**DISORDERED GROUND STATES OF CLASSICAL LATTICE MODELS***

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ABSTRACT

We use strictly ergodic dynamical systems to describe two methods for constructing short range interactions of classical statistical mechanics models with unique ground states and unusual properties of disorder; in particular, these ground states can be mixing under translations (and therefore have purely continuous spectrum), and can have positive entropy. Because of the uniqueness of the ground state the disorder is not of the usual type associated with local degeneracy.

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**1. Introduction.** The goal of this paper is twofold: to demonstrate that the mathematics of strictly ergodic dynamical systems can be useful in mathematical physics, and to use it to get new results on the "crystal problem".

The crystal problem is the attempt to understand why real materials seem to have a strong tendency to be highly ordered at low temperature and high pressure. The microscopic structure of solids can be quite complicated. The range of experimentally known structures is much reduced however by restricting attention to materials in thermal equilibrium, and rejecting glasses as nonequilibrium, which we do. Even so one is left, at least, with crystals (with perhaps large unit cells), incommensurate solids, and quasicrystals [34]. The basic problem we wish to address is to understand where these structures come from on the basis of (classical, equilibrium) statistical mechanics.

This is a notoriously hard problem; see [27] for an historical review. Much of the progress on this "crystal problem" concerns lattice (toy) models, and in particular the ground states (that is, the zero temperature states), of lattice models. (There has also been important work on continuum models, in particular by Aubry and coworkers [2].) The results have been of two kinds: there have been interesting examples – such as those based on nonperiodic tilings of space [23,25,26,27] – which exhibit the qualitative properties of quasicrystals, and there have been some generic results, concerning the existence of long range order among general classes of models [15,24,28].

In this paper we will use strictly ergodic dynamical systems to describe two methods for producing lattice models for which the ground state has previously unattainable properties of disorder. Various conventional meanings of the term "disorder" will be considered below.

**2. Notation.** It is necessary to introduce a framework which reveals the manner in which interactions influence the structure of ground states. Within mathematics our framework is called symbolic dynamical systems and within physics it is known as classical lattice gas models. We need to review some traditional terminology and results from both perspectives.

First we summarize some conventional notation from the mathematics of symbolic dynamical systems. In (topological) dynamical systems one starts with a compact metrizable space $X$, a Borel probability measure $m$ on $X$ and a group of homeomorphisms of $X$ which leave $m$ invariant. The support of $m$ is defined as the complement in $X$ of the union of all open sets of zero measure. A closed subset $Y$ of $X$, invariant under the homeomorphisms, is called minimal if it contains no proper closed invariant subset, it is called uniquely ergodic if there is only one invariant Borel probability measure with support in $Y$, and it is called strictly ergodic if it is both uniquely ergodic and minimal. A point $x \in X$ will be called minimal, uniquely ergodic or strictly ergodic if the closure of its orbit under the homeomorphisms, $\bar{O}(x)$, has the property. A dynamical system is called a d-dimensional symbolic dynamical system if the space $X$ is a subset of a product space of the form $A^{\mathbf{Z}^d}$ with the integer $d \geq 1$, the group of homeomorphisms is that implementing the translations $\{T^j \mid j \in \mathbf{Z}^d\}$, and the set $A$ is finite. By the "spectrum" of (the translations in) a symbolic dynamical system we refer to the spectrum of the unitary operators which implement the translations of $\mathbf{Z}^d$ on the complex Hilbert space $L_2(X,m)$, in the following standard sense. There is a projection valued measure $dE(\lambda)$ on $[0,1)^d$ such that for any vectors $f, g \in L_2(X,m)$,

$$< f, T^j g > = \int_{[0,1)^d} \exp(j \cdot \lambda \, 2\pi i) \; d\!< f, E(\lambda) g >, \quad j \in \mathbf{Z}^d \qquad 1)$$

where $<,>$ refers to the inner product in $L_2(X,m)$. For discussion of the above and other concepts from dynamical systems, such as the various types of mixing and entropy, see [21,5,20,29].

Next we define the type of statistical mechanical model known as a classical lattice gas (or spin) model. As physical space we take the integer lattice $\mathbf{Z}^d$, with dimension $d \geq 1$. At each site $j$ in $\mathbf{Z}^d$ we associate a copy of the finite set $A$ (representing the possible local states at each site) of cardinality $N(A) \geq 2$. (One of the elements of $A$ could denote an "unoccupied" site.) If we wish to include an extended hard core condition, the configuration space, $\Omega$, of the model is obtained from the Cartesian product $A^{\mathbf{Z}^d}$ by removing all points $\omega$ (henceforth called configurations) in which two sites, $j, k$, occupied by particles, have (Euclidean) separation $\|j - k\| \leq R$, where the hard core radius $R \geq 1$ is fixed. (See [33,32] for discussions of hard core implementation.)

We now need several definitions in order to describe the allowed interactions of our models. Let $B$ be the set of all nonempty finite subsets of $\mathbf{Z}^d$. We introduce an equivalence relation on $B$ by declaring two sets $b_1$ and $b_2$ equivalent if there is some (lattice) translation $T$ such that $T(b_1) = b_2$. From each equivalence class we select a representative which, as a set, contains the origin $\mathbf{0}$ of the lattice, and we denote the set of these representatives by $B_0$. For each $b$ in $B$ we denote by $\Omega_b$ the set of all possible restrictions $\omega^b$ to $b$ of configurations $\omega$ in $\Omega$, and by $\Omega^r$ the set of all these restrictions for all $b$ in $B_0$: $\Omega^r \equiv \bigcup_{b \in B_0} \Omega_b$. (For a singleton $b = \{j\}$ we use the abbreviation $\omega^j$ in place of $\omega^{\{j\}}$.) Using the above notation we now introduce interactions as follows [31,10]. By the (translation invariant) interaction $V$ we mean a real valued function on $\Omega^r$ – that is, an assignment of the (many-body) energy $V(\omega^b)$ to each state of occupancy of each finite set of sites – with the restriction that

$$\sum_{\omega^b \in \Omega^r} |V(\omega^b)| \; g(\omega^b) \equiv \|V\|_g < \infty \qquad 2)$$

where $g(\omega^b)$ depends only on the set $b$ on which $\omega^b$ is defined, and can be used to restrict the (allowed) class of interactions through their range and/or through the allowed strength of the many-body terms. (If one is interested in phase diagrams one should isolate the one-body energies as adjustable parameters, namely as chemical potentials.) We will require that $g(\omega^b) \geq 1$ for $\omega^b$ in $\Omega^r$; note that $g$ may depend on the diameter of $b$.

To see the connection between interactions and ground states we need some more detailed notation. For each $b$ in $B$ let $d(b)$ denote the number of elements of $\Omega_b$, and let $B_0 = \{b_j \mid j \in \mathbf{N}\}$ be some ordering of the countable set $B_0$. Let $K$ denote the set of all ordered pairs $\{(j,k) \mid j,k \in \mathbf{N},\ k \leq d(b_j)\}$. We can use $K$ to label the elements of $\Omega^r$: for each $(j,k)$ in $K$ let $\omega_{(j,k)}$ be the $k^{th}$ of the $d(b_j)$ possible restrictions to $b_j$. The space $\Omega$, as a subset of the Cartesian product $A^{\mathbf{Z}^d}$, is equipped with the usual product topology and Borel measurable sets, both generated by the (basic) cylinder sets, $\{C_{(j,k)} \mid (j,k) \in K\}$, associated with the elements $\omega_{(j,k)}$ of $\Omega^r$.

Let $F$ be the set of all translation invariant Borel probability measures on $\Omega$. Now given any measure $m$ in $F$ and interaction $V$ satisfying 2), the energy density in $m$ is just

$$< V \mid m > \equiv \sum_{(j,k) \in K} V(\omega_{(j,k)}) \; m(C_{(j,k)}). \qquad 3)$$

As is well known (see [30,33,4]), if $m_0(V) \in F$ is a "ground state for $V$" in the sense that $m_T(V)[C_{(j,k)}] \to m_0(V)[C_{(j,k)}]$ for all $(j,k) \in K$ as $T \to 0$, where $m_T(V)$ is a Gibbs state for the interaction $V$ at temperature $T$, then $m_0(V)$ satisfies

$$< V \mid m_0(V) > = \inf_{m \in F} < V \mid m > . \qquad 4)$$

(Note that 4) is necessary but not sufficient for ground states; just consider the free model, where $V(\omega_{(j,k)}) = 0$ for all $(j,k)$). It is easy to show [4,32,33] that any ground state for an interaction $V$ has its support in the set of all "ground state configurations" for $V$, namely the set of configurations defined as follows. We define the energy $E_b(\omega) = \sum_{b' : b' \cap b \neq \emptyset} V(\omega^{b'})$. The ground state configurations of $V$ are then those configurations $\omega$ such that for all $b$ in $B$

$$E_b(\omega) = \inf_{\hat{\omega} \in \Omega} \{E_b(\hat{\omega}) \mid \hat{\omega}^{b'} = \omega^{b'} \text{ for all } b' \text{ such that } b' \cap b = \emptyset\}. \qquad 5)$$

We remark that the set of all ground state configurations for a given interaction is compact.

Finally we note that for a generic $V$ the set of all ground state configurations is uniquely ergodic [24]. This strongly suggests, by reference to the Gibbs phase rule [31,10], that the set of ground state configurations of a low temperature pure thermodynamic phase is uniquely ergodic.

**3. First Method.** What concerns us in this paper is the degree to which ground states need be highly ordered. The greatest possible order occurs when $m_0(V)$ is of the form $m_\omega$ defined as follows: take any periodic element $\omega \in \Omega$, let $m'_\omega$ be the point mass concentrated at $\omega$, and let $m_\omega$ be the (finite) convex combination of point masses that you get by averaging $m'_\omega$ over translations. Aside from degenerate models, a noninteracting system being an extreme example, there are very few examples known of interactions which do not have ground states of this simple periodic type. However, there are interactions $V$, obtained from Wang tilings [23,25,26,27], where the support of $m_0(V)$ is the orbit closure, $\bar{O}(\omega)$, of a "quasiperiodic" configuration $\omega$, and contains no periodic configurations. This uniquely ergodic $m_0(V)$ is still highly ordered: for example it is not strongly mixing and has zero entropy – as holds for generic interactions [15,28]. (The spectrum of the translations is presumably pure point, as is expected for "real" quasicrystals.) The main objective of this paper is to exhibit examples of interactions $V$ for which the ground state $m_0(V)$ is uniquely ergodic on the ground state configurations but highly **disordered**; for example weakly or strongly mixing, and/or with positive entropy.

To produce such interesting ground states one can begin with any "appropriate" configuration $\omega$, where by appropriate we mean that $\bar{O}(\omega)$ is strictly ergodic; in such a case denote by $m_\omega$ the corresponding unique translation invariant probability measure on $\bar{O}(\omega)$. (This agrees with the special case treated above, namely that of periodic $\omega$.) For example [3] consider the configuration $\hat{\omega}$, for dimension $d = 1$ and with $A = \{1, -1\}$, obtained as follows. Let $\hat{\omega}^0 = 0$ and define $\hat{\omega}^j$ by the following iterative ("substitution") process for $j \geq 0$ – replace all 0's by 001, and replace all 1's by 11100 –

$$0 \rightarrow 001, \quad 1 \rightarrow 11100. \qquad 6)$$

(So after one iteration we have $\hat{\omega}^1 = 0$ and $\hat{\omega}^2 = 1$, and after the second iteration we have $\hat{\omega}^3 = 0$, $\hat{\omega}^4 = 0$, $\hat{\omega}^5 = 1$, $\hat{\omega}^6 = 1$, $\hat{\omega}^7 = 1$, $\hat{\omega}^8 = 1$, $\hat{\omega}^9 = 0$, and $\hat{\omega}^{10} = 0$; etcetera.) We could now define $\hat{\omega}^j$ for $j < 0$, but in fact this is unnecessary since the orbit closure under translations of this "partial" configuration is already uniquely defined, and has the above mentioned property that this orbit closure is the support of a unique translation invariant probability measure, $m_{\hat{\omega}}$ [3,22].

Now that we have fixed the measure $m_{\hat{\omega}}$, we need to define the interaction $V^{\hat{\omega}}$. First we prove the following simple lemma.

**Lemma**. If the compact set $X \subset \Omega$ is minimal invariant under lattice translations and supports the invariant probability measure $m$, then:

$$X = \Big\{\omega \in \Omega \,\Big|\, \{C_{(j,k)} \mid m(C_{(j,k)}) > 0\} = \{C_{(j,k)} \mid C_{(j,k)} \cap O(\omega) \neq \emptyset\}\Big\}.$$

**Proof**. Fix some point $\omega' \in X$. It follows from minimality (Prop. IV.1 in [22]) that

$$X = \{\omega \in \Omega \mid \text{for every } (j,k) \in K,\ O(\omega) \cap C_{(j,k)} \neq \emptyset \iff O(\omega') \cap C_{(j,k)} \neq \emptyset\}.$$

If $\omega \in X$ and for some cylinder set $C_{(j,k)}$ we have $O(\omega) \cap C_{(j,k)} \neq \emptyset$, then (Prop. IV.5 in [22]) we must have $m(C_{(j,k)}) > 0$. Also, if $\omega \in X$ and for some cylinder set $C_{(j,k)}$ we have $m(C_{(j,k)}) > 0$, then $\omega'' \in C_{(j,k)}$ for some $\omega'' \in X$, and therefore $O(\omega) \cap C_{(j,k)} \neq \emptyset$. So if $\omega \in X$ then $m(C_{(j,k)}) > 0$ if and only if $O(\omega) \cap C_{(j,k)} \neq \emptyset$. Finally, if $\omega \notin X$ then there is some cylinder set $C_{(j,k)}$ disjoint from $X$ such that $\omega \in C_{(j,k)}$ and so $O(\omega) \cap C_{(j,k)} \neq \emptyset$. But $m(C_{(j,k)}) = 0$ from the disjointness and so for $\omega \notin X$

$$\{C_{(j,k)} \mid m(C_{(j,k)}) > 0\} \neq \{C_{(j,k)} \mid C_{(j,k)} \cap O(\omega) \neq \emptyset\}. \ \blacksquare$$

So with the configuration $\hat{\omega}$ defined above we associate (following Aubry [2]) the interaction $V^{\hat{\omega}}$ by: $V^{\hat{\omega}}[C(j,k)] > 0$ if $m(C_{(j,k)}) = 0$, and $V^{\hat{\omega}}[C(j,k)] = 0$ if $m(C_{(j,k)}) > 0$. (This does not completely define $V^{\hat{\omega}}$ of course, but we can select the positive values of the interaction so that the interaction will belong to any of the allowed spaces of interactions associated with the norms $\|\cdot\|_g$ considered above; in particular, the interaction can have any prescribed rate of decrease with diameter or with the number of particles in the many-body terms.) It follows by the above procedure that the set of ground state configurations of $V^{\hat{\omega}}$ is uniquely ergodic, so $m_{\hat{\omega}}$ is the unique ground state of $V^{\hat{\omega}}$. We note the following features of $m_{\hat{\omega}}$ – it is weakly mixing, and so its spectrum contains only one eigenvalue, the number 1, with multiplicity 1, the rest of the spectrum being continuous [3].

We have just constructed an interaction whose ground state is associated with a certain (rather wild) prescribed configuration, $\hat{\omega}$. By this method we can in fact produce even wilder examples. It is a rather surprising theorem of Jewett, Krieger and Weiss [11,12,35] that given any invertible ergodic measure preserving transformation on a Lebesgue measure space, with finite entropy $S$, there is a measure theoretic isomorphism with a strictly ergodic symbolic dynamical system, that is with the type of lattice system we have been using, where $\exp(S) < N(A) \leq \exp(S)+1$. This is an existence theorem, which guarantees the existence of $\omega$'s with $m_\omega$'s of great variety: for example strongly mixing, and/or with nonzero entropy. Explicit examples are constructed in [7,8]. And therefore the above proves the existence of interactions with a very wide range of measures as their unique ground states. The interactions have as short a range as desired (in the sense of rate of fall-off), but of course the method does require the use of many-body energies of all orders. We summarize the above in a theorem.

**Theorem 1.** Given any representation of $\mathbf{Z}^d$ by invertible ergodic measure preserving transformations on a Lebesgue measure space, with finite entropy, and any norm $\| \cdot \|_g$, there is a short range lattice gas model with interaction $V$ satisfying $\|V\|_g < \infty$, for which the set of ground state configurations is uniquely ergodic, and for which the ground state dynamical system is measure theoretically isomorphic to the given system.

**Remarks.** The method used above for constructing an interaction $V^\omega$ associated with a given configuration $\omega$ was first defined by Aubry for the general case where the orbit closure $\bar{O}(\omega)$ is minimal [2]. It is important to emphasize at this point why it is necessary to make further restrictions on $\omega$, such as the unique ergodicity of $\bar{O}(\omega)$.

If one starts with any minimal configuration $\omega$ and uses the above method to construct an interaction $V^\omega$, it follows easily from 3) and 4) that $\bar{O}(\omega)$ contains the support of any zero temperature, translation invariant limit of finite temperature Gibbs states of the model. If $\omega$ is uniquely ergodic (but not necessarily minimal), **every** configuration in $\bar{O}(\omega)$ (including $\omega$ of course) reproduces the unique probability measure $m_\omega$ in the sense of Birkhoff's pointwise ergodic theorem; that is, if $\chi$ is the characteristic function for any cylinder set, and $T^j$ represents translation by $j \in \mathbf{Z}^d$, then as $N \to \infty$:

$$1/(2N+1)^d \sum_{|j|\leq N} \chi(T^j\omega') \to \int_{\bar{O}(\omega)} \chi \, dm_\omega. \qquad 7)$$

(uniformly in $\omega'$) for $\omega'$ in $\bar{O}(\omega)$ (Thm. 3.5 in [5]). On the other hand, if one uses an $\omega$ which is minimal but not uniquely ergodic (see section 10 in [18] for an example), then first of all one loses control of the zero temperature limits since there are now infinitely many invariant probability measures, all with the same minimal set as support. Also the configuration $\omega$ that one started with need not be well associated with **any** of these measures; in the example from [18] noted above, some averages of the type in 7) do not even exist for $\omega' = \omega$. Therefore if one wants to construct an interaction for which a given configuration $\omega$ is not only a ground state configuration in the sense of the definition 7), but is also associated in some real sense with low temperature Gibbs states, then one must assume more than minimality for $\omega$. We assume unique ergodicity, in part because it is sufficient for this purpose but also because as noted at the end of section 2 it seems appropriate for pure thermodynamic phases.

**4. Second Method.** Two of the shortcomings of the above result (namely the use of many-body energies of all orders, and the requirement that the interaction not be strictly finite range), can be avoided by a different method described in this section.

First we need to outline two techniques for defining interesting symbolic dynamical systems. In both cases we define the dynamical system $X$ as the set of all points $x = \{x^{(j^1,\cdots,j^d)} \mid (j^1,\cdots,j^d) \in \mathbf{Z}^d\} \in A^{\mathbf{Z}^d}$ for which all "blocks" $\{x^{(j^1,\cdots,j^d)} \mid J_1 \leq j^1 \leq K_1,\cdots,J_d \leq j^d \leq K_d\}$ in $x$ satisfy certain restrictions.

For the class of one dimensional dynamical systems usually associated with the term "substitution", one begins with a set of "substitution rules", that is, for each element $a \in A$ one has a finite sequence $\{a_1,\cdots,a_k\}$, where $a_j \in A$ and $k$ (the so-called length of the rule) may depend on $a$. (We used such rules in 6) to define half of the configuration $\hat{\omega}$.) Given any finite sequence $B$ of elements of $A$, we define $D(B)$ as the finite sequence obtained by replacing each of the elements of $B$ using its rule. Next define $V_0 = A$, $V_{n+1} = \bigcup_{B\in V_n} D(B)$ and $V = \bigcup_{n\geq 0} V_n$. Then $X$ is the set of all two sided sequences $x$ for which every subblock of $x$ is a subblock of an element of $V$. Finally, given a set of substitution rules defining the set $X$, we say $X$ has "unique derivation" if for every $x \in X$ there is a unique $y \in X$ (unique up to translation) such that $x$ is obtained from $y$ when the substitution rules are used to replace the elements of $y$. (Note that there is no need to keep track of the absolute coordinates $x^j$ of our two-sided sequences $x$ under substitution, as was done in section 3. This difference is an example of the change in perspective between focusing on a configuration and on a dynamical system, as discussed in section 5 below.) For example, the dynamical system associated with the rules 6) has unique derivation. An example of a dynamical system without unique derivation is the one with the rules $0 \to 010$, $1 \to 101$. See [17,22].

The second technique produces interesting **two** dimensional dynamical systems $X$, and is associated with the term "tiling". Here one begins by specifying two subsets $K_h$ and $K_v$ of $A \times A$. Both subsets are assumed to satisfy the condition:

**i)** If $(a,b)$, $(a,c)$ and $(d,c)$ are in the set, so is $(d,b)$.

We then define the dynamical system $X$ as the subset of $A^{\mathbf{Z}^2}$ such that all blocks in $X$ of the form $\{x^{(j^1,j^2)} \mid J \le j^1 \le J+1\}$ are in $K_h$ and all blocks in $X$ of the form $\{x^{(j^1,j^2)} \mid J \le j^2 \le J+1\}$ are in $K_v$.

We now show how this dynamical system is related to tilings of the plane. Think of each element of $A$ as a unit square centered over a point of $\mathbf{Z}^2$. These unit squares, henceforth called "tiles", have four edges (called "left", "right", "top" and "bottom"), and these edges will be assigned "colors". We will be restricting arrangements of the tiles in the plane by requiring that they may abut only if the overlapping edges have the same color. If we think of $K_h$ and $K_v$ as a list of the pairs of tiles that may abut horizontally and vertically, then we can define colors for the edges of the tiles by the following prescription. For tile $a$ we define the color of: the right edge to be $\{(c,b) \in K_h \mid (a,b) \in K_h\}$, the left edge to be $\{(c,b) \in K_h \mid (c,a) \in K_h\}$, the top edge to be $\{(c,b) \in K_v \mid (a,b) \in K_v\}$, and the bottom edge to be $\{(c,b) \in K_v \mid (c,a) \in K_v\}$. Condition **i)** ensures that the sets of pairs of tiles used to define colors are pairwise disjoint, so that the colors are well defined. (Although we never need to use the colors, we defined them to show that a definition of colored tiles is possible for which the allowed tilings – that is, those tilings in which each pair of abutting edges have the same color – satisfy the restrictions of $K_h$ and $K_v$.)

To summarize, the above allows us to interpret $X$, henceforth called a (two dimensional) tiling dynamical system, as the set of all tilings of the plane, by the tiles in $A$, such that abutting edges always have the same color. (For completeness we note that the above two techniques, for substitution dynamical systems and for tiling dynamical systems, can both be generalized to other dimensions [17,5].)

By a recent result of Mozes (Theorem 6.4 in [17]), given any one dimensional substitution dynamical system with unique derivation and with substitution rules all of length at least 2, one can build a two dimensional tiling dynamical system which is measure theoretically isomorphic to the product of the one dimensional substitution dynamical system with itself. The one dimensional substitution dynamical system associated with the rules 6) is easily seen to satisfy the two hypotheses.

Now given any tiling dynamical system as defined above, assuming $X$ is nonempty (which is automatic in the application below) it is easy to define a two dimensional lattice gas model, with **nearest neighbor two-body interaction**, for which the ground state dynamical system is measure theoretically isomorphic to the tiling dynamical system [23,25,26,27]. Combining this with the above application of the theorem of Mozes using 6), and general facts about the product of weakly mixing dynamical systems (Prop. 4.6 and Thm. 4.30 in [5]) we have the following example.

**Example**. There is a two dimensional classical lattice gas model with nearest neighbor two-body interaction, such that the set of ground state configurations is uniquely ergodic, and the operators representing translations have no eigenvalues other than 1.

**Remarks**. The use of the rules 6) is of course just one example of a general method for producing interesting ground states. There are some natural shortcomings of this second method also. First of all, any one dimensional substitution dynamical system has zero entropy [22] and is not strongly mixing [3,22]. While it is possible that the above method can be extended to yield two dimensional models with strongly mixing translations, in fact it is impossible to have a finite range interaction and nonzero entropy in the ground state, as we see in the following theorem proven with Jacek Miękisz. (The two notions "topological entropy" and "measure theoretic entropy" are known to coincide for strictly ergodic dynamical systems [19], and also coincide with the physical entropy for ground states [1], so we have used the simple term "entropy".)

**Theorem 2**. In any dimension $d$, if a lattice gas model has an interaction of strictly finite range and a uniquely ergodic set of ground state configurations, then the ground state has zero entropy.

Proof. For simplicity we only consider dimension $d = 2$. Assume the (topological) entropy of the ground state $m$ is $\alpha > 0$, and the range of the interaction is $R$. Define $S_N$ as the square set of $N^2$ sites in the lattice centered at the origin, $\mathbf{C}_N$ as the set of cylinder sets based in $S_N$, and $\mathbf{D}_{N,R}$ as the set of cylinder sets based in $S_{N+R}/S_N$. From the definition of (topological) entropy, the number of $C \in \mathbf{C}_N$ for which $m(C) > 0$ is $\alpha^N$ asymptotically in $N \to \infty$. The number of elements of $\mathbf{D}_{N,R}$ is $N(A)^{4R\sqrt{N}}$. Let $\omega$ be any configuration in the support of $m$, so $\omega \in C'$ for some $C' \in \mathbf{C}_N$ for which $m(C') > 0$. Let $D$ be the element of $\mathbf{D}_{N,R}$ to which $\omega$ belongs. Consider the translations of $\omega$ by fewer than $N$ units along both axes; there are fewer than $4N$ of them, and so there are fewer than $4N$ cylinders

$C \in \mathbf{C}_N$ to which these translations of $\omega$ belong. From the above, for fixed $N$ large enough there exists a cylinder $C'' \in \mathbf{C}_N$ for which $m(C'' \cap D) > 0$ and which contains none of these translations of $\omega$. We now construct the configuration $\omega''$ as follows. For each translation $T$ such that $T(\omega) \in C' \cap D$ we change $\omega$ at $N^2$ sites so that $T(\omega) \in C'' \cap D$. It is easy to see that this new configuration is also a ground state configuration. But if we let $\chi$ be the characteristic function for $C'' \cap D$, it follows from Prop. IV.5 in [22] that the left hand side of 7) has a different limit if $\omega' = \omega$ than if $\omega' = \omega''$, which is a contradiction. Therefore $\alpha = 0$. ∎

**5. Summary.** There were two goals of this article. The first was to show how the mathematics of strictly ergodic dynamical systems can play a powerful role in mathematical physics, for example in the crystal problem.

The second goal was to exhibit statistical mechanical models of traditional type with unprercedented levels of disorder in their ground state. In particular we described two methods for producing models with short range, translation invariant interactions with disordered ground states; the methods are constructive in some circumstances (when enough is known about the desired ground state, as in the examples discussed), and nonconstructive when dependent on existence theorems such as that of Jewett, Krieger and Weiss. The models have ground states exhibiting unusual spectral disorder (the translation operators have continuous spectrum) even with nearest neighbor two-body interactions; interactions having positive entropy ground states are constructed but are proven to require that the interaction not be strictly finite range.

Examples of interactions whose unique ground state has positive entropy (using [8] in our second method, for instance) are of some special interest. It is well known that models can have ground states of positive entropy, as in the Ising antiferromagnet on the triangular lattice. However in such a model the entropy is produced by degeneracy, while that is not so clear in our examples; if there is only one invariant probability measure on the ground state configurations of an interaction, in some strong sense the model is nondegenerate. So the traditional connection between positive entropy and degeneracy needs clarification. It need not be associated with **local** nonuniqueness (that is, with a freedom to pass from one ground state configuration to another by any of many strictly local changes) as in the antiferromagnet example.

There is one further aspect which is clarified by the use of uniquely ergodic dynamical systems. In statistical mechanics one tends to think of a ground state as associated with a configuration, but this is not always justifiable even when it is uniquely ergodic. The ground state is really an (invariant) probability measure, and only effectively reduces to a configuration when the ground state is periodic as described in section 3. The example of the tiling models mentioned in section 1 is apposite. For tiling models, if one ignores the existence of fault lines or planes by only considering the support of the uniquely ergodic ground state measure, one finds that all configurations in the support have precisely the same finite patterns in them (and with the same frequencies). Therefore in some physical sense they are indistinguishable; knowledge of any finite region of a configuration cannot characterize the configuration. Therefore it seems more reasonable to think of the ground state of a pure phase as the (unique, uniquely ergodic) probability measure rather than to try to associate a configuration with the ground state. An historical reason to associate configurations with ground states, and with other low temperature ordered states, is the existence in some models of translation noninvariant Gibbs states which are in some sense perturbations of some fixed configuration, as in the nearest neighbor Ising antiferromagnet on the square lattice. It should be remembered however that, in principal at least, a low temperature phase need not be distinguished by having nonunique (in particular, translation noninvariant) Gibbs states; it is conceivable that, say, a tiling model has a low temperature phase described by a unique Gibbs state. The positive temperature behavior of tiling models and others as discussed in this paper should be determined. (For tiling models see [13,14,16].) The fact that entropy can enter these models in a new manner suggests that the usual method for estimating low temperature Gibbs states, using Peierls contours [31], may need essential revision.

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## REFERENCES

[1] Aizenman, M. and Lieb, E.: The third law of thermodynamics and the degeneracy of the ground state for lattice systems. *J. Stat. Phys.* **24**, 279-297 (1981)

[2] Aubry, S.: Weakly periodic structures and example. *J. Phys. (Paris), Coll. C3, supp. no. 3* **50**, 97-106 (1989)

[3] Dekking, F.M. and Keane, M.: Mixing properties of substitutions. *Z. Wahrsch. Verw. Geb.* **42**, 23-33 (1978)

[4] Dobrushin, R.L., and Shlosman, S.B.: The problem of translation invariance of Gibbs states at low temperature. *Sov. Sci. Rev. C Math. Phys.* **5**, 53-196 (1985)

[5] Furstenberg, H.: *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton: Princeton University Press 1981

[6] Gardner, C., Miękisz, J., Radin, C. and van Enter, A.: Fractal symmetry in an Ising model. *J. Phys. A.: Math. Gen.* **22**, L1019-1023 (1989)

[7] Grillinberger, C.: Constructions of strictly ergodic systems, I. *Z. Wahrsch. Verw. Geb.* **25**, 323-334 (1973)

[8] Grillinberger, C., and Shields, P.: Constructions of strictly ergodic systems, III. *Z. Wahrsch. Verw. Geb.* **33**, 215-217 (1975)

[9] Grünbaum, B. and Shephard, G.C.: *Tilings and Patterns.* New York: Freeman 1986

[10] Israel, R.: *Convexity in the Theory of Lattice Gases.* Princeton: Princeton University Press 1979

[11] Jewett, R.I.: The prevalence of uniquely ergodic systems. *J. Math. Mech.* **19**, 717-729 (1970)

[12] Krieger, W.: On unique ergodicity. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability 1970*, pp. 327-346

[13] Miękisz, J.: Many phases in systems without periodic ground states. *Commun. Math. Phys.* **107**, 577-586 (1986)

[14] Miękisz, J.: Classical lattice gas model with a unique nondegenerate but

unstable periodic ground state configuration. *Commun. Math. Phys.* **111**, 533-538 (1987)

[15] Miękisz, J.: How low temperature causes long-range order. *J. Phys. A: Math. Gen.* **21**, L529-L531 (1988)

[16] Miękisz, J.: A microscopic model with quasicrystalline properties. *J. Stat. Phys.* **58**, 1137-1149 (1990)

[17] Mozes, S.: Tilings, substitution systems and dynamical systems generated by them. *J. d'Analyse Math.* **53**, 139-186 (1989)

[18] Oxtoby, J.: Ergodic sets. *Bull. Amer. Math. Soc.* **58**, 116-136 (1952)

[19] Parry, W.: Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.* **122**, 368-378 (1966)

[20] Parry, W.: *Topics in Ergodic Theory.* Cambridge tracts in mathematics, Vol. 75. Cambridge: Cambridge University Press 1981

[21] Petersen, K.: *Ergodic Theory.* Cambridge studies in advanced mathematics, Vol. 2. Cambridge: Cambridge University Press 1983

[22] Queffélec, M.: *Substitution Dynamical Systems - Spectral Analysis.* Lecture Notes in Mathematics, Vol. 1294. Berlin: Springer-Verlag 1987

[23] Radin, C.: Tiling, periodicity and crystals. *J. Math. Phys.* **26**, 1342-1344 (1985)

[24] Radin, C.: Correlations in classical ground states. *J. Stat. Phys.* **43**, 707-712 (1986)

[25] Radin, C.: Crystals and quasicrystals: a lattice gas model. *Phys. Lett.* **114A**, 381-383 (1986)

[26] Radin, C.: Crystals and quasicrystals: a continuum model. *Commun. Math. Phys.* **105**, 385-390 (1986)

[27] Radin, C.: Low temperature and the origin of crystalline symmetry. *Int. J. Mod. Phys. B* **1**, 1157-1191 (1987)

[28] Radin, C.: Ordering in lattice gases at low temperature. *J. Phys. A: Math. Gen.* **22**, 317-319 (1989)

[29] Riesz, F. and Sz.-Nagy, B.: *Functional Analysis.* New York: Frederick Ungar 1955

[30] Ruelle, D.: Some remarks on the ground state of infinite systems in statistical mechanics. *Commun. Math. Phys.* **11**, 339-345 (1969)

[31] Ruelle, D.: *Statistical Mechanics: Rigorous Results.* New York: Benjamin 1969

[32] Ruelle, D.: *Thermodynamic Formalism.* New York: Addison-Wesley 1978

[33] Schrader, R.: Ground states in classical lattice systems with hard core. *Commun. Math. Phys.* **16**, 247-264 (1970)

[34] Venkataraman, G., Sahoo, D. and Balakrishnan, V.: *Beyond the Crystalline State: An Emerging Perspective.* Springer Series in Solid-State Sciences, Vol. 84. Berlin: Springer-Verlag (1989)

[35] Weiss, B.: Strictly ergodic models for dynamical systems. *Bull. Amer. Math. Soc.* **13**, 143-146 (1985)