LOW TEMPERATURE PHASE DIAGRAM FOR A CLASS OF FINITE RANGE INTERACTIONS ON THE PENROSE LATTICE

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Recently, new alloys have been synthesized that exhibit X-ray diffraction patterns with unusual crystallographic symmetries, excluded by classical crystallography, like five- or ten-fold axes. It is shown that such patterns can be obtained by diffraction on quasiperiodic structures like the Penrose tiling of the plane.

To describe the quasiperiodic lattice, we use the projection method introduced in [1]. Namely, we decompose the space $\mathbf{R}^\nu$ into two orthogonal subspaces $E_\parallel$ and $E_\perp$ and denote by $\pi_\parallel$ and $\pi_\perp$ the corresponding projections. The quasiperiodic lattice is identified to a particular discrete subset of $E_\parallel$ that will be constructed in the sequel.

Denote by $\{\epsilon_1, \ldots, \epsilon_\nu\}$ an orthonormal basis of $\mathbf{R}^\nu$, by γ the unit hypercube

$$\gamma = \{\xi \in \mathbf{R}^\nu : \xi = \sum_{i=1}^{\nu} \xi_i \epsilon_i \quad \text{with} \quad \xi_i \in [0, 1[\},$$

and by K the projection of γ on $E_\perp$, *i.e.* $K = \pi_\perp(\gamma)$. Sliding γ over $E_\parallel$ creates a strip $S = E_\parallel + \gamma$. The projection of the points of the lattice $\mathbf{Z}^\nu$ that are intersected by S onto $E_\parallel$ gives the quasiperiodic lattice we are looking for. Generically, *i.e.* for almost every orientation of $E_\parallel$, the projection $\pi_\parallel$ is bijective between $\mathbf{Z}^\nu$ and the $\mathbf{Z}$-module $\pi_\parallel(\mathbf{Z}^\nu)$. Therefore, generically, the quasiperiodic lattice L can be described by

$$L = \{s \in \pi_\parallel(\mathbf{Z}^\nu) : \pi_\perp(\pi_\parallel^{-1}(s)) \in K\}.$$

For the lattice L to have unusual diffraction patterns it is necessary ν to be large enough. Thus, for Penrose tilings of the plane $\nu = 5$, $\dim(E_\parallel) = 2$ and for tilings of the three-dimensional space with rhombohedra and diffraction patterns with eikosahedral symmetry $\nu = 6$ and $\dim(E_\parallel) = 3$.

We turn now to the statistical mechanics on L. Assume that a discrete space S is given and denote by X the set of all configurations $x : L \mapsto S$ that will be called *configuration space*. For every *finite* subset A of L, denote by $\mathcal{F}_A = \mathcal{S}^A$ — $\mathcal{S}$ being the exhaustive σ-field of S — and by Φ_A a $\mathcal{F}_A$- measurable map $\Phi_A : X \mapsto \mathbf{R}$. The $\mathcal{F}_A$-mesurability condition implies that Φ_A depends only on the values of the configuration x over the set A. Such functions Φ_A are called *interaction potentials.* Assume moreover that $\Phi_A \equiv 0$

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whenever $\operatorname{diam} A > r$, where r is some fixed number. Statistical mechanics is as usual described in terms of the finite volume Hamiltonians

$$H_\Lambda(x) = \sum_A \Phi_A(x)$$

the sum extending over all finite subsets of L that intersect Λ.

In a previous work, the case of quasiperiodic interaction potentials over configurations of a regular lattice was considered and the Pirogov-Sinai theory was proved valid [2]. Here, the complementary point of view is examined: the lattice L being quasiperiodic, it is the underlying configuration space that becomes quasiperiodic. The main result obtained in [3] can be sketched as follows:

Theorem : *Assume that*

1. *the Hamiltonian admits a finite number of local ground state configurations,*
2. *the interaction potentials are invariant under the transformations of the symmetry group of* $\pi_{\|}(\mathbf{Z}^\nu)$
3. *the Peierls condition is verified*
4. *the temperature is sufficiently low*

then, the Pirogov-Sinai theory is applicable.

Among the conditions of the theorem, condition 2 needs some clarification. Let g be an element of the discrete group acting on $E_{\|}$. Denote by gA the image of the set A under the transformation g and by gx_A the coresponding image of the configuration x over A. Then, invariance under g means $\Phi_{gA}(gx_A) = \Phi_A(x_A)$. The following example provides a simple nontrivial case.

Example : Let L be the 2-dimensional Penrose lattice tiled with two kinds of rhombs of unit edge and let a and b be the two shortest lengths appearing in the structure, namely $a = 2\sin(\pi/10)$, $b = 1$. Let J_1, J_2 be two positive constants and choose as one spin space the set $\{-1, 1\}$. Then

$$\Phi_A(x) = \begin{cases} -J_1 x_s x_t & \text{if } \; A = \{s,t\} \; \text{ with } \; \|s-t\| = a \\ -J_2 x_s x_t & \text{if } \; A = \{s,t\} \; \text{ with } \; \|s-t\| = b \\ 0 & \text{otherwise,} \end{cases}$$

with $\|\cdot\|$ being the L_2 distance in $E_{\|}$, defines a a variant of Ising models for which our theorem applies.

Other different and more general situations, including many body interactions, ferromagnetic and antiferromagnetic couplings etc., can be handled by the present extension.

References

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