# APPROXIMATION OF INFINITE-DIMENSIONAL ALGEBRAS AND GEOMETRIC QUANTIZATION

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## 1. Introduction.

In the following, I like to report on joint work with M. Bordemann, J. Hoppe, and P. Schaller [1]. The starting point of our work was the observation that in the context of membrane theory the algebra $diff_A\ S^2$ ( the algebra of infinitesimal area preserving diffeomorphisms, resp. the algebra of divergence free vector fields of the sphere $S^2$) and the algebra $diff_A\ T^2$ (the analogous algebra for the torus) can be described as a $su(N)$ -limit for $N \to \infty$, in the sense that in certain specific basis of $su(N)$ the structure constants of $su(N)$ converge to the structure constants of the above mentioned algebras. Let me just cite Hoppe [2] for the sphere and Fairlie, Fletcher and Zachos [3] for the torus. For more references, see our paper [1].

Now the story got a little bit confusing, because it was tried to make a naive identification of $diff_A\ S^2$ and $diff_A\ T^2$ with the algebra $su_+(\infty)$ (or $su(\infty)$). In our paper we showed that the algebras $diff_A\ S^2$, $diff_A\ T^2$ (or better certain dense subalgebras of them) and $gl(\infty)$ are pairwise non isomorphic. It was even proved by Hoppe and Schaller [4] that there exits a infinite family of algebras, including $diff_A\ T^2$, each of them non isomorphic but nevertheless each of them can be approximated by $su(N)$, $N \to \infty$.

Note, the relation between an infinite dimensional algebra and a series of finite dimensional algebras is also important in other contexts. For example, it appears also in quantization schemes, semiclassical limit and so on.

Because the time is limited, I will not be able to cover every aspect in detail. Hence, I like to explain the main points in rough terms and concentrate on some special points later on.

The term "limit" above was a rather flabby notion. We suggested a rigourous definition of limit and approximation. This we call an $L_\alpha-$approximation. We showed that the above limits can be covered by this concept.

As already remarked, non isomorphic algebras can have the same sequence of approximating finite dimensional algebras. The situation is even more general. We start from a compact Kähler manifold. If we consider this Kähler manifold as a classical phase space we can apply a geometric quantization procedure to approximate the algebra of infinitesimal symplectic transformations by a sequence of antihermitean quantum operators. The rough idea is to realize the quantum operators as operators on the space of holomorphic sections of increasing tensor powers of the prequantum line bundle.

The algebra of infinitesimal symplectic transformations is in relation to the algebra of infinitesimal area preserving diffeomorphism (with respect to the symplectic volume). In the case of dimension 2 they are the same. We expect this approximation to be a $su(N)-$ approximation in our sense. We showed this to be true for the real $2n-$dimensional torus. The Kähler form $\omega$ defines on the torus the structure of an $n-$dimensional complex manifold and by the prequantization condition it is automatically an abelian variety (i.e. a complex tori which admits an embedding into projective space). In this case everything can be calculated in terms of theta functions.

Let me remark, that A.S. Schwarz [5] also mentions such connections and that there is an relation to Berezin's coherent states.

## 2. $L_\alpha-$Approximations

We introduced the following concept of an approxiamtion, resp. limit. Given a family of real or complex Lie algebras

$$(L_\alpha, [..,..]_\alpha, d_\alpha, \alpha \in I)$$

where we denote by $[..,..]_\alpha$ the Lie bracket in $L_\alpha$, $d_\alpha$ a metric in $L_\alpha$ and by $I$ either $\mathbb{N}$, $\mathbb{R}$ or some other subset of $\mathbb{R}$ with $\infty$ as accumulation point. Now let $(L, [..,..])$ be another Lie algebra satisfying the

**condition 1.** *(i) There exists a surjective map $p_\alpha : L \to L_\alpha$ for every $\alpha \in I$.*

*(ii) For each $x, y \in L$ the following holds: If $d_\alpha(p_\alpha(x), p_\alpha(y)) \to 0$ for $\alpha \to \infty$ then $x = y$.*

We call $(L_\alpha, [..,..]_\alpha, d_\alpha, \alpha \in I)$ an *approximating sequence* for $(L, [..,..])$ induced by $(p_\alpha, \alpha \in I)$ and $L$ an $L_\alpha-$*quasilimit* if the following condition is also valid

**Condition 2.** *For each $x, y \in L$,*

$$d_\alpha(p_\alpha[x, y], [p_\alpha x, p_\alpha y]_\alpha) \to 0 \quad (\alpha \to \infty)\ .$$

Of course the definition depends on the metrics $d_\alpha$ chosen. If we replace the family of metrics $(d_\alpha)$ by a uniformly equivalent family of metrics $(d'_\alpha)$ we obtain that the

property of beeing an approximating sequence is the same, whether we take it with respect to the metrics $d$ or $d'$.
(By an uniform equivalent family of metric we understand that there exist positive $a, b \in \mathbb{R}$ such that

$$a \cdot d_\alpha(x_\alpha, y_\alpha) \;\le\; d'_\alpha(x_\alpha, y_\alpha) \;\le\; b \cdot d_\alpha(x_\alpha, y_\alpha) \qquad \forall \alpha \in I,\ \forall x_\alpha, y_\alpha \in L_\alpha\ .)$$

We will immediately see, that the same sequence of algebras can approximate non isomorphic algebras. However we have a weak uniqueness theorem implying, that the Lie structure on the underlying vector space $L$ of an $L_\alpha-$quasilimit is unique once the linear maps $(p_\alpha, \alpha \in I)$ is specified.

In many examples also the following condition is fulfilled

**Condition 3.** *There exists a family of linear maps* $(i_\alpha : L_\alpha \to L, \alpha \in I)$ *and an index* $\alpha_0 \in I$ *such that for* $\alpha \ge \alpha_0$

$$p_\alpha \circ i_\alpha = id_\alpha \quad \textit{and} \quad i_\alpha(L_\alpha) \subseteq i_\beta(L_\beta) \quad \beta \ge \alpha\ .$$

In this case we call $(L_\alpha)$ a splitting $L_\alpha-$approximation.

Let me make an example illustrating the concept:

Let $L$ and $(L_\alpha, \alpha \in I)$ be different Lie algebras with the same underlying vector space $V$. If we choose as $p_\alpha$ and as $i_\alpha$ the identity map and as $d_\alpha$ a fixed metric $d$ on $V$ then the condition 1 and 3 are clearly fulfilled. condition 2 reads as

$$d([x,y],[x,y]_\alpha) \to 0 \quad (\alpha \to \infty)\ .$$

This reflects the approximation of the structure constants. To make the example more concrete let $L$ (and hence all $L_\alpha$) be generated by $T_n$ with $n \in \mathbb{N}$. By $\langle T_n, T_m\rangle = \delta_{n,m}$ we get a scalar product on $V$. If we choose $d(x,y) := \sqrt{\langle x-y, x-y\rangle}$ this implies for the structure constants $f^k_{nm}$, $f^{k,\alpha}_{nm}$ defined by

$$[T_n, T_m] = f^k_{nm} T_k \quad \text{resp.} \quad [T_n, T_m]_\alpha = f^{k,\alpha}_{nm} T_k\ .$$

convergency

$$\lim_{\alpha \to \infty} f^{k,\alpha}_{nm} = f^k_{nm}\ .$$

Conversely, if for fixed $n$ and $m$ the set

$$\{\, k \in \mathbb{N} \ \mid \text{there exists a } \alpha \text{ such that } f^{k,\alpha}_{n,m} \neq 0\,\}$$

is finite then the convergence of the structure constants implies condition 2.

In our paper [1] we showed that the algebra $diff'_A\ T^2 \oplus \mathbb{C}T_{00}$ and the algebra $diff'_A\ S^2 \oplus \mathbb{C}Y_{00}$ are $L_\alpha-$quasilimits having the same sequence of $gl(n)$ as approximating sequence. Here I want to concentrate on a different application of $L_\alpha-$quasilimits.

## 3. Geometric Quantization

To formulate our results, resp. our conjecture let me remind you of a few concepts in the theory of symplectic manifolds and geometric quantization (for more details see [6],[7],[8]). Let $(M,\omega)$ be a symplectic manifold of dimension $2n$, i.e. a differentiable manifold $M^{2n}$ and $\omega$ a closed nondegenerate 2-form on $M$. Then a volume form on $M$ is defined by

$$\Omega := (-1)^{\binom{n}{2}}\frac{1}{n!}\,\omega^n \ .$$

The vector fields $diff_V\ M$ on $M$ representing infinitesimal volume preserving diffeomorphisms can be given as the vector fields which are divergence free, or equivalently which obey $L_X\Omega = 0$ , where $L_X$ denotes the Lie derivative in direction of the vector field $X$.

Vector fields $X$ obeying $L_X\omega = 0$ are called locally Hamiltonian. In particular, they are divergence free. Moreover, the the space of locally Hamiltonian vector fields $LHam\ M$ is a subalgebra of the Liealgebra $diff_V\ M$ .

To each smooth real valued function $H$ an $M$ one assigns with the help of $\omega$ its Hamiltonian vector field $X_H$ by

$$i_{X_H}(\omega) = dH, \qquad \text{resp.} \quad \omega(X_H,.) = dH(.) \ .$$

Because the Lie- derivative on $p-$forms can be given as

$$L_X \ = \ i_X\circ d + d\circ i_X$$

we obtain $L_X\omega = d(i_X(\omega))$. If $X_H$ is a hamiltonian vector field we get $L_{X_H}\omega = 0$. In particular, Hamiltonian vector fields are locally Hamiltonian and divergence free. The space of Hamiltonian vector fields $Ham\ M$ is an ideal of $LHam\ M$. (This follows from the identity $[X,Y] = -X_{\omega(X,Y)}$ if $L_X\omega = L_Y\omega = 0$. )

Of course, if the dimension is equal to 2 $diff_V M$ is the algebra of locally hamiltonian vector fields and the quotient $diff_V\ M/Ham\ M$ can be identified with $H^1(M,\mathbb{R})$ via $X\mapsto i_X\omega$ .

The Poissonalgebra $\mathcal{P}(M)$ is the space of $C^\infty-$function with Liealgebra structure given by the Poisson bracket

$$\{f,g\} := df(X_g) \ .$$

The map

$$\mathcal{P}(M)\to Ham\ M, \quad f\mapsto -X_f$$

shows that $\mathcal{P}(M)$ is a central extension of the Hamiltonian vector fields. (The central elements are the constant functions.) In the case that $M$ is compact the extension will split. By this we see that one can investigate the Poisson algebra $\mathcal{P}(M)$ in order to study an essential part of $LHam\ M$ $(= diff_V M$ for $\dim M = 2)$ and simply 'omit the constants at the end'. This was the starting point of our investigation.

We shall now relate the Lie algebra $\mathcal{P}(M)$ to a geometric quantization scheme. For this we restrict $M$ to be a compact complex Kähler manifold of real dimension $2n$. The Kähler form $\omega$ is now the symplectic form. In addition we need a holomorphic line bundle $L$, a fibre metric $h$ and a covariant derivative $\nabla$ with the compatibility relations:

$$\begin{gathered} h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2) = d(h(s_1,s_2))(X), \\ \nabla_V s = 0 \\ F(X,Y)s_1 := (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s_1 = -\,\mathbf{i}\,\omega(X,Y)s_1 \ . \end{gathered}$$

Here $s_1$ and $s_2$ are smooth sections of $L$, $s$ is a holomorphic section, $X$ and $Y$ are complex vector fields on $M$, and $V$ is a complex vector field of type $(0,1)$ (an "antiholomorphic direction"). The first two conditions says that the connection is compatible with the metric and the complex structure. The third condition is the prequantum condition. Because the Chern class of the line bundle $L$ is an integer cohomology class it yields that the Kähler manifold admits an embedding into projective space by the Kodaira embedding theorem.

For every smooth real (or complex) valued function $f$ on $M$ the following prequantum operator $P_f$ acting on the complex vector space $\Gamma(M,L)$ of all smooth sections of $L$ is formed

$$P_f := -\nabla_{X_f} + \mathbf{i}\, f\cdot 1 \ .$$

This defines a map

$$P : \mathcal{P}(M) \to Op(\Gamma(M,L)), \quad f \mapsto P_f \ .$$

The prequantum condition guarantees that $P$ is an injective Lie algebra homomorphism

$$P_{\{f,g\}} = [P_f, P_g] \ .$$

After defining a scalar product $\langle ..|..\rangle$ in $\Gamma(M,L)$ by

$$\langle s_1|s_2\rangle := \int_M \Omega\, h(s_1,s_2)$$

$P_f$ becomes an antihermitean operator in $\Gamma(M,L)$ for real valued $f$. The prequantum Hilbert space $\mathcal{H}$ is then defined to be the completion of $\Gamma(M,L)$ with respect to $\langle ..|..\rangle$.

A second step in a geometric quantization scheme is the choice of a polarization. I.e. one would like to have only those wave functions in the prequantum Hilbert space $\mathcal{H}$ that depend on 'only one (certain) half of the phase space variables'. For Kähler manifolds there is a canonical concept. $L$ should be a holomorphic line bundle as was already assumed.

The quantum Hilbert space is then defined to be the subspace $\Gamma_{hol}(M,L)$ of all holomorphic sections in $\mathcal{H}$. For compact manifolds it is always finite dimensional. In order to define quantum observables or quantum operators $Q_f$ acting on $\Gamma_{hol}(M,L)$ one simply takes 'the holomorphic part' of the prequantum operators $P_f$

$$Q_f := \rho \circ P_f \circ \rho \ .$$

Here $\rho$ is the orthogonal projection

$$\rho : \mathcal{H} \to \Gamma_{hol}(M,L) \ .$$

$Q_f$ clearly is an antihermitean operator for real valued smooth functions $f$ but in general

$$Q_{\{f,g\}} \neq [Q_f, Q_g] \ .$$

To get an explicit expression for $\rho$ one can choose any orthonormal basis $|s_1\rangle, \ldots, |s_d\rangle$ ($d = \dim \Gamma_{hol}(M,L)$) of $\Gamma_{hol}(M,L)$ and set

$$Q_f := \sum_{a,b=1}^{d} |s_a\rangle\langle s_a|P_f|s_b\rangle\langle s_b| \ .$$

Hence it suffices to compute the matrix elements $\langle s_a|P_f|s_b\rangle$ of $P_f$.

In order to achieve an $L_\alpha-$approximation for $Ham\ M$ we would like to have the geometric quantization scheme dependent on a parameter $\alpha$. This can be done by fixing a holomorphic line bundle $L$, a fibre metric $h$ and a covariant derivative $\nabla$ which fulfils the compatibility conditions and then considering arbitrary $m-$fold tensor powers of $L$

$$L^m := L^{\otimes m} := L \otimes \cdots \otimes L \qquad (\ m \text{ factors }).$$

For the holomorphic line bundle $L^m$ one can now construct a canonical fibre metric $h^{(m)}$ with compatible covariant derivative $\nabla^{(m)}$ by

$$h^{(m)} := h \otimes \cdots \otimes h \quad m \text{ factors},$$
$$\nabla^{(m)} := \sum_{k=1}^{m} 1 \otimes \ldots \otimes (\nabla)_k \otimes \ldots \otimes 1,$$

where in the $k-$th summand the $\nabla$ is at the $k-$th position.

Now we can calculate the usual prequantum operators

$$P_f^{(m)} := -\nabla^{(m)}_{X_f^{(m)}} + \mathbf{i}\, f \cdot 1$$

and get

$$[P_f^{(m)}, P_g^{(m)}] = P^{(m)}_{\{f,g\}^{(m)}} = \frac{1}{m} P^{(m)}_{\{f,g\}} \ .$$

But since we are looking for a representation of $\mathcal{P}(M)$, i.e. the Poisson algebra w.r.t. $\omega$ and not w.r.t. $m\,\omega$ we have to rescale the prequantum operators as follows

$$\hat{P}_f^{(m)} := mP_f^{(m)} = -\nabla_{X_f}^{(m)} + \mathbf{i}\, mf \cdot 1$$

which yields

$$[\hat{P}_f^{(m)}, \hat{P}_g^{(m)}] = \hat{P}_{\{f,g\}}^{(m)} \ .$$

If we denote by $\mathcal{H}^{(m)}$ (resp. $\Gamma_{hol}(M, L^{(m)})$, resp. $\rho^{(m)}$) the corresponding objects introduced above (where we choose the volume form on $M$ to be equal to $\Omega$ and not $(m\,\Omega)^n$) we can form the (rescaled) quantum operators in $\Gamma_{hol}(M, L^{(m)})$

$$\hat{Q}_f^{(m)} := \rho^{(m)} \circ \hat{P}_f^{(m)} \circ \rho!BSL!sp(m) \ .$$

Now we set

$$L_m := \{ \text{ antihermitean linear operators in } \Gamma_{hol}(M, L^{(m)}) \ \}$$

$$p_m : \mathcal{P}(M) \to L_m, \quad f \to \hat{Q}_f^{(m)}$$

$$d_m : L_m \times L_m \to \mathbb{R}, \quad (A, B) \mapsto r_m \cdot \sqrt{\text{ Tr } (A-B)^+ \cdot (A-B)}$$

where the $r_m$ are positive real numbers. We formulate the following

**Conjecture.** *Let $(M, \omega)$ be a compact Kähler manifold with symplectic form $\omega$. Then there is a $\omega-$compatible complex structure $I$ in $M$, with respect to which $M$ is also a Kähler manifold, a holomorphic prequantum line bundle $L$ compatible with $I$, a fibre metric $h$ with compatible covariant derivative $\nabla$ and a sequence of positive real numbers $r_m, m \in \mathbb{N}$ such that the Poisson algebra $\mathcal{P}(M)$ admits a $(L_m, d_m)$ approximation induced by $p_m$. Here $L_m, p_m$ and $d_m$ are defined as above*

If one thinks of $m$ as $1/\hbar$ this concept can be interpreted as $\hbar \to 0$ limit. Note that we leave the complex structure to be adjustable because the main interest lies in the symplectic structure of $M$.

## 4. An Example: Complex Tori

We verified our conjecture in the case of (real) $2n-$ dimensional tori with integral symplectic form $\omega$. We introduced a complex structure compatible with it. By the integrality of $\omega$ (forced by the prequantum condition) the complex torus admits an embedding into projective space. In the language of algebraic geometry it is an abelian variety with polarization. We restrict ourselves to the case of principally polarized tori. The Torus $T$ can be given by

$$T \ = \ \mathbb{C}^n / L \ ,$$

where $L$ is a lattice in $\mathbb{C}^n$. In our situation a basis of the lattice can be given by the columns of the $n \times 2n-$ matrix $(I, Z)$ . $I$ denotes the $n \times n$ identity matrix, $Z$ a $n \times n$ complex, symmetric matrix with positive definite imaginary part. All objects on the torus can be described as objects on $\mathbb{C}^n$ with a certain "automorphic " behaviour.

As prequantum line bundle we take the theta bundle $L$. The space $\Gamma_{hol}(T, L)$ is $1-$dimensional and is generated by the Riemann $\Theta-$function

$$\Theta(v) = \sum_{l \in \mathbb{Z}^n} \exp\big(\pi \mathbf{i}\, l^{tr} \cdot Z \cdot l + 2\pi \mathbf{i}\, l^{tr} \cdot v\big) \ .$$

For arbitrary tensor powers $L^m$ of $L$ the vector spaces $\Gamma_{hol}(T, L^m)$ are now $m^n-$dimensional and can be generated by certain theta functions with characteristics [9]

$$\hat{f}_a(v)^{(m)} = \sum_{l \in \mathbb{Z}^n} \exp\left(\mathbf{i}\,\pi m \cdot (l + \frac{a}{m})^{tr} \cdot Z \cdot (l + \frac{a}{m}) + 2\pi\,\mathbf{i}\, m \cdot (l + \frac{a}{m})^{tr} \cdot v\right) \ .$$

with $a = (a_1, a_2, \ldots, a_n)^{tr}$, $a_i \in \mathbb{Z}$ with $0 \le a_i < m$ for $1 \le i \le n$.

With respect to them the matrix elements $\langle \Theta_a^{(m)} | \hat{P}_{F_r}^{(m)} | \Theta_b^{(m)} \rangle$ can be calculated for the Fourier modes

$$F_r(v) = \exp\left(2\pi\,\mathbf{i} \sum_{k=1}^{n} (r_k x_k + \frac{1}{\tau_k} r_{k+n} y_k)\right)$$

where $r = (r_1, \ldots, r_{2n})$ is a $2n-$tupel of integers.

After some rescaling the (rescaled) quantum operators obey the commutation relation of the (multidimensional) sine algebra [3]

$$[T_r^{(m)}, T_s^{(m)}] = \frac{1}{2\pi\Lambda} \sin\left(2\pi\Lambda(r \times s)\right) T_{r+s}^{(m)}$$

where $\Lambda = (2m)^{-1}$ and

$$r \times s := \sum_{k=1}^{n} (r_k s_{k+n} - r_{k+n} s_k) \ .$$

In fact, we proved that with suitable norms $d_m$ and the maps

$$p_m : \ \mathcal{P}(T^{2n}) \ \to \ u(m^n), \qquad f \ \mapsto \ p_m f := (\langle \Theta_a^{(m)} | \hat{P}_f^{(m)} | \Theta_b^{(m)} \rangle)$$

$(u(m^n) \ldots., m \in \mathbb{N})$ is an approximating sequence for $(\mathcal{P}(T^{2n}), \{.., ..\})$ induced by the $(p_m)$.

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