# P-determinants and boundary values *

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**Abstract**

We show that a regularized determinant based on Hilbert's approach (wich we call the "p-determinant") of a quotient of elliptic operators defined on a manifold with boundary is equal to the "p-determinant" of a quotient of pseudodifferential operators. The last ones are entirely expressible in terms of boundary values of solutions of the original differential operators. We argue that, in the context of Quantum Field Theory, this boundary values also determine the subtractions (i.e., the counterterms) to which this regularization scheme gives rise.

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# 1 - Introduction

In a recent paper[1], R. Forman has established a relation between the quotient of $\zeta$-function regularized determinants of elliptic operators defined on a manifold with boundary and the boundary values of their solutions. In fact, under certain conditions[1], such quotient is equal to the determinant of a quotient of pseudodifferential operators, entirely expressible in terms of boundary values of the solution space of the original elliptic operators. Thus, this approach has the advantage that, if one has a good understanding of this solution space (which is independent of the boundary conditions), one can study how the determinant varies as a function of the operators and of the boundary conditions. In such way, all considerations can be restricted to the boundary, which is a manifold of dimension one less than the original one, without requiring the knowledge of eigenvalues, which are global quantities of difficult handling.

Even though succesfully employed in some applications of physical interest[1, 2], the conditions under which Forman's result is valid appear to be too restrictive. In fact, they can be addressed to the existence of the derivative of the Fredholm determinant of the quotient of elliptic operators. This clearly excludes those situations in which this Fredholm determinant does not exist, but a more general regularized determinant (which we will call "p-determinant"), based on Hilbert's approach , can be defined [3, 4].

The aim of this paper is to generalize Forman's result, without loosing its forementioned advantages.

In section 2 we establish our main result: Under certain conditions to be specified later, the p-determinant $det_p(L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0))$ (where $L_A(\mu)$ ($L_B(\mu)$) is an elliptic differential operator depending on a parameter $\mu$, $L(\mu)$, restricted to functions $f$ defined in a region $\Omega \subset \mathbf{R}^n$ and satisfying $Af = 0$ ($Bf = 0$) at the boundary $\partial\Omega$) equals the p-determinant $det_p(\Phi_{AB}^{-1}(0)\Phi_{AB}(\mu))$ (where $\Phi_{AB}(\mu)$, acting on functions living on $\partial\Omega$, is Forman's map defined in reference [1]).

We also argue that the application of this method in the context of Quantum Field Theory allows to determine the subtractions which occur in this regularization scheme (i.e., the pieces requiring renormalization) also from boundary values of solutions of the operators.

In section 3 we apply our result to some physically interesting problems. In particular, for a bosonic field at finite temperature, the difference of free energies corresponding to two boundary conditions is evaluated, carefully studying the subtractions occuring and the related problem of renormalization.

Finally, section 4 contains some comments and conclusions.

# 2 - Generalization of Forman's result

In what follows, we will be interested in elliptic linear differential operators $L(\mu)$, depending on a parameter $\mu$ and restricted to functions defined in a region $\Omega \subset \mathbf{R}^n$ and satisfying elliptic boundary conditions corresponding to the projection $Bf = 0$ at $\partial\Omega$. We will suppose that the homogeneous problem:

$$
\begin{aligned}
Lf(x) &= 0, \ \ x \in \Omega \\
Bf(x) &= 0, \ \ x \in \partial\Omega
\end{aligned} \tag{1}
$$

has no nontrivial solution. So, following reference [1], we introduce the Poisson map for the operator $L$ and the boundary condition $B$, $P_B(\mu)$, such that the unique solution of

$$
\begin{aligned}
Lf(x) &= 0, \ \ x \in \Omega \\
Bf(x) &= h, \ \ x \in \partial\Omega
\end{aligned} \tag{2}
$$

is given by

$$
f(x) = P_B(\mu)h(x). \tag{3}
$$

As shown in Lemma 1.4 in the same reference, the Poisson map satisfies:

$$
\tfrac{d}{d\mu}P_B(\mu)|_B = -L_B^{-1}(\mu)L'(\mu)P_B(\mu)|_B, \tag{4}
$$

where $L'(\mu) = \frac{d}{d\mu}L(\mu)$ and $L_B^{-1}(\mu)$ is the Green's function of $L$, satisfying $BL_B^{-1}(\mu) = 0$ at $\partial\Omega$, and $|_B$ means acting on functions projected by $B$ at the boundary.

Let us consider a second elliptic boundary condition, defined by the operator $A$, and its corresponding Green's function $L_A^{-1}(\mu)$ ($AL_A^{-1}(\mu) = 0$ at $\partial\Omega$). The second result quoted in the abovementioned Lemma is:

$$
P_A(\mu)A\left(-L_B^{-1}(\mu)\right) = L_A^{-1}(\mu) - L_B^{-1}(\mu), \tag{5}
$$

since the right hand side is the unique smooth solution of $L$ with the same $A$-boundary value as (-$L_B^{-1}(\mu)$):

$$A\left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) = A\left(-L_B^{-1}(\mu)\right). \tag{6}$$

Now, the p-determinant of $(1-M)$, where $M \in \mathcal{I}_p$ (i.e., $M^p$ is trace class), is defined as[4]:

$$\ln det_p(1-M) = -\int_\gamma dz z^{p-1} Tr\left\{M^p\left(1-zM\right)^{-1}\right\}, \tag{7}$$

where $\gamma$ is any differentiable path,

$$\gamma : [0,1] \to \mathbf{C},$$

$$\gamma(0) = 0, \gamma(1) = 1,$$

such that $(\gamma(t))^{-1}$ does not belong to the spectrum of $M$.

If $M$ is a differentiable function of a parameter $\mu$, from equation (7), one gets:

$$\tfrac{d}{d\mu}\ln det_p\left(1-M\right) = Tr\left\{M^{p-1}\left(1-M\right)^{-1}\tfrac{d}{d\mu}M\right\}. \tag{8}$$

We will now suppose that $L_A(\mu)$ and $L_B(\mu)$ are such that the operator $[L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0)]$ has a p-determinant, and

$$Tr\left\{\left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right)\left(1 - L_A(\mu)L_A^{-1}(0)L_B(0)L_B^{-1}(\mu)\right)^{p-1} L'(\mu)|_{kerL}\right\} < \infty, \tag{9}$$

where $L'$ acts on functions in the kernel of the differential operator $L$.

Then, from equation (8), it is easy to get:

$$\tfrac{d}{d\mu}\ln det_p\left(L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0)\right) =$$

$$Tr\left\{\left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right)\left(1 - L_A(\mu)L_A^{-1}(0)L_B(0)L_B^{-1}(\mu)\right)^{p-1} L'(\mu)|_{kerL}\right\}. \tag{10}$$

By repeated application of equations (5) and (6), we will prove the following

LEMMA

$$\left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) L_A(\mu) L_A^{-1}(0) L_B(0) L_B^{-1}(\mu)$$
$$= P_B(\mu) B P_A(0) A \left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) \tag{11}$$

PROOF

$$\begin{aligned}
&\left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) L_A(\mu) L_A^{-1}(0) L_B(0) L_B^{-1}(\mu) \\
&= \left(P_B(\mu) B L_A^{-1}(\mu)\right) L_A(\mu) L_A^{-1}(0) L_B(0) L_B^{-1}(\mu) \\
&= P_B(\mu) B \left(L_A^{-1}(0) - L_B^{-1}(0)\right) L_B(0) L_B^{-1}(\mu) \\
&= P_B(\mu) B \left(-P_A(0) A L_B^{-1}(0)\right) L_B(0) L_B^{-1}(\mu) \\
&= P_B(\mu) B P_A(0) A \left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right).
\end{aligned} \tag{12}$$

When replaced in equation (10), this result allows us to write:

$$\begin{aligned}
&\tfrac{d}{d\mu} \ln det_p \left(L_B^{-1}(\mu) L_A(\mu) L_A^{-1}(0) L_B(0)\right) \\
&= Tr\left\{(1 - P_B(\mu) B P_A(0) A)^{p-1} \left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) L'(\mu)|_{kerL}\right\} \\
&= Tr\left\{P_B(\mu)\left(1 - B P_A(0) A P_B(\mu)\right)^{p-1} B \left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) L'(\mu)|_{kerL}\right\}.
\end{aligned} \tag{13}$$

Notice that, using equation (4), the last factors in the trace can be rewritten as:

$$\begin{aligned}
\left(L_A^{-1}(\mu) - L_B^{-1}(\mu)\right) L'(\mu) P_B(\mu) B|_{kerL} &= P_A(\mu) A \left(-L_B^{-1}(\mu)\right) L'(\mu) P_B(\mu) B|_{kerL} \\
&= P_A(\mu) A P_B'(\mu) B|_{kerL}.
\end{aligned} \tag{14}$$

Therefore:

$$\begin{aligned}
&\tfrac{d}{d\mu} \ln det_p \left(L_B^{-1}(\mu) L_A(\mu) L_A^{-1}(0) L_B(0)\right) \\
&= Tr\left\{P_B(\mu)\left(1 - B P_A(0) A P_B(\mu)\right)^{p-1} B P_A(\mu) A P_B'(\mu) B|_{kerL}\right\} \\
&= Tr\left\{\left(1 - B P_A(0) A P_B(\mu)\right)^{p-1} \left(B P_A(\mu)\right) \left(A P_B(\mu)\right)'|_B\right\}
\end{aligned}$$

$$= \frac{d}{d\mu} \ln det_p \left( \Phi_{AB}^{-1}(0) \Phi_{AB}(\mu) \right), \tag{15}$$

where $\Phi_{AB}(\mu)$ is the operator introduced by R. Forman in reference [1]:

$$\Phi_{AB}(\mu) = AP_B(\mu). \tag{16}$$

So, we have proved the following

THEOREM

$$det_p \left( L_B^{-1}(\mu) L_A(\mu) L_A^{-1}(0) L_B(0) \right) = det_p \left( \Phi_{AB}^{-1}(0) \Phi_{AB}(\mu) \right), \tag{17}$$

as long as these p-determinants exist and equation (9) is satisfied (since both sides equal 1 at $\mu = 0$).

Notice that this is a generalization of Forman's result. Indeed, if $L_A(\mu)$ and $L_B(\mu)$ are such that $L_B(0) L_B^{-1}(\mu)$ and $L_A(\mu) L_A^{-1}(0)$ have a Fredholm determinant (1-determinant), by the well-known properties of 1-determinants[3] and their relation with $\zeta$-function regularized determinants[1], one can write:

$$\begin{aligned} \frac{Det_\zeta L_A(\mu)}{Det_\zeta L_B(\mu)} \frac{Det_\zeta L_B(0)}{Det_\zeta L_A(0)} &= det_1 \left( L_B(0) L_B^{-1}(\mu) \right) det_1 \left( L_A(\mu) L_A^{-1}(0) \right) \\ &= det_1 \left( L_B(0) L_B^{-1}(\mu) L_A(\mu) L_A^{-1}(0) \right) = det_1 \left( \Phi_{AB}^{-1}(0) \Phi_{AB}(\mu) \right), \end{aligned} \tag{18}$$

which is the integrated version of Theorem 1 in reference [1].

In this case ($p = 1$) one could also follow reference [1] and collect all $\mu$-independent factors into an integration constant (see equations (10) and (15)) which, under particularly lucky circumstances, can be evaluated[1, 2]. In the general case ($p > 1$), no such factorization takes place.

Equation (17) in the Theorem above states that the p-determinant on the left can be entirely expressed in terms of objects acting on functions defined just on the boundary, which can be evaluated from the boundary values of solutions of the elliptic operators[1, 2] (see section 3 later in this paper for an explicit example).

We wish to stress that, in the context of Quantum Field Theory, the problem of analyzing the subtractions taking place in this regularization scheme can also be reduced to the consideration of objects at the boundary. In fact, let us consider for a while that the (p-1)-determinants of the operators in equation (17) also exist. Then, from the following property of p-determinants[3]:

$$det_p\,(1-M) = \exp\left(\frac{1}{p-1}TrM^{p-1}\right) det_{p-1}(1-M), \tag{19}$$

we conclude that

$$Tr\left\{\left(1 - L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0)\right)^{p-1}\right\}$$
$$= Tr\left\{\left(1 - \Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)\right)^{p-1}\right\} \tag{20}$$

When the (p-1)-determinant does not exist, equation (20) can be considered to be a formal identity, which can be extended to all integer powers from 1 to (p-1). In such a way, the (divergent) subtractions in the present regularization scheme (and, therefore, the finite counterterms required) can be expressed entirely in terms of the operator $\Phi_{AB}(\mu)$. We will make explicit application of these arguments in the third example of Section 3.

Our Theorem concerns variations of the elliptic operator $L$. To end this section, a comment on variations of the boundary conditions is in place: Let us consider a situation in which boundary conditions, rather than operators, depend on $\mu$ through a regular transformation $\mathcal{U}(\mu)$ such that the relevant boundary values of a solution of $L$, $f$, are given by $A\mathcal{U}^{-1}(\mu)f$ $(B\mathcal{U}^{-1}(\mu)f)$. Again, we will suppose that the boundary value problem is nonsingular.

We can translate the $\mu$-dependence to the differential operator by noticing that:

$$\begin{pmatrix} L\Psi = 0 \\ A\mathcal{U}^{-1}\Psi = h \end{pmatrix} \Rightarrow \begin{pmatrix} \mathcal{U}^{-1}L\mathcal{U}\chi = 0 \\ A\chi = h \end{pmatrix}, for\ \chi = \mathcal{U}^{-1}\Psi, \tag{21}$$

and therefore apply our previous result, equation 17, to $L_A(\mu) \equiv \left(\mathcal{U}^{-1}L\mathcal{U}\right)_A$.

Since

$$\left(\mathcal{U}^{-1}L\mathcal{U}\right)_A^{-1} = \mathcal{U}^{-1}L_{A\mathcal{U}^{-1}}^{-1}\mathcal{U}, \tag{22}$$

equation (17) implies:

$$det_p\left\{\left(\mathcal{U}^{-1}L\mathcal{U}\right)_B^{-1}\left(\mathcal{U}^{-1}L\mathcal{U}\right)_A L_A^{-1}L_B\right\}$$
$$= det_p\left\{\mathcal{U}^{-1}L_{B\mathcal{U}^{-1}}^{-1}L_{A\mathcal{U}^{-1}}\mathcal{U}L_A^{-1}L_B\right\}$$

$$= det_p \left\{ \Phi_{AB}^{-1}(0)\Phi_{AB}(\mu) \right\}, \tag{23}$$

where

$$\Phi_{AB}(\mu)B\mathcal{U}^{-1}\Psi = A\mathcal{U}^{-1}\Psi, \tag{24}$$

for any solution of $L\Psi = 0$. In the next section we apply this result to some examples of physical interest.

# 3 - Some applications

## 3.1 - The Laplacian on the disc

In this section, we will explicitly construct the operator $\Phi_{AB}(\mu)$ for the case in which the differential operator is:

$$L = - \triangle + \lambda^2, \tag{25}$$

acting on functions $f(r, \theta)$ defined on a disc of radius $R$, and the boundary conditions correspond to the projections:

$$\begin{array}{rcl} Af & = & a\partial_r f(R,\theta) + f(R,\theta) \\ Bf & = & f(R,\theta), \end{array} \tag{26}$$

We will introduce a smooth function $\mathcal{U}^{-1}(r)$, such that $\mathcal{U}^{-1}(R) = 1$ and $\partial_r \mathcal{U}^{-1}(R) = -\mu$. Then, we have:

$$\begin{array}{rcl} A\mathcal{U}^{-1}f & = & a\partial_r f(R,\theta) + (1-\mu a)\, f(R,\theta) \\ B\mathcal{U}^{-1}f & = & f(R,\theta). \end{array} \tag{27}$$

In order to determine $\Phi_{AB}(\mu)$, we choose as a basis in the kernel of $L$ the system:

$$\{\Psi_k(r,\theta) = I_k\,(\lambda r) \exp\{ik\theta\}, for\ k \in \mathbf{Z}\} \tag{28}$$

where $I_k(z)$ is the modified Bessel function. The boundary values of those functions are given by:

$$\begin{array}{rcccl} h'_k(\theta) & = & A\mathcal{U}^{-1}\Psi_k & = & \{a\lambda I'_k(\lambda R) + (1-\mu a)\, I_k(\lambda R)\} \exp\{ik\theta\} \\ h_k(\theta) & = & B\mathcal{U}^{-1}\Psi_k & = & I_k(\lambda R) \exp\{ik\theta\}. \end{array} \tag{29}$$

Since, for all k:

$$h'_k(\theta) = \Phi_{AB}(\mu) h_k(\theta), \tag{30}$$

we conclude that, referred to the basis $\{\exp\{ik\theta\}, k \in \mathbf{Z}\}$, $\Phi_{AB}(\mu)$ is diagonal and given by:

$$(\Phi_{AB}(\mu))_{k',k} = \left\{(1-\mu a) + a\lambda \frac{I'_k(\lambda R)}{I_k(\lambda R)}\right\} \delta_{k',k}. \tag{31}$$

So:

$$\left(\Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)\right)_{k',k} = \left\{1 - \frac{\mu a}{1 + a\lambda \frac{I'_k(\lambda R)}{I_k(\lambda R)}}\right\} \delta_{k',k}. \tag{32}$$

From equation (7) and the asymptotic behavior of $I_k(z)$ for large $k$, which gives:

$$\left(1 - \Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)\right)_{k',k} \approx \frac{\mu R}{|k|}\delta_{k',k}, \ \ |k| \gg 1, \tag{33}$$

one can see that the Fredholm determinant of the operator in equation (32) doesn't exist, but its p-determinant for $p \geq 2$ does.

Then, the Theorem in section 2 implies that:

$$det_2\left(\mathcal{U}^{-1} L^{-1}_{B\mathcal{U}^{-1}} L_{A\mathcal{U}^{-1}} \mathcal{U} L_A^{-1} L_B\right) =$$

$$\prod_{k=-\infty}^{\infty} \left\{1 - \frac{\mu a}{1 + a\lambda \frac{I'_k(\lambda R)}{I_k(\lambda R)}}\right\} \exp\left\{\frac{\mu a}{1 + a\lambda \frac{I'_k(\lambda R)}{I_k(\lambda R)}}\right\}. \tag{34}$$

Notice that, in the present case, $L^{-1}_{B\mathcal{U}^{-1}} = L_B^{-1}$, which follows from equations (25), (26) and (27)).

If $\lambda = 0$, one must consider the basis of functions in the kernel of $L = -\triangle$ given by $\{\Psi_k = r^{|k|}\exp\{ik\theta\}, \, k \in \mathbf{Z}\}$.

One finally gets:

$$det_2\left(\Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)\right) = \prod_{k=-\infty}^{\infty} \left\{1 - \frac{\mu a}{1 + |k|a/R}\right\} \exp\left\{\frac{\mu a}{1+|k|a/R}\right\}. \tag{35}$$

Notice that choosing $\mu = 1/a$ the $k = 0$ factor vanishes, thus showing that $L_{A\mathcal{U}^{-1}}$, the Laplacian with Neumann boundary conditions, has a null eigenvalue. This is related to the property of p-determinants that $det_p(1-M) \neq 0$ if and only if $(1-M)$ is invertible[3].

### 3.2 - Bosonic field at temperature $\frac{1}{\beta} > 0$

In this section, we study a massive scalar field confined to a circle of radius $R$, at finite temperature $T = \beta^{-1} > 0$ and subject, at the spatial boundary, to the conditions defined in equation (26) of the previous example.

As is well-known, the partition function of such a system leads to the consideration of the fuctional deteminant of the differential operator:

$$L = -\triangle^2 - \partial_t^2 + m^2, \tag{36}$$

where $t \in [0, \beta]$ is the "temporal" coordinate, and $L$ acts on functions periodic in the $t$-direction and satisfying $A\mathcal{U}^{-1}f = 0$ ($B\mathcal{U}^{-1}f = 0$) at $r = R$.

A basis in the kernel of $L$ is given by:

$$\left\{\Psi_{n,k}(r, \theta, t) = I_k(\lambda_n r) \exp\{ik\theta + i\omega_n t\}, \ for \ n, k \in \mathbf{Z}\right\}, \tag{37}$$

where $\lambda_n = \sqrt{(\omega_n^2 + m^2)}$ and $\omega_n = \frac{2n\pi}{\beta}$.

To determine Forman's map, it is enough to consider projections on the spatial boundary (since $\Psi_{nk}$ is periodic in the $t$-direction):

$$h_{n,k}(\theta, t) = A\mathcal{U}^{-1}\Psi_{n,k}$$

$$= \{(1 - \mu a)I_k(\lambda_n R) + a\lambda_n I_k'(\lambda_n R)\} \exp\{ik\theta + i\omega_n t\}$$

$$h_{n,k}'(\theta, t) = B\mathcal{U}^{-1}\Psi_{n,k} = I_k(\lambda_n R) \exp\{ik\theta + i\omega_n t\} \tag{38}$$

Once more, $\Phi_{AB}(\mu)$ is diagonal when referred to the basis of functions at the boundary

$$\left\{\exp\{ik\theta + i\omega_n t\}, for \ n, k \in \mathbf{Z}\right\}, \tag{39}$$

so:

$$\left(\Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)\right)_{k',k}^{n',n} = \left\{1 - \frac{\mu a}{1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)}}\right\} \delta_{k',k}\delta^{n',n}. \tag{40}$$

Now, the asymptotic Debye expansion for Bessel functions[5, 6] at large order and/or argument:

$$z\frac{I_k'(z)}{I_k(z)} \approx \rho - \frac{z^2}{2\rho^2} + O(1/\rho), \ for \ \rho = \sqrt{k^2 + z^2}, \tag{41}$$

shows, when replaced in equation (7) that the p-determinant of the operator in equation (40) exists only for $p \geq 3$.

So, from our main result, it turns that:

$$det_3(\mathcal{U}^{-1} L^{-1}_{B\mathcal{U}^{-1}} L_{A\mathcal{U}^{-1}} \mathcal{U} L_A^{-1} L_B) = \prod_{n=-\infty}^{\infty} \prod_{k=-\infty}^{\infty}$$

$$\left\{ 1 - \frac{\mu a}{1 + a\lambda_n \frac{I'_k(\lambda_n R)}{I_k(\lambda_n R)}} \right\} \exp \left\{ \frac{\mu a}{1 + a\lambda_n \frac{I'_k(\lambda_n R)}{I_k(\lambda_n R)}} + \frac{1}{2} \left( \frac{\mu a}{1 + a\lambda_n \frac{I'_k(\lambda_n R)}{I_k(\lambda_n R)}} \right)^2 \right\}. \tag{42}$$

In order to define a quotient of partition functions of the system (subject to the different boundary conditions) in the regularization scheme provided by this approach, one should study the subtractions which take place. As pointed in section 2, this can be reduced to the consideration of the two terms subtracted in equation (42) (corresponding to the argument of the exponential in each factor of the double product). These are two (divergent) terms, which must be readded after being renormalized [7]. Notice that they are linear and quadratic in $\mu$ respectively, which suggests the introduction of two finite counterterms, given by the integral over the boundary of densities linear and quadratic in $\mathcal{U}\partial_r\mathcal{U}^{-1}$. This point will be more explicitly considered in our last example, where a variable external field will be introduced.

## 3.3 - Variable external field

In order to analyze the renormalization scheme to which the approach developed in this paper leads, we now treat the situation in which a more general external $\mathcal{U}$ field appears. We consider $\mathcal{U}(r, \theta)$ such that:

$$\mathcal{U}(R, \theta) = 1$$

$$\mathcal{U}\partial_r\mathcal{U}^{-1}(R, \theta) = -\mu \sum_{l=-\infty}^{\infty} C_l \exp\{il\theta\}. \tag{43}$$

Since only boundary conditions have changed, we take the same basis for the kernel of $L$ as in the previous example. So, the boundary values are now given by:

$$h'_{n,k}(\theta,t) = \sum_{l=-\infty}^{\infty} \left\{ \left(1 + a\lambda_n \frac{I'_k(\lambda_n R)}{I_k(\lambda_n R)}\right) \delta_{l,0} - \mu a C_l \right\} I_k(\lambda_n R) \exp\{i(k+l)\theta + i\omega_n t\}$$

$$h_{n,k}(\theta,t) = I_k(\lambda_n R) \exp\{ik\theta + i\omega_n t\}. \qquad (44)$$

In the present case, $\Phi_{AB}(\mu)$ is diagonal only in $n$ when referred to the basis in equation (39). After a direct algebra, one gets:

$$\left(\Phi^{-1}_{AB}(0)\Phi_{AB}(\mu)\right)^{n',n}_{k',k} = \left\{ \delta_{k',k} - \frac{\mu a C_{k'-k}}{1 + a\lambda_n \frac{I'_k(\lambda_n R)}{I_k(\lambda_n R)}} \right\} \delta^{n',n}. \qquad (45)$$

It can be seen that the p-determinant of the operator in equation (45) exists only for $p \geq 3$. This means that, in order to obtain a finite 3-determinant, two (divergent) traces,

$$\frac{1}{q} Tr \left\{ \left(1 - \Phi^{-1}_{AB}(0)\Phi_{AB}(\mu)\right)^q \right\}, \ for\ q = 1, 2\ (= p - 1), \qquad (46)$$

are subtracted (see equation(19)). To define (a difference of) free energies, this traces must be readded after being renormalized. So, we will write (see equation (23)):

$$2\beta \left\{ [F_A(\mu) - F_B(\mu)] - [F_A(0) - F_B(0)] \right\}$$

$$\equiv \ln det_3 \left(\Phi^{-1}_{AB}(0)\Phi_{AB}(\mu)\right) + \sum_{q=1}^{2} \frac{1}{q} Tr \left\{ \left(1 - \Phi^{-1}_{AB}(0)\Phi_{AB}(\mu)\right)^q \right\} |_{ren.}. \qquad (47)$$

From equation (45) and the development in equation (41), it can be seen that, for $q = 1$, this trace is proportional to $C_0$ and can be regularized as follows:

$$G_1 = Tr \left\{ 1 - \Phi^{-1}_{AB}(0)\Phi_{AB}(\mu) \right\} |_{reg.} =$$

$$\mu a C_0 \sum_{k,n=-\infty}^{\infty} \left\{ \frac{1}{1 + a\lambda_n \frac{I'_k(\lambda_n R)}{I_k(\lambda_n R)}} - \frac{R}{a\rho} + \frac{R^2}{a^2\rho^2}\left(1 - \frac{aR\lambda_n^2}{2\rho^2}\right) \right\} +$$

$$\mu R C_0 \; F.P.|_{s=0} \sum_{k,n=-\infty}^{\infty} \frac{1}{\rho^s} \left\{ \frac{1}{\rho} - \frac{R}{a\rho^2} \left( 1 - \frac{aR\lambda_n^2}{2\rho^2} \right) \right\}, \tag{48}$$

where $\rho = \sqrt{k^2 + (R\lambda_n)^2}$. In equation (48) the first series is absolutely convergent, while the second is analytically regularized through the introduction of a factor $1/\rho^s$, and only its finite part at $s = 0$ is retained. It can be verified that the residue at $s = 0$ is linear in $\beta$, so that only temperature-independent divergencies must be removed from the free energy[8]. This clearly requires the introduction of a (finite) local counterterm of the form:

$$K_1 = \beta\alpha_1 C_0 = \beta\alpha_1 \int_0^{2\pi} \mathcal{U} \partial_r \mathcal{U}^{-1}(R,\theta) d\theta. \tag{49}$$

For $q = 2$, the situation is more involved, since one can distinguish a finite nonlocal contribution plus a divergent local term. Indeed, one can write:

$$\left( [1 - \Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)]^2 \right)_{k,k}^{n,n} = \sum_{k'} \frac{(\mu a)^2 |C_{k-k'}|^2}{1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)}}$$

$$\left\{ \frac{1}{1 + a\lambda_n \frac{I_{k'}'(\lambda_n R)}{I_{k'}(\lambda_n R)}} - \frac{1}{1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)}} \right\} + \sum_{k'} \frac{(\mu a)^2 |C_{k-k'}|^2}{\left( 1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)} \right)^2}. \tag{50}$$

The first term in the right hand side gives a finite nonlocal contribution:

$$G_2 = \frac{-1}{2}(\mu a)^2 \sum_{n,k,k'} \frac{|C_{k-k'}|^2}{1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)}} \left\{ \frac{1}{1 + a\lambda_n \frac{I_{k'}'(\lambda_n R)}{I_{k'}(\lambda_n R)}} - \frac{1}{1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)}} \right\}. \tag{51}$$

The trace of the second one can be regularized to give:

$$G_3 = \frac{(a\mu)^2}{2} \sum_{\mathcal{K}=-\infty}^{\infty} |C_{\mathcal{K}}|^2 \sum_{k,n=-\infty}^{\infty} \left\{ \frac{1}{\left( 1 + a\lambda_n \frac{I_k'(\lambda_n R)}{I_k(\lambda_n R)} \right)^2} - \left( \frac{R}{a\rho} \right)^2 \right\}$$

$$+ \frac{(R\mu)^2}{2} \sum_{\mathcal{K}=-\infty}^{\infty} |C_{\mathcal{K}}|^2 \; F.P.|_{s=0} \sum_{k,n=-\infty}^{\infty} \frac{1}{\rho^{2+s}}. \tag{52}$$

Again, the residue is linear in $\beta$. Also in this case, the introduction of a (finite) local counterterm is needed:

$$K_2 = \beta\alpha_2 \sum_{\mathcal{K}} |C_{\mathcal{K}}|^2 = \beta\alpha_2 \int_0^{2\pi} \left(\mathcal{U}\partial_r\mathcal{U}^{-1}(R,\theta)\right)^2 d\theta. \tag{53}$$

Moreover, it is obvious from equations (26) and (27) that $F_B(\mu)$ is $\mu$-independent (since energy eigenvalues are so). Thus we finally define the difference of renormalized free energies of this bosonic system subject to the boundary condition $A$ with and without the presence of the external field as:

$$2\beta[F_A(\mu) - F_A(0)] \equiv$$

$$\ln det_3 \left(\Phi_{AB}^{-1}(0)\Phi_{AB}(\mu)\right) + G_1 + G_2 + G_3 + K_1 + K_2, \tag{54}$$

where the last two terms on the right hand side contain the undetermined constants $\alpha_1$ and $\alpha_2$.

# 4 - Conclusions

To summarize, in Section 2 we have presented a generalization of the results in reference [1] for the determinant of the quotient $L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0)$, where $L_A(\mu)$ ($L_B(\mu)$) are elliptic differential operators depending on a parameter $\mu$ ($L(\mu)$ restricted to functions $f$ satisfying $Af = 0$ ($Bf = 0$) at the boundary). Such generalization can be applied to situations in which Forman's map fails to have a Fredholm determinant, but a regularized p-determinant can be defined.

Although we have not attempted to establish the most general conditions under which our result holds, we have proved that, when equation (9) is satisfied, $det_p(L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0))$ equals $det_p(\Phi_{AB}^{-1}(0)\Phi_{AB}(\mu))$ (where $\Phi_{AB}(\mu)$ is Forman's map defined in reference [1]), provided that both p-determinants exist. Notice that $\Phi_{AB}(\mu)$ acts on functions defined on the boundary of the manifold. In principle, it can be determined just from boundary values of solutions of the homogeneous equation $Lf = 0$, regardless of boundary conditions.

By making use of well-known properties of p-determinants, we have argued that the subtractions to which our regularization scheme gives rise can also be determined from boundary values, which is an important point when applying our results in the context of Quantum Field Theory. Indeed, these are the divergent pieces which require to be renormalized.

In Section 3, we have studied some models of interest to Physics, which clearly show the need for this generalization as well as the great simplification due to the reduction to the boundary. In particular, in the third one, we have discussed the correction to the free energy of confined free bosons as a function of a variable external field coupled through boundary conditions. The study of the discarded traces, once regularized, allowed us not only to determine the required counterterms, but also to recover a nonlocal finite contribution as well as local finite temperature-dependent pieces thrown away by the definition of the p-determinant.

Notice that, even though this approach requires the introduction of two different boundary conditions, in the examples treated here, a convenient selection of one of the boundary projection operators (such that the free energy $F_B(\mu)$ it determines is $\mu$-independent) allowed us to obtain the difference of free energies with and without external field for a unique boundary condition. Another [Afreedom this method allows is the possibility of choosing also a particular value of the parameter $\mu$. This idea has been applied to evaluate the chiral correction to the free energy of a four-dimensional chiral bag, and will be presented somewhere else[9].

Finally, we wish to mention that the relationship between this regularization with $p > 1$ and the $\zeta$-function one is rather more involved than in the p=1 case, and is at present under study.

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