n+1}$, for $i = 0, \dots, k_{\sigma_n+1} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n)})$, and $\Delta_0^{(\sigma_n+1)} \subset \Delta_{q_{\sigma_n+1}}^{(n)}$, we have that the invariant measure of the complement of E_n is

$$\mu(E_n^c) \leq (2\eta_n k_{\sigma_n+1} - 2) q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}) + q_{\sigma_n} \mu(\Delta_0^{(\sigma_n+1)}) + q_{\sigma_n-1} \mu(\Delta_0^{(\sigma_n)}), \tag{3.12}$$

and, hence,

$$\mu(E_n^c) \leq 2\eta_n k_{\sigma_n+1} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}) \leq 2\eta_n q_{\sigma_n+1} \mu(\Delta_0^{(n)}) \leq 2\eta_n. \tag{3.13}$$

Here, we have also used that $q_{\sigma_n-1} \leq q_{\sigma_n}$. The claim follows. **QED**

Proposition 3.7 $\mu(E) = 1$.

Proof. Each "large" interval of partition $\mathcal{P}_i$ is partitioned into k_{i+1} "large" intervals and one "small" interval of partition $\mathcal{P}_{i+1}$. Each "small" interval of partition $\mathcal{P}_i$ is a "large" interval of partition $\mathcal{P}_{i+1}$. This partitioning occurs in an identical way as the partitioning of the whole circle $\mathbb{T}^1$, which is the only interval of partition $\mathcal{P}_0$. For a fixed $m \in \mathbb{N}$, the ratio

$$\frac{\mu(E_n^c \cap \Delta_0^{(\sigma_m)})}{\mu(\Delta_0^{(\sigma_m)})} = |\tilde{E}_{\tilde{n}}^c| \tag{3.14}$$

where $\tilde{E}_{\tilde{n}}$ is defined analogously to (3.9) and (3.8) for the rotation $R_{\tilde{\rho}}$, with the rotation number $\tilde{\rho} = [k_{\sigma_m+2}, k_{\sigma_m+3}, \dots]$, $\tilde{n} = n - m$, $\tilde{\sigma}_{\tilde{n}} = \sigma_n - \sigma_m$, and with

$$\tilde{\mathcal{I}}_{\tilde{n},0} := \{\Delta_{\tilde{q}_{\tilde{\sigma}_{\tilde{n}}-1}+i\tilde{q}_{\tilde{\sigma}_{\tilde{n}}}} : \Delta_{q_{\sigma_n-1}+iq_{\sigma_n}}^{(\sigma_n)} \in \mathcal{I}_{n,0},\ 0 \leq i \leq k_{\sigma_n+1}\}, \tag{3.15}$$

for all $\tilde{n} \in \mathbb{N}$.

Following the same reasoning as at the end of the proof of Proposition 3.6, we obtain that the Lebesgue measure of the complement of $\tilde{E}_{\tilde{n}}$, $|\tilde{E}_{\tilde{n}}^c| \leq 2\eta_n$, for $\tilde{n} \in \mathbb{N}$, this immediately gives us, for a fixed $m \in \mathbb{N}$,

$$\mu(E_n^c \cap E_m^c) = |\tilde{E}_{\tilde{n}}^c| \mu(E_m^c) \leq 2\eta_n \mu(E_m^c), \tag{3.16}$$

and, thus,

$$\mu(\cup_{j\geq n} E_j) = 1 - \mu(\cap_{j\geq n} E_j^c) \geq 1 - \prod_{j\geq n} (2\eta_j). \tag{3.17}$$

If the sequence η_j is such that the series $\sum_{j=n}^{\infty} \ln(2\eta_j)$ diverges, $\mu(\cup_{j\geq n} E_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

3.3 Distance of dynamical convergents

In the following, we consider the class C^{1+BV} of orientation-preserving homeomorphisms of a circle T, C^1-smooth outside a singularity point $\chi_0 \in \mathbb{T}^1$, with an irrational rotation number and bounded variation $V := \mathrm{Var}_{\xi\in\mathbb{T}^1} \ln T'(\xi) < \infty$. In particular, C^1-smooth circle maps with a break belong to this class. The following proposition holds for all intervals $I_0 \subset \Delta_0^{(n-1)}$ such that $I_0 \in \mathcal{P}_{n+1}$, and the corresponding intervals $I_i = T^i(I_0)$, $i \in \mathbb{Z}$. The point that defines the partitions $\mathcal{P}_n$ is chosen to be the singularity point χ_0.

Proposition 3.8 *If T is a C^{1+BV} orientation-preserving circle homeomorphism with a singularity at $\chi_0 \in \mathbb{T}^1$, with an irrational rotation number, there exists $C_1 > 0$ such that $|I_i| \le C_1 |\Delta_i^{(n-1)}| \frac{|I_0|}{|\Delta_0^{(n-1)}|}$, for all $i = 0, \dots, q_n - 1$, and all $n \in \mathbb{N}$.*

Proof. For $i = 0, \dots, q_n - 1$, there exist $\zeta_{i-1} \in I_{i-1} \subset \Delta_{i-1}^{(n-1)}$ and $\xi_{i-1} \in \Delta_{i-1}^{(n-1)}$ such that

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} = \frac{|T(I_{i-1})|}{|T(\Delta_{i-1}^{(n-1)})|} = \frac{T'(\zeta_{i-1})}{T'(\xi_{i-1})} \frac{|I_{i-1}|}{|\Delta_{i-1}^{(n-1)}|}. \tag{3.18}$$

This implies the estimate

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \le \left(1 + \frac{|T'(\zeta_{i-1}) - T'(\xi_{i-1})|}{T'(\xi_{i-1})}\right) \frac{|I_{i-1}|}{|\Delta_{i-1}^{(n-1)}|}. \tag{3.19}$$

By iterating this inequality, we obtain that, for some $\zeta_j, \xi_j \in \Delta_j^{(n-1)}$,

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \le \prod_{j=0}^{i-1} \left(1 + \frac{|T'(\zeta_j) - T'(\xi_j)|}{\min_{\xi\in\mathbb{T}^1} T'(\xi)}\right) \frac{|I_0|}{|\Delta_0^{(n-1)}|}. \tag{3.20}$$

Using the obvious inequality $1 + x \le e^x$, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \le \exp\left(\sum_{j=0}^{i-1} \frac{|T'(\zeta_j) - T'(\xi_j)|}{\min_{\xi\in\mathbb{T}^1} T'(\xi)}\right) \frac{|I_0|}{|\Delta_{i-1}^{(n-1)}|}. \tag{3.21}$$

Since, for $i = 0, \dots, q_n - 1$, the intervals $\Delta_i^{(n-1)}$ do not overlap except possibly at the end points, we have

$$\sum_{j=0}^{q_n-1} |T'(\zeta_j) - T'(\xi_j)| \le \max_{\xi\in\mathbb{T}^1} T'(\xi) \sum_{j=0}^{q_n-1} |\ln T'(\zeta_j) - \ln T'(\xi_j)| \le V \max_{\xi\in\mathbb{T}^1} T'(\xi), \tag{3.22}$$

where $V = \mathrm{Var}_{\xi\in\mathbb{T}^1} \ln T'(\xi)$. Since T' is bounded both from below and from above by positive constants, the claim follows. **QED**

Let $l_n = \max_{\xi\in\mathbb{T}^1} |T^{q_n}\xi - \xi|$. If T is a C^{1+BV} orientation-preserving circle homeomorphism, the Denjoy estimate (A) implies (see Lemma 2 in [23]) that, for some $\bar{C} > 0$,

(F) $l_n \leq \bar{C}\bar{\lambda}^n$, where $\bar{\lambda} = \frac{1}{1+e^{-2V}}$.

Proposition 3.9 *If T is a $C^{2+\varepsilon}$-smooth ($\varepsilon > 0$) circle diffeomorphism with a break of size $c \in \mathbb{R}^+\backslash\{1\}$, then there exists $C_2 > 0$ such that, for all $x \in E$, there are infinitely many $n \in \mathbb{N}$ such that*

$$|T^{q_{\sigma_n}} x - x| \leq C_2 |\Delta_j^{(\sigma_n - 1)}| (c_{\sigma_n} + \epsilon)^{\eta_n k_{\sigma_n+1}}, \tag{3.23}$$

where $\Delta_j^{(\sigma_n-1)}$ is an element of partition $\mathcal{P}_{\sigma_n}$ containing x.

Proof. For every $x \in E$, there are infinitely many n, such that $x \in E_n$. Furthermore, there exists an element I_j of partition $\mathcal{P}_{\sigma_n+1}$ inside $E_{n,j} \subset \Delta_j^{(\sigma_n-1)}$, for some $j = 0, \ldots, q_{\sigma_n} - 1$, such that $x \in I_j$. It follows from the definition of $E_{n,0}$ and Proposition 3.8 that there exists $\chi \in E_{n,j}$, such that $I_j = [\chi, T^{q_{\sigma_n}}\chi]$ and $|I_j| \leq C_1 |\Delta_j^{(\sigma_n-1)}|(c_{\sigma_n} + \epsilon)^{\eta_n k_{\sigma_n+1}}$. Therefore,

$$|x - \chi| \leq |T^{q_{\sigma_n}}\chi - \chi| \leq C_1 |\Delta_j^{(\sigma_n-1)}| (c_{\sigma_n} + \epsilon)^{\eta_n k_{\sigma_n+1}}. \tag{3.24}$$

Since, by the mean value theorem, there exists $\zeta \in I_i$ such that

$$T^{q_{\sigma_n}} x \;=\; T^{q_{\sigma_n}}\chi + (T^{q_{\sigma_n}})'(\zeta)(x - \chi), \tag{3.25}$$

using the Denjoy estimate (A) and the first inequality in (3.24), we obtain the following estimate

$$|T^{q_{\sigma_n}} x - x| \leq (T^{q_{\sigma_n}})'(\zeta)|x - \chi| + |T^{q_{\sigma_n}}\chi - \chi| + |x - \chi| \leq (e^V + 2)|T^{q_{\sigma_n}}\chi - \chi|. \tag{3.26}$$

The claim now follows using the second inequality in (3.24). **QED**

Let $x_i = T^i x$ and let $I_i := [x_{i-q_n}, x_i]$, if n is even, and $I_i := [x_i, x_{i-q_n}]$, if n is odd. Let $\chi_0 \in \mathbb{T}^1$, $\chi_j = T^j \chi_0$, and let $\Delta_j^{(n-1)}(\chi_0) := [T^{q_{n-1}}\chi_j, \chi_j]$, if n is even, and $\Delta_j^{(n-1)}(\chi_0) := [\chi_j, T^{q_{n-1}}\chi_j]$, if n is odd.

Proposition 3.10 *If T is a C^{1+BV} orientation-preserving circle homeomorphism with a singularity at $\chi_0 \in \mathbb{T}^1$, with an irrational rotation number $\rho \in (0,1)$, and $x \in \Delta_j^{(n-1)}(\chi_0)$, then there exists $C_3 \geq 1$ such that*

$$|I_i| \leq C_3 |\Delta_i^{(n-1)}(\chi_{j-q_n})| \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|}, \tag{3.27}$$

for all $i = 0, \ldots, q_n - 1$.

Proof. It follows from the mean value theorem that, for $i = 0, \ldots, q_n - 1$, $i \neq q_n - j$, there exist $\xi_i \in \Delta_i^{(n-1)}(\chi_{j-q_n}) \cup \Delta_i^{(n)}(\chi_{j-q_n})$ and $\zeta_i \in \Delta_i^{(n-1)}(\chi_{j-q_n})$, such that

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} = \frac{|T^{-1}(I_{i+1})|}{|T^{-1}(\Delta_{i+1}^{(n-1)}(\chi_{j-q_n}))|} = \frac{|I_{i+1}|}{|\Delta_{i+1}^{(n-1)}(\chi_{j-q_n})|} \frac{T'(\zeta_i)}{T'(\xi_i)}. \tag{3.28}$$

This implies the estimate

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} \le \frac{|I_{i+1}|}{|\Delta_{i+1}^{(n-1)}(\chi_{j-q_n})|}\left(1+\frac{|T'(\zeta_i)-T'(\xi_i)|}{|T'(\xi_i)|}\right). \tag{3.29}$$

For $i = q_n - j$, the same estimates hold, just that ξ_i is not necessarily a point in $\Delta_i^{(n-1)}(\chi_{j-q_n}) \cup \Delta_i^{(n)}(\chi_{j-q_n})$, but just some point $\xi_i \in \mathbb{T}^1$. Namely, if I_{q_n-j} contains the singularity point χ_0, I_{q_n-j} and I_{q_n-j+1} can be divided into two subintervals, such that the ratios of lengths of the corresponding subintervals equals the values of T' at some points in these subintervals. Therefore, the ratio $|I_{q_n-j+1}|/|I_{q_n-j}|$ is between the minimum and maximum value of T' and such a value is achieved at some point ξ_{q_n-j} on the circle.

By iterating the latter inequality, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} \le \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|}\exp\left(\sum_{k=i}^{q_n-1}\frac{|T'(\zeta_k)-T'(\xi_k)|}{\min_{\xi\in\mathbb{T}^1}|T'(\xi)|}\right). \tag{3.30}$$

Since the intervals $\Delta_i^{(n-1)}(\chi_{j-q_n})$, for $i = 0, \ldots, q_n - 1$, belong to the same partition of a circle, for $k = i, \ldots, q_n - 1$, we obtain

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} \le \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|}\exp\left(\frac{\max_{\xi\in\mathbb{T}^1}|T'(\xi)|}{\min_{\xi\in\mathbb{T}^1}|T'(\xi)|}3V\right). \tag{3.31}$$

Factor 3 appears by using again the first inequality in (3.22), using the triangle inequality taking into account all possible orderings of the points ζ_k and ξ_k (e.g. $\zeta_{i+q_{n-1}} < \xi_i < \xi_{i+q_{n-1}} < \zeta_i$), and estimating the term

$$|\ln T'(\zeta_{q_n-j}) - \ln T'(\xi_{q_n-j})| \le |\ln T'(\zeta_{q_n-j}) - \ln T'(\xi^\star_{q_n-j})| + |\ln T'(\xi^\star_{q_n-j}) - \ln T'(\xi_{q_n-j})|, \tag{3.32}$$

where $\xi^\star_{q_n-j}$ is any point in $\Delta_0^{(n-1)}(\chi_0)$. The claim follows. **QED**

Proposition 3.9, Proposition 3.10 and Denjoy estimate (A) imply the following lemma.

Lemma 3.11 *If T is $C^{2+\varepsilon}$-smooth ($\varepsilon > 0$) circle diffeomorphism with a break of size $c \in \mathbb{R}^+\backslash\{1\}$, with an irrational rotation number $\rho \in (0,1)$, then there exists $C_4 > 0$ such that, for all $x \in E$, there are infinitely many $n \in \mathbb{N}$ such that, for all $i = 0, \ldots, 2q_{\sigma_n} - 1$,*

$$|x_i - x_{i-q_{\sigma_n}}| \le C_4 l_{\sigma_n-1}(c_{\sigma_n} + \epsilon)^{\eta_n k_{\sigma_n+1}}. \tag{3.33}$$

Proof. For $i = q_{\sigma_n}$, the claim holds directly from Proposition 3.9, with $C_4 \ge C_2$. Proposition 3.9 and Proposition 3.10 together imply (3.33) for $i = 0, \ldots, q_{\sigma_n} - 1$, with $C_4 \ge C_2C_3$. Using the Denjoy estimate (A), the bound (3.33) can be extended to $i = q_{\sigma_n} + 1, \ldots, 2q_{\sigma_n} - 1$, with $C_4 \ge C_2C_3e^V$, since $|x_{i+q_{\sigma_n}} - x_i| \le e^V|x_i - x_{i-q_{\sigma_n}}|$, for $i = 1, \ldots, q_{\sigma_n} - 1$. **QED**

3.4 Convex renormalization graphs and set $\mathfrak{E}$ of full measure

In this section, we construct another set of full invariant measure for which we have appropriate control on the distances between points of an orbit and their dynamical convergents, i.e, control of the quantity $\hat{\beta}$ in (2.7), for circle maps with a break.

Let $(\sigma_n)_{n\in\mathbb{N}}$, be an increasing subsequence of $2\mathbb{N}-1$, if $c<1$, or an increasing subsequence of $2\mathbb{N}$, if $c>1$, such that the corresponding sequence k_{σ_n+1} of partial quotients diverges to infinity. In this section, we assume that such a subsequence exists. Let $(\eta_n)_{n\in\mathbb{N}}$ be any sequence of positive numbers converging to zero such that the sequence $\eta_n k_{\sigma_n+1}$ diverges to infinity as well, and $\frac{\ln \eta_n}{q_{\sigma_n}}$ converges to zero, as $n\to\infty$. Consider partitions $\mathcal{P}_n$ defined with the partitions defining point χ_0 being the break point x_{br}.

Proposition 3.12 *If T is $C^{2+\varepsilon}$-smooth circle map ($\varepsilon>0$) with a break of size $c\neq 1$ and irrational rotation number, then there exists a constant $\mathfrak{C}>1$, such that, for sufficiently large n,*

$$\mathfrak{C}^{-1}\frac{1}{\min\{i+1,k_{\sigma_n+1}-i\}^2}\le \tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})\le \mathfrak{C}\frac{1}{\min\{i+1,k_{\sigma_n+1}-i\}^2}, \tag{3.34}$$

for $i=0,\dots,k_{\sigma_n+1}-1$.

Proof. For sufficiently large n, renormalizations f_n of $C^{2+\varepsilon}$-smooth circle maps with a break, and intervals $\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})$, for $i=0,\dots,k_{\sigma_n+1}-1$, satisfy the assumptions of Lemma 3.2. Clearly $\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+(i+1)q_{\sigma_n}})=f_n(\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})$ and if follows from property (C) that, for sufficiently large n, renormalizations f_n are $C^{2+\varepsilon}$-smooth circle diffeomorphisms on $[-1,0]\supset\cup_{i=0}^{k_{\sigma_n+1}-1}\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+iq_{\sigma_n}})$. It follows from the Denjoy estimate (A) that there exists $\sigma>0$ such that the lengths of $\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}})$ and $\tau_{\sigma_n}(\Delta^{(\sigma_n)}_{q_{\sigma_n-1}+(k_{\sigma_n+1}-1)q_{\sigma_n}})$ are of the same order and at least σ, due to property (E). Condition (i) follows from property (B). Convexity property (E) assures conditions (ii) and (iii). The claim follows directly from the assertion of this lemma. **QED**

For each $n\in\mathbb{N}$, let

$$\mathfrak{E}_{n,0}:=\bigcup_{I\in J_{n,0}} I,\quad J_{n,0}:=\left\{I\in\mathcal{P}_{\sigma_n+1}|I\subset\Delta_0^{(\sigma_n-1)}\setminus\Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)|\le\frac{1}{(\eta_n k_{\sigma_n+1})^2}\right\}, \tag{3.35}$$

and let

$$\mathfrak{E}_{n,i}:=T^i(\mathfrak{E}_{n,0}),\quad \text{for}\quad i=1,\dots,q_{\sigma_n}-1. \tag{3.36}$$

We define

$$\mathfrak{E}_n:=\bigcup_{i=0}^{q_{\sigma_n}-1}\mathfrak{E}_{n,i}, \tag{3.37}$$

and

$$\mathfrak{E} := \limsup_{n\to\infty} \mathfrak{E}_n = \bigcap_{n\geq 1} \bigcup_{j\geq n} \mathfrak{E}_j. \tag{3.38}$$

Proposition 3.13 $\mu(\mathfrak{E}) = 1$.

Proof. It follows from Proposition 3.12 that, for sufficiently large n, the number of the elements I of partition $\mathcal{P}_{\sigma_n+1}$ inside of $\Delta_0^{(\sigma_n-1)}$, that do not belong to $\mathfrak{E}_{n,0}$ is bounded from above by $C_5\eta_n k_{\sigma_n+1}$, for some $C_5 > 0$. Since the invariant measure of the intervals $\tau_{\sigma_n}^{-1}([f_{\sigma_n}^{i-1}(-1), f_{\sigma_n}^{i}(-1)])$ is independent of i and equal to $\mu(\Delta_0^{(\sigma_n)})$, for $i = 1, \ldots, k_{\sigma_n+1}$, and $\Delta_0^{(\sigma_n+1)} \subset \tau_{\sigma_n}^{-1}([f_{\sigma_n}^{i-1}(-1), f_{\sigma_n}^{i}(-1)])$, for $i = k_{\sigma_n+1}+1$, we have

$$\mu(\mathfrak{E}_{n,0})/\mu(\tau_{\sigma_n}^{-1}([-1,0])) \geq 1 - \frac{C_5\eta_n k_{\sigma_n+1}\mu(\Delta_0^{(\sigma_n)})}{k_{\sigma_n+1}\mu(\Delta_0^{(\sigma_n)}) + \mu(\Delta_0^{(\sigma_n+1)})} \geq 1 - C_5\eta_n. \tag{3.39}$$

By the invariance of the measure μ, $\mu(\mathfrak{E}_{n,i})/\mu(\Delta_i^{(\sigma_n-1)}) \geq 1 - C_5\eta_n$. Since

$$\sum_{i=0}^{q_{\sigma_n}-1} \mu(\Delta_i^{(\sigma_n-1)}) + \sum_{i=0}^{q_{\sigma_n-1}-1} \mu(\Delta_i^{(\sigma_n)}) = q_{\sigma_n}\mu(\Delta_0^{(\sigma_n-1)}) + q_{\sigma_n-1}\mu(\Delta_0^{(\sigma_n)}) = 1, \tag{3.40}$$

$q_{\sigma_n-1} \leq q_{\sigma_n}$ and $\mu(\Delta_0^{(\sigma_n)}) = \mu(\tau_{\sigma_n}^{-1}([-1, f_{\sigma_n}(-1)]))$, we have

$$\mu(\mathfrak{E}_n) \geq (1 - C_5\eta_n)\frac{k_{\sigma_n+1}}{k_{\sigma_n+1}+1}. \tag{3.41}$$

Since $\mu(\cup_{j\geq n}\mathfrak{E}_j) \geq \mu(\mathfrak{E}_i)$, for any $i \geq n$, and $\mu(\mathfrak{E}_i) \to 1$ as $i \to \infty$, it follows that $\mu(\cup_{j\geq n}\mathfrak{E}_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

Repeating the steps of the previous section, analogously to Lemma 3.11, we can prove the following.

Lemma 3.14 *If T is $C^{2+\varepsilon}$-smooth circle map with a break ($\varepsilon > 0$) with an irrational rotation number $\rho \in (0,1)$, then there exists $C_6 > 0$ such that, for all $x \in \mathfrak{E}$, there are infinitely many $n \in \mathbb{N}$ such that, for all $i = 0, \ldots, 2q_{\sigma_n} - 1$,*

$$|x_i - x_{i-q_{\sigma_n}}| \leq C_6 l_{\sigma_n-1}\frac{1}{(\eta_n k_{\sigma_n+1})^2}. \tag{3.42}$$

3.5 Singular continuous phase

Proof of Theorem 1.2. If $L(E) < \alpha\max\{\frac{1}{2}\beta_{\rm br}|\ln c|, 2\beta\}$, then either $L(E) < \frac{1}{2}\alpha\beta_{\rm br}|\ln c|$ or $L(E) < 2\alpha\beta$. Assume first that $L(E) < \frac{1}{2}\alpha\beta_{\rm br}|\ln c|$. Then the rotation number ρ is such that $\beta_{\rm br} > 0$, and there is an increasing sequence σ_n, of even numbers if $c < 1$, or of odd numbers if $c > 1$, such that $\beta_{\rm br} = \lim_{n\to\infty} \frac{k_{\sigma_n+1}}{q_{\sigma_n}}$. Let $\epsilon > 0$ and $\eta < 1/2$ be such that $L(E) < \alpha\eta\beta_{\rm br}|\ln(\min\{c, c^{-1}\} + \epsilon)|$ and let $\eta_n \in (\eta, 1/2)$, for $n \in \mathbb{N}$. We use this ϵ and these sequences to construct the set E, as in section 3.2. By Proposition 3.7, $\mu(E) = 1$. For every $x \in E$, by Lemma 3.11, there are infinitely many n, such that estimate (3.33) holds. This implies $\hat{\beta} \geq \eta\beta_{\rm br}|\ln(\min\{c, c^{-1}\} + \epsilon)|$. Hence, $L(E) < \alpha\hat{\beta}$, and the claim follows from Theorem 2.3.

If $L(E) < 2\alpha\beta$, then $\beta > 0$, and there is an increasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of either odd or even numbers such that $\beta = \lim_{n\to\infty} \frac{\ln k_{\sigma_n+1}}{q_{\sigma_n}}$. If $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of even numbers if $c < 1$, or of odd numbers if $c > 1$, then $\beta_{\rm br} = \infty$ and the claim holds for the set E, as discussed above. We assume that $(\sigma_n)_{n\in\mathbb{N}}$ is an increasing sequence of odd numbers if $c < 1$, or of even numbers if $c > 1$. In that case, we choose a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero such that $\eta_n k_{\sigma_n+1}$ diverges to infinity, and $\frac{\ln \eta_n}{q_{\sigma_n}}$ converges to zero, as $n \to \infty$. We use these sequences to construct a set of full measure $\mathfrak{E}$ as in section 3.4. For every $x \in E$, by Lemma 3.14, there are infinitely many n, such that estimate (3.42) holds. This implies $\hat{\beta} \geq 2\beta$. Hence, $L(E) < \alpha\hat{\beta}$, and the claim again follows from Theorem 2.3. **QED**

4 Schrödinger operators over critical circle maps

4.1 Renormalizations of critical circle maps

A C^r-smooth critical circle maps is a C^r-smooth orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, for which there exists a point $x_c \in \mathbb{T}^1$ such that $T'(x_c) = 0$; $\mathrm{Var}_{\xi\in I} \ln T'(\xi) < \infty$, for every compact interval $I \subset \mathbb{T}^1$ such that $x_c \notin I$; and $A|t|^{\gamma-1} \leq T'(x_c+t) \leq B|t|^{\gamma-1}$, for some $A, B, \gamma, \varepsilon > 0$, and every $|t| < \varepsilon$.

To prove Theorem 1.3, we will use some properties of critical circle maps that follow from real a priori bounds. Let T be a C^3-smooth critical circle map with an irrational rotation number. The following estimates have been proved in [5].

(*a*) There exist constants $\varkappa_1, \varkappa_2 \in (0, 1)$ such that, for all $n \in \mathbb{N}$,

$$\varkappa_1 \leq \frac{\Delta_i^{(n+1)}}{\Delta_i^{(n-1)}} \leq \varkappa_2, \qquad 0 \leq i < q_n; \tag{4.1}$$

(*b*) There exists $\mathcal{K}_1 > 0$ such that $\|f_n\|_{C_3} \leq \mathcal{K}_1$, for all $n \in \mathbb{N}$;

(*c*) There exists $\mathcal{K}_2 > 0$ such that $f_n'(x) \geq \mathcal{K}_2\delta^2$, for $x \in [-1, -\delta]$, for all $n \in \mathbb{N}$;

(*d*) There exists $\mathcal{K}_3 > 0$ such that, for sufficiently large n, $Sf_n(x) \leq -\mathcal{K}_3$, for $x \in [-1, 0)$, where $Sf := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$, is the Schwarzian derivative of f.

Constants $\varkappa_1, \varkappa_2, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are universal, i.e., they do not depend of the map T, for sufficiently large n, but only on the order of the critical point. Estimate (a) (with non-universal constants $\varkappa_1$ and $\varkappa_2$), reflecting the bounded geometry of these maps, follows from Swiatek's estimates [24]. Estimate (b) follows, in part, from Denjoy's lemma, which was, for critical circle maps, proved by Yoccoz [26].

Proposition 4.1 *If T is C^3-smooth critical circle map with an irrational rotation number, for sufficiently small $\epsilon > 0$ and sufficiently large $n \in \mathbb{N}$, the set $\mathcal{F}(\epsilon) = \{z \in [-1, 0], f_n(z) - z < \epsilon\}$ is either an open interval or empty. Also, there is $\delta > 0$ such that the distances from points -1 and 0 to the set $\mathcal{F}(\epsilon)$ are larger than δ. Furthermore, there exists $\mathcal{C} > 1$ such that, for sufficiently large $n \in \mathbb{N}$,*

$$\mathcal{C}^{-1} \frac{1}{\min\{i+1, k_{n+1} - i\}^2} \leq \tau_{\sigma_n}(\Delta^{(n)}_{q_{n-1}+iq_n}) \leq \mathcal{C} \frac{1}{\min\{i+1, k_{n+1} - i\}^2}, \tag{4.2}$$

for $i = 0, \ldots, k_{n+1} - 1$.

Proof. For sufficiently small $\epsilon > 0$, the constant size intervals near -1 and 0 do not belong to $\mathcal{F}(\epsilon)$, due to (a) and (b). Assume that for some small $\epsilon > 0$, $\mathcal{F}(\epsilon)$ is not empty. For every $x \in \mathcal{F}(\epsilon)$, $f_n'(x)$ must be close to 1; otherwise, since by (a) f_n'' is bounded, the graph of f_n would intersect the diagonal, which is impossible, since the rotation number of T is irrational. Furthermore, $f_n''(x)$ must be positive and of order 1. Namely, if it were of order 1 and negative, the graph would again intersect the diagonal. If it were small, then it follows from the negative Schwarzian derivative property (d) that $f_n'''(z)$ would be negative and with magnitude of order 1 and, again, the graph would intersect the diagonal.

Clearly, $\mathcal{F}(\epsilon)$ cannot be a union of more than one interval. Namely, if this were the case, there would be some region between such two intervals where the $f_n''(x)$ is negative and consequently, there would be a point y such that $f_n''(y) = 0$ and $f_n'''(y) > 0$ (since $f_n''(x)$ changes sign from negative to positive at y). Since $y_n'(y) > 0$, due to (c), this would violate property (d).

For sufficiently large n, renormalizations f_n of C^3-smooth critical circle maps, and intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+(i+1)q_n}) = f_n(\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$, for $i = 0, \ldots, k_{n+1} - 1$, satisfy the assumptions of Lemma 3.2. We have already verified conditions (ii) and (iii). Property (b) verifies assumption (i). Properties (a) and (b) also assure that $\tau_n(\Delta^{(n)}_{q_{n-1}})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}+(k_{n+1}-1)q_n})$ are of length at least σ. Bounds (4.2) follow directly from this lemma. **QED**

4.2 Set $\mathcal{E}$ of full measure

In this section, we construct a set of full invariant measure $\mathcal{E}$ for which Theorem 1.3 holds, i.e., we have appropriate control on the distances between points of an orbit and their dynamical convergents, for critical circle maps.

Set $\mathcal{E}$ is defined analogously to set $\mathfrak{E}$ for circle maps with a break, introduced in section 3.4, with a sequence $(\sigma_n)_{n\in\mathbb{N}}$ chosen as follows. Let $(\sigma_n)_{n\in\mathbb{N}}$, be any increasing subsequence of $\mathbb{N}$ such that the corresponding sequence k_{σ_n+1} of partial quotients diverges to infinity. We will assume that such a subsequence exists since if the sequence of partial quotients is bounded, then $\beta = 0$. Let η_n be any sequence of positive numbers converging to zero such that $\eta_n k_{\sigma_n+1}$ diverges to infinity as well and the sequence $\frac{\ln \eta_n}{q_{\sigma_n}}$ converges to zero, as $n \to \infty$. Consider partitions $\mathcal{P}_n$ defined with the partitions defining point χ_0 being the critical point x_c.

For each $n \in \mathbb{N}$, let

$$\mathcal{E}_{n,0} := \bigcup_{I\in\mathcal{J}_{n,0}} I, \quad \mathcal{J}_{n,0} := \left\{ I \in \mathcal{P}_{\sigma_n+1} | I \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)| \leq \frac{1}{(\eta_n k_{\sigma_n+1})^2} \right\}, \tag{4.3}$$

and let

$$\mathcal{E}_{n,i} := T^i(\mathcal{E}_{n,0}), \qquad \text{for} \qquad i = 1, \dots, q_{\sigma_n} - 1. \tag{4.4}$$

We define

$$\mathcal{E}_n := \bigcup_{i=0}^{q_{\sigma_n}-1} \mathcal{E}_{n,i}, \tag{4.5}$$

and

$$\mathcal{E} := \limsup_{n\to\infty} \mathcal{E}_n = \bigcap_{n\geq 1} \bigcup_{j\geq n} \mathcal{E}_j. \tag{4.6}$$

Proposition 4.2 $\mu(\mathcal{E}) = 1$.

Proof. The proof is analogous to that of Proposition 3.13. **QED**

4.3 Distance of dynamical convergents

To estimate the distance between points on an orbit and their dynamical convergents for critical circle maps, we cannot apply directly the procedure of section 3.3 for maps with breaks, since the distortion is not bounded in this case.

Let $\varepsilon > 0$ be the half-width of the neighborhood around the critical point x_c, where T' has the desired power law behavior (see the beginning of section 4). We consider two classes of intervals

$$\begin{aligned} \mathcal{F}_1 &:= \{\Delta \subset \mathbb{T}^1 | \Delta \cap (x_c - \epsilon/2, x_c + \epsilon/2) = \emptyset\}\ , \\ \mathcal{F}_2 &:= \{\Delta \subset \mathbb{T}^1 | \Delta \subset (x_c - \epsilon, x_c + \epsilon)\}. \end{aligned} \tag{4.7}$$

Since the length of the intervals of partitions $\mathcal{P}_n$ decrease exponentially with n (due to (a)), for sufficiently large n, every interval of partition $\mathcal{P}_n$ belongs either to $\mathcal{F}_1$ or to $\mathcal{F}_2$.

In what follows, we will need an estimate on the number of intervals of partition $\mathcal{P}_n$ of class $\mathcal{F}_2$. For $\epsilon > 0$, let $I_\epsilon \subset \mathbb{T}^1$ be an interval of length $\epsilon > 0$, with one of the end points being the partitions defining point x_0.

Proposition 4.3 *There is $\delta = \delta(\epsilon) > 0$, approaching zero, as $\epsilon \to 0$, such that for every $n \in \mathbb{N}$, the cardinality*

$$\text{card}\ \{\Delta_i^{(n-1)} \subset I_\epsilon | i = 0, \ldots, q_n - 1\} \leq \delta q_n. \tag{4.8}$$

Proof. Let $N \in \mathbb{N}$ be the largest number such that $I_\epsilon \subset \Delta_0^{(N-1)}$. Since the partitioning of each of the q_N intervals $\Delta_i^{(N-1)}$ by the higher level partitions follows the same pattern — a "large" interval of partition $\mathcal{P}_i$ is divided into k_{i+1} "large" intervals and a "small" interval of partition $\mathcal{P}_{i+1}$; a small interval of partition $\mathcal{P}_i$ becomes a "large" interval of partition $\mathcal{P}_{i+1}$ — it is not difficult to see that, for each $n > N$, the number of intervals $\Delta_i^{(n-1)}$ of partition $\mathcal{P}_n$ inside of I_ϵ is less than q_n/q_N. Since $q_N \to \infty$, as $\epsilon \to 0$, the claim follows. **QED**

The following proposition holds for all intervals $I_0 \subset \Delta_0^{(n-1)}$ such that $I_0 \in \mathcal{P}_{n+1}$ and $I_0 \subset \mathcal{E}_{m,0}$ for sufficiently large $m \in \mathbb{N}$, and the corresponding intervals $I_i = T^i(I_0)$, $i \in \mathbb{Z}$.

Let $V_1 = V_1(\varepsilon) := \text{Var}_{\xi \in \mathbb{T}^1 \setminus (x_c - \varepsilon/2, x_c + \varepsilon/2)} \ln T'(\xi)$. Notice that $V_1 \to \infty$, as $\varepsilon \to 0$.

Proposition 4.4 *If T is a C^3-smooth critical circle map with an irrational rotation number, there exists $C_7 > 0$ and $\delta_1 = \delta_1(\varepsilon) > 0$, satisfying $\delta_1 \to 0$ as $\varepsilon \to 0$, such that*

$$\left| \ln \frac{|I_i|}{|\Delta_i^{(n-1)}|} - \ln \frac{|I_0|}{|\Delta_0^{(n-1)}|} \right| \leq V_1 + C_7 \delta_1 q_n, \tag{4.9}$$

for all $n \in \mathbb{N}$ and all $i = 0, \ldots, q_n - 1$.

Proof. For $i = 0, \ldots, q_n - 1$, there exist $\zeta_{i-1} \in I_{i-1} \subset \Delta_{i-1}^{(n-1)}$ and $\xi_{i-1} \in \Delta_{i-1}^{(n-1)}$ such that

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} = \frac{|T(I_{i-1})|}{|T(\Delta_{i-1}^{(n-1)})|} = \frac{T'(\zeta_{i-1})}{T'(\xi_{i-1})} \frac{|I_{i-1}|}{|\Delta_{i-1}^{(n-1)}|}. \tag{4.10}$$

By iterating this inequality, we obtain that, for some $\zeta_j \in I_j$ and $\xi_j \in \Delta_j^{(n-1)}$,

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} = \frac{|I_0|}{|\Delta_0^{(n-1)}|} \prod_{j=0}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} = \frac{|I_0|}{|\Delta_0^{(n-1)}|} \prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \in \mathcal{F}_1}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} \prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \in \mathcal{F}_2}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)}. \tag{4.11}$$

By taking the logarithm of this identity, we obtain

$$\left| \ln \frac{|I_i|}{|\Delta_i^{(n-1)}|} - \ln \frac{|I_0|}{|\Delta_0^{(n-1)}|} \right| \leq \sum_{\substack{j=0 \\ \Delta_j^{(n-1)} \in \mathcal{F}_1}}^{i-1} |\ln T'(\zeta_j) - \ln T'(\xi_j)| + \left| \ln \prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \in \mathcal{F}_2}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} \right|. \tag{4.12}$$

Since, for $i = 0, \ldots, q_n - 1$, the intervals $\Delta_j^{(n-1)}$ do not overlap, except possibly at the end points, we have

$$\left| \ln \frac{|I_i|}{|\Delta_i^{(n-1)}|} - \ln \frac{|I_0|}{|\Delta_0^{(n-1)}|} \right| \leq V_1 + C_7 \delta_1 q_n, \tag{4.13}$$

where $C_7, \delta_1 > 0$, and $\delta_1 \to 0$ as $\varepsilon \to 0$. Here, we have used that, for some $C_7 > 0$, and all j such that $\Delta_j^{(n-1)} \in \mathcal{F}_2$,

$$\left| \ln \frac{T'(\zeta_j)}{T'(\xi_j)} \right| \leq C_7. \tag{4.14}$$

The latter estimate follows from the following considerations. The interval I_0 is a constant fraction of $|\Delta_0^{(n-1)}|$ away from x_c, as follows from Proposition 4.1, given that $I_0 \subset \mathcal{E}_{m,0}$ for sufficiently large $m \in \mathbb{N}$. Due to the power-law behavior of T' in $(x_c - \varepsilon, x_c + \varepsilon)$, the middle value point ξ_0 is at least a constant fraction of $|\Delta_0^{(n-1)}|$ away from each of its end points (and x_c, in particular). Every other interval $\Delta_j^{(n-1)} \in \mathcal{F}_2$, for $j = 1, \ldots, q_n - 1$, is at least a constant fraction of its length away from x_c. This follows from (a) and (b), and the fact that the lengths of the intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ are of the same order. So, although the distortion of ratio is not necessarily bounded and we have no estimate on the position of ζ_j inside of $\Delta_j^{(n-1)}$, for all $j = 0, \ldots, q_n - 1$, the points ζ_j and ξ_j are comparable distances away from the critical point, i.e., there is a constant $C_8 > 0$, such that $|\zeta_j / \xi_j| \leq C_8$. Estimate (4.14) follows from power-law behavior of T' near x_c. **QED**

Let l_n be the maximal length interval of partition $\mathcal{P}_n$.

Proposition 4.5 *If T is a C^3-smooth critical circle map, then there exists $C_9 > 0$ such that, for all $x \in \mathcal{E}$, there are infinitely many $n \in \mathbb{N}$ such that*

$$|T^{q_{\sigma_n}} x - x| \leq C_9 e^{V_1 + C_7 \delta_1 q_{\sigma_n}} \frac{|\Delta_j^{(\sigma_n - 1)}|}{(\eta_n k_{\sigma_n + 1})^2}, \tag{4.15}$$

where $\Delta_j^{(\sigma_n-1)}$ *is an element of partition* $\mathcal{P}_{\sigma_n}$ *containing* x.

Proof. For every $x \in \mathcal{E}$, there are infinitely many n, such that $x \in \mathcal{E}_n$. Furthermore, there exists an element I_j of partition $\mathcal{P}_{\sigma_n+1}$ inside $\mathcal{E}_{n,j} \subset \Delta_j^{(\sigma_n-1)}$, for some $j = 0, \dots, q_{\sigma_n} - 1$, such that $x \in I_j$. It follows from the definition of $\mathcal{E}_{n,0}$ and Proposition 4.4 that there exists $\chi \in \mathcal{E}_{n,j}$, such that $I_j = [\chi, T^{q_{\sigma_n}}\chi]$ and $|I_j| \le e^{V_1 + C_7\delta_1 q_n}|\Delta_j^{(\sigma_n-1)}|(\eta_n k_{\sigma_n+1})^{-2}$. Then, $|x-\chi| \le |T^{q_{\sigma_n}}\chi - \chi| \le e^{V_1+C_7\delta_1 q_n}|\Delta_j^{(\sigma_n-1)}|(\eta_n k_{\sigma_n+1})^{-2}$.

Since there exists $\zeta \in I_j$ such that

$$T^{q_{\sigma_n}}x \;=\; T^{q_{\sigma_n}}\chi + (T^{q_{\sigma_n}})'(\zeta)(x-\chi), \tag{4.16}$$

we obtain the following estimate

$$|T^{q_{\sigma_n}}x - x| \le |T^{q_{\sigma_n}}\chi - \chi| + |x-\chi| + (T^{q_{\sigma_n}})'(\zeta)|x-\chi|. \tag{4.17}$$

If T is a C^3-smooth critical circle map, by (b), we have $(T^{q_n})'(\xi) = f_n'(\tau_n(\xi)) \le \mathcal{K}_1$, for all $\xi \in \mathbb{T}^1$. The claim follows. **QED**

Let $x_i = T^i x$ and let $I_i := [x_{i-q_n}, x_i]$, if n is even, or $I_i := [x_i, x_{i-q_n}]$, if n is odd. Let $\chi_0 \in \mathbb{T}^1$, $\chi_j = T^j\chi_0$, and let $\Delta_j^{(n-1)}(\chi_0) := [T^{q_{n-1}}\chi_j, \chi_j]$, if n is even, or $\Delta_j^{(n-1)}(\chi_0) := [\chi_j, T^{q_{n-1}}\chi_j]$, if n is odd. The following proposition holds for all intervals $I_{q_n} \subset \Delta_j^{(n-1)}(\chi_0)$ such that $I_{q_n} \in \mathcal{P}_{n+1}$ and $I_{q_n} \subset \mathcal{E}_{m,j}$ for some $m \in \mathbb{N}$, and the corresponding intervals $I_i = T^{i-q_n}(I_{q_n})$, $i \in \mathbb{Z}$.

Proposition 4.6 *If T is a C^3-smooth critical circle map with an irrational rotation number, and $x \in \Delta_j^{(n-1)}(\chi_0)$, there exists $C_{10} > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$, satisfying $\delta_2 \to 0$, as $\varepsilon \to 0$, such that*

$$\left| \ln \frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} - \ln \frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|} \right| \le V_1 + C_{10}\delta_2 q_n, \tag{4.18}$$

for all $i = 0, \dots, q_n - 1$.

Proof. It follows from the mean value theorem that, for $i = 0, \dots, q_n - 1$, there exist $\xi_i \in I_i \subset \Delta_i^{(n-1)}(\chi_{j-q_n})$ and $\zeta_i \in \Delta_i^{(n-1)}(\chi_{j-q_n})$, such that

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} = \frac{|T^{-1}(I_{i+1})|}{|T^{-1}(\Delta_{i+1}^{(n-1)}(\chi_{j-q_n}))|} = \frac{|I_{i+1}|}{|\Delta_{i+1}^{(n-1)}(\chi_{j-q_n})|}\frac{T'(\zeta_i)}{T'(\xi_i)}. \tag{4.19}$$

This implies the identity

$$\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|} = \frac{|I_{q_n}|}{|\Delta_{q_n}^{(n-1)}(\chi_{j-q_n})|}\prod_{k=i}^{q_n-1}\frac{T'(\zeta_k)}{T'(\xi_k)}. \tag{4.20}$$

Thus, we obtain

$$\left|\ln\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|}-\ln\frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|}\right| \le \sum_{\substack{k=i\\ \Delta_k^{(n-1)}(\chi_{j-q_n})\in\mathcal{F}_1}}^{q_n-1} |\ln T'(\zeta_k)-\ln T'(\xi_k)| + \sum_{\substack{k=i\\ \Delta_k^{(n-1)}(\chi_{j-q_n})\in\mathcal{F}_2}}^{q_n-1} \left|\ln\frac{T'(\zeta_k)}{T'(\xi_k)}\right|. \tag{4.21}$$

Since the intervals $\Delta_i^{(n-1)}(\chi_{j-q_n})$, for $i=0,\dots,q_n-1$, belong to the same partition of a circle, for $k=i,\dots,q_n-1$, we obtain

$$\left|\ln\frac{|I_i|}{|\Delta_i^{(n-1)}(\chi_{j-q_n})|}-\ln\frac{|I_{q_n}|}{|\Delta_j^{(n-1)}(\chi_0)|}\right| \le V_1+C_{10}\delta_2 q_n, \tag{4.22}$$

for some $C_{10},\delta_2>0$, such that $\delta_2\to 0$, as $\epsilon\to 0$. Here, we have again used estimate (4.14). The claim follows. **QED**

Proposition 4.5, Proposition 4.6 and property (b) imply the following lemma.

Lemma 4.7 *If T is a C^3-smooth critical circle map with an irrational rotation number $\rho\in(0,1)$, then there exists $C_{11}>0$, $\mathcal{V}=\mathcal{V}(\varepsilon)>0$ and $\delta=\delta(\varepsilon)>0$, satisfying $\mathcal{V}\to\infty$ and $\delta\to 0$, as $\varepsilon\to 0$, such that, for all $x\in\mathcal{E}$, there are infinitely many $n\in\mathbb{N}$ such that, for all $i=0,\dots,2q_{\sigma_n}-1$,*

$$|x_i-x_{i-q_{\sigma_n}}|\le C_{11} l_{\sigma_n-1} e^{\mathcal{V}+\delta q_{\sigma_n}}\frac{1}{(\eta_n k_{\sigma_n+1})^2}. \tag{4.23}$$

Proof. For $i=q_{\sigma_n}$, the claim holds directly from Proposition 4.5, with $C_{11}\ge C_9$, $\mathcal{V}\ge V_1$ and $\delta\ge C_7\delta_1$. Proposition 4.5 and Proposition 4.6 together imply (4.23) for $i=0,\dots,q_{\sigma_n}-1$, with $C_{11}\ge C_9$, $\mathcal{V}=2V_1$, $\delta=C_7\delta_1+C_{10}\delta_2$. Using the Denjoy estimate $(T^{q_n})'(\xi)=f_n'(\tau_n(\xi))\le\mathcal{K}_1$ (that follows from (b)), for all $\xi\in\mathbb{T}^1$, the bound (4.23) can be extended to $i=q_{\sigma_n}+1,\dots,2q_{\sigma_n}-1$, with $C_{11}\ge C_9\mathcal{K}_1$, since $|x_{i+q_{\sigma_n}}-x_i|\le\mathcal{K}_1|x_i-x_{i-q_{\sigma_n}}|$, for $i=1,\dots,q_{\sigma_n}-1$. **QED**

4.4 Singular continuous phase

Proof of Theorem 1.3. If $L(E)<2\alpha\beta$, then $\beta>0$, and there is an increasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\beta=\lim_{n\to\infty}\frac{\ln k_{\sigma_n+1}}{q_{\sigma_n}}$. Furthermore, there exist $\hat{\delta}>0$ such that $L<\alpha(2\beta-\hat{\delta})$ as well. Let η_n be any sequence of positive numbers converging to zero such that $\eta_n k_{\sigma_n+1}$ diverges to infinity and $\frac{\ln \eta_n}{q_{\sigma_n}}$ converges to zero, as $n \to \infty$. We use these sequences to construct the set $\mathcal{E}$, as in section 4.2. By Proposition 4.2, $\mu(\mathcal{E}) = 1$. For $\varepsilon > 0$, by Lemma 4.7, there exist $C_{11} > 0$, $\delta = \delta(\varepsilon) > 0$ and $\mathcal{V} = \mathcal{V}(\varepsilon) > 0$ such that, for every $x \in \mathcal{E}$, there are infinitely many n, such that estimate (4.23) holds. We assume that $\varepsilon > 0$ has been chosen such that $\delta \leq \hat{\delta}$. This implies $\hat{\beta} \geq (2\beta - \delta)$. Hence, $L(E) < \alpha\hat{\beta}$, and the claim follows from Theorem 2.3. **QED**

5 Proof of Theorem 1.4

For $C^{2+\varepsilon}$-smooth circle diffeomorphisms with a break, the claim follows from Theorem 1.2, taking into account Corollary 3.5 and Proposition 3.12. If $L(E) < \alpha\delta_{\max}$, then, there exists $\delta > 0$ such that, for every $\epsilon > 0$,

$$L(E) < \alpha(\limsup_{n\to\infty} \frac{|\ln \Theta((c_{\sigma_n} - \epsilon)^{\frac{1}{2}(1+\Theta(\epsilon))k_{\sigma_n+1}})|}{q_{\sigma_n}} - \delta), \tag{5.1}$$

where σ_n is a sequence of even numbers, if $c < 1$, or odd numbers, if $c > 1$, or $L(E) < \alpha(\limsup_{n\to\infty} \frac{|\ln \Theta(k_{\sigma_n+1}^{-2})|}{q_{\sigma_n}} - \delta)$, where σ_n is a sequence of odd numbers, if $c < 1$, or even numbers, if $c > 1$. For sufficiently small $\epsilon > 0$, either $L(E) < \frac{1}{2}\alpha\beta_{\rm br}$ or $L(E) < 2\alpha\beta$. The claim now follows from Theorem 1.2.

For C^3-smooth critical circle maps, the claim follows from Theorem 1.3, taking into account Proposition 4.1. If $L(E) < \alpha\delta_{\max}$, then $L < \alpha \limsup_{n\to\infty} \frac{|\ln \Theta(k_{n+1}^{-2})|}{q_n}$. Hence, $L(E) < 2\alpha\beta$ and the claim follows from Theorem 1.3. **QED**

A Proof of Lemma 3.2

Let ζ^* be a point such that $f'(\zeta^*) = 1$. Such a point exists, for sufficiently large k, since, by assumption, the first and the last intervals are of the same order, and on the interval $B_{K'}$ (which is non-empty for sufficiently large k), the function is convex. We will perform an affine orientation-preserving change of variables

$$y = h(z) = \frac{1}{2} f''(\zeta^*)(z - \zeta^*) \tag{A.1}$$

that maps ζ^* into 0 and normalizes the second derivative of f there. Under this change of variables f is transformed into $g = h \circ f \circ h^{-1}$ which satisfies $g'(0) = 1$ and $g''(0) = 2$. Let $\kappa := g(0) = \min_y\{g(y) - y\}$. Since f is $C^{2+\alpha}$-smooth, so is g, and from (A.1), we have

$$|g(y) - (\kappa + y + y^2)| \leq \mathfrak{C}|y|^{2+\alpha}, \qquad y \in h[-1, 0], \tag{A.2}$$

where $\mathfrak{C} > 0$.

Lemma A.1 ([16]) *Suppose that, for a sequence of real numbers* $\{s_i\}_{i\geq 0}$, *there exist* $\mathfrak{C}_1 > 0$ *and* $\alpha \in (0,1)$ *such that* $|s_{i+1} - (s_i - s_i^2)| \leq \mathfrak{C}_1 |s_i|^{2+\alpha}$, *for every* $i \geq 0$. *Then, there exist constants* $D_1 > 0$ *and* $d_1 \in (0,1)$ *such that, as long as* $s_0 \in (0, d_1]$, *the estimate*

$$\left| s_i - \frac{1}{i + s_0^{-1}} \right| \leq \frac{D_1}{(i + s_0^{-1})^{1+\alpha}} \tag{A.3}$$

holds, for every $i \geq 0$. *Moreover, there exists* $D_2 > 0$ *such that*

$$s_i - s_{i+1} = \frac{1}{(i + s_0^{-1})^2}(1 + \delta_i), \tag{A.4}$$

where $|\delta_i| \leq D_2 s_0^{\alpha}$, *for all* $i \geq 0$, *as long as* $s_0 \in (0, d_1]$.

Lemma A.2 ([16]) *Suppose that, for a sequence of real numbers* $\{s_i\}_{i\geq 0}$, *there exist* $\mathfrak{C}_2, \mathfrak{C}_3 > 0$ *and* $\kappa, \alpha \in (0,1)$ *such that*

1. $|s_0| \leq \mathfrak{C}_2 \kappa$,
2. $|s_{i+1} - (\kappa + s_i + s_i^2)| \leq \mathfrak{C}_3 |s_i|^{2+\alpha}$, *for every* $i \geq 0$.

Fix arbitrary $\mathfrak{C}_4 > 0$ *and define* $N = \kappa^{-1/2} \tan^{-1}(\mathfrak{C}_4 \kappa^{-\frac{\alpha}{2(2+\alpha)}})$. *Then, there exist constants* $D_3 > 0$ *and* $d_2 \in (0,1)$ *such that, as long as* $\kappa \in (0, d_2]$, *the following estimate holds for every* $0 \leq i \leq N$,

$$|s_i - \sqrt{\kappa}\tan(\sqrt{\kappa} i + a_0)| \leq D_3(\sqrt{\kappa}\tan\sqrt{\kappa} i)^{1+\frac{\alpha(\alpha+1)}{2}}, \tag{A.5}$$

where $a_0 = \tan^{-1}(s_0/\sqrt{\kappa})$. *Moreover, there exists* $D_4 > 0$ *such that*

$$s_{i+1} - s_i = \frac{\kappa}{(\cos\sqrt{\kappa} i)^2}(1 + \delta_i), \tag{A.6}$$

where $|\delta_i| \leq D_4 \kappa^{\frac{\alpha(\alpha+1)}{2(2+\alpha)}}$, *for all* $0 \leq i < N$, *as long as* $\kappa \in (0, d_2]$.

Proof of Lemma 3.2. Let a and b be the left and right end points of I. $t_0 = h(a)$ and $t_i = g^i(t_0)$, i.e., $t_i = h(f^i(a))$.

Since $\kappa = g(0)$, there exists a unique number i_c satisfying $0 < i_c < k$ such that $t_{i_c} \in [0, \kappa)$. Let $i_l = i_c - [\kappa^{-1/2}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}}]$ and $i_r = i_c + [\kappa^{-1/2}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}}]$. Combining $\tan^{-1}\frac{1}{x} = \frac{\pi}{2} - \tan^{-1} x$ with $\tan^{-1} x = x + \mathcal{O}(x^3)$, $x \to 0$, it is easy to derive the following asymptotic formula

$$\kappa^{-\frac{1}{2}}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}} = \frac{\pi}{2}\kappa^{-\frac{1}{2}} - \kappa^{-\frac{1}{2+\alpha}} + \mathcal{O}(\kappa^{\frac{-1+\alpha}{2+\alpha}}), \quad \kappa \to 0. \tag{A.7}$$

To obtain the desired estimates for $i_l \le i \le i_r$, we can apply Lemma A.2. To obtain the estimates for $i_l \le i < i_c$, we can apply this lemma to $s_i = -(t_{i_c-i}-\kappa)$, where $0 \le i \le i_c - i_l$. It immediately follows from this lemma that, for $i_l \le i < i_c$,

$$t_{i+1} - t_i = s_{i_c-i} - s_{i_c-i-1} = \frac{\kappa i^2}{i^2(\cos(\sqrt{\kappa}(i_c - i - 1)))^2}(1 + \delta_{i_c-i-1}). \tag{A.8}$$

It is not difficult to check that the function $\chi(\sqrt{\kappa}i) = \frac{\sqrt{\kappa}i}{\cos(\sqrt{\kappa}(i_c-i-1))}$ is monotonically increasing on $i_l \le i < i_c$. This follows from the fact that the function $\sqrt{\kappa}i\tan(\sqrt{\kappa}(i_c-i)-1)$ has maximum when $\sqrt{\kappa}i = \frac{\tan(\sqrt{\kappa}(i_c-i-1))}{1+\tan^2(\sqrt{\kappa}(i_c-i)-1)}$ and, therefore, $\chi'(\sqrt{\kappa}i) = \frac{1-\sqrt{\kappa}i\tan(\sqrt{\kappa}(i_c-i-1))}{\cos(\sqrt{\kappa}(i_c-i-1))} \ge (\cos(\sqrt{\kappa}(i_c - i - 1)))^{-1}(1 + \tan^2(\sqrt{\kappa}(i_c - i - 1))^{-1} > 0$, for $i_l \le i < i_c$. Since $i_c = \frac{k}{2} + \mathcal{O}(\kappa^{-\frac{1-\alpha}{2}}) = \frac{\pi}{2}\kappa^{-\frac{1}{2}} + \mathcal{O}(\kappa^{-\frac{1-\alpha}{2}})$ as $\kappa \to 0$ (Lemma 3.19 in [15]) and, from asymptotic formula (A.7), $i_l = \kappa^{-\frac{1}{2+\alpha}} + \mathcal{O}(\kappa^{-\frac{1-\alpha}{2}})$ and

$$\frac{\kappa i_l^2}{\cos(\sqrt{\kappa}(i_c - i_l - 1))} \to 1, \qquad \text{as} \qquad \kappa \to 0, \tag{A.9}$$

the function $\frac{\kappa i^2}{i^2(\cos(\sqrt{\kappa}(i_c-i-1)))^2}$ is bounded and the claim follows for $i_l \le i < i_c$. Here, we have also used the fact that, since the second derivative of f is bounded both from above and from below by positive constants, the lengths of the intervals $[t_{i-1}, t_i]$ and Δ_i are of the same order. Similarly, we can obtain the desired estimates for $i_c \le i \le i_r$, by applying Lemma A.2 to $s_i = t_{i_c+i}$, where $0 \le i \le i_r - i_c$.

For $0 \le i \le i_l$ and $i_r < i \le k$, we can obtain the desired estimates by applying Lemma A.1. This is a consequence of the convexity and the fact that it follows from (A.5), using the (A.7), that $t_{i_l} = \kappa^{\frac{1}{2+\alpha}} + \mathcal{O}(\kappa^{\frac{1}{2+\alpha}+\frac{\alpha(\alpha+1)}{2(2+\alpha)}})$ and, similarly, $t_{i_r} = \kappa^{\frac{1}{2+\alpha}} + \mathcal{O}(\kappa^{\frac{1}{2+\alpha}+\frac{\alpha(\alpha+1)}{2(2+\alpha)}})$. We first obtain the estimates for $0 \le i < i_l$. For $0 \le i < i_l - j$, let $s_i = -t_{i+j}$. For sufficiently large k, and some fixed large j, $s_0 \in (0, d_1]$. Since, for such i's, $\kappa <$ const.$|t_{i+j}|^{2+\alpha}$, it follows from (A.2) that s_i satisfy the assumptions of Lemma A.1. We can apply this lemma for $0 \le i < i_l - j$. The estimate (A.4) immediately gives us the desired bounds for $1 \le i < i_l$. Similarly, by defining $s_i = t_{k-j-i}$, for $0 \le i \le i_r - j$, for some large j, we again have $s_0 \in (0, d_1]$, for sufficiently large k. Since $\kappa <$ const.$|t_{k-j-i}|^{2+\alpha}$, it again follows from (A.2) that s_i satisfy the assumptions of Lemma A.1. The estimate (A.4) of Lemma A.1 immediately gives us the desired estimates for $k - j_r < i \le k$. **QED**

Acknowledgments

I am grateful to Svetlana Jitomirskaya for sparking my interest in the spectral theory of Schrödinger operators and for her hospitality during my visit to the University of

California Irvine. This material is based upon work supported in part by the National Science Foundation EPSCoR RII Track-4 # 1738834 and the University of Mississippi College of Liberal Arts Summer Research Grant.

References

[1] A. Avila, Global theory of one-frequency Schrödinger operators, *Acta Math.* **215** (2015), 1–54.

[2] A. Avila, J. You, Q. Zhou, Sharp phase transitions for the almost Mathieu operator, *Duke Math. J.* **166** 14 (2017), 2697–2718.

[3] J. Avron, B. Simon, Singular continuous spectrum for a class of almost periodic Jacobi matrices, *Bull. Amer. Math. Soc.* **6** 1 (1982), 81–85.

[4] D. Damanik, Schrödinger operators with dynamically defined potentials, *Ergodic Theory Dynam. Systems* **37** 6 (2017), 1681–1764.

[5] E. de Faria,W. de Melo, Rigidity of critical circle mappings I, *J. Eur. Math. Soc.* **1** 4 (1999), 339–392.

[6] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems, *Annales de l'Institut Henri Poincaré (B) Probab. Stat.* **33** 6 (1997), 797–815.

[7] H. Furstenberg, Strict ergodicity and transformation of the torus, *Amer. J. Math.* **83** 4 (1961), 573–601.

[8] A. Y. Gordon, The point spectrum of the one-dimensional Schrödinger operator, *Uspekhi Mat. Nauk* **31** 4 (1976), 257–258.

[9] P. Guarino, M. Martens, W. de Melo, Rigidity of critical circle maps, *Duke Math. J.* **167** 11 (2018), 2125–2188.

[10] M. R. Herman, Sur la conjugasion differentiable des difféomorphismes du cercle a de rotations, *Publ. Math. Inst. Hautes Etudes Sci.* **49** (1979), 5–234.

[11] S. Jitomirskaya, Ergodic Schrödinger operators (on one foot) *Proc. Sympos. Pure Math.* **76.2** (2007), 613–647.

[12] S. Jitomirskaya, S. Kocić, Spectral theory of Schrödinger operators over circle diffeomorphisms, *Int. Math. Res. Not.* (2020), *to appear.*

[13] S. Jitomirskaya, C.A. Marx, Dynamics and spectral theory of quasi-periodic Schrödinger-type operators, *Ergodic Theory Dynam. Systems* **37** 8 (2017), 2353–2393.

[14] K. Khanin, S. Kocić, Renormalization conjecture and rigidity theory for circle diffeomorphisms with breaks, *Geom. Funct. Anal.* **24** 6 (2014), 2002–2028.

[15] K. Khanin, S. Kocić, E. Mazzeo, C^1-rigidity of circle diffeomorphisms with breaks for almost all rotation numbers, *Ann. Sci. Éc. Norm. Supér.* **50** (2017), 1163–1203.

[16] K. Khanin, A. Teplinsky, Robust rigidity for circle diffeomorphisms with singularities, *Invent. Math.* **169** (2007), 193–218.

[17] K.M. Khanin, E. B. Vul, Circle homeomorphisms with weak discontinuities, *Adv. Sov. Math.* **3** (1991), 57–98.

[18] S. Kocić, Generic rigidity for circle diffeomorphisms with breaks, *Comm. Math. Phys.* **344** 2 (2016), 427–445.

[19] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, *Stochastic Analysis* (Katata/Kyoto, 1982), 225– 247, North-Holland Math. Library, 32. North-Holland, Amsterdam, 1984.

[20] S. Kotani, One-dimensional random Schrödinger operators and Herglotz functions, Probabilistic Methods in Mathematical Physics (Katata/Kyoto, 1985), 219–250, Academic Press, Boston, MA, 1987.

[21] S. Kotani, Generalized Floquet theory for stationary Schrödinger operators in one dimension, Chaos Solitons Fractals **8** 11 (1997), 1817–1854.

[22] Y. Last, B. Simon, Eigenvalues, transfer matrices and absolutely continuous spectrum of one-dimensional Schrödinger operators, *Invent. Math.* **135** 2 (1999), 329–367.

[23] Ya. G. Sinai, K. M. Khanin, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Uspekhi Mat. Nauk* **44** 1 (1989), 57–82.

[24] G. Swiatek, Rational rotation numbers for maps of the circle, *Comm. Math. Phys.* **119** 1 (1988), 109–128.

[25] J.-C. Yoccoz, Conjugaison differentiable des difféomorphismes du cercle donc le nombre de rotation vérifie une condition Diophantienne, *Ann. Sci. Éc. Norm. Supér.* **17** (1984), 333–361.

[26] J.-C. Yoccoz, Il n'y a pas de contre-example de Denjoy analytique, *C. R. Acad. Sc. Paris* **298**, Série I 7 (1984), 141–144.