

28 Biology and Medicine. Moreover, in Mathematics itself, Differential and Integral
 29 Calculus is used in other areas, such as Linear Algebra, Analytical
 30 Geometry, Probability, and Optimization, among others.

31 Differential and Integral Calculus was developed by Isaac Newton [1] (1643-
 32 1727) and Gottfried Wilhelm Leibniz [2] (1646-1716), independently of each
 33 other, in the 17th century and basically established three operations that are
 34 applicable to any function: calculus of limits, derivatives, and integrals.

35 The derivative concerns the instantaneous rate of change of a function.
 36 On the other hand, the integral concerns the area under the curve described
 37 by a function. Both the derivative and the integral are based on the calculus
 38 of infinitesimals through the concept of limit, and the Fundamental Theorem
 39 of Calculus formalizes the inverse operations relationship between Differential
 40 and Integral Calculus.

41 The derivative is an operation that is performed on any function $f(x)$ ¹,
 42 resulting in another function $f'(x)$ that represents the slope of the tangent
 43 line to $f(x)$ for each x . Differential Calculus uses the line as the “reference
 44 function” and its slope as the result of the derivative.

45 **Remark** *Why use only the line as the reference function and its slope as*
 46 *the result of the derivative?*

47 This paper presents the derivative performed for other reference functions
 48 different from the line and other parameters different from the slope of the
 49 line, thus generalizing the Differential Calculus.

50 Since the derivative and the integral are inverse operations, the same gen-
 51 eralization concept employed for Differential Calculus is applied to Integral
 52 Calculus.

53 1.1 The Derivative, its Generalization and the 54 Antiderivative

55 The derivative of a function can be understood as a linear interpolation process.
 56 Let \mathbb{I} a non-empty open interval, $f : \mathbb{I} \rightarrow \mathbb{R}$ a function, $y = f(x)$, $\mathbb{I} \subseteq \mathbb{R}$, $x_0 \in \mathbb{I}$
 57 and $\Delta \in \mathbb{R}$, as illustrated in figure 1.

58 Two points determine a line: from the points $(x_0, f(x_0))$ and $(x_0 + \Delta, f(x_0 +$
 59 $\Delta))$ it is possible to calculate the angular (a_1) and linear (a_0) coefficient of the
 60 linear equation $y = a_1x + a_0$ secant to the graph of the function $f(x)$. This
 61 calculation is obtained by solving the following linear system:

$$S : \begin{cases} f(x_0) = a_1x_0 + a_0 \\ f(x_0 + \Delta) = a_1(x_0 + \Delta) + a_0 \end{cases} \quad (1)$$

¹ $f(x)$ is formally defined in the remaining sections.

The resolution of 1 is:

$$a_1 = \frac{f(x_0 + \Delta) - f(x_0)}{\Delta} \tag{2}$$

$$a_0 = \frac{f(x_0)(x_0 + \Delta) - f(x_0 + \Delta)x_0}{\Delta} \tag{3}$$

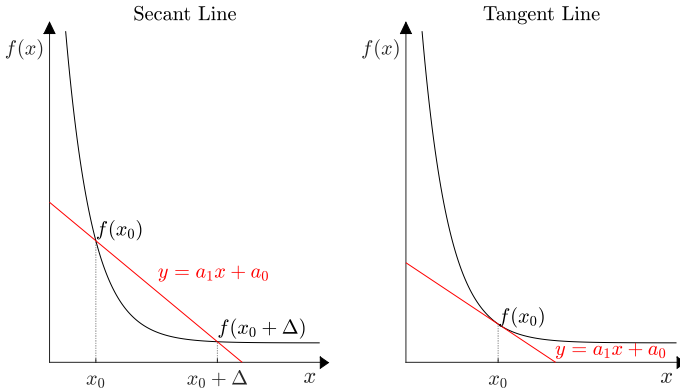


Fig. 1 Secant line (red) passing through the points $(x_0, f(x_0))$ and $(x_0 + \Delta, f(x_0 + \Delta))$ (left figure) and the tangent line at point $(x_0, f(x_0))$ (right figure) to the function (black).

62 In differential calculus, the angular coefficient (a_1) in (2) is known as
 63 Newton's Difference Quotient.

64 For small Δ values, the linear equation $y = a_1x + a_0$ will be practically
 65 tangential to the graph of the function $f(x)$ near the point x_0 , and in the limit
 66 $\Delta \rightarrow 0$, this line will be tangential to the graph of $f(x)$ at point x_0 . Applying
 67 limit of $\Delta \rightarrow 0$ in (1) is:

$$S : \begin{cases} f(x_0) = a_1x_0 + a_0 \\ f(x_0 + \Delta \rightarrow 0) = a_1(x_0 + \Delta \rightarrow 0) + a_0 \end{cases} \tag{4}$$

And, its resolution is:

$$a_1 |_{x_0} = \lim_{\Delta \rightarrow 0} \frac{f(x_0 + \Delta) - f(x_0)}{\Delta} \tag{5}$$

$$a_0 |_{x_0} = \lim_{\Delta \rightarrow 0} \frac{f(x_0)(x_0 + \Delta) - f(x_0 + \Delta)x_0}{\Delta} \tag{6}$$

68 In (5), the value of $a_1 |_{x_0}$ is the value of the derivative of $f(x)$ at the
 69 point x_0 . The value of $a_0 |_{x_0}$ in (6) is **not used** in traditional differential and
 70 integral calculus. Generalizing for any point x in the domain, $a_1^{ins} : \mathbb{I} \rightarrow \mathbb{R}$ the
 71 function a_1 instantaneous, the derivative of $f(x)$ is:

$$\frac{df(x)}{dx} = a_1^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{f(x + \Delta) - f(x)}{\Delta} \quad (7)$$

72 **Remark** *Therefore, the derivative of a function is the application of the limit of*
 73 $\Delta \rightarrow 0$ *to Newton's Difference Quotient.*

74 This process uses a linear procedure to determine the **slope** (angular coef-
 75 ficient) of the linear equation of the tangent line to any function at a given
 76 point. This linear procedure is simply a linear interpolation or a regression to
 77 the linear equation for two infinitesimally close points belonging to $f(x)$. How-
 78 ever, similarly to (7), from (6), $a_0^{ins} : \mathbb{I} \rightarrow \mathbb{R}$ the function a_0 instantaneous,
 79 one can write:

$$\begin{aligned} \mathfrak{D}\{\} &= a_1x + a_0 \\ \mathfrak{D}\{a_0\} \frac{df(x)}{dx} &= a_0^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{f(x)(x + \Delta) - f(x + \Delta)x}{\Delta} \end{aligned} \quad (8)$$

80 where,

81 $\mathfrak{D}\{\}$ is the tangent function to $f(x)$ in which the derivative is defined (the
 82 line equation in this case, as in the “classical derivative”, but it could be any
 83 other function);

84 $\mathfrak{D}\{a_0\} \frac{df(x)}{dx}$ indicates under which parameter of the $\mathfrak{D}\{\}$ the derivative $\frac{df(x)}{dx}$
 85 is defined.

In this context, as (8), the generalized notation for the “classical derivative”
 (7) is:

$$\begin{aligned} \mathfrak{D}\{\} &= a_1x + a_0 \\ \mathfrak{D}\{a_1\} \frac{df(x)}{dx} &= a_1^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{f(x + \Delta) - f(x)}{\Delta} \end{aligned} \quad (9)$$

The reasoning used in (1) to (9) can be generalized to other functions (and
 not just the linear equation) and their respective parameters. This concept can
 also be applied to the integral of a function. For example, the notation for the
 inverse operation of (9) is:

$$\begin{aligned} \mathfrak{I}\{\} &= a_1x + a_0 \\ F(x) &= \mathfrak{I}\{a_1\} \int f(x)dx \end{aligned} \quad (10)$$

86 where,

87 $\mathfrak{I}\{\}$ is the tangent function to $F : \mathbb{I} \rightarrow \mathbb{R}$ which the integral is defined;

88 $\mathfrak{I}\{a_1\} \int f(x)dx$ indicates under which parameter of the $\mathfrak{I}\{\}$ the integral of
 89 $f(x)$ is defined.

90 As a result, the following concepts can be defined:

91 • **Derivator Function** is the function $\mathfrak{D}\{\}$ that is used in the interpola-
 92 tion process (the classic derivative uses the linear equation as Derivator
 93 Function).

- 94 • **Derivator Parameter** is the parameter of interest p_k of the $\mathfrak{D}\{\}$, represented by $\mathfrak{D}\{p_k\}$, where $p_k \in \mathfrak{P}$, $\mathfrak{P} = \{p_0, p_1, p_2, \dots, p_{N-1}\}$ is the set of
- 95 $N \in \mathbb{N}$ parameters of the $\mathfrak{D}\{\}$, $k \in \mathbb{N}$ and $k < N$. (the classic derivative
- 96 uses the angular coefficient a_1 as Derivator Parameter).
- 97
- 98 • **Integrator Function** is the function $\mathfrak{I}\{\}$ that is used in the process of
- 99 obtaining the primitive function (the classic integral uses the linear equation
- 100 as Integrator Function).
- 101 • **Integrator Parameter** is the parameter of interest p_k of the $\mathfrak{I}\{\}$, represented by $\mathfrak{I}\{p_k\}$, where $p_k \in \mathfrak{Q}$, $\mathfrak{Q} = \{p_0, p_1, p_2, \dots, p_{N-1}\}$ is the set of
- 102 $N \in \mathbb{N}$ parameters of the $\mathfrak{I}\{\}$, $k \in \mathbb{N}$ and $k < N$. (the classic integral uses
- 103 the angular coefficient a_1 as Integrator Parameter).
- 104

105 Figure 2 illustrates the names of the functions and operations involved in Generalized Differential and Integral Calculus.

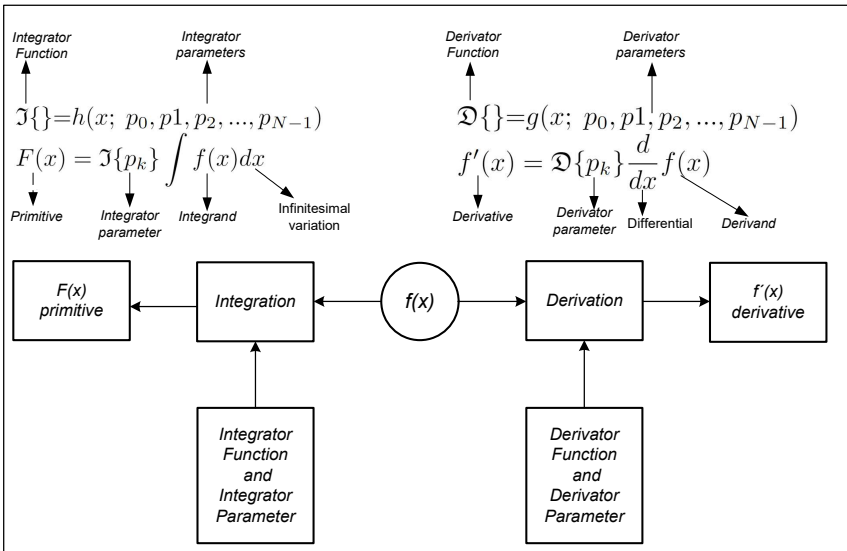


Fig. 2 Functions and operations involved in Differential and Integral Calculus.

106

107 2 Background

108 Different forms of the derivative have already been established. These forms use

109 concepts different from the foundation employed for generalizing Differential

110 and Integral Calculus presented in this article.

- 111 • Symmetric Derivative [3]

112 A simple variant form of the “classical derivative” is the Symmetric Deriva-
 113 tive, which uses Newton’s Difference Quotient in a symmetrical form. The
 114 Symmetric Derivative f'_S is defined as:

$$f'_S = \lim_{\Delta \rightarrow 0} \frac{f(x + \Delta) - f(x - \Delta)}{2\Delta} \quad (11)$$

115 Although the Symmetric Derivative uses a different form for Newton’s Dif-
 116 ference Quotient, the derivative function can still be understood as the slope
 117 of the tangent line to the function $f(x), \forall x$. In this form, the Symmetric
 118 Derivative is a different manner of defining the “classical derivative”.

119 • Fréchet Derivative [4]

120 Given \mathbb{V} and \mathbb{U} normed vectorial spaces, $\mathbb{W} \subseteq \mathbb{V}$, $f : \mathbb{W} \rightarrow \mathbb{U}$ a function Fréchet
 121 differentiable at $x \in \mathbb{V}$. If there is a bounded and linear operator $A : \mathbb{V} \rightarrow \mathbb{U}$
 122 such that:

$$\lim_{\Delta \rightarrow 0} \frac{\| f(x + \Delta) - f(x) - A\Delta \|_{\mathbb{U}}}{\| \Delta \|_{\mathbb{V}}} = 0 \quad (12)$$

123 Then, A is the derivative of f at x . The Fréchet Derivative is used on a
 124 vector-valued function of multiple real variables and to define the Functional
 125 Derivative, generalizing the derivative of a real-valued function of a single real
 126 variable.

127 • Functional Derivative [4]

128 Another form of a derivative is the Functional Derivative. Given \mathbb{V} a vec-
 129 torial (function) space, \mathbb{K} a field and F a functional, $F : \mathbb{V} \rightarrow \mathbb{K}$, $f \in \mathbb{V}$, ζ an
 130 arbitrary function, the Functional Derivative of F at f , $\frac{d(F)}{d(f)}$ is:

$$\int \frac{d(F)}{d(f)}(x)\zeta(x)dx = \lim_{\Delta \rightarrow 0} \frac{F(f + \Delta\zeta) - F(f)}{\Delta} \quad (13)$$

131 In this case, the concept of the derivative is applied to a functional and
 132 not to a function. In this paper, the concept of the derivative is generalized to
 133 functions.

134 • Fractional Derivative [5]

135 The derivative can be repeated n times over a function, resulting in the deriva-
 136 tive’s order. Thus, the order of the derivative is clearly a natural number
 137 ($n \in \mathbb{N}$). The fractional derivative generalizes the concept of derivative order
 138 so that the α order of the fractional derivative is $\alpha \in \mathbb{R}$ or even $\alpha \in \mathbb{C}$. It is
 139 then possible, under this generalization, to calculate the derivative of $f(x)$ of
 140 order $alpha = 2.5$ or $alpha = -1$ (integral of $f(x)$), for example. In this paper,
 141 the derivative is generalized to functions and not to the order of the derivative.

142 • q-Derivative [6]

The q-Derivative of a function $f(x)$ is a q-analog of the “classic derivative”. Let $q \in \mathbb{R}$, it is given by:

$$\left(\frac{d}{dx}\right)_q f(x) = \frac{f(qx) - f(x)}{qx - x} \tag{14}$$

143 For $q \rightarrow 1$, the q-Derivative is the “classic derivative”.

144 • Arithmetic Derivative [7]

Let $a.b \in \mathbb{N}$ and p a prime number, the arithmetic derivative $D(a.b)$ is such that:

$$\begin{aligned} D(0) &= D(1) = 0 \\ D(p) &= 1, \forall \text{ prime } p \\ D(a.b) &= D(a)b + a.D(b) \end{aligned} \tag{15}$$

145 The Arithmetic Derivative is a ”number derivative”, which is based on prime
146 factorization. The arithmetic derivative can be extended to rational numbers.

147 Other forms of derivatives include:

- 148 • Carlitz derivative [8]
- 149 • Covariant derivative [9]
- 150 • Dini derivative [10]
- 151 • Exterior derivative [11]
- 152 • Gateaux derivative [12]
- 153 • H derivative [13]
- 154 • Hasse derivative [14]
- 155 • Lie derivative [15]
- 156 • Pincherle derivative [16]
- 157 • Quaternionic derivative [17]
- 158 • Radon Nikodym derivative [18]
- 159 • Semi differentiability [19]
- 160 • Subderivative [20]
- 161 • Weak derivative [21]

162 All of these forms use concepts different from the foundation employed for
163 the generalization of Differential and Integral Calculus presented in this article.

164 3 Polynomials Derivators Functions

165 Let $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\forall i \in \mathbb{N}$, $P : \mathbb{R} \rightarrow \mathbb{R}$ is a Polynomial Function if:

$$P(x) = a_n x^n + a_{n-1} x^{(n-1)} + \dots + a_1 x^1 + a_0 x^0 = \sum_{i=0}^n a_i x^i \tag{16}$$

166 In (16), the value of n defines the degree of the polynomial. For $n = 1$,
167 the polynomial is a linear equation, and two points are needed to define its
168 parameters (as in (1)). For $n = 2$ and $n = 3$, the polynomial is a quadratic
169 (parabola) and cubic function, and 3 and 4 points are needed to define their

170 parameters, respectively. For other degrees of the polynomial, the reasoning is
 171 analogous; therefore, $n + 1$ points are necessary to define the parameters of a
 172 polynomial of degree n . Using a polynomial of degree 2 ($n = 2$) as derivator
 173 function, the derivative becomes an interpolation process to the quadratic
 174 function for three infinitesimally close points belonging to $f(x)$, resulting in
 175 the **Parabolic Derivative**, as shown in figure 3.

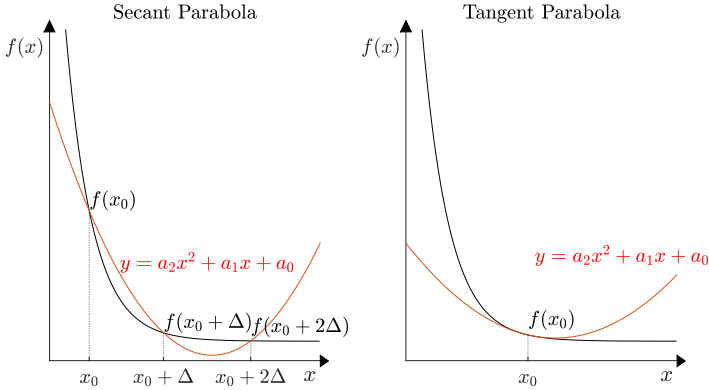


Fig. 3 Secant parabola (red) passing through the points x_0 , $x_0 + \Delta$ and $x_0 + 2\Delta$. (left figure) and tangent parabola at point x_0 (right figure) to the function (black).

176 The system is:

$$S : \begin{cases} f(x_0) = a_2x_0^2 + a_1x_0 + a_0 \\ f(x_0 + \Delta) = a_2(x_0 + \Delta)^2 + a_1(x_0 + \Delta) + a_0 \\ f(x_0 + 2\Delta) = a_2(x_0 + 2\Delta)^2 + a_1(x_0 + 2\Delta) + a_0 \end{cases} \quad (17)$$

177 Generalizing for any point x in the domain, $a_0^{ins} : \mathbb{I} \rightarrow \mathbb{R}$ the function
 178 instantaneous, $a_1^{ins} : \mathbb{I} \rightarrow \mathbb{R}$ the function a_1 instantaneous, $a_2^{ins} : \mathbb{I} \rightarrow \mathbb{R}$ the
 179 function a_2 instantaneous, and applying limit to $\Delta \rightarrow 0$, the resolution of (17)
 180 is:

$$\mathfrak{D}\{\} = a_2x^2 + a_1x + a_0 \quad (18)$$

$$\mathfrak{D}\{a_2\} \frac{df(x)}{dx} = a_2^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{1}{2} \frac{f(x) - 2f(x + \Delta) + f(x + 2\Delta)}{\Delta^2} \quad (19)$$

$$\mathfrak{D}\{a_1\} \frac{df(x)}{dx} = a_1^{ins}(x) = \lim_{\Delta \rightarrow 0} -\frac{1}{2} \frac{K_0f(x) - K_1f(x + \Delta) + K_2f(x + 2\Delta)}{\Delta^2} \quad (20)$$

where,

$$K_0 = 2x + 3\Delta$$

$$K_1 = 4x + 4\Delta$$

$$K_2 = 2x + \Delta$$

$$\mathfrak{D}\{a_0\} \frac{df(x)}{dx} = a_0^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{1}{2} \frac{K_0 f(x) + K_1 f(x + \Delta) + K_2 f(x + 2\Delta)}{\Delta^2} \quad (21)$$

where,

$$K_0 = 2\Delta^2 + x^2 + 3x\Delta$$

$$K_1 = -2x^2 - 4x\Delta$$

$$K_2 = x^2 + x\Delta$$

181 In this form, (19), (20) and (21) define the Parabolic Derivative to the
 182 a_2 , a_1 and a_0 parameters, respectively. For polynomials of other degrees, the
 183 procedure is similar to that performed in (17). For example, the generalized
 184 polynomial derivative (polynomial derivator function) of $f(x) = cx^m$, $c, m \in \mathbb{R}$
 185 for the highest degree parameter $n \in \mathbb{N}^*$ of the polynomial derivator function
 186 is:

$$\mathfrak{D}\{f\} = a_n x^n + a_{n-1} x^{(n-1)} + \dots + a_1 x^1 + a_0 x^0 \quad (22)$$

$$\begin{aligned} \mathfrak{D}\{a_n\} \frac{d(cx^m)}{dx} &= c \left(\prod_{i=0}^{n-1} \frac{m-i}{i+1} \right) x^{(m-n)} = \\ &= \frac{c}{n!} \left(\prod_{i=0}^{n-1} (m-i) \right) x^{(m-n)} \end{aligned} \quad (23)$$

187 The Antiderivative of (23) is:

$$\mathfrak{I}\{f\} = \mathfrak{D}\{f\} = a_n x^n + a_{n-1} x^{(n-1)} + \dots + a_1 x^1 + a_0 x^0 \quad (24)$$

$$\begin{aligned} \mathfrak{I}\{a_n\} \int cx^m dx &= \left(\frac{cx^{(m+n)}}{(m+n) \prod_{i=1}^{n-1} \frac{m+i}{i+1}} \right), \forall (m+n) \neq 0 \wedge n \geq 2 \\ \mathfrak{I}\{a_1\} \int cx^m dx &= \left(\frac{cx^{(m+1)}}{(m+1)} \right), m \neq -1 \end{aligned} \quad (25)$$

188 For the other functions $f(x)$ and/or other derivator parameters, the
 189 reasoning is analogous to (17), (23) and (25).

190 3.1 Vanishing Terms and Primitives

The “classic derivative” uses the linear equation as the derivator function and the angular coefficient as the derivator parameter (as in (9)). However the derivative in this form does not model the linear coefficient of the derivator

function, and therefore this term, if it exists in the derivand $f(x)$, “vanishes” for the differential operator, not influencing the derivative. Using the linear equation as the derivator function and the linear coefficient as the derivator parameter as in (8), the derivative in this form does not model the first-degree term (angular coefficient) of the derivator function. Therefore, this term “vanishes” for the differential operator, not influencing the derivative. The following example is suitable for showing this case. Considering:

$$f(x) = x^2 + 2x + 3 \quad (26)$$

191 for $\mathfrak{D}\{ \} = a_1x + a_0$, their derivatives are:

$$\mathfrak{D}\{a_1\} \frac{df(x)}{dx} = 2x + 2 \quad (27)$$

192 and

$$\mathfrak{D}\{a_0\} \frac{df(x)}{dx} = -x^2 + 3 \quad (28)$$

193 The term +3 and $2x$ in (26) vanishes in (27) and (28), respectively. For
194 $\mathfrak{I}\{ \} = a_1x + a_0$, the antiderivatives, respectively, for (27) and (28) are:

$$\mathfrak{I}\{a_1\} \int (2x + 2)dx = x^2 + 2x \quad (29)$$

and

$$\mathfrak{I}\{a_0\} \int (-x^2 + 3)dx = x^2 + 3 \quad (30)$$

195 Since (27) and (28) do not model the terms a_0 and a_1x , respectively, the
196 antiderivatives (29) and (30) do not return in (26) and must be added by the
197 following terms (k_0 and k_1x , with k_0 and k_1 constants):

$$\mathfrak{I}\{a_1\} \int (2x + 2)dx = x^2 + 2x + k_0 \quad (31)$$

and

$$\mathfrak{I}\{a_0\} \int (-x^2 + 3)dx = x^2 + k_1x + 3 \quad (32)$$

198 The addition of the terms k_0 and k_1x in (31) and (32) is necessary because,
199 **independently of k_0 and k_1x** , their derivatives are the same:

$$\mathfrak{D}\{a_1\} \frac{d(x^2 + 2x + k_0)}{dx} = 2x + 2 \quad (33)$$

200 and

$$\mathfrak{D}\{a_0\} \frac{d(x^2 + k_1x + 3)}{dx} = -x^2 + 3 \quad (34)$$

201 **3.2 Integrals without Antiderivatives**

202 The Fundamental Theorem of Calculus (FTC) [22] establishes the relationship
 203 between differential calculus and integral calculus, as **inverse operations**
 204 (with reservations). The FTC is divided into two parts. Part 1 shows that
 205 the derivative of the integral of $f(x)$ is equal to $f(x)$: this is perfect! Part 2
 206 reverses the order, that is, the integral of the derivative of $f(x)$ is equal to
 207 $f(x)$, however, plus a constant k , that is, $f(x) + k$: this is perfect too, but the
 208 exact return to function $f(x)$ does not occur when the derivative is performed
 209 first and then the integral. Thus, in formal terms, the FTC states that the
 210 operations of derivation and integration are inverse, apart from a constant
 211 value.

$$\frac{d \int f(x)dx}{dx} = f(x) \neq \int \frac{df(x)}{dx} dx = f(x) + k \tag{35}$$

212 This problem occurs simply because the “classic derivative” only gives the
 213 instantaneous rate of change of a function for its domain. This rate, as seen
 214 in (5), is the angular coefficient for the linear equation when used as deriva-
 215 tor function. Obviously, the linear equation cannot be defined by its angular
 216 coefficient a_1 alone. The linear coefficient a_0 also needs to be defined for the
 217 linear equation to be complete.

218 The derivation process is carried out by applying the concept of limit to
 219 Newton’s quotient. On the other hand, the integration process does not have
 220 a specific form, and this is obtained, in practice, through the calculation of
 221 antiderivatives. Nonetheless, a function can be defined by applying the inte-
 222 grator function $\mathcal{I}\{\}$ to the generalized derivatives for a given derivator function
 223 $\mathcal{D}\{\}$ for **all** their respective parameters, with $\mathcal{I}\{\} = \mathcal{D}\{\}$.

224 For the integrator function $\mathcal{I}\{\} : a_1x + a_0$ (linear equation) and $\mathcal{I}\{\} = \mathcal{D}\{\}$,
 225 the primitive $f(x)$ is:

$$f(x) = \mathcal{D}\{a_1\} \frac{df(x)}{dx} x + \mathcal{D}\{a_0\} \frac{df(x)}{dx} \tag{36}$$

226 It is important to emphasize that $f(x)$ was obtained from its generalized
 227 derivatives without the conventionally used integration process (antiderivative)
 228 in “classical integral calculus”.

For example, from the functions (27) and (28) (derivatives of $f(x) = x^2 + 2x + 3$), the primitive $f(x)$ is:

$$f(x) = (2x + 2)x - x^2 + 3 = x^2 + 2x + 3 \tag{37}$$

229 The **exact** return to $f(x)$ is obtained ((37) equals (26)). The concept
 230 involved in obtaining the function $f(x)$ is:

Theorem Let \mathbb{I} a non-empty open interval, $\mathbb{I} \subseteq \mathbb{C}$, $h : \mathbb{I} \rightarrow \mathbb{C}$ a function, $y = h(x; p_0, p_1, p_2, \dots, p_{N-1})$ and $\mathfrak{P} = \{p_0, p_1, p_2, \dots, p_{N-1}\}$ the set of $N \in \mathbb{N}$ parameters.

If S is a system that has a unique solution for N points (x_k, y_k) , $k \in \mathbb{N}$, $k \leq N - 1$, such as:

$$S : \begin{cases} y_0 = h(x_0; p_0, p_1, p_2, \dots, p_{N-1}) \\ y_1 = h(x_1; p_0, p_1, p_2, \dots, p_{N-1}) \\ y_2 = h(x_2; p_0, p_1, p_2, \dots, p_{N-1}) \\ \dots \\ y_{N-1} = h(x_{N-1}; p_0, p_1, p_2, \dots, p_{N-1}) \end{cases} \quad (38)$$

231 $\mathfrak{J}\{\} = \mathfrak{D}\{\} = h(x; p_0, p_1, p_2, \dots, p_{N-1})$, $f(x)$ differentiable on $\mathfrak{D}\{\}$, $\forall x \in \mathbb{I}$, then $f(x)$
 232 can be described by $\mathfrak{J}\{\}$ whose parameters are given by their generalized derivatives in
 233 their respective N parameters, i.e. $f(x) = h(x; \mathfrak{D}\{p_0\}, \mathfrak{D}\{p_1\}, \mathfrak{D}\{p_2\}, \dots, \mathfrak{D}\{p_{N-1}\})$.

234 4 Exponential Derivators Functions

235 Let $A, a, b, x \in \mathbb{R}$, $a^{ins} : \mathbb{R} \rightarrow \mathbb{C}$, $b^{ins} : \mathbb{R} \rightarrow \mathbb{C}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ an exponential
 236 function as:

$$f(x) = Ae^{ax}, A \geq 0 \quad (39)$$

Making $A = e^b$, (39) is:

$$f(x) = e^b e^{ax} = e^{ax+b} \quad (40)$$

237 The following system can be written:

$$S : \begin{cases} f(x) = e^{ax+b} \\ f(x + \Delta) = e^{\alpha(x+\Delta)+b} \end{cases} \quad (41)$$

238 Solving the system and applying the limit of $\Delta \rightarrow 0$ in (41), the Exponential
 239 Derivative (derivator function is exponential) of a function $f(x)$ becomes:

$$\mathfrak{D}\{\} = e^{ax+b} \quad (42)$$

$$\mathfrak{D}\{a\} \frac{df(x)}{dx} = a^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\ln(f(x + \Delta)) - \ln(f(x))}{\Delta} \quad (43)$$

$$\mathfrak{D}\{b\} \frac{df(x)}{dx} = b^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\ln(f(x))(x + \Delta) - \ln(f(x + \Delta))x}{\Delta} \quad (44)$$

$f(x)$ can be reconstructed from its exponential derivatives as:

$$f(x) = e^{\mathfrak{D}\{a\} \frac{df(x)}{dx} x + \mathfrak{D}\{b\} \frac{df(x)}{dx}} \quad (45)$$

240 The function $e^{-i\omega x}$ (kernel of the Fourier Transform) is discussed in the
 241 section 6.2.

242 5 Trigonometric Derivators Functions

243 Let $\omega, \phi \in \mathbb{R}$, the frequency and phase, respectively, and $\omega^{ins} : \mathbb{R} \rightarrow \mathbb{C}, \phi^{ins} :$
 244 $\mathbb{R} \rightarrow \mathbb{C}$ the instantaneous frequency and phase, respectively, the following
 245 system can be written:

$$S : \begin{cases} f(x) = \sin(\omega x + \phi) \\ f(x + \Delta) = \sin(\omega(x + \Delta) + \phi) \end{cases} \quad (46)$$

246 Solving the system and applying the limit of $\Delta \rightarrow 0$ in (46), the sinusoidal
 247 derivative (derivator function is sinusoidal) of a function $f(x)$ becomes:

$$\mathfrak{D}\{f\} = \sin(\omega x + \phi) \quad (47)$$

$$\begin{aligned} \mathfrak{D}\{\omega\} \frac{df(x)}{dx} &= \omega^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\arcsin(f(x + \Delta)) - \arcsin(f(x))}{\Delta} = \\ &= \lim_{\Delta \rightarrow 0} \frac{\arcsin(f(x + \Delta))\sqrt{1 - f(x)^2} - f(x)\sqrt{1 - f(x + \Delta)^2}}{\Delta} \end{aligned} \quad (48)$$

$$\mathfrak{D}\{\phi\} \frac{df(x)}{dx} = \phi^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\arcsin(f(x))(x + \Delta) - \arcsin(f(x + \Delta))x}{\Delta} \quad (49)$$

248 The same can be written to cosine and tangent functions:

$$\mathfrak{D}\{f\} = \cos(\omega x + \phi) \quad (50)$$

$$\begin{aligned} \mathfrak{D}\{\omega\} \frac{df(x)}{dx} &= \omega^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\arccos(f(x + \Delta)) - \arccos(f(x))}{\Delta} = \\ &= \lim_{\Delta \rightarrow 0} \frac{\arccos(f(x + \Delta))f(x) + \sqrt{(1 - f(x + \Delta)^2)(1 - f(x)^2)}}{\Delta} \end{aligned} \quad (51)$$

$$\mathfrak{D}\{\phi\} \frac{df(x)}{dx} = \phi^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\arccos(f(x))(x + \Delta) - \arccos(f(x + \Delta))x}{\Delta} \quad (52)$$

$$\mathfrak{D}\{f\} = \tan(\omega x + \phi) \quad (53)$$

$$\begin{aligned} \mathfrak{D}\{\omega\} \frac{df(x)}{dx} &= \omega^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\arctan(f(x + \Delta)) - \arctan(f(x))}{\Delta} = \\ &= \frac{\arctan\left(\frac{f(x + \Delta) - f(x)}{1 + f(x + \Delta)f(x)}\right)}{\Delta} \end{aligned} \quad (54)$$

$$\mathfrak{D}\{\phi\} \frac{df(x)}{dx} = \phi^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{\arctan(f(x))(x + \Delta) - \arctan(f(x + \Delta))x}{\Delta} \quad (55)$$

249 If $f(x) \in [-1, 1]$, (48) to (52) $\in \mathbb{R}$, otherwise (48) to (52) $\in \mathbb{C}$.

$f(x)$ can be reconstructed from its sinusoidal, cosinusoidal and tangential derivatives, respectively, as:

$$f(x) = \sin \left(\mathfrak{D}\{\omega\} \frac{df(x)}{dx} x + \mathfrak{D}\{\phi\} \frac{df(x)}{dx} \right) \quad (56)$$

$$f(x) = \cos \left(\mathfrak{D}\{\omega\} \frac{df(x)}{dx} x + \mathfrak{D}\{\phi\} \frac{df(x)}{dx} \right) \quad (57)$$

$$f(x) = \tan \left(\mathfrak{D}\{\omega\} \frac{df(x)}{dx} x + \mathfrak{D}\{\phi\} \frac{df(x)}{dx} \right) \quad (58)$$

250 6 Instantaneous Frequency and Heisenberg's 251 Uncertainty Principle

Let a phase function [23] $\Omega(x) : \mathbb{R} \rightarrow \mathbb{R}$, and a waveform (signal, Wave Function [24]) $\varphi(x, \omega)$ given by:

$$\varphi(x, \omega) = \sin(\Omega(x)) \quad (59)$$

252 If $\Omega(x)$ is known, the determination of the instantaneous frequency $\omega(x)$
253 presents no difficulty and is determined by:

$$\omega(x) = \frac{d\Omega(x)}{dx} \quad (60)$$

254 However, in many real applications, $\Omega(x)$ is not known, but only the wave-
255 form $\varphi(x, \omega)$ and then, determining $\omega(x)$ or $x(\omega)$ precisely, from $\varphi(x, \omega)$ is not
256 a possible task, according to Heisenberg's Uncertainty Principle [25], [26].

257 Heisenberg's Uncertainty Principle was first proposed for Quantum
258 Mechanics [27]. However, it is used to demonstrate that there is a limit to
259 the accuracy with which the pair of canonically conjugate variables [28] in
260 phase space, (x, ω) or (x, p) , where p is the momentum, e.g., can be measured
261 simultaneously.

262
263 *"...Thus, the more precisely the position is determined, the less precisely
264 the momentum is known, and conversely..."*

265 *Heisenberg, 1927*

266 De Broglie's [29] relation establishes the undulatory nature of the particle
267 (matter) by:

$$k = \frac{p}{\hbar}, \quad (61)$$

268 where, k is the wavenumber (spatial frequency), p is the momentum and \hbar is
 269 the reduced Planck's constant.

270 Thus, one can understand that determining the momentum p as a function
 271 of position x is equivalent to determining the wavenumber k as a function of
 272 position x or even the (temporal) frequency ω as a function of time x (x , in
 273 this case, is the “position” in time) - Instantaneous Frequency.

274 The classical mathematical operation that changes the domain of a function
 275 from time to frequency (and vice versa) is the Fourier Transform, but a non-
 276 zero function $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ and its Fourier Transform $F(\omega) : \mathbb{R} \rightarrow \mathbb{C}$ cannot
 277 both be sharply localized [26].

278 The figure 4 shows a function $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ (a sinusoidal wave with
 279 frequency equal 2 Hz) and $|F(\omega)| : \mathbb{R} \rightarrow \mathbb{R}$ in x (time) and ω (frequency)
 280 domain. $F(\omega)$ has no information about x and $f(x)$ has no information about
 281 ω .

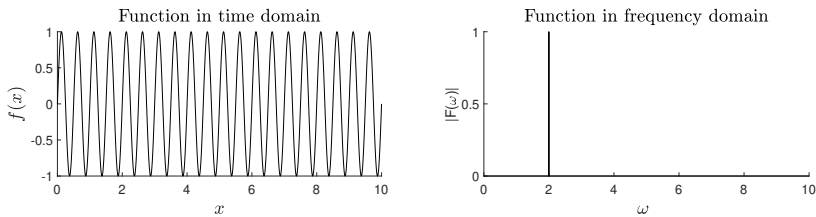


Fig. 4 Function in time and frequency domain

282 A Window Function $g(x) : \mathbb{R} \rightarrow \mathbb{R}$ that is “well localized” in the time
 283 is used to localize the frequency in time. Figure 5 shows the wide (above)
 284 and narrow (below) window function and its respective Fourier Transform
 285 Magnitude $|G(\omega)| : \mathbb{R} \rightarrow \mathbb{R}$ (narrow (above) and wide (below)).

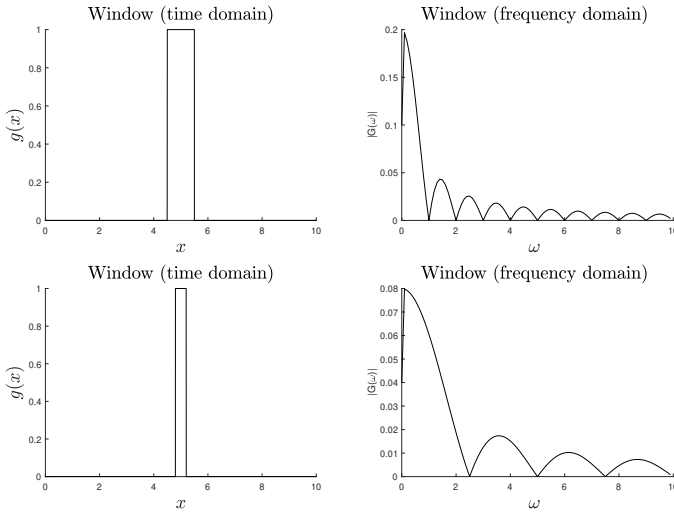


Fig. 5 Window Functions in time and frequency domain.

286 The function $f(x)$ is multiplied by the Window Function $g(x)$. Figure 6
 287 shows the wide (above) and narrow (below) Windowed Function and its respective
 288 Fourier Transform Magnitude $|F(\omega) * G(\omega)|$ (narrow (above) and wide
 289 (below)).

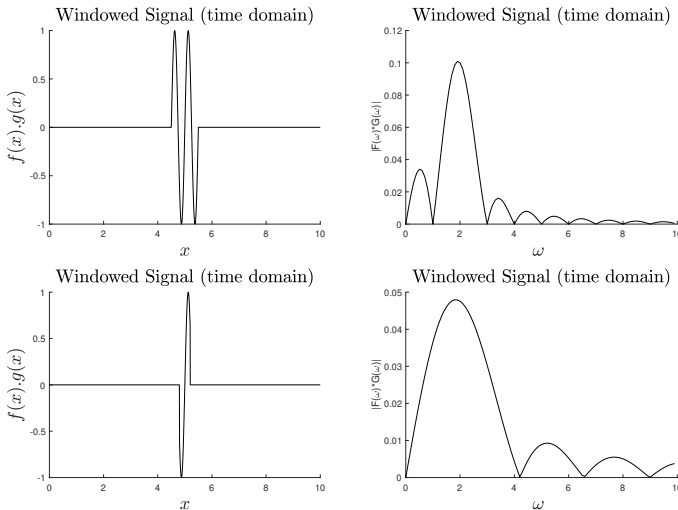


Fig. 6 Windowed Functions in time and frequency domain.

290 The Windowed Function in the frequency domain should have only one
 291 component (at 2 Hz), but it has components at several frequencies with non-
 292 zero amplitudes. This fact is due to the Windowed Function in the frequency

293 domain being the result of the convolution of the function by the Window
 294 Function in the frequency domain (in an analogous form, one can use the
 295 concept of wave packets in order to locate a wave in space [30]). So it is
 296 impossible to identify whether a particular frequency component is due to the
 297 function or the Window Function.

298 To measure the frequency as a function of the time, it was necessary to
 299 “locate” the wave in time using the Window function. However, this fact goes
 300 beyond the classical concept in physics of the observer effect [31], [32], in which
 301 to make a measurement, it is necessary to interfere with the measurement
 302 (which causes uncertainty). As everything that exists is a wave (wave nature
 303 of matter), Heisenberg’s Uncertainty Principle states that uncertainty occurs
 304 not only due to the measurement of an experiment (observer effect) but due
 305 to the impossibility of locating a wave sharply in the time and frequency
 306 (wavenumber, momentum, among others) domain simultaneously.

307 Analytically, Heisenberg’s Uncertainty Principle can be demonstrated con-
 308 sidering $\psi(x)$ and $\Psi(p)$ wave functions and Fourier Transform ² of each other
 309 for position x and momentum p , respectively.

310 Born’s rule [33] states that $|\psi(x)|^2$ and $|\Psi(p)|^2$ are probability density
 311 functions and then the variances of position σ_x^2 and momentum σ_p^2 are:

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx = \int_{-\infty}^{\infty} (x\psi(x))^* x\psi(x) dx \quad (62)$$

$$\sigma_p^2 = \int_{-\infty}^{\infty} p^2 |\Psi(p)|^2 dp = \int_{-\infty}^{\infty} (p\Psi(p))^* p\Psi(p) dp \quad (63)$$

Let $f(x) = x\psi(x)$:

$$\sigma_x^2 = \int_{-\infty}^{\infty} f^*(x)f(x) dx = \int_{-\infty}^{\infty} |f(x)|^2 dx = \langle f|f \rangle \quad (64)$$

Let $\mathcal{F}\{\cdot\}$ the Fourier transform, $-i\hbar \frac{d\psi(x)}{dx}$ the momentum operator in posi-
 tion space, $G(p) = p\Psi(p)$, $g(x) = \mathcal{F}\{G(p)\}$ and applying the Parseval’s
 theorem [34]:

$$\sigma_p^2 = \int_{-\infty}^{\infty} G^*(p)G(p) dp = \int_{-\infty}^{\infty} |G(p)|^2 dp = \int_{-\infty}^{\infty} |g(x)|^2 dx = \langle g|g \rangle \quad (65)$$

Using the Cauchy–Schwarz inequality [35]:

$$\langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2 \quad (66)$$

$$|\langle f|g \rangle|^2 \geq \text{Im}(|\langle f|g \rangle|^2) = \left(\frac{\langle f|g \rangle - \langle g|f \rangle}{2i} \right)^2 \quad (67)$$

² $\psi(x)$ and $\Psi(p)$ are functions in two corresponding orthonormal bases in Hilbert space and, therefore, are Fourier Transform of each other and x and p are conjugate variables.

$$\begin{aligned} & \langle f|g\rangle - \langle g|f\rangle = \\ = & \int_{-\infty}^{\infty} x\psi^*(x) \left(-i\hbar \frac{d\psi(x)}{dx}\right) dx - \int_{-\infty}^{\infty} \left(-i\hbar \frac{d\psi^*(x)}{dx}\right) x\psi(x) dx = i\hbar \end{aligned} \quad (68)$$

312 Applying 68 in 67:

$$|\langle f|g\rangle|^2 = \left(\frac{i\hbar}{2i}\right)^2 = \frac{\hbar^2}{4} \quad (69)$$

313 Applying 64, 65 and 69 in 66:

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4} = \sigma_x \sigma_p \geq \frac{\hbar}{2} \quad (70)$$

314 The demonstration of the uncertainty principle is strictly mathemati-
315 cal. Any pair of variables conjugated will produce the same results as this
316 demonstration.

317 Following Kennard's consideration [36], $\Delta x = \sigma_x$ the uncertainty in posi-
318 tion x (proportional to the width of the Window Function in time or space
319 domain), $\Delta p = \sigma_p$ the uncertainty in momentum, h the Plank's constant,
320 Heisenberg's Uncertainty Principle is normally presented as:

$$\Delta x \Delta p \geq \frac{h}{4\pi} \quad (71)$$

In the frequency domain, let Δk be the uncertainty in wavenumber (spa-
tial frequency) or $\Delta \omega$ the uncertainty in (temporal) frequency (Δk or $\Delta \omega$
are proportional to the width of the Window Function in frequency domain).
Through de Broglie's relation 61, 71 can be written as:

$$\Delta x \Delta k \geq \frac{1}{2\pi} \quad \text{or} \quad \Delta x \Delta \omega \geq \frac{1}{2\pi} \quad (72)$$

321 Another way to understand Heisenberg's Uncertainty Principle (and per-
322 haps the simplest) is through the Fourier Transform of the Gaussian function.
323 Let $f(x)$ be a Gaussian function in the space (time) domain x , $F(\omega)$ is its
324 Fourier transform in the frequency domain ω and is also a Gaussian function.
325 Then, the standard deviation σ can be understood as a measure of precision,
326 and this occurs inversely in $f(x)$ and $F(\omega)$. Thus, if the uncertainty is small
327 in one domain, it is large in the other domain.

$$f(x) = e^{-\sigma x^2} \Leftrightarrow F(\omega) = \frac{1}{\sqrt{2\sigma}} e^{-\frac{\omega^2}{4\sigma}} \quad (73)$$

328 A time-frequency representation³ is used when it is necessary to "localize"
329 ω in x (instantaneous frequency) and vice-versa. This representation is also

³Time-frequency is a representation with a two-dimensional domain (x, ω) , and is used to represent any pair of canonically conjugate coordinates, such as time-frequency, position-wavenumber, position-momentum, among others.

330 known as a spectrogram, generally obtained through the Short Time Fourier
 331 Transform [37] (or other transforms, such as Wavelet Transform [38], Wigner-
 332 Ville distribution function [39], etc).

333 A classic example of a signal whose frequency varies with time (non-
 334 stationary signal) is the Chirp Signal [40].

335 6.1 Chirp Derivative

A Chirp Signal can be defined with the following waveform (signal):

$$f(x) = \sin(2\pi \int \omega(x)dx + \phi), \quad (74)$$

336 where, $\omega(x)$ is the instantaneous frequency function and $\Omega(x) =$
 337 $2\pi \int \omega(x)dx + \phi$ is the phase function.

338

339 The following example is suitable for showing the determination of the
 340 instantaneous frequency. Considering a Chirp Signal with instantaneous
 341 frequency function given by:

$$\omega(x) = x^2 + 2x + 1 \quad (75)$$

342 For $\phi = 0$, the waveform is;

$$f(x) = \sin\left(2\pi\left(\int (x^2 + 2x + 1) dx\right) + 0\right) = \sin\left(2\pi\left(\frac{x^3}{3} + x^2 + x\right)\right) \quad (76)$$

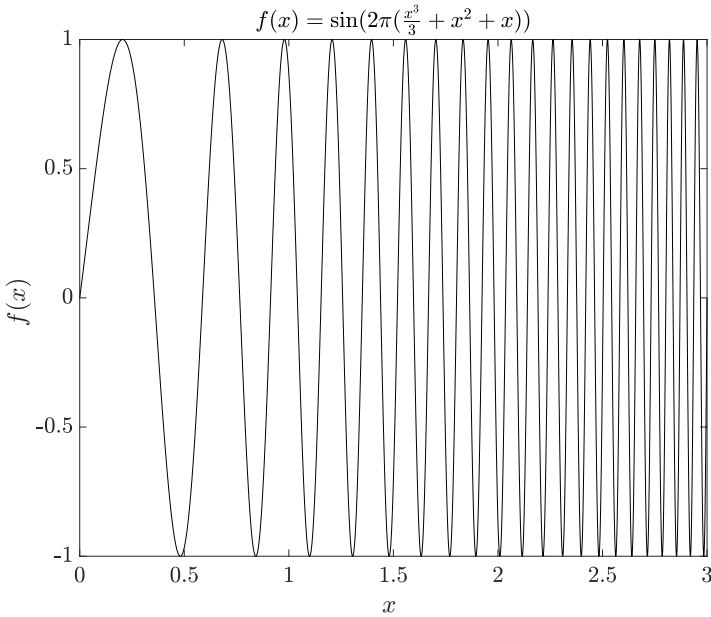


Fig. 7 Quadratic Chirp Signal - $\omega(x) = x^2 + 2x + 1$; $f(x) = \sin\left(2\pi\left(\frac{x^3}{3} + x^2 + x\right)\right)$

343 Figure 8 shows the spectrogram for the signal sampled at 100 samples/u-
 344 nit of x (100Hz if x is given in seconds) obtained through Short Time Fourier
 345 Transform with Gaussian window function (Gabor Transform [37]) and stan-
 346 dard deviation equal to 1. The $|F(x, \omega)|$ values (z-axis) are proportional to
 347 the energy in the signal at (x, ω) . For each x (or ω) value there is a range of
 348 ω (or x) values whose function $|F(x, \omega)|$ is non-zero. These intervals at (x, ω)
 349 domain represent the uncertainty in determining the instantaneous frequency
 350 in this signal representation.

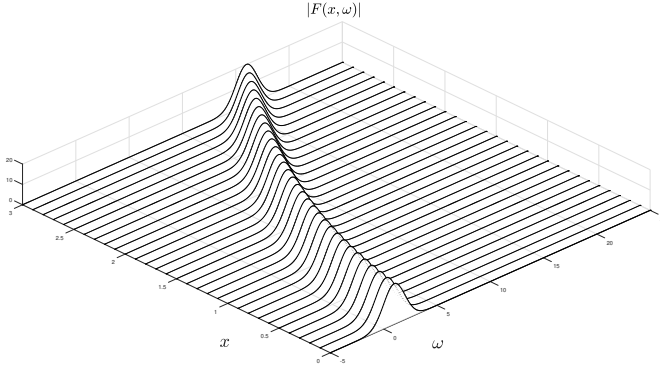


Fig. 8 Short Time Fourier Transform - $|F(x, \omega)|$

With:

$$\omega(x) = \omega_1 x + \omega_0 \tag{77}$$

(78) is a Linear Chirp (or Quadratic Phase Signal), with initial frequency ω_0 (at $x = 0$) and rate chirp ω_1 :

$$f(x) = \sin\left(2\pi\left(\int (\omega_1 x + \omega_0) dx\right) + \phi\right) = \sin\left(2\pi\left(\frac{1}{2}\omega_1 x^2 + \omega_0 x\right) + \phi\right) \tag{78}$$

351

The resolution of the system,

$$S : \begin{cases} f(x) = \sin\left(2\pi\left(\frac{1}{2}\omega_1 x^2 + \omega_0 x\right) + \phi\right) \\ f(x + \Delta) = \sin\left(2\pi\left(\frac{1}{2}\omega_1 (x + \Delta)^2 + \omega_0 (x + \Delta)\right) + \phi\right) \\ f(x + 2\Delta) = \sin\left(2\pi\left(\frac{1}{2}\omega_1 (x + 2\Delta)^2 + \omega_0 (x + 2\Delta)\right) + \phi\right) \end{cases} \tag{79}$$

352

applying limit to $\Delta \rightarrow 0$, is:

$$\mathfrak{D}\{\omega_1\} \frac{df(x)}{dx} = \omega_1^{ins}(x) = \lim_{\Delta \rightarrow 0} \frac{K_0 - K_1 + K_2}{2\pi\Delta^2} \tag{80}$$

where,

$$\begin{aligned} K_0 &= \arcsin(f(x)) \\ K_1 &= \arcsin(2f(x + \Delta)) \\ K_2 &= \arcsin(f(x + 2\Delta)) \end{aligned}$$

$$\mathfrak{D}\{\omega_0\} \frac{df(x)}{dx} = \omega_0^{ins}(x) = \lim_{\Delta \rightarrow 0} -\frac{1}{2} \frac{K_0 - K_1 + K_2}{2\pi\Delta^2} \tag{81}$$

where,

$$\begin{aligned} K_0 &= (2x + 3\Delta) \arcsin(f(x)) \\ K_1 &= (4x + 4\Delta) \arcsin(f(x + \Delta)) \\ K_2 &= (2x + \Delta) \arcsin(f(x + 2\Delta)) \end{aligned}$$

$$\mathfrak{D}\{\phi\} \frac{df(x)}{dx} = \phi(x) = \lim_{\Delta \rightarrow 0} \frac{1}{2} \frac{K_0 + K_1 + K_2}{\Delta^2} \tag{82}$$

where,

$$\begin{aligned} K_0 &= (2\Delta^2 + x^2 + 3x\Delta) \arcsin(f(x)) \\ K_1 &= (-2x^2 - 4x\Delta) \arcsin(f(x + \Delta)) \\ K_2 &= (x^2 + x\Delta) \arcsin(f(x + 2\Delta)) \end{aligned}$$

According to (77), the instantaneous frequency function is obtained as:

$$\omega(x) = \mathfrak{D}\{\omega_1\} \frac{df(x)}{dx} x + \mathfrak{D}\{\omega_0\} \frac{df(x)}{dx} \tag{83}$$

$\mathfrak{D}\{\omega_1\}$, $\mathfrak{D}\{\omega_0\}$ ($\mathfrak{D}\{\phi\}$ is not needed in this example) are:

$$\mathfrak{D}\{\omega_1\} \frac{df(x)}{dx} = \frac{2 \cos\left(\frac{2\pi x(x^2+3x+3)}{3}\right)^3 (x+1)}{\left(\cos\left(\frac{2\pi x(x^2+3x+3)}{3}\right)^2\right)^{\frac{3}{2}}} \tag{84}$$

$$\mathfrak{D}\{\omega_0\} \frac{df(x)}{dx} = -\frac{\cos\left(\frac{2\pi x(x^2+3x+3)}{3}\right)^3 (x^2-1)}{\left(\cos\left(\frac{2\pi x(x^2+3x+3)}{3}\right)^2\right)^{\frac{3}{2}}} \tag{85}$$

353 The (84) and (85) have positive and negative values, which are therefore
 354 associated with positive and negative frequency values. In absolute values, (84)
 355 and (85) are, respectively:

$$\left| \mathfrak{D}\{\omega_1\} \frac{df(x)}{dx} \right| = |2x + 2| \tag{86}$$

$$\left| \mathfrak{D}\{\omega_0\} \frac{df(x)}{dx} \right| = |-x^2 + 1| \tag{87}$$

However, negative frequency values can be neglected, and therefore the modulus functions at (86) and (87) can be removed without loss of generality. According to (83), the frequency function $\omega_{qr}(x)$ can be reconstructed by:

$$\omega_{qr}(x) = (2x + 2)x - x^2 + 1 = x^2 + 2x + 1 \tag{88}$$

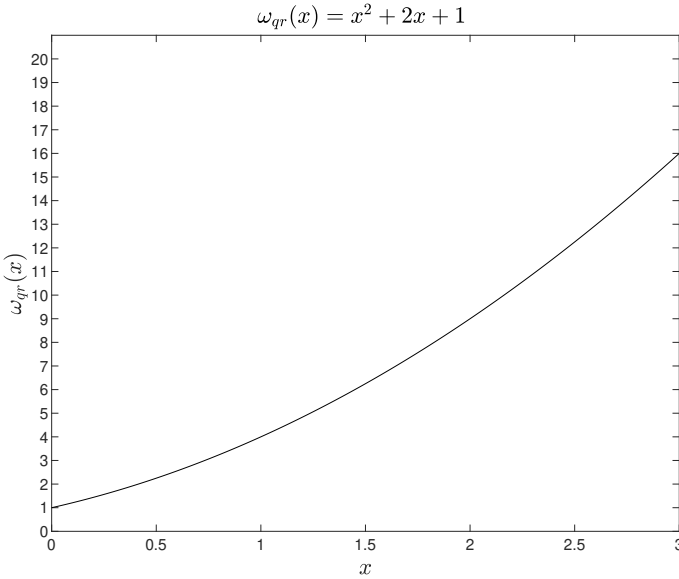


Fig. 9 $\omega_{qr}(x) = x^2 + 2x + 1$, obtained from $f(x) = \sin\left(2\pi\left(\frac{x^3}{3} + x^2 + x\right) + \phi\right)$

356 It is important to note that (88) is obtained from (76) and not from the
 357 (60), i.e., the exact instantaneous frequency is obtained from waveform (wave
 358 function or signal) ($f(x)$) and not from phase function ($\Omega(x)$) and **there is**
 359 **no uncertainty**.

360 **Remark** *Heisenberg's Uncertainty Principle was not respected.*

From the instantaneous frequency function $\omega(x)$, the amplitude spectrum $F(\omega)$ can be calculated as:

$$F(\omega) = \frac{1}{\omega} \int_{-\infty}^{\infty} \omega(x)h(x)dx, \tag{89}$$

with,

$$h(x) = \begin{cases} 1; & \text{if } \omega(x) = \omega; \\ 0; & \text{otherwise} \end{cases} \tag{90}$$

361 6.2 Fourier Derivative

362 An important exponential function is $e^{-i\omega x}$, with $i = \sqrt{-1}$ and $x, \omega, \in \mathbb{R}$,
 363 which is the kernel of the Fourier Transforms [34]. Adding a scaling factor
 364 (as in (39)) to this kernel, and proceeding analogously to (41), the **Fourier**
 365 **Derivative** of function $f(x)$ is:

$$\mathfrak{D}\{\} = e^{-i\omega x+b} \tag{91}$$

$$\mathfrak{D}\{\omega\} \frac{df(x)}{dx} = \omega(x) = \lim_{\Delta \rightarrow 0} \frac{\ln(f(x+\Delta)) - \ln(f(x))}{-i\Delta} \tag{92}$$

$$\mathfrak{D}\{b\} \frac{df(x)}{dx} = b(x) = \lim_{\Delta \rightarrow 0} \frac{\ln(f(x))(x+\Delta) - \ln(f(x+\Delta))x}{\Delta} \tag{93}$$

$f(x)$ can be reconstructed from its Fourier derivatives as:

$$f(x) = e^{-i\mathfrak{D}\{\omega\} \frac{df(x)}{dx} x + \mathfrak{D}\{b\} \frac{df(x)}{dx}} \tag{94}$$

366 The parameter ω is the frequency in the kernel of the Fourier Transform,
 367 and therefore (92) is the instantaneous frequency ($\omega(x) \in \mathbb{C}$).

368 Let $A \in \mathbb{R}$, a wave function of the type $\psi(x, \omega) = Ae^{-i\omega x}$. Considering, as
 369 example, $A = 2$, $\omega(x) = x^3 + 2x$ (the frequency ω varies with x , i.e. $\omega(x)$) and
 370 $\Omega(x)$ the phase function (as in Chirp Signal (74)), the wave function $\psi(x, \omega)$
 371 becomes:

$$\psi(x) = 2e^{-i\Omega(x)} = 2e^{-i\left(\frac{x^4}{4} + x^2\right)} \tag{95}$$

where,

$$\Omega(x) = \int (x^3 + 2x)dx = \left(\frac{x^4}{4} + x^2\right) \tag{96}$$

372 Applying 92 and 93 in 95:

$$\mathfrak{D}\{\omega\} \frac{d\psi(x)}{dx} = \omega(x) = x^3 + 2x \tag{97}$$

$$\mathfrak{D}\{b\} \frac{d\psi(x)}{dx} = b(x) = i\left(\frac{3x^4}{4} + x^2\right) + \ln 2 \tag{98}$$

373 Applying 94 in 97 and 98, the wave function $\psi(x)$ (95), can be reconstructed
 374 as:

$$\psi(x) = e^{-i(x^3+2x)x+i\left(\frac{3x^4}{4}+x^2\right)+\ln 2} = 2e^{-i\left(\frac{x^4}{4}+x^2\right)} \tag{99}$$

375 **There is no uncertainty.**

376 **Remark** *Heisenberg's Uncertainty Principle was not respected.*

377 7 Conclusion

378 In a simplified and summarized manner, Differential Calculus is based on
 379 applying a limit tending to zero for Newton's Difference Quotient applied under
 380 any function $f(x)$.

381 This operation determines another function (the derivative) whose values
 382 represent the instantaneous angular coefficients of the tangent lines to the
 383 function $f(x)$.

384 This paper showed that the Differential and Integral Calculus could be
 385 applied to other parameters of other functions called derivator and integrator
 386 functions.

387 All the theories presented can be applied to two or more dimensions (partial
 388 derivatives and multiple integrals), in addition to well-established operations
 389 in classical differential and integral calculus such as the chain rule, product
 390 and division derivatives and integrals, differential and integral equations, and
 391 others, and this is suggested as future work.

392 Some examples were presented, with emphasis on the determination of
 393 the instantaneous frequency. Although Heisenberg's Uncertainty Principle is
 394 formalized as a property of waves, this paper has shown that uncertainty occurs
 395 due to the methodology employed for determining the instantaneous frequency
 396 in a function (wave function or signal).

397 Heisenberg's Uncertainty Principle is based on the use of integral trans-
 398 forms (such as Fourier Transform and similar wave packets), for a function
 399 in the time (or space) domain to obtain its representation in the frequency
 400 domain and vice versa.

401 An integral transform is obviously based on the calculation of integrals.
 402 Hence, the integral is suitable for measuring general quantities associated with
 403 the whole function domain, such as an area, expected value, norm, autocorrela-
 404 tion, and even frequency distribution (spectral density), but not instantaneous
 405 quantities.

406 Integral transforms (or wave packets) will produce uncertainty in the phase
 407 space of canonically conjugate variables.

408 Nevertheless, why use a mathematical operation based on integral to try
 409 to determine instantaneous quantities?

410 In turn, the derivative is suitable for measuring instantaneous quantities in
 411 a function. This paper presented a form to obtain the instantaneous frequency
 412 of a function given in the time (or space) domain using derivatives (and not
 413 integrals).

414 The Fourier, Trigonometric, and Chirp Derivatives are examples of different
 415 forms to obtain the instantaneous frequency sharply.

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