THE RATE OF ACCUMULATION OF NEGATIVE EIGENVALUES TO ZERO AND THE ABSOLUTELY CONTINUOUS SPECTRUM

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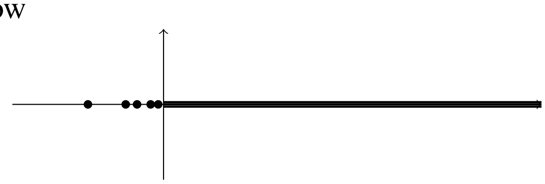
ABSTRACT. For a bounded real-valued function V on $\mathbb{R}^d$, we consider two Schrödinger operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$. We prove that if the negative spectra H_+ and H_- are discrete and the negative eigenvalues of H_+ and H_- tend to zero sufficiently fast, then the absolutely continuous spectra cover the positive half-line $[0, \infty)$.

1. MAIN RESULTS

Being discrete and being continuous are two opposite properties of a set in the plane. However, there are situations in which the fact that one part of the set is discrete implies that the other part is continuous. The set below



has two parts: the discrete part (to the left of the vertical arrow), and the continuous one (to the right of the arrow). In general, one part is not related to the other. That is no longer true if this picture represents the spectrum of a Schrödinger operator!

There is a relation between the two parts of the spectrum. It is particularly simple if the potential $V(x)$ in the Schrödinger equation is bounded and negative. In this case, if the left part of the spectrum is discrete, then the right part is continuous. Moreover, the continuous part coincides with the half-line $[0, \infty)$. In the general case, one has to consider two Schrödinger operators, one of which is obtained from the other by flipping the sign of the electric potential $V(x)$ at every point x.

Theorem 1. *Let V be a real-valued bounded measurable function on $\mathbb{R}^d$. If the negative spectra of the two Schrödinger operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ are discrete, then both spectra contain every point of the interval $[0, \infty)$.*

This theorem admits mathematical assumptions of the form $V \in L^p_{loc}(\mathbb{R}^d)$ that allow usual singularities of V appearing in physics (see [2]).

The rate of accumulation of eigenvalues to zero determines certain properties of the positive spectrum. If the negative eigenvalues tend to zero sufficiently fast, we can talk about absolute continuity of the positive part. Absolute continuity is a mathematical notion that is not easy to describe. An absolutely continuous spectrum can be seen in a rainbow in which one color is consecutively followed by another. The colors change from red to violet so gradually and smoothly, that one gets an impression that this passage is ”absolutely continuous”.

Theorem 2. *Let V be a real-valued bounded measurable function on $\mathbb{R}^d$. Assume that the negative spectra of both operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ consist of isolated eigenvalues $\{\lambda_j^+\}_{j=1}^\infty$ and $\{\lambda_j^-\}_{j=1}^\infty$ satisfying the condition*

$$\sum_j |\lambda_j^+|^{1/2} + \sum_j |\lambda_j^-|^{1/2} < \infty.$$

Then the absolutely continuous spectrum of each of the two operators is essentially supported on the positive half-line $[0, \infty)$.

The last line of the theorem should be understood in the sense that the density of the spectrum is positive almost everywhere on $[0, \infty)$. Namelly, for each $f \in L^2(\mathbb{R}^d)$, there is a unique non-negative measure $\mu_\pm$ on $\mathbb{R}$ having the property

$$\big((H_\pm - z)^{-1} f, f\big) = \int_{\mathbb{R}} \frac{d\mu_\pm(t)}{(t-z)}, \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

The measure $\mu_\pm$ is said to be of maximal spectral type for $H_\pm$ provided that any condition of the form $\mu_\pm(\delta) = 0$ implies that the spectral projection $E_{H_\pm}(\delta)$ of $H_\pm$ corresponding to the same Borel set $\delta \subset \mathbb{R}$ is zero. By a density of the spectrum we mean the derivative of a spectral measure $\mu_\pm$ of the maximal spectral type. The theorem says that

$$\mu'_\pm > 0 \qquad \text{almost everywhere on} \qquad [0, \infty).$$

A complete proof of Theorem 1 can be found in our joint paper [2] with R. Killip and S. Molchanov. The case $d = 1$ of Theorem 2 was studied by D. Damanik and Ch. Remling. The corresponding proof for $d = 1$ can be found in [1].

The main goal of our paper is to present a better proof of Theorem 2 than the unsatisfactory sketch given in [4]. This proof is different from the one written for $d = 3$ in [5] , because it covers all dimensions.

2. Estimates of the potential

The following theorem tells us that the rate of accumilation of negative eigenvalues to zero might determine some properties of the potential.

Theorem 3. *Let $W \geq 0$ be a bounded function on $\mathbb{R}^d$ having the property*

$$\int_{\mathbb{R}^d} \frac{W(x)}{|x|^{d-1}} dx < \infty.$$

Let V be a real-valued bounded function on $\mathbb{R}^d$ and let $\lambda_j^\pm$ be the negative eigenvalues of the Schrödinger operator $H_\pm = -\Delta + W \pm V$. Suppose that

$$\sum_j \Big(\sqrt{|\lambda_j^+|} + \sqrt{|\lambda_j^-|} \Big) < \infty.$$

Then V is representable in the form

$$V(x) = \tilde{W}(x) + \operatorname{div} A(x) + |A(x)|^2,$$

where the vector potential $A:\mathbb{R}^d\to\mathbb{R}^d$ and the function $\tilde W:\mathbb{R}^d\to\mathbb{R}$ satisfy the conditions

$$A\in L^\infty_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d)\cap\mathcal{H}^1_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d),\qquad \tilde W\in L^\infty_{\rm loc}(\mathbb{R}^d),$$
$$\int_{\mathbb{R}^d}\frac{(|\tilde W(x)|+|A(x)|^2)}{|x|^{d-1}}dx<\infty.$$

Remark. The theorem does not say that the functions $\tilde W$ and A have to be bounded or have to decay at infinity.

The next statement can be proved by integration by parts.

Lemma 2.1. *Let ϕ be a real-valued bounded function with bounded derivatives of first order defined on a domain $\Omega\subset\mathbb{R}^d$. Suppose that $\psi\in\mathcal{H}^2(\Omega)$ is a real-valued solution of*

$$-\Delta\psi+(W\pm V)\psi=\lambda\psi$$

and the product $\phi\psi$ vanishes on the boundary of the domain $\{a<|x|<b\}\subset\Omega$ with $a>0$. Then

$$\int_{a<|x|<b}\Big(|\nabla(\phi\psi)|^2+(W\pm V)|\phi\psi|^2\Big)dx=\int_{a<|x|<b}\Big(|\nabla\phi|^2\psi^2+\lambda|\phi\psi|^2\Big)dx.$$

Before stating a very important lemma, we introduce the notion of the inner size (width) $d(G)$ of a spherical layer $G=\{a\le|x|\le b\}$ by setting it equal to $d(G)=b-a$. For two spherical layers $\tilde G=\{\tilde a\le|x|\le\tilde b\}$ and $G=\{a\le|x|\le b\}$, we say that G encloses $\tilde G$, if $\tilde b\le a$.

By the Schrödinger operator $-\Delta+W\pm V$ on a domain $\Omega\subset\mathbb{R}^d$ we always mean an operator with the Dirichlet boundary conditions. We will sometimes denote these operators by $H_+\Big|_\Omega$ and $H_-\Big|_\Omega$. More often we will denote them by H_+ and H_-, but in this case, we will provide a verbal description mentioning the domain Ω.

Lemma 2.2. *Assume that the lowest eigenvalue of $H_\pm$ on the domain $\{a<|x|<b\}$ is the number $-\gamma^2$ where $\gamma>0$. Suppose that $b-a\ge 6\gamma^{-1}$. Then there is a spherical layer $\Omega\subset\{a<|x|<b\}$ with $d(\Omega)=6\gamma^{-1}$ such that the lowest eigenvalue of $H_\pm$ on Ω is not higher than $-\gamma^2/2$.*

Proof. Let ψ be the real eigenfunction corresponding to the eigenvalue $-\gamma^2$ for the problem on the domain $\{a<|x|<b\}$ with the Dirichlet boundary conditions. Put $L=\gamma^{-1}$ and pick a number $c>0$ giving the maximum to the function $f(c)=\int_{||x|-c|<L}|\psi|^2dx$ on the interval $[a,b]$. Define ϕ as by

$$\phi(x)=\begin{cases}1 & \text{if}\quad ||x|-c|<L,\\ 0 & \text{if}\quad ||x|-c|\ge 3L,\\ 3/2-||x|-c|/(2L), & \qquad\text{otherwise.}\end{cases}$$

By the choice of the number c,

$$\int_{a<|x|<b}|\nabla\phi|^2\psi^2dx\le\frac{\gamma^2}{2}\int_{a<|x|<b}|\phi\psi|^2dx. \tag{1}$$

Indeed, $|\nabla\phi|$ vanishes everywhere except for the two sperical layers of width $2L$, where it equals $\gamma/2$. Consequently,

$$\int_{a<|x|<b} |\nabla\phi|^2\psi^2 dx \le \frac{\gamma^2}{2}\int_{||x|-c|<L} |\psi|^2 dx = \frac{\gamma^2}{2}\int_{||x|-c|<L} |\phi\psi|^2 dx.$$

Therefore, by Lemma 2.1 and the inequality (1),

$$\int_{a<|x|<b}\Big(|\nabla(\phi\psi)|^2 + (W\pm V)|\phi\psi|^2\Big)dx \le -\frac{\gamma^2}{2}\int_{a<|x|<b} |\phi\psi|^2 dx.$$

That proved the result with Ω defined as the intersection of the support of ϕ with the layer $\{a < |x| < b\}$. If $d(\Omega) < 6\gamma^{-1}$, then we enlarge Ω until its width becomes equal to $6\gamma^{-1}$. The bottom of the spectrum of the corresponding operator will not move up in this process. □

Lemma 2.3. *Let V and $W \ge 0$ be two real valued bounded potentials on $\mathbb{R}^d$. Let $H_\pm = -\Delta \pm V + W$ be two Schrödinger operators acting on $L^2(\mathbb{R}^d)$. Suppose that the negative spectra of the operators $H_\pm$ are discrete and consist of eigenvalues $\{\lambda_j^\pm\}$ satisfying*

$$\sum_j \Big(\sqrt{|\lambda_j^+|} + \sqrt{|\lambda_j^-|}\Big) < \infty.$$

Then there is a sequence of spherical layers $\Omega_n = \{x \in \mathbb{R}^d : \; a_n \le |x| \le b_n\}$ and a monotone sequence of numbers $\epsilon_n > 0$ having the properties:

(1) $\sum_n \epsilon_n^{1/2} < \infty$ and the widths $d(\Omega_n)$ of Ω_n are estimated by

$$d(\Omega_n) \le 42\epsilon_n^{-1/2}, \qquad \forall n > 1. \tag{2}$$

(2) $H_\pm \ge 0$ on the set $\mathbb{R}^d \setminus \cap_n\Omega_n$. Moreover,

$$H_\pm \ge -\epsilon_n \quad on \quad \Omega_n \cup \big(\mathbb{R}^d \setminus \cup_{j<n}\Omega_j\big), \qquad \forall n. \tag{3}$$

(3) If $\Omega_j \cap \Omega_n \ne \emptyset$, then the width of the intersection $\Omega_j \cap \Omega_n$ is bounded below by $6\epsilon_k^{-1/2}$

$$d\Big(\Omega_j \cap \Omega_n\Big) \ge 6\epsilon_k^{-1/2},$$

where $k = \min\{j, n\}$.

(4) For each index n, there are at most two sets Ω_j intersecting Ω_n and

$$\operatorname{dist}\Big(\Omega_n, \cup_{m<j(n)}\Omega_m\Big) \ge 6\epsilon_{j(n)}^{-1/2},$$

where $j(n)$ is the smallest index $j < n$ for which the intersection $\Omega_j \cap \Omega_n$ is not empty.

(5) Any ball $B_r = \{x \in \mathbb{R}^d : \; |x| \le r\}$ of a finite radius $r > 0$ intersects only a finite number of sets Ω_j.

Proof. We will construct the sets

$$\Omega_n = \{x \in \mathbb{R}^d : \; a_n \le |x| \le b_n\}$$

inductively. We will also construct auxiliary sets $\omega_n = \{x \in \mathbb{R}^d : \; \alpha_n \le |x| \le \beta_n\} \subset \Omega_n$ whose description will take a lot of space in this proof. First, set

$$\omega_0 = \{x \in \mathbb{R}^d : \; |x| \le 6\epsilon_0^{-1/2}\} \qquad \text{and} \quad \Omega_0 = \{x \in \mathbb{R}^d : \; |x| \le 12\epsilon_0^{-1/2}\}$$

where $-\epsilon_0$ is the the lowest of the eigenvalues $\{\lambda_j^\pm\}$.

Suppose the sets $\omega_n \subset \Omega_n$ and the numbers ϵ_n are already constructed for $n < N$. Consider the set

$$S = \mathbb{R}^d \setminus \cup_{n<N}\Omega_n$$

and define $-\epsilon_N$ as the lowest of the eigenvalues of H_+ and H_- on S. By construction,

$$\epsilon_j \geq \epsilon_{j+1}.$$

Define $\omega \subset S$ to be the spherical layer on which one of the operators $H_\pm$ has spectrum below $-\epsilon_n/2$, i.e.

$$\inf \sigma\Big(H_\pm\Big|_\omega\Big) \leq -\epsilon_N/2 \quad \text{either for + or -},$$

while the width of ω is not larger than $L = 6\epsilon_N^{-1/2}$. We assume that one can not enlarge ω preserving the properties described above. The existence of this set is proved in Lemma 2.2.

Let α and β be the non-negative numbers defined by

$$\omega = \{x \in \mathbb{R}^d : \ \alpha \leq |x| \leq \beta\}.$$

Choose the index l so that a_l is the smallest of the numbers $\{a_n\}_{n<N}$ having the property

$$\beta \leq a_n.$$

After that, choose the index k so that b_k is the largest of the numbers $\{b_n\}_{n<N}$ having the property

$$b_n \leq \alpha.$$

Note that the number l might not exist. However, the case where l does not exist can be dealt with as if a_l was infinite.

Case 1. If $a_l - b_k < 2\max\{L_-, L_+\}$ where $L_- = 6\epsilon_k^{-1/2}$ and $L_+ = 6\epsilon_l^{-1/2}$, then we replace Ω_k and Ω_l by two larger sets so that the width of the intersection will be equal to

$$d(\Omega_k \cap \Omega_l) = \min\{L_-, L_+\}$$

For instance, if $L_- \leq L_+$, then we replace Ω_k by $\{a_k \leq |x| \leq b_k + L_-\}$ and replace Ω_l by $\{b_k \leq |x| \leq b_l\}$. This operation would not change the property

$$H_\pm\Big|_{\Omega_n} \geq -\epsilon_n \quad \text{for} \quad n < N,$$

because of the claim 2) of the lemma.

After we redefine the two sets Ω_k and Ω_l, we start the process over with a new collection of the sets $\{\Omega_n\}_{n<N}$.

Case 2. If both $a_l - \beta > L$ and $\alpha - b_k > L$, then we set

$$\Omega_N = \{x \in \mathbb{R}^d : \ \alpha - L \leq |x| \leq \beta + L\}$$

and $\omega_N = \omega$.

Case 3. If $a_l - b_k \geq 2\max\{L_-, L_+\}$, but $\alpha - b_k \leq L$ and $a_l - \beta \leq L$, then we set $\omega_N = \omega$,

$$\Omega_N = \{x \in \mathbb{R}^d : \ \ b_k \leq |x| \leq a_l\},$$

and we replace Ω_k and Ω_l by the sets

$$\{x \in \mathbb{R}^d : \ \ a_k \leq |x| \leq b_k + L_-\} \quad \text{and} \quad \{x \in \mathbb{R}^d : \ \ a_l - L_+ \leq |x| \leq b_l\}$$

correspondingly.

Case 4. Finally, consider the case where $a_l - b_k \geq 2\max\{L_-, L_+\}$, but only one of the numbers $\alpha - b_k$ and $a_l - \beta$ is not larger than L. Let us assume that $\alpha - b_k \leq L$ but $a_l - \beta > L$. In this case, we set $\omega_N = \omega$,

$$\Omega_N = \{x \in \mathbb{R}^d : \quad b_k \leq |x| \leq \beta + L\},$$

and we replace Ω_k by the set $\{x \in \mathbb{R}^d : \quad a_k \leq |x| \leq b_k + L_-\}$.

We see that initially the width of Ω_N does not exceed $3L$. However, we might change Ω_N by $2L$ at the next step of the process. Since the number of the steps at which one set Ω_n can be changed is at most two, the width of Ω_n does not exceed $42\epsilon_n^{-1/2}$. That proves (2).

Since the set $\Omega_N \cup \big(\mathbb{R}^d \setminus \cup_{j<N}\Omega_j\big)$ is contained in S, we the relation (3) holds for $n = N$. Therefore it holds for any n after the construction of the sets Ω_n is completed.

Obviously, we extended the sets Ω_n so that the claim 3) holds. Since ω_k and ω_l have not been changed, the are at least distance L_- and L_+ apart from Ω_N. Since $L_\pm \geq 6\epsilon_{j(N)}^{-1/2}$, we obtain the clam 4) is true.

The sets ω_n are disjoint and one of the operators $H_\pm$ on ω_n has an eigenvalue below $-\epsilon_n/2$ for $n \geq 1$. Consequently,

$$\sum_{n=1}^{\infty} \epsilon_n^{1/2} \leq \sqrt{2} \sum_n \Big(\sqrt{|\lambda_n^+|} + \sqrt{|\lambda_n^-|} \Big).$$

It is also clear that a ball B_r of finite radius $r > 0$ can intersect only a finite number of the disjoint sets ω_n. Otherwise the spectrum of one of the operators $H_\pm\Big|_{B_r}$ would accumulate to zero, which can never occur on a bounded domain due to one of Sobolev's embedding theorems. This implies the fifth claim of the lemma.

The fact that, for each N,

$$\mathbb{R}^d \setminus \cup_n \Omega_n \subset \mathbb{R}^d \setminus \cup_{n<N} \Omega_n$$

implies that $H_\pm \geq -\epsilon_N$ on $\mathbb{R}^d \setminus \cup_n\Omega_n$. Consequently, $H_\pm \geq 0$ on $\mathbb{R}^d \setminus \cup_n\Omega_n$. □

Lemma 2.3 allows one to estimate the potential V on the union $\cup_n\Omega_n$. However, these sets might not cover the whole space $\mathbb{R}^d$, so we have to consider the case

$$\mathbb{R}^d \setminus \cup_n\Omega_n \neq \emptyset.$$

Lemma 2.4. *Enlarging some of the sets Ω_n from Lemma 2.3, one can achieve that*

$$\mathbb{R}^d = \Big(\cup_n\Omega_n\Big) \cup \Big(\cup_n\Lambda_n\Big)$$

where $\Lambda_n = \{x \in \mathbb{R}^d : \quad \alpha_n < |x| < \beta_n\}$ are spherical layers with the properties:

(1) both operators H_+ and H_- are positive on Λ_n

(2) each bounded layer Λ_m intersects exactly two sets Ω_n

(3) if Λ_n intersects Ω_{n_1} and Ω_{n_2}, then

$$d(\Lambda_n) \geq 6\epsilon_{n_1}^{-1/2} + 6\epsilon_{n_2}^{-1/2},$$

and

$$d(\Lambda_n \cap \Omega_{n_j}) = 6\epsilon_{n_1}^{-1/2}, \quad j = 1, 2,$$

where $d(G)$ denotes the width of G.

(4) all claims of Lemma 2.3 hold for the sets Ω_n except for inequality (2) *which should be replaced by*

$$d(\Omega_n) \le 67\epsilon_n^{-1/2}, \qquad \forall n > 1. \tag{4}$$

Proof. Let the collection of sets $\{\Omega_n\}$ be the same as in Lemma 2.3. The set $\mathbb{R}^d \setminus \cup_n \Omega_n$ is a disjoint union of spherical layers on which both operators $H_\pm$ are positive. If a spherical layer Λ is a connected component of $\mathbb{R}^d \setminus \cup_n \Omega_n$ then there are two sets Ω_{n_1} and Ω_{n_2} whose boundaries intersect the boundary of Λ. In this case, the width of Λ should be compared with $6\epsilon_{n_1}^{-1/2} + 6\epsilon_{n_2}^{-1/2}$. If $d(\Lambda)$ is smaller than this number, we enlarge Ω_{n_1} and Ω_{n_2} so that the gap between them will disappear. For instance, if $d(\Lambda) < 6\epsilon_{n_1}^{-1/2} + 6\epsilon_{n_2}^{-1/2}$, and $\epsilon_{n_1}^{-1/2} \le \epsilon_{n_2}^{-1/2}$, then we replace Ω_{n_2} by the union $\Omega_{n_2} \cup \bar{\Lambda}$ and give the piece of width $6\epsilon_{n_1}^{-1/2}$ to the set Ω_{n_1}. Otherwise, if if $d(\Lambda) \ge 6\epsilon_{n_1}^{-1/2} + 6\epsilon_{n_2}^{-1/2}$, we keep Λ as a member of the collection $\{\Lambda_n\}$. In this case, we enlarge both sets Ω_{n_1} and Ω_{n_2} giving them the pieces of Λ of the width $6\epsilon_{n_1}^{-1/2}$ and $6\epsilon_{n_2}^{-1/2}$, correspondingly.

Since the width of Ω_n in this process might change at most by $24\epsilon_n^{-1/2}$, we obtain the inequality (4). □

In order to obtain the required estimates of the potential V we need the following elementary statement.

Lemma 2.5. *Let both $H_+ \ge -\gamma^2$ and $H_+ \ge -\gamma^2$ on a bounded spherical layer $\Omega = \{a < |x| < b\}$, $a > 0$. Then $W + V + \gamma^2 = \operatorname{div} A + |A|^2$ on Ω, where the vector potential $A \in L^\infty_{\rm loc}(\Omega; \mathbb{R}^d) \cap \mathcal{H}^1_{\rm loc}(\Omega; \mathbb{R}^d)$ satisfies the estimate*

$$\frac{1}{2}\int_{a<|x|<b} |\phi|^2 |A(x)|^2 dx \le \gamma^2 \int_{a<|x|<b} |\phi|^2 dx + \int_{a<|x|<b} W|\phi|^2 dx + 3\int_{a<|x|<b} |\nabla\phi|^2 dx. \tag{5}$$

for any function $\phi \in C_0^\infty(\Omega)$.

Proof. Let u be a positive solution of the equation $-\Delta + (W+V)u = -\gamma^2 u$. Then $A = u^{-1}\nabla u$ is a vector potential obeying

$$W + V = -\gamma^2 + \operatorname{div} A + |A|^2 \quad \text{on} \quad \Omega.$$

This step is justified in my paper [**?**]. Now, the condition $H_+ \ge -\gamma^2$ can be written in the form

$$\int_{a<|x|<b} \Big(|\nabla\phi|^2 + (W - V)|\phi|^2 dx \Big) \ge -\gamma^2 \int_{a<|x|<b} |\phi|^2 dx.$$

The latter leads to the inequality (5) due to the estimate

$$\int_{a<|x|<b} \operatorname{div} A |\phi|^2 dx \le \frac{1}{2}\int_{a<|x|<b} |A|^2|\phi|^2 dx + 2\int_{a<|x|<b} |\nabla\phi|^2 dx.$$

The proof is completed. □

Since the functions ϕ in Lemma 2.5 must vanish at the boundary of Ω, this lemma allows one to estimate A only inside the domain.

Corollary 2.1. *Let both $H_+ \geq -\gamma^2$ and $H_+ \geq -\gamma^2$ on a bounded spherical layer $\Omega = \{a < |x| < b\}$, where a, $\gamma > 0$ and $b - a \leq 67/\gamma$. Let also*

$$\tilde{\Omega} = \{\tilde{a} < |x| < \tilde{b}\},$$

where $a < \tilde{a} < \tilde{b} < b$. Then

$$W + V + \gamma^2 = \operatorname{div} A + |A|^2 \tag{6}$$

on Ω, where the vector potential $A \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^d) \cap \mathcal{H}^1_{\rm loc}(\Omega;\mathbb{R}^d)$ satisfies the estimate

$$\begin{aligned} &\frac{1}{2}\int_{\tilde{\Omega}} |A(x)|^2 |x|^{1-d} dx \leq \\ 67\gamma + \int_\Omega \bigl(W + 6|x|^{-2}\bigr)|x|^{1-d} dx &+ 6\Bigl((\tilde{a} - a)^{-1}) + (b - \tilde{b})^{-1}\Bigr). \end{aligned} \tag{7}$$

Proof. The inequality (7) follows from (5) in which one has to set $\phi(x) = \theta(|x|)|x|^{(1-d)/2}$, where θ is a continuous function on $\mathbb{R}$ defined by

$$\theta(t) = \begin{cases} 0, & \text{if} \quad t \notin [a,b]; \\ 1, & \text{if} \quad t \in [\tilde{a},\tilde{b}]; \\ \text{is linear on } [a,\tilde{a}]; \\ \text{is linear on } [\tilde{b},b]. \end{cases}$$

The proof is completed. □

Obviously, Corollary 2.1 holds for $\gamma = 0$.

Corollary 2.2. *Let both $H_+ \geq 0$ and $H_+ \geq 0$ on a bounded spherical layer $\Lambda = \{\alpha < |x| < \beta\}$, where $\alpha > 0$. Let also*

$$\tilde{\Lambda} = \{\tilde{\alpha} < |x| < \tilde{\beta}\},$$

where $\alpha < \tilde{\alpha} < \tilde{\beta} < \beta$. Then

$$W + V = \operatorname{div} A + |A|^2 \tag{8}$$

on Λ, where the vector potential $A \in L^\infty_{\rm loc}(\Lambda;\mathbb{R}^d) \cap \mathcal{H}^1_{\rm loc}(\Lambda;\mathbb{R}^d)$ satisfies the estimate

$$\begin{aligned} &\frac{1}{2}\int_{\tilde{\Lambda}} |A(x)|^2 |x|^{1-d} dx \leq \\ \int_\Lambda \bigl(W + 6|x|^{-2}\bigr)|x|^{1-d} dx &+ 6\Bigl((\tilde{\alpha} - \alpha)^{-1}) + (\beta - \tilde{\beta})^{-1}\Bigr). \end{aligned} \tag{9}$$

We can now use the information obtained in the two preceding corollaries to prove the following statement.

Lemma 2.6. *Let V and $W \geq 0$ be two real valued bounded potentials on $\mathbb{R}^d$. Assume that*

$$\int_{\mathbb{R}^d} \frac{W}{|x|^{d-1}} dx < \infty.$$

Suppose that the negative spectra of the operators $H_\pm = -\Delta \pm V + W$ are discrete and consist of eigenvalues $\{\lambda_j^\pm\}$ satisfying

$$\sum_j \Big(\sqrt{|\lambda_j^+|} + \sqrt{|\lambda_j^-|}\Big) < \infty.$$

Let Ω_n, Λ_n and ϵ_n be the same as in Lemma 2.4. Assume that $\Omega_n \subset \{|x| \geq 6\epsilon_0^{-1/2}\}$ for all $n \geq 1$. Then there is a sequence of $\mathcal{H}^1$-functions $\phi_n \geq 0$ supported inside Ω_n and a sequence of $\mathcal{H}^1$-functions $\psi_n \geq 0$ supported inside Λ_n such that

$$\sum_n \phi_n(x) + \sum_n \psi_n(x) = 1, \tag{10}$$

$$\sum_n \int_{\mathbb{R}^d} \big(|\nabla\phi_n(x)|^2 + |\nabla\psi_n(x)|^2\big)|x|^{1-d}dx \leq 72 \sum_n \epsilon_n^{1/2} \tag{11}$$

Moreover, one can find vector potentials A_n and $\tilde{A}_n$ such that

$$V + W + \epsilon_n = \operatorname{div} A_n + |A_n|^2 \quad on \quad \Omega_n, \qquad V + W = \operatorname{div} \tilde{A}_n + |\tilde{A}_n|^2 \quad on \quad \Lambda_n. \tag{12}$$

and

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^\infty \Big(\int_{\operatorname{supp}\phi_n} |A_n|^2|x|^{1-d}dx + \int_{\operatorname{supp}\psi_n} |\tilde{A}_n|^2|x|^{1-d}dx\Big) \leq \\ &\qquad (|\mathbb{S}_d| + 500)\sum_n \epsilon_n^{1/2} + \int_{\mathbb{R}^d} \frac{W}{|x|^{d-1}}dx, \end{aligned} \tag{13}$$

where $|\mathbb{S}_d|$ is the area of the unit sphere in $\mathbb{R}^d$.

Proof. According to Lemma 2.3, the width of a non-empty set of the form $\Omega_j \cap \Omega_n \neq \emptyset$ is bounded from below by $6\epsilon_k^{-1/2}$, where $k = \min\{j, n\}$. Also, according to Lemma 2.4, if $\Lambda_j \cap \Omega_n \neq \emptyset$, then the width of the intersection $\Lambda_j \cap \Omega_n$ is not less than $6\epsilon_n^{-1/2}$. Let

$$\{r_n < |x| < R_n\}$$

be the enumeration of the interiors of all such intersections that has the property $R_n \leq r_{n+1}$ for all n. Define the functions θ_n so that they are continuous on $\mathbb{R}$ and are linear on the middle thirds

$$\Big[r_n + \frac{(R_n - r_n)}{3}, R_n - \frac{(R_n - r_n)}{3}\Big] \quad \text{and} \quad \Big[r_{n+1} + \frac{(R_{n+1} - r_{n+1})}{3}, R_{n+1} - \frac{(R_{n+1} - r_{n+1})}{3}\Big]$$

of the intervals

$$[r_n, R_n] \quad \text{and} \quad [r_{n+1}, R_{n+1}],$$

correspondingly. We define θ_n to be identically zero outside of

$$[r_n + \frac{(R_n - r_n)}{3}, R_{n+1} - \frac{(R_{n+1} - r_{n+1})}{3}].$$

Finally, we define θ_n to be identically equal to one on the interval

$$[R_n - \frac{(R_n - r_n)}{3}, r_{n+1} + \frac{(R_{n+1} - r_{n+1})}{3}].$$

Now for each index n, we set $\phi_n(x) = \theta_j(|x|)$, where j is the index for which the support of the function $\theta_j(|\cdot|)$ is contained in Ω_n. Also, for each index n, we set $\psi_n(x) = \theta_l(|x|)$, where l is the index for which the support of the function $\theta_l(|\cdot|)$ is contained in Λ_n.

Observe that

$$\int_{\Omega_j\cap\Omega_n} |\nabla\phi_n|^2|x|^{1-d}dx \le 18\epsilon_k^{1/2}, \qquad k = \min\{n, j\}.$$

$$\int_{\Lambda_j\cap\Omega_n} |\nabla\phi_n|^2|x|^{1-d}dx \le 18\epsilon_n^{1/2}.$$

These relations imply (11). Moreover, $\phi_n + \phi_j = 1$ on the set $\Omega_j \cap \Omega_n$ and $\phi_n + \psi_j = 1$ on the set $\Lambda_j \cap \Omega_n$. The latter properties imply (10).

The representations (12) as well as the integral estimates for A_n and $\tilde{A}_n$ follow from Corollaries 2.1 and 2.2, because $H_\pm \ge -\epsilon_n$ on Ω_n, and both operators $H_\pm$ are positive on Λ_n. We also use the fact that

$$6\int_{|x|>6\epsilon_0^{-1/2}} |x|^{-2}|x|^{1-d}dx = |\mathbb{S}_d|\epsilon_0^{1/2}.$$

□

The end of the proof of Theorem 3. Let us define

$$A = \sum_{n=1}^\infty \big(\phi_n A_n + \psi_n\tilde{A}_n\big), \quad p(x) = -\sum_{n=1}^\infty \epsilon_n\phi_n(x), \quad V_1 = p + \operatorname{div} A + |A|^2.$$

Note that

$$\int_{\mathbb{R}^d} |p(x)|\,|x|^{1-d}dx \le 42\sum_n \epsilon_n^{1/2} < \infty, \quad \int_{\mathbb{R}^d} |A(x)|^2\,|x|^{1-d}dx \le$$
$$2\sum_{n=1}^\infty\Big(\int_{\operatorname{supp}\phi_n} |A_n(x)|^2\,|x|^{1-d}dx + \int_{\operatorname{supp}\psi_n} |\tilde{A}_n(x)|^2\,|x|^{1-d}dx\Big) < \infty. \tag{14}$$

The relations (12) imply

$$\phi_n(V + W + \epsilon_n) = \phi_n(\operatorname{div} A_n + |A_n|^2), \qquad \psi_n(V + W) = \psi_n(\operatorname{div}\tilde{A}_n + |\tilde{A}_n|^2),$$

Taking the sum over all n and using the property that $\{\phi_n\}$ and $\{\psi_n\}$ is a partition of the unity, we obtain the relation

$$V + W - p = \sum_{n=0}^\infty \phi_n(\operatorname{div} A_n + |A_n|^2) + \sum_{n=0}^\infty \psi_n(\operatorname{div}\tilde{A}_n + |\tilde{A}_n|^2).$$

Consequently,

$$V + W = V_1 - \sum_{n=0}^\infty (A_n\nabla\phi_n + \tilde{A}_n\nabla\psi_n) - |A|^2 + \sum_{n=0}^\infty (\phi_n|A_n|^2 + \psi_n|\tilde{A}_n|^2).$$

This representation implies that

$$\int_{\mathbb{R}^d} |V + W - V_1|\,|x|^{1-d}dx < \infty, \tag{15}$$

because the gradients of ϕ_n and ψ_n obey the condition (11).

It remains to set $\tilde{W} = V - V_1 + p$. Then $V = \tilde{W} + \operatorname{div} A + |A|^2$,

$$\int_{\mathbb{R}^d} |A(x)|\,|x|^{1-d}dx < \infty, \quad \text{and} \quad \int_{\mathbb{R}^d} |\tilde{W}|\,|x|^{1-d}dx < \infty$$

due to (14) and (15). The proof is completed.

3. Absolute continuity of the spectum for potentials of a special form

According to Theorem 3 proved in the preceding section, Theorem 2 follows from the statement formulated below.

Theorem 4. *Let V be a real-valued bounded measurable function on $\mathbb{R}^d$ representable in the form*

$$V(x) = \tilde{W}(x) + \operatorname{div} A(x) + |A(x)|^2, \tag{16}$$

where the vector potential $A : \mathbb{R}^d \to \mathbb{R}^d$ and the function $\tilde{W} : \mathbb{R}^d \to \mathbb{R}$ satisfy the conditions

$$\begin{gathered} A \in L^\infty_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{H}^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d), \quad \tilde{W} \in L^\infty_{\text{loc}}(\mathbb{R}^d), \\ \int_{\mathbb{R}^d} \frac{(|\tilde{W}(x)| + |A(x)|^2)}{|x|^{d-1}} dx < \infty. \end{gathered} \tag{17}$$

Assume that the negative spectrum of the operator $H = -\Delta + V$ consists of eigenvalues $\{\lambda_j\}$ obeying the condition

$$\sum_j \sqrt{|\lambda_j|} < \infty.$$

Then the absolutely continuous spectrum of the operator $H = -\Delta + V$ is essentially supported on $[0, \infty)$.

Theorem 3 is a consequence of a certain estimate of the entropy of the spectral measure corresponding to an element $f \in L^2(\mathbb{R}^d)$. This measure is defined as a unique non-negative measure μ on $\mathbb{R}$ having the property

$$\big((H - z)^{-1} f, f\big) = \int_{\mathbb{R}} \frac{d\mu(t)}{(t - z)}, \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

Theorem 5. *Let the conditions of Theorem 3 be fulfilled. Then there is a vector $f \in L^2(\mathbb{R}^d)$ such that, for any $0 < a < b < \infty$,*

$$\int_a^b \log\big(\mu'(\lambda)\big)\lambda^{-1/2} d\lambda \ge -C_d\Big(\int_{\mathbb{R}^d} (\tilde{W} + |A|^2)|x|^{1-d}dx + \sum_j \sqrt{|\lambda_j|}\Big) - \alpha_d(a, b; \|V\|_\infty), \tag{18}$$

where the constant $C_d > 0$ depends only on the dimension d, while $\alpha_d(a, b; \|V\|_\infty)$ depends on a, b, the dimension d and the norm $\|V\|_\infty$.

If the right hand side of (18) is finite, then $\mu'(\lambda) > 0$ for almost every $\lambda > 0$. Therefore (18) implies Theorem 4.

An important part of the proof of this theorem is related to approximations of the spectral measure of the operator $-\Delta + W + V$ by spectral measures of similar operators with compactly supported potentials. We have to consider several cases, one of which is the case where the potential is unbounded. The operator in this case can be defined in the sense of quadratic forms.

Let us recall certain facts of this theory. Let $a[u,v]$ be a closed semi-bounded sesquilinear form in a Hilbert space $\mathfrak{H}$. Semi-boundedness means that

$$a[u,u] \geq -C\|u\|^2, \qquad \forall u \in \mathrm{Dom}\,[a],$$

with some positive constant $C > 0$. Closedness means that for any $\tau > C$, the domain $\mathrm{Dom}\,[a]$ of the form is a complete Hilbert space with respect to the inner product

$$a[u,v] + \tau(u,v).$$

There is a unique self-adjoint operator A corresponding to the form a, such that $\mathrm{Dom}\,A \subset \mathrm{Dom}\,[a]$ and

$$(Au, v) = a[u,v] \qquad \forall u, v \in \mathrm{Dom}\,[a].$$

A vector $u \in \mathfrak{H}$ belongs to $\mathrm{Dom}\,A$ if and only if there is a vector $w \in \mathfrak{H}$ such that

$$a[u,v] = (w,v) \qquad \forall v \in \mathrm{Dom}\,[a].$$

In this case, $Au = w$.

First consider a Schrodinger operator $-\Delta + \tilde{W}_- + V$, where V and $\tilde{W}_- \geq 0$ obey the conditions

$$V \in L^\infty(\mathbb{R}^d), \quad \tilde{W}_- \in L^\infty_{\text{loc}}(\mathbb{R}^d), \tag{19}$$

$$\int_{\mathbb{R}^d} \frac{\tilde{W}_-}{|x|^{d-1}} dx < \infty. \tag{20}$$

We define $-\Delta + \tilde{W}_- + V$ as the operator corresponding to the quadratic form

$$\int_{\mathbb{R}^d} \Big(|\nabla u|^2 + (\tilde{W}_- + V)|u|^2\Big) dx.$$

The domain of this quadratic form consists of all $\mathcal{H}^1(\mathbb{R}^d)$-functions that are square integrable with respect to the measure $\tilde{W}_- dx$.

Proposition 3.1. *Let $f \in L^2(\mathbb{R}^d)$ and let V and $\tilde{W}_- \geq 0$ satisfy* (19). *Assume that $u \in \mathrm{Dom}\,(-\Delta + \tilde{W}_- + V)$ is a solution of the equation*

$$-\Delta u + (\tilde{W}_- + V - z)u = f, \qquad \mathrm{Im}\, z \neq 0.$$

Then

$$\|u\|_{\mathcal{H}^1} \leq C\|f\|_{L^2}$$

with

$$C = \sqrt{\Big((3/2 + |\mathrm{Re}\, z| + \|V\|_\infty)/|\mathrm{Im}\, z|^2 + 1/2\Big)}.$$

Proof. Since

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} (\tilde{W}_- + V - z)|u|^2 dx = \int_{\mathbb{R}^d} f\bar{u} dx,$$

we conclude that

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} (\tilde{W}_- + V - \mathrm{Re}\, z)|u|^2 dx = \mathrm{Re} \int_{\mathbb{R}^d} f\bar{u} dx,$$

Consequently,

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \leq (1/2 + \|V\|_\infty + |\mathrm{Re}\, z|) \int_{\mathbb{R}^d} |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} |f|^2 dx.$$

It remains to note that $\|u\|_{L^2} \leq (\mathrm{Im}\, z)^{-1} \|f\|_{L^2}$. $\square$

Let V be a real-valued bounded measurable function on $\mathbb{R}^d$ representable in the form (16), where the vector potential $A : \mathbb{R}^d \to \mathbb{R}^d$ and the function $\tilde{W} : \mathbb{R}^d \to \mathbb{R}$ satisfy the conditions (17). Let θ be a smooth real-valued function on $\mathbb{R}$ having the property

$$\theta(t) = \begin{cases} 1 & \text{if} \quad t < 0, \\ 0 & \text{if} \quad t > 1. \end{cases} \tag{21}$$

For a natural number n, we define θ_n by

$$\theta_n(x) = \theta(|x| - n), \qquad x \in \mathbb{R}^d. \tag{22}$$

After that, we set

$$V_n = \theta_n(\tilde{W}_- + V) + |\nabla\theta_n \cdot A| + \nabla\theta_n \cdot A - \chi_R \tilde{W}_-, \tag{23}$$

where $\tilde{W}_- = \frac{1}{2}(|\tilde{W}| - \tilde{W})$ is the negative part of the function $\tilde{W}$ and χ_R is the characteristic function of the ball $\{x \in \mathbb{R}^d : \ |x| < R\}$.

Now, for a fixed function $f \in L^2(\mathbb{R}^d)$, define the non-negative measures μ_n and μ on $\mathbb{R}$ by

$$\Big((-\Delta + V_n - z)^{-1} f, f\Big) = \int_{\mathbb{R}} \frac{d\mu_n(t)}{t - z}, \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}, \tag{24}$$

and

$$\Big((-\Delta + (1 - \chi_R) W_- + V - z)^{-1} f, f\Big) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}, \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}. \tag{25}$$

Proposition 3.2. *Let μ_n and μ be the measures defined by* (24) *and* (25). *Then the sequence μ_n converges to μ in the weak-$*$ topology, i.e. for any compactly supported continuous function $\phi \in C(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\, d\mu_n(t) \to \int_{\mathbb{R}} \phi(t)\, d\mu(t), \qquad \textit{as} \quad n \to \infty.$$

Proof. Since any compactly supported function $\phi \in C(\mathbb{R})$ can be approximated by finite linear combinations of functions of the form $\phi_z(t) = \mathrm{Im}\,\big(1/(t - z)\big)$, it is sufficient to show that

$$\int_{\mathbb{R}} \frac{d\mu_n(t)}{t - z} \to \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \qquad \text{as} \quad n \to \infty, \quad \forall z \in \mathbb{C} \setminus \mathbb{R},$$

uniformly on compact sets in $\mathbb{C} \setminus \mathbb{R}$, which is the same as showing that

$$\Big((-\Delta + V_n - z)^{-1} f, f\Big) \to \Big((-\Delta + (1 - \chi_R)\tilde{W}_- + V - z)^{-1} f, f\Big) \qquad \text{as} \quad n \to \infty, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

Using the Heine-Borel lemma, one can reduce it to one point $z \in \mathbb{C} \setminus \mathbb{R}$. In order to establish the required convergence, we use Hilbert's identity saying that

$$\Big((-\Delta + V_n - z)^{-1} f, f\Big) - \Big((-\Delta + (1-\chi_R)\tilde{W}_- + V - z)^{-1} f, f\Big) =$$
$$\Big(((1-\chi_R)\tilde{W}_- + V - V_n)(-\Delta + V_n - z)^{-1} f, (-\Delta + (1-\chi_R)\tilde{W}_- + V - \bar{z})^{-1} f\Big).$$

It becomes clear that to prove the proposition, one needs to show that

$$\int_{\mathbb{R}^d} ((1-\chi_R)\tilde{W}_- + V - V_n) u_n(x)\bar{u}(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty,$$

where

$$u_n = (-\Delta + V_n - z)^{-1} f \qquad \text{and} \quad u = (-\Delta + (1-\chi_R)\tilde{W}_- + V - \bar{z})^{-1} f.$$

Let us first establish the relation

$$\int_{\mathbb{R}^d} (1-\theta_n)(\tilde{W}_- + V) u_n(x)\bar{u}(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty. \tag{26}$$

According to Proposition 3.1,

$$\sup_n \|u_n\|_{\mathcal{H}^1} < \infty, \qquad \text{and} \qquad \|u\|_{\mathcal{H}^1} < \infty. \tag{27}$$

On the other hand, for $n > R$,

$$\begin{gathered} \int_{\mathbb{R}^d} \Big((-\nabla\theta_n) u_n + (1-\theta_n)\nabla u_n\Big)\nabla\bar{u}\, dx + \\ + \int_{\mathbb{R}^d} (1-\theta_n(x))(\tilde{W}_- + V - z) u_n(x)\bar{u}(x)\, dx = \int_{\mathbb{R}^d} (1-\theta_n(x)) u_n(x) f(x)\, dx. \end{gathered} \tag{28}$$

Thus (26) follows from (28) by (27).

Since $(1-\chi_R)\tilde{W}_- + V - V_n = (1-\theta_n)(\tilde{W}_- + V) - |\nabla\theta_n \cdot A| - \nabla\theta_n \cdot A$, it remains to show that

$$\int_{\mathbb{R}^d} (|\nabla\theta_n \cdot A| + \nabla\theta_n \cdot A) u_n(x)\bar{u}(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty. \tag{29}$$

Replacing $1-\theta_n$ by $(1-\theta_{n-1})\theta_n$ and $-\nabla\theta_n$ by $\nabla(1-\theta_{n-1})\theta_n$ in (28), one can easily show that

$$\int_{\mathbb{R}^d} (1-\theta_{n-1})\theta_n(\tilde{W}_- + V) u_n(x)\bar{u}(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty. \tag{30}$$

Using the equality

$$\begin{gathered} \int_{\mathbb{R}^d} \nabla u_n \Big((-\nabla\theta_{n-1})\bar{u} + (1-\theta_{n-1})\nabla\bar{u}\Big)\, dx + \\ + \int_{\mathbb{R}^d} (1-\theta_{n-1}(x))(V_n - z) u_n(x)\bar{u}(x)\, dx = \int_{\mathbb{R}^d} (1-\theta_{n-1}(x)) f(x)\bar{u}(x)\, dx, \end{gathered}$$

one also obtains

$$\int_{\mathbb{R}^d} (1-\theta_{n-1}) V_n(x) u_n(x)\bar{u}(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty. \tag{31}$$

Since $V_n = \theta_n(\tilde W_- + V) + |\nabla\theta_n \cdot A| + \nabla\theta_n \cdot A - \chi_R \tilde W_-$ and $\nabla\theta_n = (1-\theta_{n-1})\nabla\theta_n$, the relation (29) follows from (30) and (31). $\square$

Let V be representable in the form (16), where $A : \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde W : \mathbb{R}^d \to \mathbb{R}$ satisfy the conditions (17). Let χ_n be the characteristic function of the ball $\{x \in \mathbb{R}^d : \ |x| < R\}$. This time, we set

$$V_n = (1-\chi_n)\tilde W_- + V,$$

where $\tilde W_- = \frac{1}{2}(|\tilde W| - \tilde W)$ is the negative part of the function $\tilde W$. For a fixed function $f \in L^2(\mathbb{R}^d)$, define the non-negative measures μ_n and μ on $\mathbb{R}$ by

$$\Big((-\Delta + V_n - z)^{-1} f, f\Big) = \int_{\mathbb{R}} \frac{d\mu_n(t)}{t-z}, \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}, \tag{32}$$

and

$$\Big((-\Delta + V - z)^{-1} f, f\Big) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}, \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}. \tag{33}$$

Proposition 3.3. *Let μ_n and μ be the measures defined by* (32) *and* (33). *Then the sequence μ_n converges to μ in the weak-$*$ topology, i.e. for any compactly supported continuous function $\phi \in C(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\, d\mu_n(t) \to \int_{\mathbb{R}} \phi(t)\, d\mu(t), \qquad \textit{as} \quad n \to \infty.$$

Proof. It suffices to show that

$$\Big((-\Delta + V_n - z)^{-1} f, f\Big) \to \Big((-\Delta + V - z)^{-1} f, f\Big) \qquad \text{as} \quad n \to \infty, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

In order to establish the required convergence, we use Hilbert's identity saying that

$$\Big((-\Delta + V_n - z)^{-1} f, f\Big) - \Big((-\Delta + V - z)^{-1} f, f\Big) =$$
$$\Big((V - V_n)(-\Delta + V_n - z)^{-1} f, (-\Delta + V - \bar z)^{-1} f\Big).$$

It becomes clear that to prove the proposition, one needs to show that

$$\int_{\mathbb{R}^d} (V - V_n) u_n(x) \bar u(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty,$$

where

$$u_n = (-\Delta + V_n - z)^{-1} f \qquad \text{and} \quad u = (-\Delta + V - \bar z)^{-1} f.$$

Put differently, we have to establish the relation

$$\int_{\mathbb{R}^d} (1-\chi_n)\tilde W_- u_n(x) \bar u(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty. \tag{34}$$

According to Proposition 3.1,

$$\sup_n \|u_n\|_{\mathcal{H}^1} < \infty, \qquad \text{and} \qquad \|u\|_{\mathcal{H}^1} < \infty. \tag{35}$$

On the other hand,

$$\int_{\mathbb{R}^d} \nabla u_n \Big((-\nabla \theta_{n-1})\bar{u} + (1-\theta_{n-1})\nabla \bar{u} \Big)\, dx + \tag{36}$$
$$+ \int_{\mathbb{R}^d} (1-\theta_{n-1}(x))(V_n - z) u_n(x)\bar{u}(x)\, dx = \int_{\mathbb{R}^d} (1-\theta_{n-1}(x)) f(x)\bar{u}(x)\, dx,$$

where θ_n are defined by (21) and (22). Combining (36) and (35), we obtain that

$$\int_{\mathbb{R}^d} (1-\theta_{n-1}(x)) V_n u_n(x)\bar{u}(x)\, dx \to 0, \qquad \text{as} \quad n \to \infty.$$

The latter relation implies (34), because $(1-\chi_n)\tilde{W}_- = (1-\theta_{n-1})(V_n - V)$. $\square$

4. The entropy of a measure

Let μ be an arbitrary non-negative finite Borel measure on the real line $\mathbb{R}$. It can be decomposed into the sum of three terms

$$\mu = \mu_{ac} + \mu_{pp} + \mu_{sc}$$

where the first term is absolutely continuous, the second term is pure point, and the last term is singular continuous with respect to the Lebesgue measure. The limit

$$\mu'(\lambda) = \lim_{\varepsilon \to 0} \frac{\mu(\lambda - \varepsilon, \lambda + \varepsilon)}{2\varepsilon}$$

exists and coincides with $\mu'_{ac}(\lambda)$ for almost every $\lambda \in \mathbb{R}$. Therefore the fact that $\mu' > 0$ almost everywhere on $\mathbb{R}_+ = [0, \infty)$ implies that the support of the absolutely continuous part of the measure contains $\mathbb{R}_+$. A useful tool that often allows to understand the structure of the set

$$\{\lambda \in \mathbb{R} : \ \mu'(\lambda) > 0\}$$

is the entropy of one measure with respect to the other.

Definition. Let ρ and ν be finite Borel measures on a compact Hausdorff space X. We define the entropy of the measure ρ relative to ν by

$$S(\rho|\nu) = \begin{cases} -\infty, & \text{if} \quad \rho \text{ is not } \ \nu - \text{ac} \\ -\int_X \log(\frac{d\rho}{d\nu}) d\rho, & \text{if} \quad \rho \text{ is } \ \nu - \text{ac}. \end{cases}$$

The following result was proved in the remarkable paper [3] by Killip and Simon.

Theorem 6. *The entropy is jointly upper semi-continuous in ρ and ν with respect to the weak-$*$ topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$, then*

$$S(\rho|\nu) \geq \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

The weak-$*$ convergence in this theorem means convergence of the sequence of integrals of an arbitrary continuous function on X with respect to the measures ρ_n and ν_n. The definition of the weak-$*$ convergence of measures on $\mathbb{R}$ involves integrals of continuous functions on $\mathbb{R}$ which can not be viewed as a compact space X.

Corollary 4.1. *Let μ_n be a sequence of finite Borel measures on the real line $\mathbb{R}$ converging to a finite Borel measure μ in the weak-$*$ sense. That is*

$$\int_{\mathbb{R}} \phi(\lambda)d\mu_n(\lambda) \to \int_{\mathbb{R}} \phi(\lambda)d\mu_n(\lambda), \qquad \textit{as} \quad n\to\infty,$$

for any compactly supported continuous function ϕ on $\mathbb{R}$. Then for any $0<a<b<\infty$,

$$\int_a^b \log(\mu'(\lambda))\lambda^{-1/2}d\lambda \geq \limsup_{n\to\infty}\int_a^b \log(\mu_n'(\lambda))\lambda^{-1/2}d\lambda. \tag{37}$$

Proof. Choose $\varepsilon>0$ so that $a-\varepsilon>0$. Set $X=[a-\varepsilon,b+\varepsilon]$, $d\rho=\chi_{[a,b]}(\lambda)\lambda^{-1/2}d\lambda$ and $d\nu_n=\theta(\lambda)d\mu_n$, where θ is a continuous function on $\mathbb{R}$ vanishing outside of X and equal to 1 on $[a,b]$. The notation $\chi_{[a,b]}$ is used for the characteristic function of the interval $[a,b]$. Consider ρ and ν_n as measures on X. According to Theorem 6,

$$\int_a^b \log\Big(\mu'(\lambda)\lambda^{1/2}\Big)\lambda^{-1/2}d\lambda \geq \limsup_{n\to\infty}\int_a^b \log\Big((\mu_n'(\lambda)\lambda^{1/2}\Big)\lambda^{-1/2}d\lambda.$$

This inequality is equivalent to (37). □

5. A "TRACE-TYPE" ESTIMATE FOR THE SPECTRAL MEASURE

Let T be the operator defined by

$$\big[Tu\big](r) = -\frac{d^2u}{dr^2}(r)+Q(r)u(r), \qquad r>1, \tag{38}$$

where $Q(r)$ is a selfadjoint $n\times n$-matrix for each $r>1$. The domain of T consists of all $\mathcal{H}^2([1,\infty);\mathbb{C}^n)$-functions vanishing at the point $r=1$. We will assume that Q is a continuous compactly supported function.

Let $e_0\in\mathbb{C}^n$ be the vector whose first component is 1 and all other components are equal to zero. Set $f(r)=\chi_{[1,2]}(r)e_0$, where $\chi_{[1,2]}$ is the characteristic function of the interval $[1,2]$. We define the measure μ as the unique non-negative measure on $\mathbb{R}$ obeying

$$\Big((T-z)^{-1}f,f\Big) = \int_{\mathbb{R}}\frac{d\mu(t)}{t-z} \qquad \text{for all} \quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{39}$$

Theorem 7. *Let $\{\lambda_j\}$ be the negative eigenvalues of the operator* (38)*. Let μ be defined by* (39)*. Assume that*

$$Q(r)e_0 = 0 \qquad \textit{for all} \qquad r\leq 2.$$

Then for any $0<a<b<\infty$,

$$\begin{aligned}\int_a^b \log\Big(\mu'(\lambda)\Big)\lambda^{-1/2}d\lambda + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2}d\lambda \\ \geq -\frac{\pi}{2}\int_2^\infty \Big(Q(r)e_0,e_0\Big)dr - 2\pi\sum_j\sqrt{|\lambda_j|} - 2\pi\|Q_-\|_\infty^{1/2},\end{aligned} \tag{40}$$

where $Q_-(r)=\frac{1}{2}(|Q(r)|-Q(r))$.

Except for the replacement of $\|Q\|_\infty$ by $\|Q_-\|_\infty$, the proof of (40) repeats word by word the proof of Theorem 2.1 from my paper [5]. It was overlooked in [5] that the bottom of the spectrum of a Schrödinger operator with the potential Q can be estimated by $\|Q_-\|_\infty$ instead of $\|Q\|_\infty$.

Let

$$\tilde H = -\Delta + V \tag{41}$$

be the operator on $L^2(\mathbb{R}^d \setminus B_1)$ with the Dirichlet condition on the boundary of the unit ball $B_1 = \{x \in \mathbb{R}^d : \quad |x| \le 1\}$. Let $f(x) = |\mathbb{S}|^{-1/2}\chi_{[1,2]}(|x|)|x|^{-(d-1)/2}$ for all $x \in \mathbb{R}^d \setminus B_1$, where $\chi_{[1,2]}$ is the characteristic function of the interval $[1,2]$ and $|\mathbb{S}|$ is the area of the unit sphere in $\mathbb{R}^d$. We define the measure $\tilde\mu$ as the unique non-negative measure on $\mathbb{R}$ obeying

$$\Big((\tilde H - z)^{-1} f, f\Big) = \int_{\mathbb{R}} \frac{d\tilde\mu(t)}{t-z} \qquad \text{for all} \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{42}$$

Corollary 5.1. *Let V be a continuous real-valued function on $\{x \in \mathbb{R}^d : \quad |x| \ge 1\}$ having the property*

$$V(x) = \frac{-(d-1)(d-3)}{4|x|^2} \qquad \textit{for} \quad 1 \le |x| \le 2.$$

Let $\tilde\mu$ be defined by (42) *where $\tilde H$ is the operator defined by* (41)*. Assume that V is compactly supported. Then for any $0 < a < b < \infty$,*

$$\begin{aligned} &\int_a^b \log\Big(\tilde\mu'(\lambda)\Big)\lambda^{-1/2} d\lambda + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2} d\lambda \\ &\ge -\frac{\pi}{2|\mathbb{S}|}\int_{|x|>2} \frac{V(x)}{|x|^{d-1}} dx - 2\pi\sum_j \sqrt{|\lambda_j|} - 2\pi\Big(\|V_-\|_\infty + \frac14\Big)^{1/2} - \frac{(d-1)(d-3)}{8}, \end{aligned} \tag{43}$$

where $V_-(x) = \frac12(|V(x)| - V(x))$ and $|\mathbb{S}|$ is the area of the unit sphere in $\mathbb{R}^d$.

Proof. Let r and θ be the polar coordinates in $\mathbb{R}^d$. For each natural number n, we define P_n to be the orthogonal projection in $L^2(\mathbb{R}^d \setminus B_1)$ onto the space of functions of the form $v(r)Y_n(\theta)$, where $Y_n(\theta)$ is the n-th eigenfunction of the Laplace-Beltrami operator $-\Delta_\theta$ on the unit sphere. Define also $\tilde P_n$ by

$$\tilde P_n = \sum_{j=1}^n P_j.$$

Then $\tilde P_n \to I$ strongly as $n \to \infty$. Using this property, one can easily show that

$$\tilde P_n V \tilde P_n u \to Vu \qquad \text{as} \quad n \to \infty \qquad \text{for each} \quad u \in L^2(\mathbb{R}^d \setminus B_1).$$

Consequently,

$$\Big((-\Delta + \tilde P_n V \tilde P_n - z)^{-1} f, f\Big) \to \Big((-\Delta + V - z)^{-1} f, f\Big), \qquad \text{as} \quad n \to \infty,$$

for each $z \in \mathbb{C}\setminus\mathbb{R}$ and $f \in L^2(\mathbb{R}^d \setminus B_1)$. This relation implies the weak-$*$ convergence of the corresponding spectral measures:

$$\tilde\mu_n \to \tilde\mu, \qquad \text{as} \quad n \to \infty, \tag{44}$$

where $\tilde{\mu}_n$ is defined by

$$\Big((-\Delta + \tilde{P}_n V \tilde{P}_n - z)^{-1} f, f\Big) = \int_{\mathbb{R}} \frac{d\tilde{\mu}_n(t)}{t-z} \qquad \text{for all} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

On the other hand, the measure $\tilde{\mu}_n$ coincides with the spectral measure (39) of the operator (38) with the potential Q

$$Q = \tilde{P}_n V \tilde{P}_n + \frac{(d-1)(d-3)}{4r^2} \tilde{P}_n - \frac{1}{r^2} \Delta_\theta \tilde{P}_n.$$

This matrix-valued potential Q can be also approximated by compactly supported matrix-valued potentials

$$Q_l = \chi_{[1,l]}(r) Q,$$

where $\chi_{[1,l]}$ is the characteristic function of the interval $[1,l]$. Since

$$\|Q - Q_l\|_\infty \to 0, \qquad \text{as} \quad l \to \infty.$$

we obtain that

$$\Big((-d^2/dr^2 + Q_l - z)^{-1} f, f\Big) \to \Big((-d^2/dr^2 + Q - z)^{-1} f, f\Big), \qquad \text{as} \quad l \to \infty,$$

for each $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore the sequence of the corresponding measures ν_l constructed for the operators $-d^2/dr^2 + Q_l$ converges to $\tilde{\mu}_n$ in the weak-$*$ topology as $l \to \infty$. Theorem 7 tells us that

$$\int_a^b \log\Big(\tilde{\nu}_l'(\lambda)\Big)\lambda^{-1/2} d\lambda + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2} d\lambda$$
$$\geq -\frac{\pi}{2|\mathbb{S}|} \int_{|x|>2} \frac{V(x)}{|x|^{d-1}} dx - 2\pi \sum_j \sqrt{|\tilde{\lambda}_j|} - 2\pi\Big(\|V_-\|_\infty + \frac{1}{4}\Big)^{1/2} - \frac{(d-1)(d-3)}{8},$$

where $\tilde{\lambda}_j$ are the negative eigenvalues of the operator $-d^2/dr^2 + Q_l$. Hence, by Corollary 4.1, the following inequality holds for the measure $\tilde{\mu}_n$ and the negative eigenvalues $\{\Lambda_j\}$ of the operator $-\Delta + \tilde{P}_n V \tilde{P}_n$:

$$\int_a^b \log\Big(\tilde{\mu}_n'(\lambda)\Big)\lambda^{-1/2} d\lambda + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2} d\lambda$$
$$\geq -\frac{\pi}{2|\mathbb{S}|} \int_{|x|>2} \frac{V(x)}{|x|^{d-1}} dx - 2\pi \sum_j \sqrt{|\Lambda_j|} - 2\pi\Big(\|V_-\|_\infty + \frac{1}{4}\Big)^{1/2} - \frac{(d-1)(d-3)}{8}, \tag{45}$$

Using Corollary 4.1 one more time, we infer (43) from (44) and (45). $\square$

6. Eigenvalue sums stay bounded

Let V_n be the sequence of potentials defined by (23). Assume that $A(x) = 0$ for $|x| < 2$. Then

$$\int_{|x|>2} |x|^{1-d} V_n dx = \int_{|x|>2} |x|^{1-d} \Big(\theta_n(\tilde{W}_+ + |A|^2) + |\nabla\theta_n \cdot A| - \chi_R \tilde{W}_-\Big) dx,$$

where $\tilde{W}_+ = \frac{1}{2}(|\tilde{W}| + \tilde{W})$ is the positive ppart of $\tilde{W}$. Consequently,

$$\int_{|x|>2} |x|^{1-d} V_n dx \leq \int_{\mathbb{R}^d} |x|^{1-d} \Big(|\tilde{W}| + |A|^2\Big) dx + c\Big(\int_{|x|>n} |x|^{1-d} |A|^2 dx\Big)^{1/2}$$

with some universal constant $c > 0$. It is also easy to see that $\|(V_n)_-\|_\infty \leq \|V_-\|_\infty$ for $n > R$. It is more difficult to prove that the eigenvalue sums $\sum_j |\lambda_j(V_n)|^{1/2}$ for the operators $-\Delta + V_n$ have an upper bound independent of n. This fact follows from the proposition stated below.

Proposition 6.1. *There are numbers $N \in \mathbb{N}$ and $C > 0$ such that each operator $-\Delta + V_n$ has at most N negative eigenvalues $\{\lambda_j(V_n)\}$ and all of them obey the condition $|\lambda_j(V_n)| \leq C$.*

Proof. The quadratic form of the operator $-\Delta + V_n$ can be estimated from below by the functional

$$\int_{\mathbb{R}^d} |\nabla u - \theta_n A u|^2 dx - \int_{\mathbb{R}^d} \chi_R \tilde{W}_-(x)|u|^2 dx, \qquad u \in \mathcal{H}^1(\mathbb{R}^d).$$

For $n > R$, the value of this funcional at $u \in \mathcal{H}^1(\mathbb{R}^d)$ does not exceed

$$\int_{B_R} |\nabla u - Au|^2 dx - \int_{B_R} \tilde{W}_-(x)|u|^2 dx,$$

where $B_R = \{x \in \mathbb{R}^d : \ |x| < R\}$ is the ball of radius $R > 0$ centered at the origin.

Since

$$\int_{B_R} |\nabla u - Au|^2 dx \geq \int_{B_R} \Big(\frac{1}{2}|\nabla u|^2 - |Au|^2\Big) dx,$$

we conclude that the iegenvalues of $-\Delta + V_n$ can be estimated from below by eigenvalues of the operator $-\Delta/2 - |A|^2 - \tilde{W}_-$ on the ball B_R. It remains to note that the spectrum of the latter operator is discrete and semi-bounded. $\Box$

Proposition 6.2. *Both Propositions 3.2 and 3.3 hold in the case where the operator $-\Delta$ on $\mathbb{R}^d$ is replaced by the operator $-\Delta$ on the domain $\mathbb{R}^d \setminus B_1$ with the Dirichlet boundary conditions on the unit sphere.*

Proof. The arguments used in the proofs of Propositions 3.2 and 3.3 are suitable for the operators on $\mathbb{R}^d \setminus B_1$. $\Box$

Corollary 6.1. *Let V be a real-valued measurable function on $\mathbb{R}^d$ representable in the form*

$$V(x) = (1 - \chi_R)\tilde{W}_-(x) + \tilde{W}(x) + \operatorname{div} A(x) + |A(x)|^2, \tag{46}$$

where the vector potential $A : \mathbb{R}^d \to \mathbb{R}^d$ and the function $\tilde{W} : \mathbb{R}^d \to \mathbb{R}$ satisfy the conditions

$$\begin{gathered} A \in L^\infty_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{H}^1_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d), \quad \tilde{W} \in L^\infty_{\rm loc}(\mathbb{R}^d), \\ \int_{\mathbb{R}^d} \frac{(|\tilde{W}(x)| + |A(x)|^2)}{|x|^{d-1}} dx < \infty. \end{gathered} \tag{47}$$

Assume that $A(x) = 0$ for $|x| < 2$ and that

$$\tilde{W}(x) = \frac{-(d-1)(d-3)}{4|x|^2} \qquad \text{for} \quad 1 \leq |x| \leq 2.$$

Let $\tilde{\mu}$ be defined by (42) *where $\tilde{H}$ is the operator defined by* (41)*. Finally, let $\{\lambda_j\}$ be the negative eigenvalues of $\tilde{H}$. Then for any $0 < a < b < \infty$,*

(48)
$$\int_a^b \log\Big(\tilde{\mu}'(\lambda)\Big)\lambda^{-1/2}d\lambda + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2}d\lambda$$
$$\geq -\frac{\pi}{2|\mathbb{S}|}\int_{|x|>2}\frac{|\tilde{W}(x)|+|A(x)|^2}{|x|^{d-1}}dx - 2\pi\sum_j\sqrt{|\lambda_j|} - 2\pi\Big(\|V_-\|_\infty + \frac{1}{4}\Big)^{1/2} - \frac{(d-1)(d-3)}{8},$$

where $V_-(x) = \frac{1}{2}(|V(x)| - V(x))$ and $|\mathbb{S}|$ is the area of the unit sphere in $\mathbb{R}^d$.

Proof. This statement is a consequence of Corollaries 4.1, 5.1 and Propositions 6.1, 6.2. □

Theorem 8. *Let V be a real-valued measurable function on $\mathbb{R}^d$ representable in the form*

(49)
$$V(x) = \tilde{W}(x) + \operatorname{div} A(x) + |A(x)|^2,$$

where the vector potential $A : \mathbb{R}^d \to \mathbb{R}^d$ and the function $\tilde{W} : \mathbb{R}^d \to \mathbb{R}$ satisfy the conditions (47)*. Assume that*

(50)
$$A(x) = 0 \qquad \text{for } |x| < 2, \quad \text{and that} \qquad \tilde{W}(x) = \frac{-(d-1)(d-3)}{4|x|^2} \qquad \text{for} \quad 1 \leq |x| \leq 2.$$

Let $\tilde{\mu}$ be defined by (42) *where $\tilde{H}$ is the operator defined by* (41) *with V representable in the form* (49)*. Finally, let $\{\lambda_j\}$ be the negative eigenvalues of $\tilde{H}$. Then for any $0 < a < b < \infty$,*

(51)
$$\int_a^b \log\Big(\tilde{\mu}'(\lambda)\Big)\lambda^{-1/2}d\lambda + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2}d\lambda$$
$$\geq -\frac{\pi}{2|\mathbb{S}|}\int_{|x|>2}\frac{|\tilde{W}(x)|+|A(x)|^2}{|x|^{d-1}}dx - 2\pi\sum_j\sqrt{|\lambda_j|} - 2\pi\Big(\|V_-\|_\infty + \frac{1}{4}\Big)^{1/2} - \frac{(d-1)(d-3)}{8},$$

where $V_-(x) = \frac{1}{2}(|V(x)| - V(x))$ and $|\mathbb{S}|$ is the area of the unit sphere in $\mathbb{R}^d$.

Proof. This theorem follows from Corollary 4.1, Proposition 6.2 and Corollary 6.1. The inequality (51) is obtained by passing to the upper limit as $R \to \infty$ on both sides of (48). One only needs to observe that negative egenvalues of the Schrödinger operator with the potential $(1-\chi_R)\tilde{W}_- + V$ are monotone functions of R. Therefore they lie higher than negative eigenvalues of the operator with the potential V. □

Let $H = -\Delta + V$ be the Schrödinger operator on the whole space $\mathbb{R}^d$ with an arbitrary bounded potential of the form (16). Assume that A and $\tilde{W}$ obey (47). Define the function $\tilde{V}$ by

$$\tilde{V}(x) = \frac{-(d-1)(d-3)\theta_2(x)}{4|x|^2} + (1-\theta_2(x)\tilde{W}(x) + \operatorname{div}\big(\theta_2(x)A(x)\big) + |\theta_2(x)A(x)|^2,$$

where θ_2 is defined by (21) and (22) with $n = 2$. After that, consider the operator $H_1 = -\Delta + \tilde{V}$ on $\mathbb{R}^d \setminus B_1$ with the Dirichlet boundary conditions on the unit sphere. Since $\tilde{V}$ satisfies the conditions of Theorem 8 imposed on V, an inequaity of the form (51) holds for the spectral measure of the operator H_1 corresponding to some $f \in L^2(\mathbb{R}^d \setminus B_1)$ that belongs to the absolutely continuous subpace for the operator H_1. By rather standard arguments of scattering theory, the absolutely continuous parts of operators H and H_1 are unitary equaivalent. Therefore (18) also holds with $C_d = \pi/(2|\mathbb{S}|) + 2\pi$ and

$$\alpha_d(a, b, \|V_-\|_\infty) = 2\pi\Big(\|V_-\|_\infty + \frac{1}{4}\Big)^{1/2} + \frac{(d-1)(d-3)}{8} + \int_a^b \log\Big(\frac{(1-\cos(\sqrt{\lambda}))^2}{4\pi\lambda^{3/2}}\Big)\lambda^{-1/2} d\lambda$$

for some $f \in L^2(\mathbb{R}^d)$.

References

[1] Damanik, D., Remling, C.: *Schrödinger operators with many bound states*. Duke Math. J. **136**, 51–80 (2007)

[2] Killip, R., Molchanov, S., Safronov, O.: *A relation between the positive and negative spectra of elliptic operators*. Lett. Math. Phys. **107**, (10), 1799–1807 (2017)

[3] Killip, R., Simon, B.: *Sum rules for Jacobi matrices and their applications to spectral theory*. Annals Math. **158** (1), 253 (2003)

[4] Safronov, O.: *Multi-dimensional Schrödinger operators with some negative spectrum*. J. Funct. Anal. **238** (1), 327–339 (2006)

[5] Safronov, O.: *Absolutely Continuous Spectrum of a Dirac Operator in the Case of a Positive Mass*. Ann. Henri Poincare **18**, 1385–1434 (2017)

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