ANALYTIC GENERICITY OF DIFFUSING ORBITS IN *A PRIORI* UNSTABLE HAMILTONIAN SYSTEMS

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ABSTRACT. We study the problem of instability in the following *a priori* unstable Hamiltonian system with a time-periodic perturbation

$$\mathscr{H}_\varepsilon(p,q,I,\varphi,t) = h(I) + \sum_{i=1}^{n} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i)\right) + \varepsilon H_1(p,q,I,\varphi,t),$$

where $(p,q) \in \mathbb{R}^n \times \mathbb{T}^n$, $(I,\varphi) \in \mathbb{R}^d \times \mathbb{T}^d$ with $n,d \geq 1$, V_i are Morse potentials, and ε is a small non-zero parameter. Using geometric methods we prove that Arnold diffusion occurs for generic analytic perturbations H_1. Indeed, the set of admissible H_1 is C^ω dense and C^3 open. The proof also works for arbitrarily small V_i. Our perturbative technique for the genericity is valid in the C^k topology for all $k \in [3,\infty) \cup \{\infty,\omega\}$.

1. INTRODUCTION

The goal of this paper is to study the Arnold diffusion problem for analytic perturbations of a given *a priori* unstable Hamiltonian system. Arnold diffusion is a phenomenon of instability in Hamiltonian systems with more than two degrees of freedom. This problem arises in the study of the effect of small perturbations on integrable systems, and has attracted a lot of attention both in mathematics and in physics due to its importance for the applications.

For a nearly integrable Hamiltonian system, the celebrated Kolmogorov-Arnold-Moser (KAM) theory asserts that the most part (in the measure-theoretic sense) of the phase space is filled with KAM invariant tori carrying quasi-periodic dynamics. Arnold diffusion asks for the large scale motions in the complement of KAM tori. The first example is constructed by V. I. Arnold in [1]. He also conjectured that the diffusive phenomena occur for generic systems: `the typical case in a multidimensional problem is topological instability: through an arbitrarily small neighborhood of any point there passes a phase trajectory along which the action variables go away from the initial values by a quantity of order one` [2, Chapter 6]. One of the main problems in this conjecture is the genericity in some appropriate function space (e.g. C^r-differentiable, analytic).

In the present paper we focus on the C^ω-genericity of Arnold diffusion in *a priori* unstable Hamiltonian systems. The *a priori* unstable Hamiltonian system consists of a rotor-pendulum system plus a time periodic perturbation, and it can be viewed as a scaled approximation on the dynamics near simple resonances of the *a priori* stable systems [15].

Note that the proof presented in this paper works even if the functions $V_i(q_i) : \mathbb{T} \to \mathbb{R}$, $i = 1, \cdots, n$ in the unperturbed part are arbitrarily small (weak hyperbolicity), so our result applies to some *a priori* stable systems. Our approach for the proof follows a recent geometric mechanism established in [36]. This mechanism relies on the presence of normally hyperbolic invariant manifold (NHIM), with transverse intersection between the associated stable and unstable manifolds. Indeed, the Melnikov method will be used to show transverse homoclinic orbits in the perturbed system. In this setting, we can then use the theory of scattering maps to compute the effect of homoclinic excursions. Heuristically, the scattering map gives the future asymptotic of an orbit as a function of its past asymptotic [17, 21]. By shadowing the pseudo-orbits of this map, it allows to show instability for the original dynamics.

Here, we give a brief overview of the previous works and approaches on the genericity problem of Arnold diffusion. The scattering map has become an effective tool to study the phenomena of instability in concrete examples or generic systems. By exploiting this geometric tool, Arnold diffusion has been proved to occur for generic perturbations in the C^r-differentiable topology, see for instance [17, 22, 24, 36].

In particular, the geometric mechanism developed in [36] requires almost no information of the inner dynamics on the NHIM. Only recurrence of the motion in the NHIM is needed, and it is automatically satisfied in the Hamiltonian case by Poincaré recurrence theorem if the motions in the NHIM are bounded (Of course, if the motions in the NHIM are not bounded, one has diffusion in the NHIM!).

The main hypothesis of the mechanism of [36] is some explicit transversality conditions, which are implied by checking that some Melnikov-type functions have non-degenerate critical points. Sometimes the dynamics of a single scattering map may have difficulties moving long distances. But if several scattering maps are available, one can iterate these scattering maps to find large scale motions. For applications of the mechanism of [37] in Celestial mechanics see e.g. [9, 26].

Another mechanism assuming mainly recurrence – but assuming some separation of time scales appears in [34]. Geometric methods that use NHIM but assume that there are some other invariant objects in the NHIM (e.g secondary tori) appear in [17, 18, 19, 24, 20, 23, 22] and applications to celestial mechanics and other concrete models appear in [24, 28, 27]. Another important geometric method based on separatrix maps to study Arnold diffusion can be found in [49, 50, 51], *etc.* It is worth mentioning that the variational method is also an effective approach to study the diffusion problem. The techniques and ideas developed by J. Mather [46, 47] have significant influence. Applying global variational methods to convex Hamiltonian systems, several authors have established the genericity in the C^r-differentiable topology, see [13, 4, 14, 41, 11, 5, 12, 42]. Note, however that in some of these papers, the notion of genericity is redefined and that, of course, convexity is not generic in the standard sense.

This paper does not aim to review the rich history on this very active area. There are many other related works, we mention here [8, 7, 6, 16, 43, 32, 40, 52, 34, 38, 45, 10, 27, 33] and references therein.

Thus, as mentioned above, the genericity of Arnold diffusion in the C^r-differentiable category has been well studied. However, the most difficult case is the analytic genericity, which is still an open

problem. The difficulty lies in that most of the previous works require the use of non-analytic techniques (e.g. bump functions) for perturbations.

In this paper, a new perturbative technique is introduced to solve the analytic genericity of Arnold diffusion. Following [36], the scattering map is used as an essential tool. We will take advantage of the Poincaré-Melnikov method and the family of periodic potential functions to verify the genericity of some transversality hypotheses. As will see below, the novelty of our technique is as follows: (I) *It is valid in both the C^r-differentiable topology and the C^ω topology*; (II) *We can also obtain the genericity in the sense of Mañé* [44], *that is, the genericity (actually, C^ω dense and C^3 open) in the space of periodic potential functions.*

The idea of our method is very simple. The work of [36] shows that it suffices to verify some transversality conditions for the zeros of a rather explicit Melnikov integral. Following the standard procedure in transversality, we show that, if there are some degenerate situations, more or less arbitrary perturbations break the degeneracy. See Section 4.

Note that the families of perturbations we choose are rather arbitrary. Hence, the result is stronger than density. We show that the transversality – and hence the diffusion – can only fail in an infinite codimension set. See Remark 10. For practical applications, we note that the only condition to check is a very explicit (and rapidly convergent) integral, so that given a concrete system (e.g. in celestial mechanics), one can verify the result with a finite precision calculation and obtain quantitative information on the location of the diffusing orbits.

2. SETUP AND MAIN RESULT

2.1. **Notation and assumptions.** For positive integers $d \geq 1$ and $n \geq 1$, we let $\mathscr{B} \subset \mathbb{R}^d$ and $\mathscr{D} \subset \mathbb{R}^n$ be two open domains with compact closures $\overline{\mathscr{B}}$ and $\overline{\mathscr{D}}$ respectively. Without loss of generality, we may suppose $\mathscr{B} = \{x \in \mathbb{R}^d : \|x\| < R_1\}$ and $\mathscr{D} = \{z \in \mathbb{R}^n : \|z\| < R_2\}$, where R_1 and R_2 are suitably large, and $\|\cdot\|$ is the standard Euclidean norm.

We consider the following *a priori* unstable system with a time-periodic perturbation:

$$\mathcal{H}_\varepsilon = H_0(p, q, I) + \varepsilon H_1(p, q, I, \varphi, t), \qquad (p, q, I, \varphi) \in \mathscr{D} \times \mathbb{T}^n \times \mathscr{B} \times \mathbb{T}^d. \tag{2.1}$$

Here, $p = (p_1, \cdots, p_n)$ and $q = (q_1, \cdots, q_n)$ are symplectically conjugate variables, $I = (I_1, \cdots, I_d)$ and $\varphi = (\varphi_1, \cdots, \varphi_d)$ are symplectically conjugate variables, and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The unperturbed Hamiltonian H_0 is given by

$$H_0(p, q, I) = h(I) + \sum_{i=1}^{n} \pm \left[\frac{1}{2}p_i^2 + V_i(q_i)\right]. \tag{2.2}$$

Here, the symbol $\pm$ in (2.2) means that one can take either the plus sign " $+$ " or the minus sign " $-$ " in front of each pendulum $\frac{1}{2}p_i^2 + V_i(q_i)$. The whole perturbation term εH_1 is assumed to be real analytic and periodic on time t with a period 2π. The unperturbed part H_0 represents a d-degree-of-freedom rotator plus n pendulums.

Throughout this paper, we assume the following conditions on H_0:

(H1) $h(I)$ and each $V_i(q_i)$ with $i \in \{1, \cdots, n\}$ are of class C^r with the extended integer $r \in [3, \infty) \cup \{\infty, \omega\}$.

(H2) For each i, the function $V_i : \mathbb{T} \to \mathbb{R}$ has a unique maximum point which is non-degenerate in the sense of Morse. Without loss of generality and to simplify the notation, we may always assume the maximum point $q_{\max} = 0$.

Condition **(H2)** tells us that $V_i'(0) = 0$ and $V_i''(0) < 0$ for each $i \in \{1, \cdots, n\}$. It is clear that the non-degeneracy condition **(H2)** is C^2-open and C^ω-dense. We also remark that the approach used in this paper is also applicable to those systems whose unperturbed part is $h(I) + \sum_{i=1}^n P_i(p_i, q_i)$, as long as each P_i has a hyperbolic equilibrium and a homoclinic orbit.

Of course, the potentials satisfying Morse non-degeneracy are generic.

The phase space is $\mathcal{M} := \mathcal{D} \times \mathbb{T}^n \times \mathcal{B} \times \mathbb{T}^d$, endowed with the standard symplectic form. The corresponding Hamilton's equations are

$$\dot{p} = -\frac{\partial H_0}{\partial q} - \varepsilon \frac{\partial H_1}{\partial q}, \qquad \dot{I} = -\varepsilon \frac{\partial H_1}{\partial \varphi},$$

$$\dot{q} = \frac{\partial H_0}{\partial p} + \varepsilon \frac{\partial H_1}{\partial p}, \qquad \dot{\varphi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}. \tag{2.3}$$

It is clear that the dynamics of the unperturbed system H_0 is integrable. Hence, the diffusion phenomena may occur only if $\varepsilon \neq 0$. Denoting the extended phase space

$$\widetilde{\mathcal{M}} := \mathcal{D} \times \mathbb{T}^n \times \mathcal{B} \times \mathbb{T}^d \times \mathbb{T},$$

the perturbation function H_1 in (2.1) is assumed to be real analytic on $\widetilde{\mathcal{M}}$, which means the analyticity can extend to a complex neighborhood of $\widetilde{\mathcal{M}}$.

For each $\kappa > 0$ we denote by $\widetilde{\mathcal{M}}_\kappa$ the set of all points $(p, q, I, \varphi, t) \in \mathbb{C}^n \times \mathbb{C}^n/(2\pi\mathbb{Z})^n \times \mathbb{C}^d \times \mathbb{C}^d/(2\pi\mathbb{Z})^d \times \mathbb{C}/(2\pi\mathbb{Z})$ satisfying

$$\mathrm{dist}(p, \mathcal{D}) < \kappa, \quad \mathrm{dist}(I, \mathcal{B}) < \kappa, \quad |\mathrm{Im}\, \iota| < \kappa, \, \iota = q_i, \varphi_i, t.$$

$\widetilde{\mathcal{M}}_\kappa$ is an open domain in the complex space. In order to discuss the genericity of Arnold diffusion in the real analytic category, we introduce the following space of bounded analytic functions on $\widetilde{\mathcal{M}}_\kappa$,

$$C^\omega(\widetilde{\mathcal{M}}_\kappa) := \left\{ f : \widetilde{\mathcal{M}}_\kappa \to \mathbb{C} \,\middle|\, f \text{ is analytic}, \, \sup_{x \in \widetilde{\mathcal{M}}_\kappa} |f(x)| < \infty, \, f(\widetilde{\mathcal{M}}) \subset \mathbb{R} \right\}.$$

Note that $f \in C^\omega(\widetilde{\mathcal{M}}_\kappa)$ is real-valued on $\widetilde{\mathcal{M}}$. Clearly, $C^\omega(\widetilde{\mathcal{M}}_\kappa)$ with the sup-norm ($\|f\|_\kappa := \sup_{z \in \widetilde{\mathcal{M}}_\kappa} |f(z)|$) is a Banach space. Sometimes, for simplicity, we use C^ω_κ instead of $C^\omega(\widetilde{\mathcal{M}}_\kappa)$ when there is no confusion.

2.2. **Main Result.** Now, we are ready to state our main result on the genericity of Arnold diffusion. In what follows, Hamiltonian system (2.1) is always assumed to satisfy conditions **(H1)**–**(H2)**.

Theorem 2.1. *Given $\kappa > 0$, $I_0 \in \mathcal{B}$ and a small neighborhood V_{I_0} of I_0. Then there exists an open and dense set $\mathcal{U} \subset C^\omega_\kappa$, and for each $H_1 \in \mathcal{U}$ we can find $\varepsilon_0 = \varepsilon_0(H_1) > 0$ and $\rho = \rho(H_1) > 0$ satisfying the following property: for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, the Hamiltonian flow of $\mathscr{H}_\varepsilon = H_0 + \varepsilon H_1$ admits a trajectory whose action variables $I(t)$ satisfy*

$$\sup_{t > 0} \|I(t) - I(0)\| \geq \rho,$$

where the initial condition $I(0) \in V_{I_0}$.

Remark 1. As will see in Section 4, the set of H_1 admitting the diffusion is indeed C_κ^ω dense and C^3 open. Theorem 2.1 implies that for generic systems, through an arbitrarily small neighborhood of a given point there passes a trajectory whose action coordinates go away from the initial values by $O(1)$ with respect to the size of the perturbation.

Remark 2. κ stands for the size of analytic extension. It is worth noting that our result of analytic genericity holds for any $\kappa > 0$.

Remark 3. As we will see from the proof in the following sections, our result allows that V_i are arbitrarily small (weakly hyperbolic case), that is to say V_i can be replaced by δV_i with $\delta > 0$ being small.

Of course, in that case the threshold value $\varepsilon_0(H_1) \ll \delta$ shrinks to zero as δ tends to zero. This is, very typical of the approches to diffusions near integrable systems. Of course one expects that making ε larger will generate more diffusion.

We can also interpret the genericity result stated above for system $H_0 + H_1$ without using the parameter ε. More precisely, let $\mathfrak{S}$ be the unit sphere in the space $(C_\kappa^\omega, \|\cdot\|_\kappa)$ with $\kappa > 0$. Then there exists a non-negative function $\epsilon_0 : \mathfrak{S} \to [0, +\infty)$ taking positive values on an open-dense subset of $\mathfrak{S}$, such that for each H_1 in the ϵ_0-ball

$$\mathfrak{B} = \left\{ \lambda P \,\middle|\, P \in \mathfrak{S}, \, \lambda \in \big(0, \epsilon_0(P)\big) \right\},$$

the Hamiltonian $H_0 + H_1$ admits Arnold diffusion.

Let $C^\omega := \bigcup_{\kappa > 0} C_\kappa^\omega$. It is exactly the set of all bounded real analytic functions on $\widetilde{\mathcal{M}}$. Note that C^ω is a Fréchet space, we then have the following immediate consequence:

Corollary 2.2. *Given a point* $I_0 \in \mathfrak{B}$ *and a small neighborhood* V_{I_0} *of* I_0. *Then there exists an open and dense set* $\mathcal{V} \subset C^\omega$, *and for each* $H_1 \in \mathcal{V}$ *we can find* $\varepsilon_0 = \varepsilon_0(H_1) > 0$ *and* $\rho = \rho(H_1) > 0$ *satisfying the following property: for each* $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, *the Hamiltonian flow of* $\mathscr{H}_\varepsilon = H_0 + \varepsilon H_1$ *admits a trajectory whose action variables* $I(t)$ *satisfy*

$$\sup_{t > 0} \|I(t) - I(0)\| \geq \rho,$$

where the initial condition $I(0) \in V_{I_0}$.

As we will show in Section 4, the genericity are verified by taking advantage of perturbation functions depending only on (q, φ, t). Thanks to the work of [36], we will see that two hypotheses formulated below as **(H3a), (H3b)** for a specific integral, imply diffusion. Hence, for us, it suffices to show that **(H3a), (H3b)** are generic. Therefore, we can even establish the genericity *in the sense of Mañé* [44], namely, the diffusive phenomenon occurs under generic periodic potential perturbations. More precisely, we denote by $C_\kappa^\omega(\mathbb{T}^{n+d+1})$ the set of all real analytic functions which can extend analytically to the complex neighborhood $\{(q, \varphi, t) \in \mathbb{C}^{n+d+1}/(2\pi\mathbb{Z})^{n+d+1} : |\text{Im } \iota| < \kappa, \iota = q_i, \varphi_i, t\}$. Then we have

Theorem 2.3. *Given* $\kappa > 0$, $I_0 \in \mathfrak{B}$ *and a small neighborhood* V_{I_0} *of* I_0. *Then there exists an open and dense set* $\mathcal{W} \subset C_\kappa^\omega(\mathbb{T}^{n+d+1})$, *and for each* $P \in \mathcal{W}$ *we can find* $\varepsilon_0 = \varepsilon_0(P) > 0$ *and* $\rho = \rho(P) > 0$ *satisfying the*

following property: for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, the Hamiltonian flow of $\mathcal{H}_\varepsilon = H_0 + \varepsilon P$ admits a trajectory whose action variables $I(t)$ satisfy

$$\sup_{t>0} \| I(t) - I(0) \| \geq \rho,$$

where the initial condition $I(0) \in V_{I_0}$.

We end this section by giving a concluding remark on our result and approach.

(1) Our perturbative technique for the genericity is valid in both the C^r-differentiable ($3 \leq r \leq \infty$) and the C^ω topologies. Also, it applies to the genericity in the sense of Mañé.

(2) The unperturbed part H_0 is only assumed to be C^r smooth with $r \geq 3$. We do not require the inner dynamics to satisfy a twist condition, and the diffusion mechanism used in the present paper only relies on the outer dynamics since invariant objects (e.g. primary KAM tori, Aubry-Mather sets) of the inner map are not used at all.

(3) Both the phase space of the rotator and the phase space of the pendulums can be of arbitrary dimensions.

2.3. **Organization of the paper.** In Section 3, we first review the results we use on the normally hyperbolic invariant manifolds and the scattering maps for the *a priori* unstable system (2.1). Then, we review the geometric program established in [36]. It allows us to obtain Arnold diffusion for the original dynamics by shadowing the pseudo-orbits of the scattering map. We provide more details for this geometric mechanism in Appendix C for the reader's convenience. Section 4 is devoted to the proofs of our results on analytic genericity. Appendix A and Appendix B give general introductions to the theory of NHIMs and the theory of scattering maps. The perturbative argument to break the possible degeneracies of the conditions in [36] is described in Section 4.

3. SCATTERING MAPS AND GEOMETRIC MECHANISM OF ARNOLD DIFFUSION

The main characteristic of an *a priori* unstable Hamiltonian system is that there exists a normally hyperbolic invariant manifold (NHIM) with unstable and stable invariant manifolds. The presence of these invariant objects plays an important role in the Arnold diffusion problem. The scattering map of the NHIM is an effective tool to quantify the homoclinic excursions. This map associates the orbit asymptotic in the past to the orbit asymptotic in the future. Using the perturbation theory and the Melnikov method one can estimate the effect of the perturbation on all the variables of the scattering map. See Appendix A and Appendix B for general introductions.

In this section, we first give some important results on the NHIM and the scattering map for our *a priori* unstable system $\mathcal{H}_\varepsilon = H_0 + \varepsilon H_1$. Then, we review a recent geometric mechanism of Arnold diffusion established in [36].

Recall that $\mathcal{H}_\varepsilon$ satisfies conditions **(H1)–(H2)**. From now on, it is convenient to fix two closed balls (suitably large) $\mathscr{D}_* \subset \mathscr{D}$ and $\mathscr{B}_* \subset \mathscr{B}$, and study the dynamics on the following domain

$$(p, q, I, \varphi) \in \mathscr{D}_* \times \mathbb{T}^n \times \mathscr{B}_* \times \mathbb{T}^d.$$

3.1. **Normal hyperbolicity of the unperturbed system.** As the unperturbed system H_0 is given by

$$H_0(p,q,I) = h(I) + \sum_{i=1}^{n} \pm\left[\frac{1}{2}p_i^2 + V_i(q_i)\right],$$

we use $\Phi_{t,0}$ to denote the corresponding autonomous C^{r-1} Hamiltonian flow on $\mathcal{M} = \mathcal{D} \times \mathbb{T}^n \times \mathcal{B} \times \mathbb{T}^d$. Here, the subscript "0" represents $\varepsilon = 0$.

Condition **(H2)** implies that each pendulum $\frac{1}{2}p_i^2 + V_i(q_i)$ has two homoclinic orbits. For each i, we choose and fix one homoclinic orbit $(p_i^0(t), q_i^0(t))$. It converges exponentially to the hyperbolic equilibrium $(0,0)$ with characteristic exponent

$$\lambda_i := \sqrt{-V_i''(0)} > 0. \tag{3.1}$$

This is equivalent to say

$$\text{dist}\left(\left(p_i^0(t), q_i^0(t)\right), (0,0)\right) \le Ce^{-\lambda_i|t|}, \quad \text{as } t \longrightarrow \pm\infty.$$

The autonomous flow $\Phi_{t,0}$ has a $2d$-dimensional invariant manifold Λ_0 with boundary,

$$\Lambda_0 = \{(0,0,I,\varphi) : (I,\varphi) \in \mathcal{B}_* \times \mathbb{T}^d\}.$$

Λ_0 is foliated completely by invariant tori, and hence the dynamics restricted on Λ_0 is integrable. Sometimes, we need to work in the extended space $\widetilde{\mathcal{M}} = \mathcal{M} \times \mathbb{T}$, which yields a $(2d+1)$-dimensional manifold

$$\widetilde{\Lambda}_0 = \{(0,0,I,\varphi,t) : (I,\varphi,t) \in \mathcal{B}_* \times \mathbb{T}^d \times \mathbb{T}\} \subset \widetilde{\mathcal{M}}.$$

$\widetilde{\Lambda}_0$ is invariant under the extended flow $\widetilde{\Phi}_{t,0}$.

$\widetilde{\Lambda}_0$ is a normally hyperbolic invariant manifold (see Appendix A). To verify it, we use (3.1) and take the normal exponents

$$\lambda_s = -\max_{i=1,\cdots,n} \lambda_i, \quad \mu_s = -\min_{i=1,\cdots,n} \lambda_i; \quad \lambda_u = \min_{i=1,\cdots,n} \lambda_i, \quad \mu_u = \max_{i=1,\cdots,n} \lambda_i, \tag{3.2}$$

For the central exponents, we can take $-\lambda_c = \mu_c$ with the positive exponent μ_c as close as desired to the value 0 since the dynamics on $\widetilde{\Lambda}_0$ is completely integrable. Consequently, for every $\tilde{x} \in \widetilde{\Lambda}_0$ we have the invariant splitting of the tangent bundle $T_{\tilde{x}}\widetilde{\mathcal{M}} = T_{\tilde{x}}\widetilde{\Lambda}_0 \oplus E_{\tilde{x}}^s \oplus E_{\tilde{x}}^u$, and

$$v \in E_{\tilde{x}}^s \Longleftrightarrow C^{-1}e^{t\lambda_s}\|v\| \le \|D\widetilde{\Phi}_{t,0}(\tilde{x})v\| \le Ce^{t\mu_s}\|v\|, \quad t \ge 0,$$

$$v \in E_{\tilde{x}}^u \Longleftrightarrow C^{-1}e^{t\mu_u}\|v\| \le \|D\widetilde{\Phi}_{t,0}(\tilde{x})v\| \le Ce^{t\lambda_u}\|v\|, \quad t \le 0, \tag{3.3}$$

$$v \in T_{\tilde{x}}\widetilde{\Lambda}_0 \Longleftrightarrow C^{-1}e^{|t|\lambda_c}\|v\| \le \|D\widetilde{\Phi}_{t,0}(\tilde{x})v\| \le Ce^{|t|\mu_c}\|v\|, \quad t \in \mathbb{R},$$

where the constant $C > 1$. The stable (resp. unstable) space $E_{\tilde{x}}^s$ (resp. $E_{\tilde{x}}^u$) is just the direct sum of the stable (resp. unstable) spaces at the hyperbolic equilibrium of each pendulum. In particular, Λ_0 is also a NHIM of the flow $\Phi_{t,0}$, with the same exponents $\lambda_s \le \mu_s < \lambda_c < 0 < \mu_c < \lambda_u \le \mu_u$.

On the other hand, there is also a family of homoclinic orbits parameterized by

$$p^0(\tau + t\bar{1}) = \left(p_1^0(\tau_1 + t), \cdots, p_n^0(\tau_n + t)\right), \qquad q^0(\tau + t\bar{1}) = \left(q_1^0(\tau_1 + t), \cdots, q_n^0(\tau_n + t)\right), \tag{3.4}$$

where $\tau = (\tau_1, \cdots, \tau_n) \in \mathbb{R}^n$ and $\bar{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Each parameter τ_i, $i \in \{1, \cdots, n\}$, represents the time shift for the i-th homoclinic orbit. $\left(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1})\right)$ is asymptotic to $(0,0)$ in the future with an

exponential rate at least μ_s, and in the past with an exponential rate at least λ_μ. Moreover, these homoclinic orbits form the stable manifold $W^s_{\widetilde{\Lambda}_0}$ and the unstable manifold $W^u_{\widetilde{\Lambda}_0}$ of the NHIM $\widetilde{\Lambda}_0$. In particular, the unstable and stable manifolds coincide, that is $W^s_{\widetilde{\Lambda}_0} = W^u_{\widetilde{\Lambda}_0}$.

3.2. **Persistence of normally hyperbolic invariant manifolds.** In the theory of normally hyperbolic invariant manifolds, it is well known that the NHIM along with its stable and unstable manifolds persist under small perturbations [29, 31, 39]. In general, the NHIM will only be finitely differentiable. The optimal regularity depends on the ratio of the normal exponents and the central exponents.

For the perturbed system $\mathcal{H}_\varepsilon = H_0 + \varepsilon H_1$ with $\varepsilon \neq 0$, we have the non-autonomous Hamilton's equations (2.3). By supplementing equations (2.3) with the equation $\dot{s} = 1$, we can consider the extended flow, denoted as $\widetilde{\Phi}_{t,\varepsilon}$, associated with the Hamiltonian $\mathcal{H}_\varepsilon(p, q, I, \varphi, s)$. Then, $\widetilde{\Phi}_{t,\varepsilon}$ becomes a C^{r-1} autonomous flow on $\widetilde{\mathcal{M}}$.

Following Appendix A.1.2, we set

$$\ell = \min\{\ell_u, \ell_s\} \tag{3.5}$$

where

$$\ell_u = \max\left\{k = 1, \cdots, r-1 \,:\, k < \frac{\mu_s}{\lambda_c}\right\} \quad \text{and} \quad \ell_s = \max\left\{k = 1, \cdots, r-1 \,:\, k < \frac{\lambda_u}{\mu_c}\right\}$$

and the exponents $\lambda_s \leq \mu_s < \lambda_c < 0 < \mu_c < \lambda_u \leq \mu_u$ are given in (3.3).

Remark 4. Note that ℓ_s and ℓ_u are only finite even when $r = \infty$ or ω. Taking the central exponents λ_c and μ_c sufficiently small if necessary, we can always let the indices $\ell_s \geq 2, \ell_u \geq 2$, and hence

$$\ell \geq 2.$$

In particular, in the case of $r \in [3, \infty)$, we can have $\ell = \ell_s = \ell_u = r - 1$ for ε sufficiently small since λ_c and μ_c can be chosen as close as desired to 0.

The argument of [36], is a transversality argument that only requires a few derivatives of the invariant manifolds and the perturbations involved.

Proposition 3.1. *Let $\mathcal{H}_\varepsilon$ satisfy conditions* **(H1)–(H2)**. *Then there exists $\varepsilon_0 > 0$ sufficiently small such that for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the flow $\widetilde{\Phi}_{t,\varepsilon}$ has a normally hyperbolic locally invariant manifold $\widetilde{\Lambda}_\varepsilon$ with the associated stable manifold $W^s_{\widetilde{\Lambda}_\varepsilon}$ and unstable manifold $W^u_{\widetilde{\Lambda}_\varepsilon}$. Moreover, the manifolds $\widetilde{\Lambda}_\varepsilon$ and $W^{u,s}_{\Lambda_\varepsilon}$ are C^ℓ differentiable with the index ℓ given in (3.5), and $\widetilde{\Lambda}_\varepsilon$ is diffeomorphic to $\widetilde{\Lambda}_0$.*

We give a sketch of the proof of Proposition 3.1 for the reader's convenience. We also refer to [19, 22, 35] for more details.

To prove Proposition 3.1, we first point out that the NHIM $\widetilde{\Lambda}_0$ of the unperturbed equations has non-empty boundary on which the flow $\widetilde{\Phi}_{t,0}$ is invariant, but the invariance on the boundary will be destroyed under perturbations in general. Then, just as pointed out in [29, 30], a standard treatment is to consider a slightly modified Hamiltonian. More precisely, we take two open domains U_1 and U_2 close enough to $\mathscr{D}$ and $\mathscr{B}$ respectively, and $\mathscr{D}_* \subset U_1 \subset \mathscr{D}$ and $\mathscr{B}_* \subset U_2 \subset \mathscr{B}$. Let $\rho(p, I) : \mathbb{R}^n \times \mathbb{R}^d \longrightarrow [0, 1]$

be a C^∞ smooth bump function such that $\rho|_{U_1 \times U_2} \equiv 1$, and $\rho(p, I) = 0$ for those points (p, I) outside $\mathcal{D} \times \mathcal{B}$. Then we define the modified Hamiltonian G_ε as follows

$$G_\varepsilon := H_0 + \varepsilon \rho H_1.$$

Clearly, $G_\varepsilon = H_0$ for $(p, I) \notin \mathcal{D} \times \mathcal{B}$, and $G_\varepsilon = \mathcal{H}_\varepsilon$ on $U_1 \times \mathbb{T}^n \times U_2 \times \mathbb{T}^d \times \mathbb{T}$. We use $\widetilde{\Phi}_{t,G_\varepsilon}$ to denote the associated Hamiltonian flow. Then, for $\varepsilon = 0$ the manifold $\widetilde{\Lambda}_{G_0} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}\}$ is normally hyperbolic and invariant under the flow $\widetilde{\Phi}_{t,G_0}$. Note that $\widetilde{\Lambda}_{G_0}$ has no boundary. Thus we can apply Theorem A.2 to the perturbed system G_ε to obtain a unique NHIM $\widetilde{\Lambda}_{G_\varepsilon}$ of the flow $\widetilde{\Phi}_{t,G_\varepsilon}$.

In general, the manifold $\widetilde{\Lambda}_{G_\varepsilon}$ constructed in this way depends on the nature of the modification on the boundary. Anyway, orbits that never pass through the modified region behave identically to those of the unmodified Hamiltonian $\mathcal{H}_\varepsilon$. Since $\widetilde{\Phi}_{t,G_\varepsilon}$ agrees with the original flow $\widetilde{\Phi}_{t,\varepsilon}$ on the domain $U_1 \times \mathbb{T}^n \times U_2 \times \mathbb{T}^d \times \mathbb{T}$, we therefore obtain a normally hyperbolic manifold $\widetilde{\Lambda}_\varepsilon$ that is locally invariant under $\widetilde{\Phi}_{t,\varepsilon}$.

The next step is to check the smoothness of $\widetilde{\Lambda}_\varepsilon$. Observe that $\widetilde{\Lambda}_\varepsilon$ is normally hyperbolic with slight changes on the normal and central exponents given in (3.3). We denote the perturbed exponents by

$$\lambda_{s,\varepsilon} \le \mu_{s,\varepsilon} < \lambda_{c,\varepsilon} < 0 < \mu_{c,\varepsilon} < \lambda_{u,\varepsilon} \le \mu_{u,\varepsilon}. \tag{3.6}$$

They are $O(\varepsilon)$-close to those in (3.3). This implies that the index ℓ defined in (3.5) would remain unchanged as long as ε is small enough. Hence, $\widetilde{\Lambda}_\varepsilon$ is C^ℓ smooth.

Finally, with the persistent manifold $\widetilde{\Lambda}_\varepsilon$, we obtain the local stable manifold $W^{s,loc}_{\widetilde{\Lambda}_\varepsilon}$ and the local unstable manifold $W^{u,loc}_{\widetilde{\Lambda}_\varepsilon}$, which can be prolonged to $W^s_{\widetilde{\Lambda}_\varepsilon}$ and $W^u_{\widetilde{\Lambda}_\varepsilon}$, respectively. This finishes the proof of Proposition 3.1.

3.3. **Transversal intersections and scattering maps.** The theory of scattering maps is developed to quantify the homoclinic excursions. To define the scattering map, some transversal intersection hypotheses are needed, see (B.2)-(B.3) in Appendix B. We can use the Melnikov method to measure these transversal intersection property in the perturbed system.

In our model, for $\varepsilon = 0$ the stable and unstable manifolds of the flow $\widetilde{\Phi}_{t,0}$ coincide:

$$W^s_{\widetilde{\Lambda}_0} = W^u_{\widetilde{\Lambda}_0} = \{(p^0(\tau), q^0(\tau), I, \varphi, s) : \tau \in \mathbb{R}^n, (I, \varphi, s) \in \mathcal{B}_* \times \mathbb{T}^{d+1}\}$$

In the case of $\varepsilon \ne 0$, $W^u_{\widetilde{\Lambda}_\varepsilon}$ and $W^s_{\widetilde{\Lambda}_\varepsilon}$ do not coincide in general, and possibly do not intersect transversely along homoclinic manifolds. To measure the splitting of manifolds we introduce the *Poincaré function* (or *Melnikov potential*),

$$L(\tau, I, \varphi, s) := -\int_{-\infty}^{\infty} \left[H_1\left(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \varphi + \omega(I)t, s + t\right) - H_1\left(0, 0, I, \varphi + \omega(I)t, s + t\right) \right] dt, \tag{3.7}$$

where $\omega(I) = (\omega_1(I), \cdots, \omega_n(I)) = Dh(I) \in C^{r-1}$, $\bar{1} = (1, \dots, 1) \in \mathbb{R}^n$ and the orbits (p^0, q^0) is given in (3.4). It is a convergent improper integral of the perturbation evaluated along homoclinic orbits of the unperturbed system. We stress that this integral is absolutely convergent, because $\left(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1})\right)$ converges exponentially fast to $(0, 0)$ as $t \to \pm\infty$.

By definition it is easily seen that

$$L(\tau + \sigma \bar{1}, I, \varphi, s) = L(\tau, I, \varphi - \omega(I)\sigma, s - \sigma), \quad \text{for all } \sigma \in \mathbb{R} \tag{3.8}$$

In particular, for the lower-dimensional case $n = 1$, $L(\tau, I, \varphi, s) = L(0, I, \varphi - \omega(I)\tau, s - \tau)$ for all $\tau \in \mathbb{R}$, which implies that the Poincaré function $L(\tau, I, \varphi, s)$ is periodic or quasi-periodic with respect to τ.

Remark 5 (**Regularity of** L). Note that the integral (3.7) is evaluated not on the perturbed homoclinic orbit but only on the unperturbed one. As H_1 is real analytic, it is not difficult to check that $L(\tau, I, \varphi, s)$ is C^{r-1} smooth. More precisely, the dependence on the variable τ is C^r, the dependence on the variable I is C^{r-1} and the dependence on the variables (φ, s) is C^ω. In particular, in the case when $r = \infty$ (resp. ω), the function $L(\tau, I, \varphi, s)$ is also C^∞ (resp. C^ω).

To verify the mechanism in [36], it suffices to verify two assumptions on L. Assumption **(H3a)** predicts that the perturbations generate homoclinic intersections and assumption **(H3b)** implies that the homoclinic intersections indeed generate changes in the action. Rather remarkably, both assumptions amount to properties of L.

The non-degenerate critical points of L would yield the existence of transverse homoclinic orbits for the perturbed system.

Given a point $(I, \varphi, s) \in \mathscr{B}_* \times \mathbb{T}^d \times \mathbb{T}$ and assume the map $\tau \in \mathbb{R}^n \longmapsto L(\tau, I, \varphi, s)$ has a non-degenerate critical point at τ^*, that is

$$\frac{\partial L}{\partial \tau}(\tau^*, I, \varphi, s) = 0, \quad \frac{\partial^2 L}{\partial \tau^2}(\tau^*, I, \varphi, s) \neq 0.$$

Then for $0 < |\varepsilon| \leq \varepsilon_0$ small enough, there exists a locally unique z^* of the form

$$z^* = z^*(\tau^*, I, \varphi, s) = (p^0(\tau^*) + O(\varepsilon), q^0(\tau^*) + O(\varepsilon), I, \varphi, s)$$

such that the unstable manifold $W^u_{\tilde{\Lambda}_\varepsilon}$ and the stable manifold $W^s_{\tilde{\Lambda}_\varepsilon}$ intersect transversally at z^*, i.e.

$$T_{z^*}\widetilde{\mathcal{M}} = T_{z^*}W^u_{\tilde{\Lambda}_\varepsilon} + T_{z^*}W^s_{\tilde{\Lambda}_\varepsilon}.$$

This therefore leads us to formulate the following assumption **(H3a)**.

(H3a) there exists an open set $U^- := \mathscr{I} \times \mathscr{J}$ with $\mathscr{I}$ a ball in $\mathscr{B}_*$ and $\mathscr{J}$ an open set in $\mathbb{T}^d \times \mathbb{T}$, such that for each point $(I, \varphi, s) \in U^-$, the map

$$\tau \in \mathbb{R}^n \longmapsto L(\tau, I, \varphi, s)$$

has a non-degenerate critical point τ^*. By the implicit function theorem, τ^* is locally given by a C^{r-1} function

$$\tau^* = \tau^*(I, \varphi, s).$$

We stress that in the present paper the size of the domain U^- in **(H3a)** does not need to be too large, but it is independent of ε since the expression we need to study does not involve ε. In applications, it suffices to verify the existence of non-degenerate critical point for a fixed point (I_0, φ_0, s_0), then by the implicit function theorem there is a neighborhood U^- of (I_0, φ_0, s_0) whose size is independent of ε, such that **(H3a)** holds.

In the case when $n = 1$, the identity $L(\tau, I, \varphi, s) = L(0, I, \varphi - \omega(I)\tau, s - \tau)$ holds. This implies that the one-dimensional map $\tau \longmapsto L(\tau, I_0, \varphi_0, s_0)$ always has critical points for each fixed (I_0, φ_0, s_0). In the case when $n > 1$, for each I_0 there exists (φ_0, s_0) such that the map $\tau \in \mathbb{R}^n \longmapsto L(\tau, I_0, \varphi_0, s_0)$ has critical points [25, 35]. Hence, the only content of **(H3a)** is that some of these critical points are non-degenerate.

With Proposition 3.1, the following result is well known. See for instance [19, 22, 35] for more details.

Proposition 3.2. *Let the Hamiltonian* (2.1) *satisfy conditions* **(H1)**, **(H2)** *and* **(H3a)**. *Then for each $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough, there exists a C^ℓ homoclinic manifold $\widetilde{\Gamma}_\varepsilon \subset W^s_{\widetilde{\Lambda}_\varepsilon} \cap W^u_{\widetilde{\Lambda}_\varepsilon}$, which can be parameterized by*

$$\widetilde{\Gamma}_\varepsilon = \left\{ z^* \left(\tau^*(I, \varphi, s), I, \varphi, s \right) \, : \, (I, \varphi, s) \in U^- \right\},$$

and the stable and unstable manifolds $W^{s,u}_{\widetilde{\Lambda}_\varepsilon}$ intersect transversally along $\widetilde{\Gamma}_\varepsilon$. Also, $\widetilde{\Gamma}_\varepsilon$ is transverse to the foliations of the stable/unstable manifolds. Moreover, $\widetilde{\Gamma}_\varepsilon$ can C^ℓ-smoothly extend to a manifold $\widetilde{\Gamma}_0 := \left\{ (p^0(\tau^*), q^0(\tau^*), I, \varphi, s) : (I, \varphi, s) \in U^- \right\}$ as $\varepsilon \to 0$.

As we can see from Remark 13 in Appendix B, taking U^- suitably small if necessary we can ensure the transversality conditions (B.2)–(B.3) are satisfied along $\widetilde{\Gamma}_\varepsilon$. Meanwhile, we consider the C^ℓ wave maps

$$\Omega^\varepsilon_+ : W^s_{\widetilde{\Lambda}_\varepsilon} \longrightarrow \widetilde{\Lambda}_\varepsilon \quad \text{and} \quad \Omega^\varepsilon_- : W^u_{\widetilde{\Lambda}_\varepsilon} \longrightarrow \widetilde{\Lambda}_\varepsilon,$$

which are projections along the stable and unstable leaves (see Appendix B). Then, the wave maps restricted on $\widetilde{\Gamma}_\varepsilon$, namely $\Omega^\varepsilon_\pm|_{\widetilde{\Gamma}_\varepsilon}$, are diffeomorphisms. In addition, by recalling the normal exponents in (3.6), for each $\tilde{x} \in \widetilde{\Gamma}_\varepsilon$ we can find a unique point $\tilde{x}_+ = \Omega^\varepsilon_+(\tilde{x})$ and a unique point $\tilde{x}_- = \Omega^\varepsilon_-(\tilde{x})$ satisfying

$$\text{dist}\left(\widetilde{\Phi}_{t,\varepsilon}(\tilde{x}), \widetilde{\Phi}_{t,\varepsilon}(\tilde{x}_+) \right) \le C e^{t \mu_{s,\varepsilon}} \quad \text{as } t \longrightarrow +\infty, \qquad \text{dist}\left(\widetilde{\Phi}_{t,\varepsilon}(\tilde{x}), \widetilde{\Phi}_{t,\varepsilon}(\tilde{x}_-) \right) \le C e^{t \lambda_{u,\varepsilon}} \quad \text{as } t \longrightarrow -\infty.$$

Consequently, each $\widetilde{\Gamma}_\varepsilon$ with $0 < |\varepsilon| \le \varepsilon_0$ is a *homoclinic channel*. This enables us to define the scattering map $\widetilde{\sigma}_\varepsilon := \sigma^{\widetilde{\Gamma}_\varepsilon}$ associated to the homoclinic channel $\widetilde{\Gamma}_\varepsilon$, that is

$$\widetilde{\sigma}_\varepsilon = \Omega^\varepsilon_+\Big|_{\widetilde{\Gamma}_\varepsilon} \circ \left(\Omega^\varepsilon_-\Big|_{\widetilde{\Gamma}_\varepsilon} \right)^{-1} \, : \, \Omega^\varepsilon_-(\widetilde{\Gamma}_\varepsilon) \longrightarrow \Omega^\varepsilon_+(\widetilde{\Gamma}_\varepsilon)$$

$$\tilde{x}_- \longmapsto \tilde{x}_+.$$

The scattering map $\widetilde{\sigma}_\varepsilon$ above is a C^ℓ diffeomorphism. By Proposition 3.2 the homoclinic manifold $\widetilde{\Gamma}_\varepsilon$ can extend smoothly to a limiting manifold $\widetilde{\Gamma}_0$. Even $\widetilde{\Gamma}_0$ is not a transversal intersection, $\widetilde{\sigma}_\varepsilon$ can still C^ℓ-smoothly extend to the identity map $\widetilde{\sigma}_0 = \text{Id}$, as $\varepsilon \to 0$.

3.4. **Perturbative formulas for the scattering maps.** The Melnikov method can also be used to estimate the effect of the perturbations on the scattering map. As was shown in [21], the symplectic property allows to give perturbative formulas for the Hamiltonian which generates the deformation of a family of symplectic scattering maps. This requires the dimension of the NHIM to be even while our $\widetilde{\Lambda}_\varepsilon$ mentioned above is of odd dimensions. To overcome this difficulty, it is standard to consider an autonomous Hamiltonian defined by

$$\widetilde{\mathcal{H}}_\varepsilon(p, q, I, \varphi, A, s) = A + \mathcal{H}_\varepsilon(p, q, I, \varphi, s) = A + H_0(p, q, I) + \varepsilon H_1(p, q, I, \varphi, s), \tag{3.9}$$

where $(A, s) \in \mathbb{R} \times \mathbb{T}$ are symplectically conjugate variables. The extended phase space is endowed with the standard symplectic structure $\widetilde{\omega} = \sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{i=1}^{d} dI_i \wedge d\varphi_i + dA \wedge ds$. Then, the motions of the conjugate variables (A, t) are governed by

$$\dot{A} = -\partial_s \mathcal{H}_\varepsilon(p, q, I, \varphi, s), \qquad \dot{s} = 1.$$

However, the variable A does not play any dynamical role, because A does not appear in any of the ODEs for any of the coordinates, including itself. Consequently, by abuse of notation, we continue to use $\widetilde{\Phi}_{t,0}$ to denote the unperturbed flow and use

$$\widetilde{\Lambda}_0 = \{(0, 0, I, \varphi, A, s) \ : \ (I, \varphi, s) \in \mathscr{B}_* \times \mathbb{T}^d \times \mathbb{T}, A \in \mathbb{R}\},$$

to denote the normally hyperbolic invariant manifold, which is $(2d+2)$-dimensional. Since A does not play any dynamical role, the results obtained in the previous sections remain true for $\widetilde{\mathcal{H}}_\varepsilon$. Then we continue to use $\widetilde{\Lambda}_\varepsilon$ and $W^{s,u}_{\widetilde{\Lambda}_\varepsilon}$, respectively, to denote the NHIM and the associated stable and unstable manifolds for the flow $\widetilde{\Phi}_{t,\varepsilon}$. Also, we have the scattering map $\widetilde{\sigma}_\varepsilon := \sigma^{\widetilde{\Gamma}_\varepsilon}$ associated to the homoclinic channel $\widetilde{\Gamma}_\varepsilon$. Now that $\widetilde{\Lambda}_\varepsilon$ has even dimensions, the map $\widetilde{\sigma}_\varepsilon$ is symplectic (see [21]).

The perturbed NHIM $\widetilde{\Lambda}_\varepsilon$ can be described in terms of the coordinates $(I, \varphi, A, s) \in \widetilde{\Lambda}_0$. In fact, there is a unique C^ℓ-smooth family of symplectic parametrization $\widetilde{k}_\varepsilon : \widetilde{\Lambda}_0 \to \widetilde{\Lambda}_\varepsilon$, with $\widetilde{k}_0 = \mathrm{Id}$, satisfying

$$\widetilde{k}_\varepsilon^* \widetilde{\omega} = \widetilde{k}_0^* \widetilde{\omega}, \qquad \frac{d}{d\varepsilon} k_\varepsilon \in \mathrm{Span}(E^{s,\varepsilon} \oplus E^{u,\varepsilon}).$$

Then we can express the scattering map $\widetilde{\sigma}_\varepsilon$ on the reference manifold $\widetilde{\Lambda}_0$ by:

$$\widetilde{s}_\varepsilon = \widetilde{k}_\varepsilon^{-1} \circ \widetilde{\sigma}_\varepsilon \circ \widetilde{k}_\varepsilon \ : \ \widetilde{k}_\varepsilon^{-1}\big(\Omega_-^\varepsilon(\widetilde{\Gamma}_\varepsilon)\big) \subset \widetilde{\Lambda}_0 \longrightarrow \widetilde{k}_\varepsilon^{-1}\big(\Omega_+^\varepsilon(\widetilde{\Gamma}_\varepsilon)\big) \subset \widetilde{\Lambda}_0.$$

It is clear that $\widetilde{s}_\varepsilon$ are the expression of $\widetilde{\sigma}_\varepsilon$ in the same coordinate system $\widetilde{\Lambda}_0$, and hence $\widetilde{s}_\varepsilon \in C^\ell$ with $\ell \geq 2$. Moreover, using the deformation theory this family of symplectic maps $\widetilde{s}_\varepsilon$ can be generated by a Hamiltonian vector field [21]: there exists a Hamiltonian function $\mathscr{S}_\varepsilon$ such that

$$\frac{d}{d\varepsilon} \widetilde{s}_\varepsilon = \mathscr{X}_\varepsilon \circ \widetilde{s}_\varepsilon, \qquad \iota_{\mathscr{X}_\varepsilon} \widetilde{\omega}_0 = d\mathscr{S}_\varepsilon,$$

where $\widetilde{\omega}_0 := \widetilde{k}_0^* \widetilde{\omega} = \sum_{i=1}^{d} dI_i \wedge d\varphi_i + dA \wedge ds$. The Hamiltonian $\mathscr{S}_\varepsilon = \mathscr{S}_0 + O(\varepsilon)$ and

$$\mathscr{S}_0(I, \varphi, A, s) := -L\big(\tau^*(I, \varphi, s), I, \varphi, s\big).$$

Here, L is the Melnikov potential and τ^* is given in **(H3a)**. The function $L(\tau^*(I, \varphi, s), I, \varphi, s) : U^- \to \mathbb{R}$ is defined in a domain U^- whose size is independent of ε. Thus, we compute the perturbed scattering map $\widetilde{s}_\varepsilon$ up to the first order with respect to the size of the perturbation:

$$\widetilde{s}_\varepsilon(I, \varphi, A, s) = (I, \varphi, A, s) + \varepsilon \mathbf{J} \nabla \mathscr{S}_0 + O(\varepsilon^2), \qquad (I, \varphi, s) \in U^-, \ A \in \mathbb{R} \tag{3.10}$$

with $\mathbf{J}$ the canonical matrix of the symplectic form $\widetilde{\omega}_0$. See [21].

We infer from $\frac{\partial L}{\partial \tau}(\tau^*(I, \varphi, s), I, \varphi, s) = 0$ that

$$\frac{\partial \mathscr{S}_0}{\partial I}(I, \varphi, A, s) = -\frac{\partial L}{\partial \tau}(\tau^*, I, \varphi, s)\frac{\partial \tau^*}{\partial I} - \frac{\partial L}{\partial I}(\tau^*, I, \varphi, s) = -\frac{\partial L}{\partial I}(\tau^*, I, \varphi, s).$$

Similarly,

$$\frac{\partial \mathscr{S}_0}{\partial \varphi} = -\frac{\partial L}{\partial \varphi}(\tau^*, I, \varphi, s), \qquad \frac{\partial \mathscr{S}_0}{\partial A} = 0, \qquad \frac{\partial \mathscr{S}_0}{\partial s} = -\frac{\partial L}{\partial s}(\tau^*, I, \varphi, s).$$

Here and subsequently, we use $\frac{\partial L}{\partial I}$ to denote the partial derivative of the Poincaré function $L(\tau, I, \varphi, s)$ with respect to the second variable, $\frac{\partial L}{\partial \varphi}$ to denote the partial derivative with respect to the third variable, and $\frac{\partial L}{\partial s}$ to denote the partial derivative with respect to the fourth variable. Now, formula (3.10) becomes

$$\widetilde{s}_\varepsilon(I, \varphi, A, s) = \left(I + \varepsilon \frac{\partial L}{\partial \varphi}(\tau^*, I, \varphi, s), \varphi - \varepsilon \frac{\partial L}{\partial I}(\tau^*, I, \varphi, s), A + \frac{\partial L}{\partial s}(\tau^*, I, \varphi, s), s\right) + O(\varepsilon^2), \tag{3.11}$$

for $(I, \varphi, s) \in U^-$ and $A \in \mathbb{R}$.

Next, we introduce the auxiliary function $\mathscr{L}(I, \varphi, s) := L(\tau^*(I, \varphi, s), \varphi, s)$. We claim that

$$\mathscr{L}(I, \varphi - \omega(I)s, 0) = \mathscr{L}(I, \varphi, s), \qquad (I, \varphi, s) \in U^-. \tag{3.12}$$

Indeed, if $\tau^*(I, \varphi, s)$ is a critical point of the map $\tau \in \mathbb{R}^n \longmapsto L(\tau, I, \varphi, s)$, then $\tau^*(I, \varphi, s) - s\bar{1}$ with $\bar{1} = (1, \ldots, 1) \in \mathbb{R}^n$ is a critical point of the map $\tau \in \mathbb{R}^n \longmapsto L(\tau + s\bar{1}, I, \varphi, s)$. Thanks to (3.8), $L(\tau + s\bar{1}, I, \varphi, s) = L(\tau, I, \varphi - \omega(I)s, 0)$, which implies

$$\tau^*(I, \varphi - \omega(I)s, 0) = \tau^*(I, \varphi, s) - s\bar{1}.$$

Then our claim (3.12) follows immediately. This enables us to introduce the new variables

$$\theta = \varphi - \omega(I)s,$$

and define the *reduced Poincaré function*,

$$\mathscr{L}^*(I, \theta) := \mathscr{L}(I, \varphi - \omega(I)s, 0) = \mathscr{L}(I, \varphi, s).$$

The reduced Poincaré function is well defined on the following $2d$-dimensional set

$$\mathrm{Dom}(\mathscr{L}^*) = \{(I, \theta) \in \mathbb{R}^d \times \mathbb{T}^d : \text{ there exists } s \in \mathbb{T} \text{ such that } (I, \theta + \omega(I)s, s) \in U^-\}.$$

Remark 6. By what we have shown above,

$$\mathscr{L}^*(I, \theta) = L(\tau^*(I, \theta, 0), I, \theta, 0) : \mathrm{Dom}(\mathscr{L}^*) \longrightarrow \mathbb{R}.$$

Since $\tau^*(I, \theta, 0)$ is a critical point for the map $\tau \mapsto L(\tau, I, \theta, 0)$, we deduce

$$\frac{\partial \mathscr{L}^*}{\partial I}(I, \theta) = \frac{\partial L}{\partial I}(\tau^*(I, \theta, 0), I, \theta, 0) \quad \text{and} \quad \frac{\partial \mathscr{L}^*}{\partial \theta}(I, \theta) = \frac{\partial L}{\partial \theta}(\tau^*(I, \theta, 0), I, \theta, 0).$$

The reduced Poincaré function $\mathscr{L}^*(I, \theta)$ plays a crucial role in understanding the scattering map of the time-2π map of the Hamiltonian flow, see Proposition 3.3 below. To explain this, we recall the extended Hamiltonian $\widetilde{\mathscr{H}}_\varepsilon = A + \mathscr{H}_\varepsilon$ in (3.9) which is 2π-periodic in the variable s. Fix an energy $\widetilde{\mathscr{H}}_\varepsilon = E$ for some E, and then restrict to a Poincaré section $\{s = 2\pi\}$ for the Hamiltonian flow $\widetilde{\Phi}_{t,\varepsilon}$. Hence we obtain a $2(n+d)$-dimensional space in $\mathscr{M}$ with the coordinates (p, q, I, φ), and use f_ε to denote the first return map. f_ε is exactly the time-2π map for the flow of the Hamiltonian $\mathscr{H}_\varepsilon$, so f_ε is symplectic. The $2d$-dimensional manifold $\Lambda_0 = \{(0, 0, I, \varphi) : (I, \varphi) \in \mathscr{B}_* \times \mathbb{T}^d\}$ is a NHIM for f_0. Under perturbations, there are manifolds Λ_ε and symplectic parameterizations k_ε such that

$$k_\varepsilon : \Lambda_0 \to \Lambda_\varepsilon, \quad k_\varepsilon(\Lambda_0) = \Lambda_\varepsilon.$$

In fact, Λ_ε is exactly, by omitting the variable A since it does not play any dynamical role, the intersection of $\widetilde{\Lambda}_\varepsilon$ and the section $\{s = 2\pi\}$. k_ε is exactly the restriction of $\widetilde{k}_\varepsilon$ to the section $\{s = 2\pi\}$. Thus, Λ_ε is the normally hyperbolic (locally) invariant manifold for the map f_ε.

Analogously, f_ε has the corresponding homoclinic channel Γ_ε and the scattering map $\sigma_\varepsilon := \sigma^{\Gamma_\varepsilon}$: $\Omega_-^\varepsilon(\Gamma_\varepsilon) \longrightarrow \Omega_+^\varepsilon(\Gamma_\varepsilon)$. As mentioned before, it is more convenient to describe the scattering map in the same coordinate (I, θ) via:

$$s_\varepsilon := k_\varepsilon^{-1} \circ \sigma_\varepsilon \circ k_\varepsilon : \ k_\varepsilon^{-1}\big(\Omega_-^\varepsilon(\Gamma_\varepsilon)\big) \subset \Lambda_0 \longrightarrow k_\varepsilon^{-1}\big(\Omega_+^\varepsilon(\Gamma_\varepsilon)\big) \subset \Lambda_0. \tag{3.13}$$

Clearly, s_ε are still symplectic, and one can choose a common domain $\mathrm{Dom}(\mathscr{L}^*)$ for all s_ε. Therefore, combining formula (3.11) and Remark 6 we conclude the following result:

Proposition 3.3. *The C^ℓ parameterized scattering maps $s_\varepsilon(I, \theta) : \mathrm{Dom}(\mathscr{L}^*) \subset \Lambda_0 \longrightarrow \Lambda_0$ given in (3.13), have the expansion:*

$$s_\varepsilon(I, \theta) = \left(I + \varepsilon \frac{\partial \mathscr{L}^*}{\partial \theta}(I, \theta), \ \theta - \varepsilon \frac{\partial \mathscr{L}^*}{\partial I}(I, \theta)\right) + O(\varepsilon^2). \tag{3.14}$$

where the O symbol means estimates in the $C^{\ell-1}$ sense of the reminder.

See [19, 21] for more details. In view of Proposition 3.3 we can formulate the following assumption:

(H3b) let $\tau^*(I, \theta, 0)$ be the C^{r-1} function determined in **(H3a)**. Assume the reduced Poincaré function satisfies that $\mathbf{J}\nabla\mathscr{L}^*(I, \theta)$ is transverse to the level set $\{I = I_0\}$ at some point $(I_0, \theta_0) \in \mathrm{Dom}(\mathscr{L}^*)$. That is equivalent to saying

$$\frac{\partial \mathscr{L}^*}{\partial \theta}(I_0, \theta_0) \neq 0. \tag{3.15}$$

It is worth noting that assumption **(H3b)** ensures that the vector field $\dot{x} = \mathbf{J}\nabla\mathscr{L}^*(x)$ always has a trajectory along which the action variables I move a quantity independent of ε.

It is remarkable that both assumptions **(H3a)** and **(H3b)** amount to properties of the Melnikov potential L in (3.7). The gist of the genericity argument is that, if they happen to fail for some H_1, a small modification of the H_1 will make them true. Both can be considered as transversality properties on the functions. We also remark that the perturbations needed to restore **(H3a), (H3b)** are themselves rather arbitrary. Hence, the assumptions can only fail for H_1 inside a manifold of infinite codimension in the space of maps.

3.5. **Geometric construction of the diffusing orbits.** In this paper, the construction of diffusing orbits is based on the geometric mechanism in [36]. This mechanism differs from earlier works, because it relies only on the outer dynamics. There are almost no assumptions on the inner dynamics (only the Poincaré recurrence is needed), because its invariant objects (e.g., primary and secondary tori, Aubry-Mather sets) on the NHIM are not used at all. The basic idea of this new mechanism is as follows: Assume that the Poincaré recurrence holds. Given any pseudo-orbit, generated by the successive iterates of the scattering map s_ε, that moves $O(1)$ with respect to the perturbation, then there exists a true orbit moving a quantity of $O(1)$ for the original Hamiltonian system. Of course, if the inner dynamics in the NHIM has no Poincaré recurrence, then there exist diffusing orbits determined just by the inner dynamics.

Through the parameterization k_ε, the restriction of f_ε to Λ_ε can be expressed on the same reference manifold Λ_0. We use $\widehat{f}_\varepsilon|_{\Lambda_0}$ to denote this restriction map:

$$\widehat{f}_\varepsilon|_{\Lambda_0} = (k_\varepsilon)^{-1} \circ f_\varepsilon|_{\Lambda_\varepsilon} \circ k_\varepsilon \; : \; \Lambda_0 \to \Lambda_0.$$

Theorem 3.4. *For $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ we have the scattering map given in* (3.14)*:*

$$s_\varepsilon(I, \theta) = \left(I + \varepsilon \frac{\partial \mathscr{L}^*}{\partial \theta}(I, \theta), \; \theta - \varepsilon \frac{\partial \mathscr{L}^*}{\partial I}(I, \theta) \right) + O(\varepsilon^2).$$

Assume that there is a point $x_0 = (I_0, \theta_0) \in Dom(\mathscr{L}^*) \subset \Lambda_0$ and

$$\nabla \mathscr{L}^*(I_0, \theta_0) \neq 0.$$

Let $\gamma : [0,1] \to \Lambda_0$ be an integral curve through x_0 induced by the Hamiltonian vector field $\dot{x} = \mathbf{J} \nabla \mathscr{L}^*(x)$. Suppose that there is a neighborhood $V \subset Dom(\mathscr{L}^*)$ of $\gamma([0,1])$ such that almost every point in V is recurrent for $\widehat{f}_\varepsilon|_{\Lambda_0}$. Denote by $\gamma_\varepsilon := k_\varepsilon \circ \gamma$ the corresponding curve in Λ_ε.

Then there exist a small $\varepsilon_1 = \varepsilon_1(H_1) \in (0, \varepsilon_0)$ and a constant $K = K(H_1) > 0$, such that for every $0 < |\varepsilon| < \varepsilon_1$ and every $\delta > 0$, there is an orbit $\{z_i\}_{i=0,\cdots,m}$ of the diffeomorphism f_ε where $m = O(1/\varepsilon)$ and

$$z_{i+1} = f_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i \in \mathbb{Z}^+,$$

and

$$\mathrm{dist}(z_i, \gamma_\varepsilon(t_i)) < \delta + K\varepsilon,$$

where $t_i = i \cdot \varepsilon$ and $0 = t_0 < t_1 < \cdots < t_m \leq 1$.

We provide a sketch of the proof in Appendix C for the reader's convenience. One can also refer to [36, Theorem 3.11] for a complete proof.

Since the recurrence assumption is satisfied automatically in our Hamiltonian model if there are no unbounded orbits in the manifold, under hypotheses **(H3a), (H3b)**, that either there are unbounded orbits in the NHIM or that there are sequences of homoclinic excursions that follow the sequence. In either of the two sides of the alternative, there is diffusion.

Theorem 3.5. *Suppose that $\mathscr{H}_\varepsilon = H_0 + \varepsilon H_1$ satisfies conditions **(H1), (H2), (H3a)** and **(H3b)**, and I_0 is the point given in* (3.15)*. Then there exist $\varepsilon_1 = \varepsilon_1(H_1) > 0$ and $\rho = \rho(H_1) > 0$, such that for any $0 < |\varepsilon| < \varepsilon_1$ and any $\delta > 0$, the Hamiltonian flow admits a trajectory $\big(p(t), q(t), I(t), \varphi(t)\big)$ whose I-component satisfy*

$$\sup_{t>0} \| I(t) - I(0) \| \geq \rho,$$

and $\| I(0) - I_0 \| < \delta + K\varepsilon$. The constant $K = K(H_1)$ does not depend on ε.

The proof is postponed to Appendix C, which comes mainly from [36]. Finally, we also refer to [37] for the result using accessibility and several scattering maps.

4. PROOF OF MAIN RESULT

This section aims to prove our main theorems given in Section 2. Thanks to Theorem 3.5, it is sufficient to show that the non-degeneracy assumptions **(H3a)**–**(H3b)** are generic properties in the analytic category. Note that assumption **(H3b)** depends on the choice of the function τ^* given in assumption **(H3a)**.

Now, we first verify that **(H3a)** is a generic property in the analytic category. According to (3.7), the Melnikov potential is

$$L(\tau, I, \varphi, s) = - \int_{-\infty}^{\infty} \left[H_1(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \varphi + \omega(I)t, s + t) - H_1(0, 0, I, \varphi + \omega(I)t, s + t) \right] dt. \quad (4.1)$$

where $p^0(\tau + t\bar{1}) = \left(p_1^0(\tau_1 + t), \cdots, p_n^0(\tau_n + t) \right)$ and $q^0(\tau + t\bar{1}) = \left(q_1^0(\tau_1 + t), \cdots, q_n^0(\tau_n + t) \right)$.

Theorem 4.1. *Let $\kappa > 0$. For each point $(I_0, \varphi_0, s_0) \in \mathscr{B}_* \times \mathbb{T}^d \times \mathbb{T}$, the set of perturbations H_1 satisfying the following non-degeneracy property **(R)**, is C_κ^ω dense and C^3 open (which also implies C_κ^ω-openness).*

(R) *there exists an open set $U^- := \mathscr{I} \times \mathscr{J}$ containing the point (I_0, φ_0, s_0) with $\mathscr{I}$ a ball in $\mathscr{B}_*$ and $\mathscr{J}$ an open set in $\mathbb{T}^d \times \mathbb{T}$, such that for each point $(I, \varphi, s) \in U^-$ the map*

$$\tau \in \mathbb{R}^n \longmapsto L(\tau, I, \varphi, s)$$

has a non-degenerate critical point τ^*, i.e.,

$$\frac{\partial L}{\partial \tau}(\tau^*, I, \varphi, s) = 0, \quad \det \left[\frac{\partial^2 L}{\partial \tau_i \tau_j}(\tau^*, I, \varphi, s) \right]_{1 \le i, j \le n} \ne 0.$$

By the implicit function theorem, τ^* is locally given by

$$\tau^* = \tau^*(I, \varphi, s).$$

Before proving it, we need some lemmas.

Lemma 4.2. *For each $\kappa > 0$ and each $a \in \mathbb{R} \setminus \{0\}$, there is a sequence of real analytic functions $f_l \in C_\kappa^\omega(\mathbb{T})$, $l = 1, \cdots, n$, satisfying*

$$\int_{-\infty}^{\infty} \left[f_l(q_l^0(t)) - f_l(0) \right] \cdot e^{iat} \, dt \ne 0,$$

where $e^{iat} = \cos at + i \sin at$, and $q_l^0(t)$ is the q_l-coordinate of the unperturbed homoclinic orbit (p^0, q^0).

Proof. For simplicity we only verify it for $l = 1$ and the others are similar. It is sufficient to prove that there exists a function f_1 in the set $X := \{\cos kx, \sin kx : k \in \mathbb{Z}\}$, such that

$$\int_{-\infty}^{\infty} \left[f_1(q_1^0(t)) - f_1(0) \right] \cdot e^{iat} \, dt \ne 0.$$

Suppose that for all $f \in X$, $\int_{-\infty}^{\infty} [f(q_1^0(t)) - f(0)] \cdot e^{iat} \, dt = 0$, then using the theory of Fourier analysis it is not difficult to prove that

$$\int_{-\infty}^{\infty} \left[\phi(q_1^0(t)) - \phi(0) \right] \cdot e^{iat} \, dt = 0, \quad \text{for each } \phi \in C^1(\mathbb{T}). \quad (4.2)$$

On the other hand, recalling that $q_1^0(t)$ converges exponentially to 0 as t tends to $\pm\infty$, we can find a small closed interval $J \subset \mathbb{T} \setminus \{0\}$, such that the orbit $q_1^0(t)$ passes through J *when and only when $t \in$* $[\bar{t} - \sigma, \bar{t} + \sigma]$ for some $\bar{t} \in \mathbb{R}, \sigma > 0$. By narrowing the interval J if necessary, we can let $\sigma < \frac{\pi}{4a}$.

Let us pick a non-negative function $h \in C^1(\mathbb{T})$ satisfying $h(0) = 0$ and the support $\text{supp}\, h = J$, then

$$\int_{-\infty}^{\infty} \left[h(q_1^0(t)) - h(0) \right] \cdot e^{iat} \, dt = \int_{\bar{t}-\sigma}^{\bar{t}+\sigma} h(q_1^0(t)) \cdot e^{iat} \, dt = e^{ia\bar{t}} \int_{-\sigma}^{\sigma} h(q_1^0(t+\bar{t})) \cdot e^{iat} \, dt \neq 0,$$

since $h(q_1^0(t+\bar{t}))$ and $\cos at$ are both positive when $t \in (-\sigma, \sigma)$. This contradicts (4.2). Now the lemma follows immediately from what we have proved. $\qquad\square$

Lemma 4.3. *For every point $(I, \varphi, s) \in \mathscr{B}_* \times \mathbb{T}^d \times \mathbb{T}$, the Melnikov potential $L(\tau, I, \varphi, s)$ satisfies*

$$\inf_{\tau \in \mathbb{R}^n} \left\| \frac{\partial L}{\partial \tau}(\tau, I, \varphi, s) \right\| = 0, \tag{4.3}$$

where the norm $\left\| \frac{\partial L}{\partial \tau} \right\| := \sum_{i=1}^{n} \left| \frac{\partial L}{\partial \tau_i} \right|$.

Proof. Fix a point $(\hat{I}, \hat{\varphi}, \hat{s}) \in \mathscr{B}_* \times \mathbb{T}^d \times \mathbb{T}$. If $\frac{\partial L}{\partial \tau}(\tau, \hat{I}, \hat{\varphi}, \hat{s}) = 0$ is attained at some point $\tau \in \mathbb{R}^n$, then we have finished the proof.

For the case where $\frac{\partial L}{\partial \tau}(\tau, \hat{I}, \hat{\varphi}, \hat{s})$ is always non-zero, we assume by contradiction that there is $\delta > 0$ such that

$$\left\| \frac{\partial L}{\partial \tau}(\tau, \hat{I}, \hat{\varphi}, \hat{s}) \right\| \geq \delta, \quad \text{for all } \tau \in \mathbb{R}^n. \tag{4.4}$$

Observe that the homoclinic orbit (p^0, q^0) is contained in a compact and convex set $D \subset \mathbb{R}^n \times \mathbb{T}^n$, so we can define

$$M(\hat{I}) := \max_{i=1,\cdots,n} \left\{ \sup_{(p,q)\in D, \, (\varphi,s)\in\mathbb{T}^d\times\mathbb{T}} \left| \frac{\partial H_1}{\partial p_i}(p, q, \hat{I}, \varphi, s) \right| + \left| \frac{\partial H_1}{\partial q_i}(p, q, \hat{I}, \varphi, s) \right| \right\},$$

which is finite and bounded. Using the mean value theorem to (4.1), for every $\tau \in \mathbb{R}^n$ we have

$$|L(\tau, \hat{I}, \hat{\varphi}, \hat{s})| \leq \sum_{i=1}^{n} \int_{-\infty}^{\infty} M(\hat{I}) \cdot |p_i^0(t+\tau_i)| + M(\hat{I}) \cdot |q_i^0(t+\tau_i)| \, dt$$
$$= M(\hat{I}) \cdot \sum_{i=1}^{n} \int_{-\infty}^{\infty} |p_i^0(t)| + |q_i^0(t)| \, dt \leq CM(\hat{I}), \tag{4.5}$$

where C is a constant, and the last inequality is a consequence of the fact that $(p_i^0(t), q_i^0(t))$ converges exponentially to $(0, 0)$ as $t \to \pm\infty$. Similarly, as the derivative $(\dot{p}_i^0(t), \dot{q}_i^0(t))$ also converges exponentially to $(0, 0)$ as $t \to \pm\infty$, we deduce that

$$\left\| \frac{\partial L}{\partial \tau}(\tau, \hat{I}, \hat{\varphi}, \hat{s}) \right\| = \sum_{i=1}^{n} \left| \frac{\partial L}{\partial \tau_i}(\tau, \hat{I}, \hat{\varphi}, \hat{s}) \right|$$
$$\leq M(\hat{I}) \cdot \sum_{i=1}^{n} \int_{-\infty}^{\infty} |\dot{p}_i^0(t+\tau_i)| + |\dot{q}_i^0(t+\tau_i)| \, dt \tag{4.6}$$
$$= M(\hat{I}) \cdot \sum_{i=1}^{n} \int_{-\infty}^{\infty} |\dot{p}_i^0(t)| + |\dot{q}_i^0(t)| \, dt \leq C'M(\hat{I}),$$

where $C' > 0$ is a constant.

Recall that $(\hat{I}, \hat{\varphi}, \hat{s})$ is fixed, then we consider an auxiliary differential equation:

$$\dot{x} = \frac{\partial L}{\partial \tau}(x, \hat{I}, \hat{\varphi}, \hat{s}), \quad x \in \mathbb{R}^n.$$

From (4.6) we see that the vector field above is bounded, which implies the flow is complete, that is all solutions are well defined for $t \in \mathbb{R}$. We pick one solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ and consider the one-dimensional function $t \longmapsto L(x(t), \hat{I}, \hat{\varphi}, \hat{s})$. Then, for any $T > 0$ we obtain

$$
\begin{aligned}
L(x(T), \hat{I}, \hat{\varphi}, \hat{s}) - L(x(0), \hat{I}, \hat{\varphi}, \hat{s}) &= \int_0^T \sum_{i=1}^n \left| \frac{\partial L}{\partial \tau_i}(x(t), \hat{I}, \hat{\varphi}, \hat{s}) \right|^2 dt \\
&\geq \frac{1}{n} \int_0^T \left\| \frac{\partial L}{\partial \tau}(x(t), \hat{I}, \hat{\varphi}, \hat{s}) \right\|^2 dt \\
&\geq \frac{\delta^2}{n} T,
\end{aligned}
\tag{4.7}
$$

Here, we have used (4.4) in the last inequality. As T can be arbitrarily large, the estimate (4.7) yields $\sup_{\tau \in \mathbb{R}^n} |L(\tau, \hat{I}, \hat{\varphi}, \hat{s})| = +\infty$. This contradicts the boundedness estimate (4.5). Therefore, we finish the proof of (4.3). $\qquad\square$

Remark 7. The estimates (4.5)–(4.6) also imply that the Melnikov potential L and its partial derivative $\partial L / \partial \tau$ are uniformly bounded in (τ, φ, s), that is for each I there is a constant $C(I) > 0$ such that

$$|L(\tau, I, \varphi, s)| \leq C(I), \quad \left\| \frac{\partial L}{\partial \tau}(\tau, I, \varphi, s) \right\| \leq C(I), \quad \text{for all } (\tau, \varphi, s) \in \mathbb{R}^n \times \mathbb{T}^d \times \mathbb{T}.$$

Remark 8 (**Existence of critical points**). We stress that in the case when $n = 1$, the identity $L(\tau, I, \varphi, s) = L(0, I, \varphi - \omega(I)\tau, s - \tau)$ holds, so the one-dimensional map $\tau \longmapsto L(\tau, I_0, \varphi_0, s_0)$ always has critical points for each fixed (I_0, φ_0, s_0). See [19].

In the case when $n > 1$, for each I_0 there exists (φ_0, s_0) such that the map $\tau \in \mathbb{R}^n \longmapsto L(\tau, I_0, \varphi_0, s_0)$ has critical points. See [35].

Now, we proceed to prove Theorem 4.1.

Proof of Theorem 4.1. The openness is evident. In fact, a non-degenerate critical point for the map $\tau \mapsto L(\tau, I, \varphi, s)$ remains non-degenerate after a sufficiently small C^2 perturbation, which therefore gives C^3-openness and also C_κ^ω-openness. Here, C^3-smoothness is necessary because H_1 needs to satisfy the lowest regularity (i.e. C^3) of the unperturbed system H_0 so that all the results obtained in Section 3 are still valid.

Then it remains to show that the existence of non-degenerate critical points is a dense property in the C_κ^ω topology. The proof splits into two steps.

Step 1: Fix the point (I_0, φ_0, s_0). If the map $\tau \longmapsto L(\tau, I_0, \varphi_0, s_0)$ has no critical points, we will show that there is an arbitrarily small perturbation to H_1 to create critical points. For this purpose, we take a small number $\delta > 0$ and add a small perturbation $\delta_2 H_2$ to H_1 with $\delta_2 \in (0, \delta/2)$, and H_2 is of the form

$$H_2(q, t) = \sum_{i=1}^n f_i(q_i) \cdot \cos(t + b_i) \in C_\kappa^\omega,$$

the coefficients $\{b_i\}_{i=1}^n$ and the analytic functions $\{f_i \in C_\kappa^\omega(\mathbb{T})\}_{i=1}^n$ will be determined later. By multiplying a constant if necessary, we can always let $\|H_2\|_\kappa < 1$. Hence the new Melnikov potential, denoted by L^{δ_2}, associated to the Hamiltonian $H_0 + \varepsilon(H_1 + \delta_2 H_2)$ is

$$
\begin{aligned}
L^{\delta_2}(\tau, I, \varphi, s) &= L(\tau, I, \varphi, s) - \delta_2 \sum_{i=1}^n \int_{-\infty}^\infty \left(f_i(q_i^0(t + \tau_i)) - f_i(0) \right) \cdot \cos(s + t + b_i)\, dt \\
&= L(\tau, I, \varphi, s) - \delta_2 \sum_{i=1}^n \int_{-\infty}^\infty \left(f_i(q_i^0(t)) - f_i(0) \right) \cdot \cos(s - \tau_i + b_i + t)\, dt \\
&= L(\tau, I, \varphi, s) + \delta_2 \sum_{i=1}^n \left[A_{i,1} \cdot \cos(s - \tau_i + b_i) - A_{i,2} \cdot \sin(s - \tau_i + b_i) \right] \\
&= L(\tau, I, \varphi, s) + \delta_2 \sum_{i=1}^n \sqrt{A_{i,1}^2 + A_{i,2}^2} \cdot \cos(s - \tau_i + b_i + \alpha_i),
\end{aligned}
\tag{4.8}
$$

where for each $i = 1, \cdots, n$, the constants $A_{i,1}, A_{i,2}$ are given by

$$
A_{i,1} = -\int_{-\infty}^\infty \left(f_i(q_i^0(t)) - f_i(0) \right) \cos t\, dt, \quad A_{i,2} = -\int_{-\infty}^\infty \left(f_i(q_i^0(t)) - f_i(0) \right) \sin t\, dt
$$

and the angle $\alpha_i = \arccos(A_{i,1}/\sqrt{A_{i,1}^2 + A_{i,2}^2})$. We can ensure each $A_{i,1}^2 + A_{i,2}^2 \neq 0$ by suitably choosing f_i (see Lemma 4.2). This gives rise to

$$
\frac{\partial L^{\delta_2}}{\partial \tau}(\tau, I, \varphi, s) = (x_1, \cdots, x_n)^\top
$$

with

$$
x_i = x_i(\tau, I, \varphi, s) = \frac{\partial L}{\partial \tau_i}(\tau, I, \varphi, s) + \delta_2 \sqrt{A_{i,1}^2 + A_{i,2}^2} \cdot \sin(s - \tau_i + b_i + \alpha_i).
$$

In particular, for the point (I_0, φ_0, s_0) we can invoke Lemma 4.3 to find a point τ^* satisfying

$$
\left\| \frac{\partial L}{\partial \tau}(\tau^*, I_0, \varphi_0, s_0) \right\| < \min_{1 \le i \le n} \delta_2 \sqrt{A_{i,1}^2 + A_{i,2}^2}.
$$

Then for each i we can find $b_i \in [0, 2\pi]$ such that $x_i(\tau^*, I_0, \varphi_0, s_0) = 0$, which therefore yields

$$
\frac{\partial L^{\delta_2}}{\partial \tau}(\tau^*, I_0, \varphi_0, s_0) = 0.
$$

Step 2: We have already shown the existence of critical points is a dense property. In the above argument, we may pick $H_2 = 0$ whenever the map $\tau \to L(\tau, I_0, \varphi_0, s_0)$ already has critical points.

Now, let us turn to check the non-degeneracy of the critical points. If $(\tau^*, I_0, \varphi_0, s_0)$ is a degenerate critical point of L^{δ_2}, i.e.,

$$
\frac{\partial L^{\delta_2}}{\partial \tau}(\tau^*, I_0, \varphi_0, s_0) = 0, \quad \det \left[\frac{\partial^2 L^{\delta_2}}{\partial \tau_i \tau_j}(\tau^*, I_0, \varphi_0, s_0) \right]_{1 \le i,j \le n} = 0.
\tag{4.9}
$$

then we have to continue to add a small perturbation to create non-degeneracy. Indeed, we may pick a perturbation $\delta_3 H_3$ to $H_1 + \delta_2 H_2$ where $\delta_3 \in (0, \delta/2)$ and

$$
\delta_3 H_3(q, t) = \delta_3 \sum_{i=1}^n g_i(q_i) \cdot \cos(t + c_i).
$$

The value δ_3, the coefficients $\{c_i \in \mathbb{R}\}_{i=1}^{n}$ and the analytic functions $\{g_i \in C_\kappa^\omega(\mathbb{T})\}_{i=1}^{n}$ will be determined later. Without loss of generality we let $\|H_3\|_\kappa < 1$. Using arguments analogous to (4.8), the new Melnikov potential, denoted by L^{δ_2,δ_3}, associated to the Hamiltonian $H_0 + \varepsilon(H_1 + \delta_2 H_2 + \delta_3 H_3)$ is

$$
\begin{aligned}
L^{\delta_2,\delta_3}(\tau, I, \varphi, s) &= L^{\delta_2}(\tau, I, \varphi, s) + \delta_3 \sum_{i=1}^{n} \Big[B_{i,1} \cdot \cos(s - \tau_i + c_i) - B_{i,2} \cdot \sin(s - \tau_i + c_i) \Big], \\
&= L^{\delta_2}(\tau, I, \varphi, s) + \delta_3 \sum_{i=1}^{n} \sqrt{B_{i,1}^2 + B_{i,2}^2} \cdot \cos(s - \tau_i + c_i + \beta_i),
\end{aligned}
\tag{4.10}
$$

where the constants $B_{i,1}, B_{i,2}$ are

$$
B_{i,1} = -\int_{-\infty}^{\infty} \Big(g_i(q_i^0(t)) - g_i(0) \Big) \cos t \, dt, \quad B_{i,2} = -\int_{-\infty}^{\infty} \Big(g_i(q_i^0(t)) - g_i(0) \Big) \sin t \, dt
$$

and $\beta_i = \arccos(B_{i,1}/\sqrt{B_{i,1}^2 + B_{i,2}^2})$. Here, we can ensure each $B_{i,1}^2 + B_{i,2}^2 \neq 0$ by suitably choosing g_i, see Lemma 4.2.

For each $i = 1, \cdots, n$ we take

$$
c_i := -s_0 + \tau_i^* - \beta_i,
\tag{4.11}
$$

then the Hessian matrix of the map $\tau \longmapsto L^{\delta_2,\delta_3}(\tau, I_0, \varphi_0, s_0)$ is

$$
\left[\frac{\partial^2 L^{\delta_2,\delta_3}}{\partial \tau_i \tau_j}(\tau^*, I_0, \varphi_0, s_0) \right]_{1 \leq i,j \leq n} = \left[\frac{\partial^2 L^{\delta_2}}{\partial \tau_i \tau_j}(\tau^*, I_0, \varphi_0, s_0) \right]_{1 \leq i,j \leq n} + \begin{bmatrix} \delta_3\lambda_1 & & & \\ & \delta_3\lambda_2 & & \\ & & \ddots & \\ & & & \delta_3\lambda_n \end{bmatrix}
$$

where the second term on the right-hand side is a diagonal matrix, and as a result of (4.11),

$$
\lambda_i = -\sqrt{B_{i,1}^2 + B_{i,2}^2} \cdot \cos(s_0 - \tau_i^* + c_i + \beta_i) = -\sqrt{B_{i,1}^2 + B_{i,2}^2} \neq 0.
\tag{4.12}
$$

Denoting

$$
\nu(\delta_3) := \det \left[\frac{\partial^2 L^{\delta_2,\delta_3}}{\partial \tau_i \tau_j}(\tau^*, I_0, \varphi_0, s_0) \right]_{1 \leq i,j \leq n}
$$

In particular, $\nu(0) = 0$ as a consequence of (4.9). It is not difficult to check that the one-dimensional function $\nu : \delta_3 \to \mathbb{R}$ is a polynomial function of degree n, and the leading term of ν is $\left(\prod_{i=1}^{n} \lambda_i \right) \delta_3^n$. Thanks to (4.12), the leading coefficient is non-zero, which implies that the polynomial ν has at most n zeros. Consequently, we can choose arbitrarily small $\delta_3 > 0$ such that

$$
\nu(\delta_3) \neq 0.
$$

In conclusion, we have constructed a perturbation $\delta_2 H_2 + \delta_3 H_3$ to H_1 where

$$
\|\delta_2 H_2 + \delta_3 H_3\|_\kappa \leq \delta_2 \|H_2\|_\kappa + \delta_3 \|H_3\|_\kappa \leq \delta_2 + \delta_3 < \delta.
$$

As $\delta > 0$ can be arbitrarily small, the existence of non-degenerate critical points for $\tau \mapsto L(\tau, I_0, \varphi_0, s_0)$ is a dense property in the C_κ^ω topology. Finally, using the implicit function theorem we can obtain an open neighborhood $U^- = \mathscr{I} \times \mathscr{J} \subset \mathscr{B}_* \times \mathbb{T}^{d+1}$ of the point (I_0, φ_0, s_0), such that for each $(I, \varphi, s) \in U^-$ the map

$$
\tau \in \mathbb{R}^n \longmapsto L(\tau, I, \varphi, s)
$$

has a non-degenerate critical point $\tau^* = \tau^*(I, \varphi, s)$. This finishes our proof. $\qquad\square$

We have provided a constructive proof for Theorem 4.1. Next, we proceed to show the genericity of assumption **(H3b)**.

Theorem 4.4. *Given $\kappa > 0$. The set of perturbations H_1 satisfying the non-degeneracy assumption* **(H3b)** *is C_κ^ω dense and C^3 open (which therefore implies C_κ^ω-openness).*

Proof. Since assumption **(H3a)** has already been proved to be open and dense, in the following proof we restrict our discussions to the case where assumption **(H3a)** always holds. As mentioned previously, $\tau^*(I, \theta, 0) \in \mathbb{R}^n$ denotes the non-degenerate critical point for the map $\tau \longmapsto L(\tau, I, \theta, 0)$. Just like assumption **(H3a)**, it is easy to see that the non-degeneracy assumption **(H3b)** is open in the C^3 topology, which directly gives C_κ^ω-openness. Thus, the only thing left is to verify the density of **(H3b)**.

In what follows, we fix a point $(\hat{I}, \hat{\theta}) \in \mathrm{Dom}(\mathscr{L}^*)$. If $\partial \mathscr{L}^* / \partial \theta(\hat{I}, \hat{\theta}) \neq 0$, then we have finished.

Suppose now

$$\frac{\partial \mathscr{L}^*}{\partial \theta}(\hat{I}, \hat{\theta}) = 0, \tag{4.13}$$

we will add a small perturbation to create non-degeneracy. More precisely, we add a perturbation δH_2 to H_1 where the number $\delta > 0$ is small enough and the analytic function H_2 will be determined later. Then the new Melnikov potential is

$$L^\delta = L + \delta \widetilde{L}$$

where

$$\widetilde{L}(\tau, I, \varphi, s) = -\int_{-\infty}^{\infty} \left[H_2(p^0(\tau + t\bar{1}), q^0(\tau + t\bar{1}), I, \varphi + \omega(I)t, s + t) - H_2(0, 0, I, \varphi + \omega(I)t, s + t) \right] dt. \tag{4.14}$$

As **(H3a)** holds for L, it follows from the implicit function theorem that near τ^* there is a unique and non-degenerate critical point $\tau^{*,\delta} = \tau^{*,\delta}(I, \theta, 0)$ for the map $\tau \longmapsto L^\delta(\tau, I, \theta, 0)$. This critical point has an expansion in δ as follows,

$$\tau^{*,\delta}(I, \theta, 0) = \tau^*(I, \theta, 0) + \delta \widetilde{\tau}^*(I, \theta, 0) + O(\delta^2).$$

Since $\tau^{*,\delta}(I, \theta, 0)$ solves the equation $\partial L^\delta / \partial \tau(\tau^{*,\delta}, I, \theta, 0) = 0$, we obtain

$$\frac{\partial L}{\partial \tau}(\tau^*, I, \theta, 0) + \delta \left[\frac{\partial^2 L}{\partial \tau^2}(\tau^*, I, \theta, 0)\widetilde{\tau}^* + \frac{\partial \widetilde{L}}{\partial \tau}(\tau^*, I, \theta, 0) \right] + O(\delta^2) = 0$$

for all small δ. Then, owing to $\partial L / \partial \tau(\tau^*, I, \theta, 0) = 0$ we obtain

$$\widetilde{\tau}^*(I, \theta, 0) = -\left(\frac{\partial^2 L}{\partial \tau^2}(\tau^*, I, \theta, 0) \right)^{-1} \frac{\partial \widetilde{L}}{\partial \tau}(\tau^*, I, \theta, 0),$$

where the matrix $\partial^2 L / \partial \tau^2(\tau^*, I, \theta, 0)$ is invertible as a result of **(H3a)**. Hence the new reduced Poincaré function $(\mathscr{L}^\delta)^*$ has an expansion in δ as follows:

$$\begin{aligned} (\mathscr{L}^\delta)^*(I, \theta) &= L^\delta(\tau^{*,\delta}, I, \theta, 0) \\ &= L(\tau^*, I, \theta, 0) + \delta \left[\frac{\partial L}{\partial \tau}(\tau^*, I, \theta, 0)\, \widetilde{\tau}^* + \widetilde{L}(\tau^*, I, \theta, 0) \right] + O(\delta^2) \\ &= \mathscr{L}^*(I, \theta) + \delta \widetilde{L}(\tau^*, I, \theta, 0) + O(\delta^2). \end{aligned} \tag{4.15}$$

Here, the last equality follows from the definition of τ^* at which $\partial L/\partial\tau(\tau^*,I,\theta,0) = 0$.

We want to show that $\frac{\partial(\mathscr{L}^\delta)^*}{\partial\theta}$ is non-zero at the point $(\hat{I},\hat{\theta})$. In view of (4.13) and (4.15), it suffices to prove

$$\frac{\partial\widetilde{L}}{\partial\theta}(\tau^*,\hat{I},\hat{\theta},0) \neq 0.$$

To achieve this, we choose H_2 as follows

$$H_2(q,\varphi) = F(q_1)\sum_{i}^{d}\cos(\varphi_i + c_i), \tag{4.16}$$

where the one-dimensional function $F \in C_\kappa^\omega(\mathbb{T})$ and the sequence of numbers $\{c_i\}_{i=1}^d$ will be determined later. Then we deduce from (4.14) that

$$\widetilde{L}(\tau,I,\varphi,s) = \sum_{i=1}^{d} A_i(I,\tau)\cdot\cos(\varphi_i + c_i) - B_i(I,\tau)\cdot\sin(\varphi_i + c_i),$$

where the coefficients

$$A_i(I,\tau) = -\int_{-\infty}^{\infty}\left(F(q_1^0(\tau_1 + t)) - F(0)\right)\cdot\cos\omega_i(I)t\,dt = -\int_{-\infty}^{\infty}\left(F(q_1^0(t)) - F(0)\right)\cdot\cos\left(\omega_i(I)\cdot(t - \tau_1)\right)dt,$$

$$B_i(I,\tau) = -\int_{-\infty}^{\infty}\left(F(q_1^0(\tau_1 + t)) - F(0)\right)\cdot\sin\omega_i(I)t\,dt = -\int_{-\infty}^{\infty}\left(F(q_1^0(t)) - F(0)\right)\cdot\sin\left(\omega_i(I)\cdot(t - \tau_1)\right)dt.$$

Note that $A_i(I,\tau)$ and $B_i(I,\tau)$ do not depend on $\tau_2,\cdots,\tau_n$. In particular, for $\tau^* = \tau^*(I,\theta,0)$ we have

$$\widetilde{L}(\tau^*,I,\theta,0) = \sum_{i=1}^{d} A_i(I,\tau^*)\cdot\cos(\theta_i + c_i) - B_i(I,\tau^*)\cdot\sin(\theta_i + c_i)$$

$$= \sum_{i=1}^{d} C_i(I,\tau^*)\cdot\cos\left(\theta_i + c_i + \alpha_i(I,\tau^*)\right).$$

Here, $C_i(I,\tau^*) = \sqrt{A_i^2(I,\tau^*) + B_i^2(I,\tau^*)}$. The angle $\alpha_i(I,\tau^*) = \arccos\left(A_i(I,\tau^*)/C_i(I,\tau^*)\right)$ if the value $C_i(I,\tau^*) \neq 0$. We choose $\alpha_i(I,\tau^*) = 0$ whenever $C_i(I,\tau^*) = 0$.

For the fixed point $(\hat{I},\hat{\theta})$, we can invoke Lemma 4.2 to choose a one-dimensional function $F(q_1)$ such that

$$C_1(\hat{I},\tau^*) = \sqrt{A_1^2(\hat{I},\tau^*) + B_1^2(\hat{I},\tau^*)} \neq 0, \qquad \text{where } \tau^* = \tau^*(\hat{I},\hat{\theta},0).$$

As we will see below, the values of $C_i(\hat{I},\tau^*)$ for $i = 2,3,\cdots,d$ play no role in the following proof.

Observe that

$$\frac{\partial\widetilde{L}}{\partial\theta}(\tau^*,\hat{I},\hat{\theta},0) = (y_1,\cdots,y_d)^\top,$$

where

$$y_i = -C_i(\hat{I},\tau^*)\cdot\sin\left(\hat{\theta}_i + c_i + \alpha_i(\hat{I},\tau^*)\right), \qquad i = 1,\cdots,d.$$

Now we can choose

$$c_1 = -\hat{\theta}_1 - \alpha_1(\hat{I},\tau^*) + \frac{\pi}{2}, \quad \text{and} \quad c_i = -\hat{\theta}_i - \alpha_i(\hat{I},\tau^*) \quad \text{for } i = 2,\cdots,d$$

which therefore gives

$$\frac{\partial\widetilde{L}}{\partial\theta}(\tau^*,\hat{I},\hat{\theta},0) = (-C_1(\hat{I},\tau^*), 0,\cdots,0)^\top \neq 0. \tag{4.17}$$

Together with (4.13) and (4.15), this implies

$$\frac{\partial(\mathscr{L}^\delta)^*}{\partial\theta}(\hat{I},\hat{\theta}) = \delta\,\frac{\partial\widetilde{L}}{\partial\theta}(\tau^*,\hat{I},\hat{\theta},0) + O(\delta^2) \neq 0$$

as long as $\delta > 0$ is sufficiently small. This completes the proof. $\square$

Remark 9. In fact, the proof of Theorem 4.4 also shows that for each fixed point $(I_0,\theta_0) \in \mathrm{Dom}(\mathscr{L}^*)$, the set of perturbations H_1 satisfying

$$\frac{\partial\mathscr{L}^*}{\partial\theta}(I_0,\theta_0) \neq 0.$$

is C_κ^ω open and dense.

   Therefore, Theorem 4.1 together with Theorem 4.4 and Remark 9 leads to the following result.

Theorem 4.5. *Let $\kappa > 0$ and $I_0 \in \mathscr{B}_*$. Then the set of all perturbations H_1, satisfying assumption* **(H3a)** *with $\{I = I_0\} \cap U^- \neq \emptyset$ and assumption* **(H3b)** *with $\frac{\partial\mathscr{L}^*}{\partial\theta}(I_0,\theta_0) \neq 0$ for some $\theta_0 \in \mathbb{T}^d$, is dense and open in the C_κ^ω topology. Actually, it is also C^3 open.*

   Now, we are ready to prove our main result in Section 2.

Proof of Theorem 2.1. Let $\kappa > 0$ and the point I_0 be fixed, we can invoke the above Theorem 4.5 to find a C_κ^ω open and dense set $\mathscr{U}$, such that for each $H_1 \in \mathscr{U}$ the Hamiltonian $\mathscr{H}_\varepsilon = H_0 + \varepsilon H_1$ satisfies the non-degeneracy assumptions **(H3a)–(H3b)**. In particular, the reduced Poincaré function $\mathscr{L}^*$ satisfies $\frac{\partial\mathscr{L}^*}{\partial\theta}(I_0,\theta_0) \neq 0$ for some $\theta_0 \in \mathbb{T}^d$.

   On the other hand, as we have already assumed that H_0 satisfies conditions **(H1)–(H2)**, it follows from Theorem 3.5 that there exist $\varepsilon_0 = \varepsilon_0(H_1) > 0$ and $\rho = \rho(H_1) > 0$, such that for every $\varepsilon \in (-\varepsilon_0,\varepsilon_0) \setminus \{0\}$ and every $\delta > 0$, the Hamiltonian flow admits a trajectory whose action variables I satisfy

$$\sup_{t>0} \|I(t) - I(0)\| \geq \rho, \quad \|I(0) - I_0\| < \delta + K\varepsilon.$$

   As for the neighborhood V_{I_0} of I_0, taking δ and ε suitably small we can ensure that the initial condition $I(0) \in V_{I_0}$. This finishes the proof. $\square$

   Note that the Fréchet space $C^\omega = \cup_{\kappa>0}C_\kappa^\omega = \cup_{m\in\mathbb{Z}^+}C_{1/m}^\omega$. Then by the definition of Fréchet topology, Corollary 2.2 follows directly from Theorem 2.1.

Proof of Theorem 2.3. As we can see from the proof of Theorem 4.1, the perturbation functions constructed by us depend only on $(q,t) \in \mathbb{T}^n \times \mathbb{T}$. Meanwhile, in the proof of Theorem 4.4, the perturbation functions constructed by us depend only on $(q,\varphi) \in \mathbb{T}^n \times \mathbb{T}^d$. This implies that the genericity of assumptions **(H3a)** and **(H3b)** are established by constructing potential perturbations that are independent of p and I. Therefore, assumptions **(H3a)–(H3b)** are also open-dense in the $C_\kappa^\omega(\mathbb{T}^{n+d+1})$ space. The remaining proof is just the same as that of Theorem 2.1. $\square$

Remark 10. We have verified that, in case that there is some degeneracy, it can be removed by adding some cos functions.

Verifying that perturbations of this kind remove the degeneracy amounted to a determinant being non-zero. Clearly, if we modify the cos slightly, this condition will remain true. This justifies the observation that the degeneracy can only fail of H_1 in a submanifold of infinite codimension.

We will not pursue this line of reasoning, but it seems that indeed, the set of directions transversal to the manifold containing all the H_1 where diffusion fails is not only infinite dimensional, but also dense. This is indeed a very strong form of genericity.

APPENDIX A. NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

Normally hyperbolic invariant manifold (NHIM) can be viewed as a natural generalization of hyperbolic set. The NHIM has not only stable and unstable directions, but also central directions (tangent to the manifold itself). The theory of normal hyperbolicity and the theory of partial hyperbolicity are closely related in their results and methods. We refer the reader to the standard references [29, 30, 39, 48]. In this appendix, we only review some classical results, including the existence of NHIMs, the existence of the stable and unstable manifolds and their invariant foliations, and the smoothness and the persistence of these manifolds. The definition of normal hyperbolicity that we adopt below is based on [39].

A.1. **The continuous case.** Let M be a smooth Riemannian manifold and Φ_t be an autonomous C^{r_0} ($1 \leq r_0 \leq \infty$) flow defined on M. A Φ_t-invariant submanifold (probably with boundary) $N \subset M$ is called a *normally hyperbolic invariant manifold* if for every $x \in N$ there is an invariant splitting

$$T_x M = T_x N \oplus E_x^s \oplus E_x^u, \qquad \text{with} \quad D\Phi_t E_x^s = E_{\Phi_t(x)}^s, \quad D\Phi_t E_x^u = E_{\Phi_t(x)}^u, \quad \forall \ t \in \mathbb{R},$$

such that

$$
\begin{aligned}
v \in E_x^s &\Longleftrightarrow C^{-1} e^{t\lambda_s} \|v\| \leq \|D\Phi_t(x)v\| \leq C e^{t\mu_s} \|v\|, \quad t \geq 0, \\
v \in E_x^u &\Longleftrightarrow C^{-1} e^{t\mu_u} \|v\| \leq \|D\Phi_t(x)v\| \leq C e^{t\lambda_u} \|v\|, \quad t \leq 0, \\
v \in T_x N &\Longleftrightarrow C^{-1} e^{|t|\lambda_c} \|v\| \leq \|D\Phi_t(x)v\| \leq C e^{|t|\mu_c} \|v\|, \quad t \in \mathbb{R},
\end{aligned}
\tag{A.1}
$$

where the constant $C > 1$, and the rates

$$\lambda_s \leq \mu_s < \lambda_c < 0 < \mu_c < \lambda_u \leq \mu_u.$$

The superscripts c, u and s stand for "center", "unstable" and "stable", respectively.

Remark 11. For the Hamiltonian model considered in this paper, the rates λ_c, μ_c are close to zero.

For a NHIM, the expansion and contraction in the central directions are weaker than those in the normal directions. Then the dynamics on N is approximately neutral and the dynamics in the normal directions is hyperbolic. That is why we call it normally hyperbolic.

We point out that the manifold M is not necessarily compact. As remarked in [39, 3], it suffices to assume that Φ_t is C^{r_0} in a neighborhood of N with all the derivatives of order up to r_0 uniformly continuous and uniformly bounded.

A.1.1. *Stable and unstable manifolds.* Let us consider a small tubular neighborhood U of the NHIM N. In both [29] and [39] the existence of local stable and unstable manifolds in U, denoted by $W_N^{s,loc}$ and $W_N^{u,loc}$, are obtained by using the method of Hadamards's graph transform. In addition, $W_N^{s,loc}$ (resp. $W_N^{u,loc}$) consists of points for which all forward (resp. backward) iterates lie in U and approach N. Then the (global) stable and unstable manifolds can be defined by , respectively,

$$W_N^s = \bigcup_{t \leq 0} \Phi_t(W_N^{s,loc}), \qquad W_N^u = \bigcup_{t \geq 0} \Phi_t(W_N^{u,loc}).$$

The global stable and unstable manifolds can have a topological characterization:

$$W_N^s = \{y \in M \mid \text{dist}(\Phi_t(y), N) \to 0, \text{ as } t \to +\infty\}, \quad W_N^u = \{y \in M \mid \text{dist}(\Phi_t(y), N) \to 0, \text{ as } t \to -\infty\}, \quad \text{(A.2)}$$

where "dist$(\cdot, \cdot)$" is the distance induced by the Riemannian metric on M. In fact, the distance convergence in (A.2) is exponential.

For each $x \in N$, we can also construct the stable and unstable manifolds with basepoint x:

$$\begin{aligned} W_x^s &= \{y \in M \mid \text{dist}(\Phi_t(x), \Phi_t(y)) \leq \widetilde{C}_y \, e^{(\mu_s + \widetilde{\varepsilon})t}, \text{ for } t \geq 0\}, \\ W_x^u &= \{y \in M \mid \text{dist}(\Phi_t(x), \Phi_t(y)) \leq \widetilde{C}_y \, e^{(\lambda_u - \widetilde{\varepsilon})t}, \text{ for } t \leq 0\}. \end{aligned} \qquad \text{(A.3)}$$

where $\widetilde{C}_y$ is a constant, and $\widetilde{\varepsilon} > 0$ is any small number satisfying

$$\mu_s + \widetilde{\varepsilon} < \lambda_c, \qquad \lambda_u - \widetilde{\varepsilon} > \mu_c.$$

This tells us that the trajectories starting on W_x^s or W_x^u satisfy certain asymptotic growth rate conditions, and the growth rate shall be greater than that of the trajectories on N.

It is important to realize that the manifolds $W_N^{s,u}$ are Φ_t-invariant while $W_x^{s,u}$ are not. Anyway, W_N^s and W_N^u can be foliated by the stable and unstable manifolds of points respectively, i.e.,

$$W_N^s = \bigcup_{x \in N} W_x^s, \qquad W_N^u = \bigcup_{x \in N} W_x^u. \qquad \text{(A.4)}$$

For $x \neq x'$, one has $W_x^s \cap W_{x'}^s = \emptyset$ and $W_x^u \cap W_{x'}^u = \emptyset$.

A.1.2. *Smoothness.* The manifolds $W_x^{s,u}$ are as smooth as the flow Φ_t, so $W_x^{s,u}$ are C^{r_0} $(1 \leq r_0 \leq \infty)$ with $T_x W_x^s = E_x^s$ and $T_x W_x^u = E_x^u$. Nevertheless, the stable manifold W_N^s and the unstable manifold W_N^u have limited regularity even if the flow is C^∞. Their regularity is dictated by the ratio of the normal hyperbolicity and the central hyperbolicity. More precisely, we introduce the following integers (see [48, Chapter 5] or [39]):

$$\ell_u := \max\left\{k = 1, \cdots, r_0 \ : \ k < \frac{\mu_s}{\lambda_c}\right\}, \quad \ell_s := \max\left\{k = 1, \cdots, r_0 \ : \ k < \frac{\lambda_u}{\mu_c}\right\}, \quad \ell := \min\{\ell_u, \ell_s\}. \qquad \text{(A.5)}$$

Clearly, ℓ_u, ℓ_s and ℓ are all finite values even when $r_0 = \infty$.

Proposition A.1. [39] *The following properties hold:*

(I) W_N^s *and* W_N^u *are at least C^ℓ differentiable. In fact, W_N^s is C^{ℓ_s} and W_N^u is C^{ℓ_u}. Then $N = W_N^s \cap W_N^u$ is C^ℓ differentiable.*

(II) *The stable foliation* $\{W_x^s : x \in N\}$ *is* C^ℓ *in the sense that* $\bigcup_{x\in N} T_x^k W_x^s$ *is a continuous bundle for each* $1 \le k \le \ell$, *where* T^k *denotes the* k-*th order tangent. Analogous result holds for the unstable foliation.*

The index ℓ means that $D\Phi_t$ expands E^u (resp. contracts E^s) at rates at least ℓ times of its expansion (resp. contraction) rate in TN. Thus, such a manifold N is also called ℓ-*normally hyperbolic*.

Property (II) implies that the invariant bundle $x \to E_x^{s,u}$ is $C^{\ell-1}$. In general, $W_N^{s,u}$ and $W_x^{s,u}$ are immersed manifolds and may fail to be embedded manifolds.

A.2. **The discrete case.** For the Hamiltonian model considered in this paper, sometimes it is convenient to study the time-T maps of the Hamiltonian flow. Hence, the study of NHIMs for maps is also needed. The definition of NHIM for diffeomorphisms is complete analogous to the definition for flows. In fact, we can replace the continuous variable t by a discrete variable without substantially changing any construction. Here, we only recall the definition.

Let $f : M \to M$ be a $C^{r_0} (1 \le r_0 \le \infty)$ diffeomorphism, and $N \subset M$ be an invariant submanifold (probably with boundary). Assume that all the derivatives of order up to r of f are uniformly continuous and uniformly bounded in a neighborhood of Λ. Then N is called a *normally hyperbolic invariant manifold* for f if for every $x \in N$ there is an invariant splitting $T_x M = T_x N \oplus E_x^s \oplus E_x^u$, with $Df E_x^s = E_{f(x)}^s$ and $Df E_x^u = E_{f(x)}^u$, such that

$$
\begin{aligned}
v \in E_x^s &\Longleftrightarrow C^{-1}\alpha_s^k \|v\| \le \|Df^k(x)\,v\| \le C\beta_s^k \|v\|, \quad k \ge 0, \\
v \in E_x^u &\Longleftrightarrow C^{-1}\beta_u^k \|v\| \le \|Df^k(x)\,v\| \le C\alpha_u^k \|v\|, \quad k \le 0, \\
v \in T_x N &\Longleftrightarrow C^{-1}\alpha_c^{|k|} \|v\| \le \|Df^k(x)\,v\| \le C\beta_c^{|k|} \|v\|, \quad k \in \mathbb{Z},
\end{aligned}
\tag{A.6}
$$

where the constant $C > 1$, and the rates $0 < \alpha_s \le \beta_s < \alpha_c < 1 < \beta_c < \alpha_u \le \beta_u$.

A.3. **Persistence and dependence on parameters.** For applications, it is important to study the persistence of the NHIM under perturbations, and if the persistent manifold depends smoothly on the perturbation parameter.

Theorem A.2. [29, 39, 3] *Let $N \subset M$ be a submanifold without boundary and N is a ℓ-normally hyperbolic invariant manifold (ℓ is defined in (A.5)) for the C^{r_0} flow Φ_t generated by the vector field X. Then for the vector field Y which is C^1-close to X, there exists a unique normally hyperbolic and Φ_t^Y-invarnat manifold N_Y, which, is C^ℓ diffeomorphic and close to N. In particular, N_Y is also ℓ-normally hyperbolic. The local stable manifold $W_{N_Y}^{s,loc}$ and local unstable manifold $W_{N_Y}^{u,loc}$ are C^ℓ close to those of N.*

Remark 12. The C^1-closeness between the vectors is enough because the change in the rates λ_ι, μ_ι, $\iota = s, c, u$, can be controlled by the C^1 distance. The persistent manifold N_Y is still C^ℓ for the reason that the exponents in (A.1) for N_Y is very close to those of N, and hence the index ℓ in (A.5) remains unchanged.

For the case where the submanifold N has non-empty boundary, a locally invariant and normally hyperbolic manifold persists. To prove it, one can construct a slightly modified system for which the

normally hyperbolic manifolds are globally invariant under the modified flow. Then, by Theorem A.2 there is a unique persistent NHIM. This invariant manifold for the slightly modified system would be a locally invariant manifold for the original system. It also implies the locally invariant manifold is not unique in general. Nevertheless, there are some cases for which the uniqueness holds. For example, the locally invariant manifold has KAM tori bounding them.

A.3.1. *Smooth parameter dependence.* The persistent manifold depends smoothly on the perturbation parameter. In fact, this can be achieved by considering a extended system.

More precisely, let $\Phi_{t,\varepsilon} : M \to M$ be a family of C^{r_0} flows depending smoothly on the parameter $\varepsilon \in [0,1]$, and assume N_0 is a NHIM of the flow $\Phi_{t,0}$ satisfying (A.5), which means N_0 is C^ℓ differentiable. Then, we can define an extended space $\widehat{M} = M \times [0,1]$ and an extended manifold $\widehat{N}_0 = N_0 \times [0,1]$. To define the extended flow, we introduce an external scaling parameter $\alpha \geq 0$, and construct a C^{r_0} flow $\widehat{\Psi}_t^\alpha$ on the manifold $\widehat{M}$ by

$$\widehat{\Psi}_t^\alpha(x,\varepsilon) := \big(\Phi_{t,\alpha\varepsilon}(x),\varepsilon\big) \quad \text{for all } (x,\varepsilon) \in M \times [0,1]$$

For $\alpha = 0$, $\widehat{N}_0$ is a C^ℓ NHIM of the flow $\widehat{\Psi}_t^0 = \Phi_{t,0} \times \text{Id}$. When α is fixed and sufficiently small, the flow $\widehat{\Psi}_t^\alpha$ is close enough to $\Phi_{t,0}$, then the persistence result implies that the flow $\widehat{\Psi}_t^\alpha$ has a NHIM $\widehat{N}_\alpha \subset \widehat{M}$, and $\widehat{N}_\alpha$ is C^ℓ close and diffeomorphic to $\widehat{N}_0$. Note that $\widehat{N}_\alpha$ can be decomposed into

$$\widehat{N}_\alpha = \coprod_{\varepsilon \in [0,1]} N_{\alpha\varepsilon} \times \{\varepsilon\},$$

where each $N_{\alpha\varepsilon}$ is invariant under $\Phi_{t,\alpha\varepsilon}$ and depends C^ℓ-smoothly on the parameter $\varepsilon \in [0,1]$. Also, it is not difficult to check that each $N_{\alpha\varepsilon}$ is a NHIM. Therefore, we conclude that for $\varepsilon \in [0,\alpha]$, N_ε is a normally hyperbolic and $\Phi_{t,\varepsilon}$-invariant manifold depending C^ℓ-smoothly on the parameter ε.

APPENDIX B. THE SCATTERING MAP

The scattering map is used to describe homoclinic excursions. This map is introduced explicitly in [17] and enjoys remarkable geometric properties [21].

Recall that W_N^s and W_N^u are, respectively, foliated by the stable leaves W_x^s and the unstable leaves W_x^u, $x \in N$. For any point $x \in W_N^s$ (resp. $x \in W_N^u$), there is a unique point $x_+ \in N$ (resp. $x_- \in N$) such that $x \in W_{x_+}^s$ (resp. $x \in W_{x_-}^u$). Then we can define the *wave maps* which are projections along the the leaves:

$$\Omega_+ : W_N^s \longrightarrow N, \quad x \longmapsto x_+; \qquad \Omega_- : W_N^u \longrightarrow N, \quad x \longmapsto x_-. \tag{B.1}$$

The wave maps are C^ℓ smooth as a result of the C^ℓ-foliation property (see Proposition A.1).

To define the scattering map, we need the following transversality conditions:

(1) W_N^s and W_N^u have a transversal intersection along a homoclinic manifold Γ, i.e., for each $z \in \Gamma$,

$$T_z M = T_z W_N^u + T_z W_N^s, \quad T_z \Gamma = T_z W_N^u \cap T_z W_N^s. \tag{B.2}$$

(2) Γ is transverse to the foliations of the stable/unstable manifolds at each point $z \in \Gamma$, that is

$$T_z W_N^s = T_z \Gamma \oplus T_z W_{x_+}^s, \quad T_z W_N^u = T_z \Gamma \oplus T_z W_{x_-}^u, \tag{B.3}$$

where $x_\pm$ are the uniquely defined points in N satisfying $z \in W^s_{x_+} \cap W^u_{x_-}$.

Remark 13. Obviously, $\dim(\Gamma) = \dim(N)$. If (B.2)–(B.3) hold at some point $z^* \in W^u_N \cap W^s_N$, then by the implicit function theorem the transversality conditions are also satisfied for all $z \in W^u_N \cap W^s_N$ close to z^*. Hence we can find a locally unique manifold $\Gamma \ni z^*$ satisfying (B.2)–(B.3). In addition, Γ is C^ℓ.

Consider the wave maps restricted to the C^ℓ manifold Γ, denoted by

$$\Omega_+|_\Gamma : \Gamma \to \Omega_+(\Gamma) \quad \text{and} \quad \Omega_-|_\Gamma : \Gamma \to \Omega_-(\Gamma).$$

They are C^ℓ local diffeomorphisms in general. Even if $\Omega_\pm|_\Gamma$ are locally invertible, they could fail to be invertible in a domain with non-contractible loops, see [17, 20] for more examples.

Following [21], we say $\Gamma \subset W^u_N \cap W^s_N$ a *homoclinic channel* if it satisfies (B.2)–(B.3) and $\Omega_\pm|_\Gamma$ are C^ℓ diffeomorphisms. Then, the *scattering map* σ^Γ associated to the homoclinic channel Γ is

$$\sigma^\Gamma = \Omega_+|_\Gamma \circ \left(\Omega_-|_\Gamma\right)^{-1} : \Omega_-(\Gamma) \longrightarrow \Omega_+(\Gamma). \tag{B.4}$$

Note that σ^Γ is C^ℓ smooth. Clearly, the definition of scattering map depends on the homoclinic channel. Sometimes we will omit Γ from the notation when there is no confusion. One can also expect to have infinitely many Γ, and each of which has a different scattering map.

Remark 14. We shall note that there is no actual orbit of the flow Φ_t starting from x_- to x_+. If $x_+ = \sigma^\Gamma(x_-)$, then we infer from (A.3) that there is a point $z \in \Gamma$ satisfying

$$\text{dist}\left(\Phi_t(z), \Phi_t(x_+)\right) \le \widetilde{C} e^{(\mu_s + \widetilde{\varepsilon})t}, \quad t \to +\infty; \quad \text{dist}\left(\Phi_t(z), \Phi_t(x_-)\right) \le \widetilde{C} e^{(\lambda_u - \widetilde{\varepsilon})t}, \quad t \to -\infty.$$

APPENDIX C. PROOFS OF THEOREM 3.4 AND THEOREM 3.5

For the reader's convenience we repeat the relevant material from [36] to give a sketch of the proof of Theorems 3.4–3.5.

Proof of Theorem 3.4. Let $N = [1/\varepsilon]$ be the integer part of $1/\varepsilon$ and consider the following orbit of s_ε:

$$x_i = s_\varepsilon(x_{i-1}), \quad x_0 = (I_0, \theta_0), \quad i = 1, \cdots, N. \tag{C.1}$$

By assumption we have a curve $\gamma : [0,1] \to \Lambda_0$ with $\gamma(0) = x_0$ which is a solution to

$$\dot{x} = \mathbf{J}\nabla\mathscr{L}^*(x), \quad x \in \Lambda_0, \tag{C.2}$$

and the set $V \subset \text{Dom}(\mathscr{L}^*)$ is a neighborhood of $\gamma([0,1])$.

We claim that all points of the sequence $\{x_i\}_{i=0}^N$ lies in V as long as ε is small enough. Let us take a sequence $y_i := \gamma(i\varepsilon)$ with $i = 0, \cdots, N$. Denoting by ϕ_t the Hamiltonian flow associated to the equation (C.2), we invoke the Gronwall inequality to (C.2) to find a constant $C_1 > 0$ such that

$$\|\phi_\varepsilon(x) - \phi_\varepsilon(x')\| \le e^{C_1\varepsilon}\|x - x'\|, \quad x, x' \in \Lambda_0. \tag{C.3}$$

Recalling the scattering map s_ε is, up to $O(\varepsilon^2)$, the ε-flow of the Hamiltonian $\mathscr{L}^*(I, \theta)$, we obtain

$$\|s_\varepsilon(x) - \phi_\varepsilon(x)\| \le C_2\varepsilon^2, \quad x \in \Lambda_0, \tag{C.4}$$

where $C_2 > 0$ is a constant. We also remark that C_1 and C_2 depend on H_1.

As $y_i = \gamma(i\varepsilon) = \phi_\varepsilon(y_{i-1})$ for each $i = 1, \ldots, N$, it follows from inequalities (C.3)–(C.4) that

$$\|x_i - y_i\| = \|s_\varepsilon(x_{i-1}) - \phi_\varepsilon(y_{i-1})\| \leq \|s_\varepsilon(x_{i-1}) - \phi_\varepsilon(x_{i-1})\| + \|\phi_\varepsilon(x_{i-1}) - \phi_\varepsilon(y_{i-1})\|$$
$$\leq C_2 \varepsilon^2 + e^{C_1 \varepsilon} \|x_{i-1} - y_{i-1}\|.$$

Denoting $w_i := \|x_i - y_i\| + \frac{C_2 \varepsilon^2}{e^{C_1 \varepsilon} - 1}$, the inequality above implies that

$$w_i \leq e^{C_1 \varepsilon} w_{i-1}, \quad \text{for } i = 1, \cdots, N$$

Since $x_0 = y_0$, we get $w_0 = \frac{C_2 \varepsilon^2}{e^{C_1 \varepsilon} - 1}$ and

$$w_i \leq w_0 \left(e^{C_1 \varepsilon}\right)^i.$$

Consequently,

$$\|x_i - y_i\| \leq \frac{C_2 \varepsilon^2}{e^{C_1 \varepsilon} - 1} \left[\left(e^{C_1 \varepsilon}\right)^i - 1 \right] \leq \frac{C_2 \varepsilon}{C_1} e^{C_1}, \quad \text{for } i = 1, \cdots, N, \tag{C.5}$$

where the last inequality follows from $e^{C_1 \varepsilon} - 1 \geq C_1 \varepsilon$ and $N = [1/\varepsilon]$. Denoting $K := C_2 e^{C_1}/C_1$, inequality (C.5) becomes

$$\mathrm{dist}(x_i, \gamma(i\varepsilon)) \leq K\varepsilon.$$

Finally, as the distance $d := \mathrm{dist}\big(\gamma([0,1]), \partial V\big) > 0$, there exists a small $\varepsilon_1 = \varepsilon_1(H_1) \in (0, d/K)$ such that for each $0 < |\varepsilon| < \varepsilon_1$, the sequence $\{x_i\}_{i=0}^N$ lies in the set V. This proves our claim.

As $\gamma_\varepsilon = k_\varepsilon \circ \gamma \subset \Lambda_\varepsilon$, we set $x_i^\varepsilon = k_\varepsilon \circ x_i \in \Lambda_\varepsilon$. By enlarging K if necessary, we have $\mathrm{dist}(x_i^\varepsilon, \gamma_\varepsilon(i\varepsilon)) \leq K\varepsilon$, and the sequence $\{x_i^\varepsilon\}_{i=0}^N$ lies in the neighborhood $V_\varepsilon := k_\varepsilon \circ V$ of the curve γ_ε. Since almost every point in V_ε is recurrent for $\widehat{f}_\varepsilon|_{\Lambda_0}$, our theorem follows immediately from the shadowing lemma C.1 below. $\qquad\square$

Lemma C.1 (Shadowing Lemma for pseudo-orbits). *For a diffeomorphism $f : M \to M$, we let Λ be a NHIM and σ be a scattering map. Assume that f preserves a measure absolutely continuous with respect to the Lebesgue measure on Λ, and that σ sends positive measure sets to positive measure sets.*

Let $x_i = \sigma(x_{i-1})$, $i = 1, \cdots, N$ be a finite orbit of the scattering map, which is contained in an open set V with almost every point of V recurrent for f. Then for every $\delta > 0$, there is an orbit $\{z_i\}_{i=0}^N$ of f satisfying $z_{i+1} = f^{k_i}(z_i)$ for some integer $k_i > 0$, and $\mathrm{dist}(z_i, x_i) < \delta$ for all $i = 0, \cdots, N$.

See [36, Theorem 3.6] for a complete proof of the lemma above.

Proof of Theorem 3.5. By assumption **(H3b)**, $\mathbf{J}\nabla\mathscr{L}^*(I, \theta)$ is transverse to the level set $\{I = I_*\}$ at some point $(I_*, \theta_*) \in \mathrm{Dom}(\mathscr{L}^*) \subset \Lambda_0$, which implies that there exist two closed balls

$$D_I = \{I \in \mathbb{R}^d \; : \; \|I - I_*\| \leq r_1\}, \quad D_\theta = \{\theta \in \mathbb{T}^d \; : \; \|\theta - \theta_*\| \leq r_1\},$$

such that $\mathbf{J}\nabla\mathscr{L}^*(I, \theta)$ is still transverse to the level set $\{I = I'\}$ at each point $(I', \theta') \in D_I \times D_\theta \subset \mathrm{Dom}(\mathscr{L}^*)$. Here, we stress that the size of radius r_1 does not depend on ε.

Let $\gamma : [0,1] \to \Lambda_0$ be an integral curve, starting from (I_*, θ_*), induced by the vector field $\mathbf{J}\nabla\mathscr{L}^*$, then there is a time $t_* \in (0,1)$ independent of ε, such that $\gamma([0, t_*]) \subset D_I \times D_\theta$. It follows that the curve $\gamma(t) = \big(I(\gamma(t)), \theta(\gamma(t))\big)$ is transverse to every level set $\{I = I(\gamma(t))\}$ for each $t \in [0, t_*]$. Thus,

$$\|I(\gamma(t_*)) - I(\gamma(0))\| = \|I(\gamma(t_*)) - I_*\| \geq 2\rho, \tag{C.6}$$

for some constant $\rho > 0$ independent of ε.

Recalling that in Theorem 3.4 we use f_ε to denote the time-2π map for the flow of the Hamiltonian $\mathcal{H}_\varepsilon$, and use $\widehat{f_\varepsilon}|_{\Lambda_0} = (k_\varepsilon)^{-1} \circ f_\varepsilon|_{\Lambda_\varepsilon} \circ k_\varepsilon$ to denote the parameterized map defined on Λ_0. Denoting

$$V := \bigcup_{k \geq 0} \widehat{f_\varepsilon}^{\,k}(D_I \times D_\theta),$$

it is a $\widehat{f_\varepsilon}$-invariant set in Λ_0, i.e. $\widehat{f_\varepsilon}(V) \subset V$. The measure of V is either finite or $+\infty$.

Case 1: The measure of V is finite. By the $\widehat{f_\varepsilon}$-invariance we infer that almost every point in V is recurrent. As V is a neighborhood of the curve $\gamma|_{[0,t_*]}$, we can apply Theorem 3.4. Indeed, let $\varepsilon_1 = \varepsilon_1(H_1) > 0$ and $K > 0$ be given in Theorem 3.4, then for every small $\delta > 0$ and every $\varepsilon \in (-\varepsilon_1, \varepsilon_1) \setminus \{0\}$, there exists an orbit $\{z_i\}_{i=0,\cdots,m}$ of the diffeomorphism f_ε and for each $i = 0, \cdots, m-1$,

$$z_{i+1} = f_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i \in \mathbb{Z}^+; \qquad d(z_i, \gamma_\varepsilon(t_i)) < \delta + K\varepsilon$$

where $0 = t_0 < t_1 < \cdots < t_m = t_*$ and $\gamma_\varepsilon = k_\varepsilon \circ \gamma \subset \Lambda_\varepsilon$. In particular, $d(z_0, \gamma_\varepsilon(0)) < \delta + K\varepsilon$ and $d(z_m, \gamma_\varepsilon(t_*)) < \delta + K\varepsilon$. Using (C.6) we obtain

$$\|I(z_m) - I(z_0)\| > 2\rho - 2(\delta + K\varepsilon).$$

Suppose $\varepsilon < \rho/(4K)$ if necessary, we can choose $\delta < \rho/4$ such that

$$\|I(z_m) - I(z_0)\| > \rho.$$

Moreover, $\|I(z_0) - I_*\| < \delta + K\varepsilon$.

Case 2: The measure of V is $+\infty$. Then the existence of diffusing orbits is evident. Indeed, it implies that for every $\rho > 0$, there is an orbit $\{\hat{z}_i : i = 0, \cdots, m\} \subset \Lambda_0$ of the map $\widehat{f_\varepsilon}$, such that $\hat{z}_0 \in D_I \times D_\theta$ and $\|I(\hat{z}_m) - I(\hat{z}_0)\| > 2\rho$. Thus, taking $z_i = k_\varepsilon \circ \hat{z}_i$ for each i, the sequence $\{z_i\}_{i=0,\cdots,m}$ is exactly an orbit of f_ε, and $\|I(z_m) - I(z_0)\| > 2\rho - O(\varepsilon) > \rho$. Finally, to ensure $I(z_0)$ is $(\delta + K\varepsilon)$-close to I_*, it suffices to assume that the radius r_1 of D_I satisfies $r_1 \leq \delta/4$.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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