ALGEBRAS OF VECTOR FIELDS ON RIEMANN SURFACES (GENERALIZED KRICHEVER - NOVIKOV - ALGEBRAS)

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1. Introduction.

In the framework of Conformal Field Theory the Virasoro Algebra and its representations have been discovered to be of fundamental importance. Later on mathematicians studied the Virasoro algebra as an example of a œœsimple" infinite dimensional Lie algebra which does not fit into the classification scheme of Kac – Moody algebras. By the recent very fruitful contact of quantum field theory and mathematics it plays a more and more important role in mathematics. Just to name one example: the work of Frenkel, Lepowski and Meurman explaining the relation between the largest sporadic simple group (the Monster or Friendly Giant) and the coefficients of of the elliptic modular function j.

The Virasoro algebra can be described as the universal central extension of the subalgebra W of complex valued analytic vector fields on the the sphere S^1 which have only a finite number of Fourier modes. For our considerations a different viewpoint is preferable. The above vector field algebra W is isomorphic to the algebra of meromorphic vector fields on $\mathbb{P}^1$ which are holomorphic on $\mathbb{P}^1 \setminus \{0, \infty\}$.

It is not only the algebra itself which is of importance, but also its graded structure. To obtain such a graded structure one chooses the special vector fields

$$l_n := z^{n+1} \frac{\partial}{\partial z} \ .$$

The set $\{ \ l_n \mid n \in \mathbb{Z} \}$ is a basis of the vector field algebra W. By defining as homogeneous elements of degree n the non-zero elements of $W_n = \mathbb{C} \cdot l_n$ we introduce a grading (with respect to the order of the vector fields at the point $z = 0$). We obtain for the commutator

$$[l_n, l_m] = (m - n) \, l_{m+n}, \qquad \text{resp.} \quad [W_n, W_m] \subseteq W_{n+m} \ .$$

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By this, we see that the Lie product respects the grading. In other words, W is a graded Lie algebra.

By taking the Lie derivative of the forms, the vector fields operate on the forms of weight λ, which are meromorphic on $\mathbb{P}^1$ and holomorphic on $\mathbb{P}^1 \setminus \{0, \infty\}$. To obtain highest weight representations one can now construct œœsemi-infinite wedge products” of the forms of weight λ. (I will introduce them later on in more detail.) Alas, one does not obtain a (Lie-)representation of the vector field algebra anymore but only a representation of the universal central extension of W, the Virasoro algebra V:

$$0 \longrightarrow \mathbb{C} \longrightarrow V \longrightarrow W \longrightarrow 0 \ .$$

If $L_n,\ n \in \mathbb{Z}$ denote suitable lifts of the basis elements l_n and $c \neq 0$ is a central element then they generate the Virasoro algebra. The structure equations are

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n} \cdot c$$

$$[c, L_n] = 0 \ .$$

The central element c operates as a scalar multiple of the identity on the wedge products. If we normalize the elements L_n in such a way that the same central extension operates simultanously on every wedge product (i.e. for all weight then c operates as

$$-2\,(6\lambda^2 - 6\lambda + 1) \cdot id \ .$$

This expression resembles very much a formula given by Mumford in connection with divisor classes on the moduli space of algebraic curves. Indeed there is a deep relation, as was shown by Arbarello, deConcini, Kac and Procesi [1].

¿From the point of view of string theory (where a S^1 is moving in $D-$dimensional space-time) the above situation describes the twice punctured world sheet of genus zero represented as a cylinder (Fig. 1). The punctures representing incoming free and outgoing free string

FIGURE 1

Now there are also other world sheets possible, like for example the following (Fig. 2): one free incoming string, one free outgoing string and nontrivial self interaction.

FIGURE 2

In 1987 Krichever and Novikov [**2,3**] generalized the above mentioned concept to this situation. In my work [**4-7**] I generalized this to the case: several incoming and several outgoing strings. The Virasoro case and the Krichever-Novikov case are special cases. I was able to show that after choosing a suitable grading (this time it is only a generalized one) most of the constructions which could be done in the Virasoro case work here too. In the following, I will give a short review of the results.

Let me mention, that the generalized situation was also studied by Rainer Dick [**8**] and by a chinese group (Guo, Na, Shen, Wang, Yu)[9]. They were working in a different direction and did not introduce such a grading, which is necessary to make the construction with the semi-infinite wedge product.

2. The Set-up

Let X be a (compact) Riemann surface with a fixed complex structure. We denote by A a set of points, divided into two disjoint, non-empty subset I and O

$$A = I \cup O, \qquad \#I = k \geq 1, \quad \#O = l \geq 1, \quad N = k + l \ .$$

The points lying in I I call œœin-points", the points lying in O I call œœout-points". Let ρ be a meromorphic differential with poles of exact order 1 at the points of A, positive (real) residues at I and negative residues at O. Of course, the sum of the residues has to be zero. To fix ρ uniquely I require for ρ that it has only purely imaginary periods. Hence (after choosing a base point $Q \in X \setminus A$)

$$u(P) = \mathsf{Re} \int_Q^P \rho$$

is a well defined harmonic function on $X \setminus A$. The level lines (for $\tau \in \mathbb{R}$)

$$C_\tau = \{ \ P \in X \setminus A \mid u(P) = \tau \ \}$$

of the function u represent in the interpretation of string theory the string configuration at œœproper time” τ. The set of all level lines $\{ C_\tau \mid \tau \in \mathbb{R} \}$ sweeps out the whole of $X \setminus A$. For $\tau \ll 0$ the C_τ are collections of circles around the in-points, for $\tau \gg 0$ circles around the out-points. Hence the interpretation: I corresponds to free incoming strings, O corresponds to free outgoing strings and the condition œœ$\sum_P \operatorname{res}_P(\rho) = 0$” corresponds to the conservation of momentum. Let me show you a picture (Fig.3):

Figure 3

Let K be the canonical bundle, i.e. its sections are the differentials $f(z)\,dz$. Let $\mathcal{F}^\lambda(A)$ denote the $\mathbb{C}-$vector space of meromorphic sections of the tensor power K^λ, which are holomorphic on $X \setminus A$. For the following I assume λ to be an integer. It is possible to work with more general λ but technical difficulties are involved. For example, for halfinteger λ we would have to choose a square root of the canonical bundle, fix it and work with respect to this fixed additional structure on X.

For $\lambda = -1$ we obtain the vector fields. We use the notation $\mathcal{KN}(A) = \mathcal{F}^{-1}(A)$.

If $e \in \mathcal{KN}(A)$ and $f \in \mathcal{F}^\lambda(A)$ then the Lie derivative $L_e(f)$ is again in $\mathcal{F}^\lambda(A)$. This one can see immediately by considering the local representation

$$e(z)_| = \alpha(z)\frac{\partial}{\partial z}, \qquad f(z)_| = \beta(z)\, dz^\lambda$$
$$e\,.\,f_| := L_e(f)_| = \left(\alpha(z)\frac{\partial \beta}{\partial z}(z) + \lambda \cdot \beta(z)\frac{\partial \alpha}{\partial z}(z)\right)\, dz^\lambda \ .$$

For $\lambda = -1$ this is the usual Lie bracket of vector fields, and it defines a Lie algebra structure on $\mathcal{KN}(A)$. By taking the Lie derivative $\mathcal{F}^\lambda(A)$ becomes a a Lie module over $\mathcal{KN}(A)$. I call $\mathcal{KN}(A)$ the (generalized) Krichever – Novikov algebra and $\mathcal{F}^\lambda(A)$ the (generalized) Krichever – Novikov module of weight λ.

Of course, $\mathcal{F}^\lambda(A)$ is also a module over the associative algebra $\mathcal{F}^0(A)$ (these are the functions). With the commutator $\mathcal{F}^0(A)$ becomes an abelian Liealgebra $L\mathcal{F}^0(A)$.

The algebra of differential operator (of degree ≤ 1) is

$$\mathcal{D}^1(A) = \mathcal{K}N(A) \oplus L\mathcal{F}^0(A)$$

with Lie product (the semi-direct product)

$$[(e,g),(f,h)] = ([e,f], L_e h - L_f g) \ .$$

$\mathcal{F}^\lambda(A)$ gets now a Lie module over $\mathcal{D}^1(A)$. For almost everything what follows there are results for the whole algebra of differential operators. However, due to lack of time I will only discuss the subalgebra $\mathcal{K}N(A)$.

Finally, there is a pairing between the spaces $\mathcal{F}^\lambda(A)$ and $\mathcal{F}^{1-\lambda}(A)$. This pairing is given by

$$(f,h) = \frac{1}{2\pi\,\mathbf{i}} \oint_{C_\tau} f \otimes h \ .$$

Here C_τ is a non-singular level line, and $f \otimes h$ is a meromorphic 1-differential. The value of the integral does not dependent on the level line chosen.

3. The Grading

For the definition of $\mathcal{K}N(A)$ and $\mathcal{F}^\lambda(A)$ the decomposition of A into I and O was of no importance. This will change now. For the following, let the points in A be generic. We denote

$$I = \{P_1, P_2, \dots, P_k\}, \qquad O = \{Q_1, Q_2, \dots, Q_l\}$$

and choose a local coordinate z_i at P_i, $i = 1, \dots, k$. To start with let $k = l$. We define

$$M(\lambda) = (2\lambda - 1)(g-1) - 1 \ .$$

Let $\lambda \in \mathbb{Z}, \lambda \ne 0, 1$ then there exists for every $p = 1, 2, \dots, k$ and every $n \in \mathbb{Z}$ exactly one $f_{n,p}(\lambda) \in \mathcal{F}^\lambda(A)$ with

$$\begin{aligned}
\operatorname{ord}_{P_i}(f_{n,p}(\lambda)) &= n - \delta_{i,p}, \quad i = 1, \dots, k \\
\operatorname{ord}_{Q_i}(f_{n,p}(\lambda)) &= -n, \quad i = 1, \quad i = 1, \dots, l-1 \\
\operatorname{ord}_{Q_l}(f_{n,p}(\lambda)) &= -n + 1 + M(\lambda) \\
f_{n,p}(\lambda)_{|} &= z_p^{n-1}(1 + O(z_p)) dz^\lambda
\end{aligned} \tag{*}$$

This is proven by using the Riemann Roch theorem together with the fact that for $\lambda \ne 0, 1$ the divisor K^λ is non-special and the the points in A are generic.

For the exceptional λ there are modifications necessary for finitely many values of n. These modifactions involve only the behaviour at the Points Q_i. For $k \ne l$ there are also modifications at the points Q_i necessary.

Let me mention that there exists explicit representations of the elements $f_{n,p}(\lambda)$ in terms of rational function, Weierstraß $\sigma-$funktion and theta functions.

Theorem. *(a) The elements*

$$f_{n,p}(\lambda), \quad n \in \mathbb{Z}, \quad p = 1, 2, \dots, k$$

are a basis of $\mathcal{F}^\lambda(A)$.

(b) We have the duality relation

$$\frac{1}{2\pi\,\mathbf{i}} \oint_{C_\tau} f_{n,p}(\lambda) \cdot f_{1-m,r}(1-\lambda) = \delta_{n,m} \cdot \delta_{p,r} \ .$$

(c) There are numbers $C^{\cdots}_{\dots,\dots}(\lambda) \in \mathbb{C}$ *and* $L \in \mathbb{Z}$ *such that*

$$L_{e_{n,p}}(f_{m,r}(\lambda)) = e_{n,p} \cdot f_{m,r}(\lambda) = \sum_{h=n+m-2}^{n+m+L} \sum_{s=1}^{k} C^{(h,s)}_{(n,p),(m,r)}(\lambda) f_{h,s}(\lambda) \ .$$

L does only depend on g, k, l, λ.

The main technique to show this is to make local calculations, resp. to calculate residues. Please note, that the Virasoro case and the Krichever - Novikov case are contained as special case ($k = l$ and index shift).

We can reformulate (c) . For this we define $\deg(f_{n,p}(\lambda)) = n$ and obtain a grading of $\mathcal{F}^\lambda(A)$.

$$\mathcal{F}^\lambda(A) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^\lambda_n(A), \qquad \mathcal{F}^\lambda_n(A) = \langle\ f_{n,p}(\lambda) \mid p = 1, \dots, k\ \rangle$$

is now the decomposition into (finite dimensional) homogeneous spaces. Now $\mathcal{F}^\lambda(A)$ is not necessarily a graded module over $\mathcal{KN}(A)$, but by (c) a generalized graded module.

If we change the numbering of points in the set I we do not change the grading. If we change the numbering of the elements in O we change the grading. Nevertheless the filtration

$$\mathcal{F}^\lambda_{(n)}(A) = \bigoplus_{m \geq n} \mathcal{F}^\lambda_n(A)$$

induced by the grading will be the same.

Hence, the decomposition of A fixes a filtration. A different decomposition yields a non equivalent filtration.

Let me add here that it is possible to weight each in or out point in a different manner (for example, one can apply a constant integer shift to the order at certain points or multiply the order at certain points with a positive integer) as long as the overall effect cancels in a suitable manner. The filtration induced by the new grading will be equivalent to the filtration induced by the prescription (*) above.

4. Wedge-Product Representations

To obtain representations with certain features (in particular, such which are of interests in physics) we want to extend the Lie representation of $\mathcal{KN}(A)$ on $\mathcal{F}^\lambda(A)$ to the œœhighest exterior power" of $\mathcal{F}^\lambda(A)$. Of course, written as this it is nonsense, because of the infinite dimensionality of $\mathcal{F}^\lambda(A)$. Exploiting the grading, we can make sense out of this and obtain so called œœhighest weight representations" (for simplicity, I do not distinguish between highest weight and lowest weight). The representation space are the vector space $\mathcal{H}^\lambda(A)$ of semi-infinite formes of weight λ with respect to the grading induced by the basis elements.

To construct these spaces, we start with the basis $f_{n,p}, n \in \mathbb{Z}, p = 1, 2, \dots, k$ of the space $\mathcal{F}^\lambda(A)$. I suppress the λ in the notation. A basis of $\mathcal{H}^\lambda(A)$ is given by the elements

$$\psi = f_{i_1,p_1} \wedge f_{i_2,p_2} \wedge \cdots \wedge f_{m,1} \wedge f_{m,2} \wedge \cdots \wedge f_{m+1,1} \wedge \cdots \ .$$

Here the indices are in strictly increasing lexikographical order and starting with one index (depending on the base element ψ) all following indices will occur. Special base elements are

$$\Phi_T = f_{T,1} \wedge f_{T,2} \wedge \cdots \wedge f_{T,k} \wedge f_{T+1,1} \wedge f_{T+1,2} \wedge \cdots \ .$$

This vectors I call vacuum vector of level T. Now the action of $\mathcal{KN}(A)$ on $\mathcal{F}^\lambda(A)$ should be extended to the œœhighest exterior power" $\mathcal{H}^\lambda(A)$. Again guided by the finite dimensional situation we try the naiv Ansatz

$$e \,.\, \psi = (e \,.\, f_{i_1}) \wedge f_{i_2} \wedge \cdots + f_{i_1} \wedge (e \,.\, f_{i_2}) \wedge \cdots + \cdots \ . \qquad (**)$$

For simplicity with respect to the notation I restricted myself to the case $k = 1$ and suppressed the 2. index. The wedge sign gives the rules how the result has to be written in terms of the bases. The rules are: (1) it is linear in every argument, (2) if 2 factors have to be exchanged, then the expression changes its sign (i.e. it is zero if two factors are identical). Note, this is in fact the definition which works if the vector space one starts with is finite dimensional.

Unfortunately, this naiv definition makes only sense if in the sum (**) only finitely many summands will be nonzero. This is always the case if the vector field e is a element of either one of the following subalgebras

$$\mathcal{KN}^+(A) = \langle e_{n,p} \mid n \geq 3,\ p = 1, \dots, k \rangle$$
$$\mathcal{KN}^-(A) = \langle e_{n,p} \mid n \leq -1 - L, ,\ p = 1, \dots, k \rangle$$

This works due to the generalized graded structure.

For the (vector space) complement between these two algebras, the action is not well defined. To give an example: we try to calculate $e_2 \,.\, \Phi_T$. We obtain

$$e_2 \,.\, f_m = ((m-1) + \lambda)\, f_m + \text{ higher order terms } .$$

Alltogether

$$e_2 \,.\, \Phi_T \diamond\diamond ='' \sum_{m=T}^{\infty} ((m-1)+\lambda)\,\Phi_T \ .$$

Hence, the action has to be modified. The Physicists call this modification regularization. Unfortunately, the regularized action will only be a projective Lie action, not a linear Lie action any more. This defines a Lie action only for a central extension $\widehat{\mathcal{K}N}(A)$ of the vector field algebra. Before I want to say a few words on the mathematical technique of this process, let me give the results. Let t be the central element. If we require the cocycle defining the central extension to be independent of the weight λ (i.e. we obtain one central extension operating on all $\mathcal{H}^\lambda(A)$ for every λ) then t operates as multiplication with the scalar

$$c_\lambda = -2\,(6\lambda^2 - 6\lambda + 1) \ .$$

Starting from the vacuum vector Φ_T we obtain a submodule $\widehat{\mathcal{K}N}(A)\,.\,\Phi_T$ which generalizes the highest weight modules in the Virasoro case. If $E_{2,p}$ is a suitable lift of $e_{2,p}$ and if we consider $\mathcal{K}N^+(A)$ as submodule of $\widehat{\mathcal{K}N}(A)$ we obtain

$$\begin{aligned} \mathcal{K}N^+(A)\,.\,\Phi_T &= 0 \\ E_{2,p}\,.\,\Phi_T &= -\frac{1}{2}(T-1)(T-2+2\lambda)\Phi_T \\ t\,.\,\Phi_T &= -2\,(6\lambda^2-6\lambda+1)\Phi_T \ . \end{aligned}$$

There are two different possibilites to obtain this central extension. The first one is a power series method, the second one is a method which works by embedding into the algebra of infinite matrices. By these methods one obtains cohomologous extensions. Here I will concentrate on the 2. method because it is a generalization of the method which was applied in the Virasoro case [10],[11].

Let $\overline{gl}(\infty)$ be the Lie algebra of infinite matrices with only finitely many diagonals. A matrix $A = (a_{ij})$ with $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is a element of this algebra if there is a number $d = d(A)$ with $a_{ij} = 0$ if $|i-j| > d$. The action of $\mathcal{K}N(A)$ on $\mathcal{F}^\lambda(A)$ defines with respect to the basis $f_{n,p}$ an embedding of $\mathcal{K}N(A)$ into $\overline{gl}(\infty)$. For this, the generalized grading is essential. For $\overline{gl}(\infty)$ there exists a well defined modification of the action on $\mathcal{H}^\lambda(A)$ and a well defined 2-cocycle defining a central extension $\widehat{gl}(\infty)$ acting on $\mathcal{H}^\lambda(A)$ (see [12],[13],[14]). By pullback one obtains a central extension $\widehat{\mathcal{K}N}(A)$ and an action of $\widehat{\mathcal{K}N}(A)$ on $\mathcal{H}^\lambda(A)$.

The spaces $\mathcal{H}^\lambda(A)$ carry a very rich structure.

(1) Of course, there is also a central extension of the algebras of differential operators (of degree ≤ 1) acting on $\mathcal{H}^\lambda(A)$.

(2) It is possible to introduce a natural pairing between right semi-infinite forms of weight λ (the forms introduced above) and left semi-infinite forms of weight $(1-\lambda)$

induced by the pairing of $\mathcal{F}^\lambda(A)$ and $\mathcal{F}^{1-\lambda}(A)$. With respect to this pairing the $E_{n,p}$ are selfadjoint operators.

(3) $\mathcal{F}^\lambda(A)$ operates by œœwedging" and $\mathcal{F}^{1-\lambda}$ operates by œœcontracting" on $\mathcal{H}^\lambda(A)$. The operators defined by this procedure, generate a Clifford algebra. Physicists call this a $b-c$ system.

(4) There are very interesting compatibility relations of all these structures which deserve further investigation.

5. Central Extensions

There is a way to obtain central extensions of the above algebras by purely geometric means. Let me say a few words on central extensions. A central extension $\widehat{\mathcal{K}N}(A)$ of the algebra $\mathcal{K}N(A)$ can be given as a short exact sequence of Liealgebras

$$0 \longrightarrow \mathbb{C} \longrightarrow \widehat{\mathcal{K}N}(A) \xrightarrow{\psi} \mathcal{K}N(A) \longrightarrow 0 . \qquad (***)$$

The central extensions are classified (up to œœequivalence") by the elements of $H^2(\mathcal{K}N(A), \mathbb{C})$. These cohomology classes can be represented by 2-cocycles, i.e. by bilinear maps

$$\chi : \mathcal{K}N(A) \times \mathcal{K}N(A) \to \mathbb{C}$$

with

$$\chi(f,g) = -\chi(g,f)$$
$$\chi([f,g],h) + \chi([g,h],f) + \chi([h,f],g) = 0 .$$

A 2-cocycle χ is a coboundary if it can be written as

$$\chi(f,g) = \kappa([f,g]),$$

where κ is a linear form. The connection with (***) can be obtained by choosing a linear lift ϕ of the Lie homomorphismus ψ. The 2-cocycle χ can be given by

$$\chi(f,g)\cdot t = [\phi(f),\phi(g)] - \phi([f,g]),$$

where t is the image of 1 in $\mathcal{K}N(A)$.

In particular, if the elements $e_{n,p}$ are a basis of $\mathcal{K}N(A)$, then a basis of $\widehat{\mathcal{K}N}(A)$ can be given by $E_{n,p} = \phi(e_{n,p})$ and the central element t. The structure equations for $\widehat{\mathcal{K}N}(A)$ read as

$$[E_{n,p}, E_{m,r}] = \sum_{h=n+m-2}^{n+m+L} \sum_{s=1}^{k} C_{(n,p),(m,r)}^{(h,s)}(-1) E_{h,s} + \chi(e_{n,p}, e_{m,r}) \cdot t .$$

To obtain such 2-cocycle I use a method which generalizes what was done by Krichever and Novikov in the 2 point case [3]. If

$$e_| = f(z)\frac{\partial}{\partial z}, \qquad h_| = t(z)\frac{\partial}{\partial z}$$

are local forms of the vector fields and R is a projective holomorphic connection on X then the following defines a 2-cocycle

$$\chi(e,h) = \frac{c}{24\pi\,\mathbf{i}}\int_{C_\tau}\frac{1}{2}\left((f'''t - ft''') - R\cdot(f't - ft')\right)dz.$$

(1) In the case $g = 0$ and 2 points this cocycle reduces to the Virasoro cocycle.

(2) R is necessary to make the definition independent on the coordinates chosen (this is of course necessary for higher genus)

(3) The difference of 2 projective connections is a quadratic differential. If we choose another connection the 2-cocycle will change only by a coboundary. Hence, the cohomology class will not depend on the projective connection chosen.

(4) The cocycle is local (with respect to the introduced grading), i.e. there are constants A and B such that

$$\chi(e_{n,p}, e_{m,r}) = 0, \quad \text{if} \quad n+m < A \quad \text{or} \quad n+m > B\ .$$

The defining cocycle for the extension $\widehat{\mathcal{KN}}(A)$ operating on $\mathcal{H}^\lambda(A)$ is also local, and hence there is the question whether it can be given by such a χ as above. By some modification of the arguments of Krichever – Novikov in [3] it should be possible to prove the following conjecture, which answers the question to the positive.

conjecture. *Every local cocycle is cohomologous to such a χ with a suitable projective connection.*

The details of the œœproof" I have not yet worked out.

It is possible to construct such geometric cocycles for $L\mathcal{F}^0(A)$ and $\mathcal{D}^1(A)$. For the former one the cocycle can be given by

$$\gamma(f,g) = \frac{1}{2\pi\,\mathbf{i}}\oint_{C_\tau} f dg\ .$$

I call the algebra $\widehat{\mathcal{F}^0}(A)$ a generalized Heisenberg algebra. Starting with this algebra I can introduce generalization of the affine Kac-Moody algebras.

For $\mathcal{D}^1$ the cocycles χ and γ also define central extensions. Beside them, there is another (linearly independent) cocycle

$$\beta(e,t) = \frac{1}{2\pi\,\mathbf{i}}\oint_{C_\tau}(ft'' + T\cdot ft')\,dz\ .$$

Here T is a meromorphic connection, which is allowed to have poles at the points of A. In the Virasoro case these 3 cocycles are a basis for $H^2(\mathcal{D}^1(A),\mathbb{C})$, as was proven in [1].

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