Nonlinear Waves and the KAM Theorem: Nonlinear Degeneracies *

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1. Introduction

This paper is concerned with solutions of nonlinear wave equations, and other partial differential equations that model conservative phenomena in physics and applied mathematics. As the initial value problem is increasingly well understood, the focus of our attention is on the more detailed structure of the phase space in which the evolution equations are posed. The nonlinear wave equation can be viewed as an infinite dimensional Hamiltonian system, thus it is natural to study important classes of periodic and quasiperiodic solutions in the neighborhood of equilibrium. The paper (Craig & Wayne [CW]) constructs periodic solutions for nonlinear wave equations, using a version of the Nash-Moser technique to overcome the inherent small divisor problem. In that reference, certain generic requirements of nonresonance and genuine nonlinearity are needed in the existence proof. This present paper addresses problems in which the hypotheses of genuine nonlinearity are not satisfied, where nonetheless the existence of families of periodic solutions near equilibrium are obtained. Other recent work on the subject of perturbation theory for Hamiltonian systems with infinitely many degrees of freedom include Kuksin [K], Wayne [W], Pöschel [P] and Albanese, Fröhlich and Spencer [AFS].

Some of the more interesting aspects of our approach to these problems are the ties between partial differential equations, Hamiltonian mechanics, and localization theory of mathematical physics. Indeed, the central estimates in this work were pioneered by Fröhlich and Spencer [FS] in the study of the Green's function for random Schrödinger operators. The nonlinear wave equation is not the only equation of interest which has Hamiltonian structure, for which results on periodic and quasiperiodic solutions are of interest. We expect the techniques of [CW] and of this paper to extend to the nonlinear Schrödinger equation, versions of the KdV equation and other problems with infinitely many degrees of freedom, for which the equilibrium solution is an elliptic stationary point. Moreover we expect the analysis of quasiperiodic solutions to be similar to the

* To appear in the **Proceedings of the Conference on Nonlinear Waves**; Villefranche France

analysis of periodic solutions in these resonant cases, and plan a further publication on this subject.

This paper describes the construction of periodic solutions of the nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u - g(x,u) \quad , \qquad 0 \le x \le \pi \quad , \tag{1.1}$$

where the solution $u(x,t)$ satisfies either periodic or Dirichlet boundary conditions at $x = 0, \pi$. The nonlinear term $g(x,u)$ is taken analytic, with the Taylor expansion in the variable u given by

$$g(x,u) = g_1(x)u + g_2(x)u^2 + g_3(x)u^3 + \cdots . \tag{1.2}$$

Well known examples are the sine-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - b^2 \sin(u), \tag{1.3}$$

and the φ^d-nonlinear Klein-Gordon equation,

$$\partial_t^2 u = \partial_x^2 u - b^2 u + u^{d-1}. \tag{1.4}$$

All of the above partial differential equations can be considered as Hamiltonian systems with infinitely many degrees of freedom. Indeed, we may define the Hamiltonian

$$H(p,u) = \int \frac{1}{2}p^2 + \frac{1}{2}(\partial_x u)^2 + R(x,u)\, dx \quad , \tag{1.5}$$

with $\partial_u R(x,u) = g(x,u)$. Denoting $z = (u,p)^T$, Hamilton's canonical equations read

$$\dot z = J \nabla H(z), \tag{1.6}$$

where J denotes the standard symplectic matrix. The methods of this paper are perturbative – we construct solutions near the equilibrium point $u = 0$. For the wave equation (1.1) $z = 0$ is elliptic, thus by analogy with finite dimensional problems one expects that the construction of quasiperiodic solutions encounters small divisor problems, and a form of the KAM theorem would be used. In fact the small divisor problem arises even in the construction of periodic solutions, as the presence of infinitely many degrees of freedom introduces a dense set of resonances.

In the reference [CW] the existence theory for periodic solutions is discussed, under hypotheses of nonresonance and genuine nonlinearity. The results are essentially that there is an open dense set $\mathcal{G}$ of nonlinearities such that for $g(x,\cdot) \in \mathcal{G}$, there exist families of periodic solutions of (1.1). The character of these families is typically that of a Cantor set foliated by invariant circles — a situation reminiscent of the conclusion of the KAM theorem for quasiperiodic solutions in finite dimensional Hamiltonian perturbation theory. In the results of [CW], the good set $\mathcal{G}$ depends only upon $g_1(x), g_2(x)$, and $g_3(x)$, the *3–jet* of the nonlinearity $g(x,\cdot)$.

In the present paper we extend the results of [CW] to cases which are equally nonresonant, but which are nonlinearly degenerate. These problems fail to satisfy the 'twist condition' of the previous results, thus the present work enlarges the class $\mathcal{G}$ of nonlinearities for which an existence theorem holds. For example, consider the nonlinear term

$$g(x,u) = g_1(x)u + g_M(x)u^M + \cdots \quad . \tag{1.7}$$

For $M > 3$, g is not in the set $\mathcal{G}$, for the curvature of any approximate solution branch will vanish. Among other situations this appears for the nonlinear φ^d Klein-Gordon equation with $d > 4$. We show in this paper that under more subtle conditions of nondegeneracy, again families of periodic solutions of the wave equation (1.1) can be constructed. These conditions depend upon the coefficient $g_1(x)$ of course, and if M is odd, upon the first nonzero coefficient $g_M(x)$ of the nonlinear term. If the first nonzero coefficient $g_M(x)$ has even order, then the existence criterion depends upon a certain subset of the $(2M-1)$–*jet* of $g(x,u)$, that is, upon certain of the coefficients $\{g_1(x), \cdots, g_{2M-1}(x)\}$.

We feel that all these results are quite general, and will extend to other equations and to the construction of quasiperiodic solutions as well. We point out that in the study of quasiperiodic solutions, the analysis of higher order nonlinear degeneracies has not been carried out, even in the case of finite dimensional Hamiltonian systems in the neighborhood of an elliptic stationary point.

Acknowledgements: The authors would like to thank the Université de Genève, the Université de Paris 6, MSRI–Berkeley and Oxford University for their hospitality, and the National Science Foundation and the Alfred P. Sloan Foundation for their support of our research.

2. Results

It is instructive to solve the equation linearized about $u = 0$,

$$\partial_t^2 v = \partial_x^2 v - g_1(x) v \quad . \tag{2.1}$$

This is done by the elementary method of separation of variables. Let $\{(\psi_j(x), \omega_j^2)\}_{j=1}^{\infty}$ be the complete set of eigenfunction—eigenvalue pairs for the linear Sturm-Liouville operator

$$L(g_1)\psi = \big(-\frac{d^2}{dx^2} + g_1(x)\big)\psi = \omega^2\psi,$$

imposing the proper boundary conditions, ($\psi(0) = \psi(\pi) = 0$ in the Dirichlet case, and $\psi(x) = \psi(x+\pi)$ in the periodic case.) We will assume that all ω^2 are positive with little loss of generality. Then a periodic solution to (2.1) is given by

$$\begin{aligned} v(x,t) =& r\cos(\Omega t + \xi)\psi_j(x) \\ \Omega =& \omega_j \quad . \end{aligned}$$

The general solution of (2.1) is given by sums of these solutions

$$v(x,t) = \sum_{j=1}^{\infty} r_j \cos(\omega_j t + \xi_j)\psi_j(x),$$

parametrized by angles $\{\xi_j\}_{j=1}^{\infty}$, and amplitudes $\{r_j\}_{j=1}^{\infty}$, (action variables $\{r_j^2\}_{j=1}^{\infty}$.) These are not usually periodic, but typically quasiperiodic if at most finitely many amplitudes r_j are nonzero, and in general they are almost periodic, unless a full set of rational relations (infinitely many) exist among the frequencies $\{\omega_j\}$. Thus it is a natural question to pose whether some of these periodic (or quasiperiodic, or almost periodic) solutions persist for the nonlinear problem.

Hypothesis: (i) Let $g(x,u)$ be π periodic in x, and analytic in the strip $\{|\text{Im } x| < \overline{\sigma}\}$ and in u in some neighborhood of the origin. In the case of Dirichlet boundary conditions we also ask that g be odd in the (x,u)–plane.
(ii) We assume that in (1.7), $M > 3$.

The cases $M = 2, 3$ were discussed in reference [1]. Define $m = M - 1$ if M is odd, and $m = (\min \{R; M < R \leq 2M-1,\ R \text{ odd, and } g_R(x) \neq 0\} - 1)$ if M is even. If no such R exists, set $m = 2M - 2$.

Theorem 2.1. *There exists a generic set $\mathcal{G}_M$ such that if $g \in \mathcal{G}_M$ then there are uncountably many small periodic solutions to the nonlinear equation (1.1). Furthermore*

(i) The solutions are analytic in a smaller strip $\{|\text{Im } x| < \overline{\sigma}/2\}$.

(ii) The solutions are close to the linear periodic solutions, and form a Cantor set foliated by circles. More precisely, there is a small r_0 and a Cantor set $\mathcal{C} \in (-r_0, r_0)$ such that if $r \in \mathcal{C}$ then there is an angle ξ such that

$$\begin{gathered}|u(x,t;r) - r\cos(\Omega(r)t+\xi)\psi_j(x)| \leq Cr^M, \\ |\Omega(r) - \omega_j| \leq Cr^m.\end{gathered} \tag{2.2}$$

(iii) The good set $\mathcal{G}_M$ is open and dense. If M is odd, $\mathcal{G}_M$ depends only upon the coefficients $g_1(x)$ and $g_M(x)$. If M is even, it depends upon $g_1(x)$ and $g_R(x)$, for the minimum R odd, $M < R < 2M-1$, $g_R(x) \neq 0$. If there is no such R, then $\mathcal{G}_M$ depends upon $g_1(x), g_M(x)$, and $g_{2M-1}(x)$.

For an exact description of the topology in which $\mathcal{G}_M$ is dense, see [CW] section 6.

An immediate corollary applies to a specific choice of nonlinearity. Consider the φ^d Klein-Gordon equation (1.4) on the interval $[0,\pi]$. For $d = 4$ this is addressed in [CW], however for $d > 5$ it fails to satisfy the hypothesis of genuine nonlinearity of that paper. When periodic boundary conditions are imposed, the problem can be reduced to an analysis of the phase plane for a solution $u(x-ct)$. When Dirichlet conditions are imposed this is not the case. For d even, Theorem 2.1 applies, giving the following result.

Corollary 2.2. *For an open set of parameters b^2 of full measure, (1.4) has nonlinearity within the good set $\mathcal{G}_M$, and therefore there exist families of periodic solutions, as described in Theorem 2.1.*

This particular dependence of the condition of genuine nonlinearity, and the power m on the coefficients, is natural in terms of the Birkhoff normal form for a dynamical system in the neighborhood of an elliptic stationary point. That is, odd terms in the Hamiltonian (even terms of the nonlinearity) are generically nonresonant, and do not enter the normal form at highest order. Even terms of the Hamiltonian (odd terms of the nonlinearity) are generically resonant, affecting the normal form and the frequency of the solution at highest order. Furthermore, the next to highest order corrections appear at order $2M-1$.

We remark here that for any $g_2(x), g_3(x), \ldots$, if $g_1(x) = 0$ then the conditions of nonresonance of [CW] are violated. Indeed both the Dirichlet problem and the periodic problem are infinitely resonant, as the equation linearized about $u = 0$ is

$$\partial_t^2 v = \partial_x^2 v,$$

which has an infinite dimensional null space, spanned respectively by the functions $\{\sin(\ell x)e^{\pm i\ell t}\}$, $\{\cos(2\ell x)e^{\pm 2i\ell t}, \sin(2\ell x)e^{\pm 2i\ell t}\}$. Problems which violate the nonresonance condition, with a finite but possibly large null space will be addressed in a subsequent paper. Other than solutions with rational period that are obtained by global variational methods [B,R], the infinitely resonant case has not been addressed, so far as we know.

3. A nonlinear lattice system

We will take the point of view of embedding a circle into phase space, in such a manner that it is invariant with respect to the flow determined by the wave equation (1.1). Denoting an embedded circle by

$$\begin{aligned} S(x,\xi) &= \sum_{j=1}^{\infty} s_j(\xi)\psi_j(x) \\ s_j(\xi) &= s_j(\xi + 2\pi), \end{aligned} \tag{3.1}$$

it will be invariant under flow by the wave equation, and traversed with frequency Ω, if $S(x,\xi)$ satisfies

$$\Omega^2 \partial_\xi^2 S - \partial_x^2 S + g(x,S) = 0. \tag{3.2}$$

To treat the spatial and temporal variables on an equal footing, one expands s_j in Fourier series

$$S(x,\xi) = \sum_{\substack{j=1 \\ k=-\infty}}^{\infty} \widetilde{s}(j,k)\ e^{ik\xi}\psi_j(x).$$

If $S(x,\xi)$ satisfies (3.2), the coefficients of this eigenfunction expansion of S satisfy an equation over the lattice, $(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}$,

$$\begin{aligned} 0 =& (\omega_j^2 - \Omega^2 k^2)\widetilde{s}(j,k) + W(\widetilde{s})(j,k) \\ =& V(\Omega)\widetilde{s}(j,k) + W(\widetilde{s})(j,k) \quad . \end{aligned} \tag{3.3}$$

We call this the 'mode interaction equation' of the nonlinear problem (3.2). The term $V(\Omega)$ is diagonal in the given basis, while $W(\widetilde{s})$ is nonlinear, and at least of order M for small $\widetilde{s}$. Linearizing about $\widetilde{s} = 0$, we have

$$V(\Omega)\varphi = 0,$$

with solutions $(\varphi, \Omega) = (\delta(j_0, \pm k_0), \omega_{j_0}/k_0)$ corresponding to a periodic solution of (2.1). The point spectrum of $V(\Omega)$ is typically dense in the real line, in particular it accumulates at zero; this is often called the phenomenon of small divisors. The fact that point spectra of the linearized problem approach zero is the fundamental difficulty of the problem. The technique that is presented in [CW] and this paper shows that the geometry of the lattice sites associated with the small divisors also plays an important role in the existence theory.

This lattice equation has certain symmetries which are relevant to the problem. Let $x = (j, k) \in \mathbf{Z}^+ \times \mathbf{Z}$, and write $\overline{x} = (j, -k)$. Then S is real if and only if $\widetilde{s}(\overline{x}) = \overline{\widetilde{s}}(x)$. The equation respects this condition, for $\overline{V(\Omega)(x)} = V(\Omega)(\overline{x})$, and $\overline{W(\widetilde{s}(x))} = W(\overline{\widetilde{s}}(x))$. Additionally there is the symmetry of an autonomous system; for $x = (j, k)$, define $T_\xi \widetilde{s}(x) = e^{ik\xi}\widetilde{s}(x)$. This is a unitary operator on $\ell^2(\mathbf{Z}^+ \times \mathbf{Z})$. The lattice equation is covariant with respect to T_ξ, indeed T_ξ commutes with $V(\Omega)$, and

$$T_\xi W(\widetilde{s})(x) = W(T_\xi \widetilde{s})(x).$$

Other group actions may also respect the equation (1.1), however these will not be addressed in this paper.

The existence theory is started by solving an approximate problem, given by projection of (3.3) onto a finite subregion of the lattice; $B_0 = \{x \in \mathbf{Z}^+ \times \mathbf{Z}; |x| \le L_0\}$. The approximate problem is solved under conditions of linear nonresonance. Fix a constant $\tau > m + 3$.

Definition 3.1. *Define* $\omega \equiv \omega_{j_0}/k_0$. *The frequency sequence* $\{\omega_j\}_{j=1}^{\infty}$ *is* (d_0, L_0)*–nonresonant with* ω *if* $L_0 >> |j_0| + |k_0|$, *and*

(i) for all $0 < |j| + |k| \le L_0$,

$$|k - \omega j| \ge \frac{d_0}{(|j| + |k|)^\tau}.$$

(ii) For all $(j, k) \ne (j_0, \pm k_0)$, *with* $|j| + |k| \le L_0$,

$$|\omega_j^2 - \omega^2 k^2| \ge d_0.$$

Note: If a sequence of $L_0 \to \infty$, with $d_0 = o(L_0^{-1/2})$, then an open dense set of coefficients $g_1(x)$ are (d_0, L_0)–nonresonant with ω for some L_0. This is a result from [CW].

Writing $\Pi_0 V(\Omega) = V_0(\Omega)$, and $\Pi_0 W(\Pi_0 \widetilde{s}) = W_0(\widetilde{s})$, the approximate equations on $\ell^2(B_0)$ are written

$$V_0(\Omega)\widetilde{s} + W_0(\widetilde{s}) = 0. \tag{3.4}$$

Then the linearized equation about $\widetilde{s} = 0$ is simply

$$V_0(\Omega)(\delta\widetilde{s}) = 0. \tag{3.5}$$

This linear operator has a nontrivial null space for $\Omega = \omega = \omega_{j_0}/k_0$. Since the problem is (d_0, L_0)–nonresonant, the null space is two dimensional, spanned by $\varphi(p) = p\delta(j_0, k_0) + \overline{p}\delta(j_0, -k_0)$, with $p \in \mathbf{C}$. Let $N = \{(j_0, k_0), (j_0, -k_0)\}$, the support of the null vectors, and define orthogonal projections Q onto $\ell^2(N)$, and $P = (\mathbf{1} - Q)$. Equation (3.4) is solved via a Lyapounov-Schmidt decomposition.

$$P\big(V_0(\Omega)u_0 + W_0(\varphi(p) + u_0)\big) = 0 \tag{3.6}$$

$$\begin{aligned} Q\big(V_0(\Omega)\varphi(p) + W_0(\varphi(p) + u_0)\big) &= 0 \\ u_0 &= Pu_0 \end{aligned} \tag{3.7}$$

Define spaces that account for exponential decay of sequences; $\mathcal{H}_\sigma = \{u \in \ell^2(\mathbf{Z}^+ \times \mathbf{Z}); \|u\|_\sigma^2 \equiv \sum_{x \in \mathbf{Z}^+ \times \mathbf{Z}} e^{2\sigma|x|}|u(x)|^2 < \infty\}$. These form a scale of Hilbert spaces, $\mathcal{H}_\sigma \subseteq \mathcal{H}_{\sigma-\gamma}$ for all $0 \le \gamma \le \sigma$. We ask of the nonlinear term that $W \in C^\omega(\mathcal{H}_\sigma : \mathcal{H}_{\sigma-\gamma})$ for all $0 < \gamma \le \sigma < \overline{\sigma}$, with norms

$$\begin{aligned} \|W(u)\|_{\sigma-\gamma} &\le \frac{C_W}{\gamma^{M+1}}\|u\|_\sigma^M \\ \|D_u W(u)v\|_{\sigma-\gamma} &\le \frac{C_W}{\gamma^{M+1}}\|u\|_\sigma^{M-1}\|v\|_{\sigma-\gamma} \\ \|D_u^2 W(u)[w,v]\|_{\sigma-\gamma} &\le \frac{C_W}{\gamma^{M+1}}\|u\|_\sigma^{M-2}\|w\|_\sigma\|v\|_{\sigma-\gamma} \quad . \end{aligned}$$

The Taylor expansion of W takes the form $W(u) = W^{(M)}(u) + W^{(M+1)}(u) + \cdots$, where the term $W^{(J)}$ is J–multilinear in u. We will assume that $W^{(J)}$ is J–multilinear and symmetric in u, although the symmetry is not essential for the existence theorem. The lattice nonlinearity that comes from the nonlinear wave equation satisfies the above conditions.

Lemma 3.2. *For $r_0^m < (d_0/3L_0^2)$, $\rho_0 = r_0$, the equation (3.6) has a solution $u_0(x;p,\Omega)$ which is analytic in a complex ρ_0–neighborhood of the set $\mathcal{N}_0 \equiv \{(p,\Omega); \|p\| < r_0,\ |\Omega - \omega| < r_0^m\}$. Furthermore, for $\overline{\sigma}/2 < \sigma_0 < \overline{\sigma} - 1/L_0$ there is an estimate*

$$\|u_0(x;p,\Omega)\|_{\sigma_0} \le \|p\|^M \frac{3C_W L_0}{d_0}.$$

This solution is covariant with respect to the translations T_ξ,

$$T_\xi u_0(x;p,\Omega) = u_0(x;T_\xi p,\Omega)$$

(where by notational abuse we denote rotations in the p–plane also by T_ξ.)

These sequences form a family of embedded circles, parametrized by $(\|p\|,\xi,\Omega)$, which are solutions of the approximate problem (3.6).

To finish the approximate bifurcation problem, equation (3.7) is also solved. This is in the form of a mapping, taking $(p,\Omega) \in \mathcal{N}_0 \to \mathbf{R}^2$. The zero set of the mapping consists locally of the Ω axis $\{p = 0\}$, and a surface $(p, \Omega_0(p))$ given as a graph over a neighborhood of zero in $\ell^2(N)$. A simple analysis of the Taylor expansion of this mapping determines that

$$\Omega_0(p) = \omega + \lambda_0^{(m)}\|p\|^m(1 + o(\|p\|)). \tag{3.8}$$

A straightforward perturbation expansion, which is left to the reader, will determine the constant $\lambda_0^{(m)}$. If M is odd, then $m = M - 1$ and

$$\lambda_0^{(m)} = \frac{1}{2k_0^2\omega} \frac{\langle \varphi(p) | W_0^{(M)}[(\varphi(p))^M] \rangle}{\|\varphi(p)\|^{M+1}} \quad . \tag{3.9}$$

When M is even, take R to be the least odd index, $M < R \leq 2M - 1$, such that $W_0^{(R)} \not\equiv 0$. If $R < 2M - 1$, then $m = R - 1$, and

$$\lambda_0^{(m)} = \frac{1}{2k_0^2\omega} \frac{\langle \varphi(p) | W_0^{(R)}[(\varphi(p))^R] \rangle}{\|\varphi(p)\|^{R+1}} \quad . \tag{3.10}$$

If $R = 2M - 1$, or there is no such R, then $m = 2M - 2$, and the perturbation theory determines first that

$$u_0^{(M)}(x; p, \Omega) = -\big(PV_0(\Omega)P\big)^{-1} P(W_0^{(M)}[(\varphi(p))^M]) \quad ,$$

and then

$$\begin{aligned} \Omega_0(p) =& \omega + \frac{1}{2k_0^2\omega} \frac{\langle \varphi(p) | W_0^{(2M-1)}[(\varphi(p))^{2M-1}] \rangle}{\|\varphi(p)\|^2} \\ &+ \frac{M}{2k_0^2\omega} \frac{\langle \varphi(p) | W_0^{(M)}[(\varphi(p))^{M-1}, u_0^{(M)}] \rangle}{\|\varphi(p)\|^2} + o(\|p\|^{2M-2}) \quad . \end{aligned} \tag{3.11}$$

This perturbation analysis generalizes the formal results of [KT], regarding solutions of the nonlinear Klein-Gordon equation.

The analog of the 'twist condition' of [CW] is a condition on the nonvanishing of the coefficients $\lambda_0^{(m)}$. This will ensure that the dependence of the frequency of the solution upon the amplitude is sufficiently nondegenerate.

The full nonlinear equations (3.3) are also considered in a Lyapounov-Schmidt decomposition

$$P\big(V(\Omega)u + W(\varphi(p) + u)\big) = 0, \tag{3.12}$$

$$Q\big(V(\Omega)\varphi(p) + W(\varphi(p) + u)\big) = 0. \tag{3.13}$$

The approximate solution u_0 of (3.6) is a close approximation to the full equation (3.12), for it satisfies the estimate

$$\|P\big(V(\Omega)u_0 + W(\varphi(p) + u_0)\big)\|_{\sigma_0 - \gamma_0} \leq \frac{C_W \|p\|^M}{\gamma_0^{M+1}} e^{-\gamma_0 L_0}. \tag{3.14}$$

However, to adjust this approximate solution to a full solution involves the small divisor problem. The exact solution is obtained not over all of the parameter region $\mathcal{N}_0$, but on a closed Cantor subset $\mathcal{N} \subseteq \mathcal{N}_0$, on which the resonances of the problem are under better control. The solutions are obtained using Newton iteration steps in conjunction with approximations of the lattice $\mathbf{Z}^+ \times \mathbf{Z}$ by an increasing family of finite subdomains $B_n = \{x \in \mathbf{Z}^+ \times \mathbf{Z}; |x| \leq L_0 2^n\}$. To state the existence result, fix $1/2 < \eta < 1$.

Theorem 3.3. *Assume that the sequence* $\{\omega_j\}_{j=1}^{\infty}$ *is* (d_0, L_0)*–nonresonant with* ω *for* $d_0 \geq L_0^{-\eta}$, *for* L_0 *sufficiently large. Then there is a constant* r_0, *a sequence* $u(x; p, \Omega) \in \mathcal{H}_{\overline{\sigma}/2}$ *which is* C^∞ *on* $\mathcal{N}_0 = \mathcal{N}_0(r_0)$, *and a Cantor subset* $\mathcal{N} \subseteq \mathcal{N}_0$ *such that for* $(p, \Omega) \in \mathcal{N}$, u *is a solution of (3.12). Furthermore*

$$\|u - u_0\|_{\overline{\sigma}/2} \leq C\|p\|^M e^{-\gamma_0 L_0/2}. \tag{3.15}$$

The second bifurcation equation (3.13) can also be solved, giving a C^∞, T_ξ–invariant solution surface $(p, \Omega(p))$ in addition to the trivial branch of solutions $p = 0$. This solution surface is close to the approximate surface $(p, \Omega_0(p))$, however unless we specify further conditions it will not necessarily intersect the remaining set $\mathcal{N}$, and (3.12) and (3.13) will not be simultaneously satisfied. We ask that in addition to being (d_0, L_0)– nonresonant, the approximate nonlinear problem satisfies a **twist condition**. Then the surface $(p, \Omega(p))$ will intersect $\mathcal{N}$, giving rise to solutions of the full problem (3.3). For the following we fix $0 < \nu < (1 - \eta)$.

Theorem 3.4. *If the approximate problem (3.4) also satisfies the quantitative twist condition* $|\lambda_0^{(m)}| \geq L_0^{-\nu}$, *then the solution surface* $(p, \Omega(p))$ *of (3.13) intersects* $\mathcal{N}$. *Define* $\mathcal{C} = \{0 < r < r_0; \|p\| = r,\ (p, \Omega(p)) \in \mathcal{N}\}$, *the set for which a solution of the full problem is obtained. Then* $\operatorname{meas}(\mathcal{C}) > 0$, *and is in fact of order* r_0.

The intersection points correspond to analytic solutions of the nonlinear wave equation (1.1), through their eigenfunction expansion. This proves Theorem 2.1 of the previous section.

Through exact or near resonance, the Cantor set $\mathcal{C}$ may not have $r = 0$ as an accumulation point, for $(p, \Omega) = (0, \omega)$ may be too resonant, and not in $\mathcal{N}$. However if the frequency sequence $\{\omega_j\}_{j=1}^{\infty}$ is fully nonresonant with ω, then $\mathcal{C}$ does accumulate at zero, and in addition there is an estimate of its density nearby. Let $\overline{\tau} > m + 3$ and $\overline{\alpha} > \overline{\tau} + 1$ be fixed.

Theorem 3.5. *Suppose that a* (d_0, L_0)*–nonresonant sequence* $\{\omega_j\}_{j=1}^{\infty}$ *satisfies the conditions of full nonresonance.*

(i) For all $0 < |(j, k)| < \infty$,

$$|k - \omega j| \geq \frac{d_0}{(|j| + |k|)^{\overline{\tau}}}.$$

(ii) For all $(j, k) \neq (j_0, \pm k_0)$,

$$|\omega^2 k^2 - \omega_j^2| \geq \frac{d_0}{(|j| + |k|)^{\overline{\alpha}}}.$$

Define $\mathcal{C}(r_1) = \mathcal{C} \cap [0, r_1]$. *Then there is an exponent* $\overline{\mu}$ *such that*

$$\operatorname{meas}(\mathcal{C}(r_1)) \geq r_1(1 - C r_1^{\overline{\mu}})$$

for all $0 < r_1 < r_0$.

One can additionally make an estimate of the size of $\overline{\mu}$, there are similar estimates in [CW].

4. Proof of Theorem 3.3

The proof is via a modified Newton iteration scheme, similar to the Nash-Moser method. The major difference is the presence of a null space, and the sensitive parametric dependence of the approximate solutions and the linearized operator. Thus during the iteration, an acceptable set of parameters must be chosen as well, resulting ultimately in the Cantor set $\mathcal{N}$ on which the first bifurcation equation (3.12) is solved. The second bifurcation equation is a finite dimensional mapping. The zero set corresponding to a nontrivial solution is given by a graph $(p, \Omega(p))$, which gives a relationship between the action and the frequency of a solution, called the **frequency map**. This exhibits one of the differences of the present technique from the more classical versions of the KAM theorem, in which the problem is assumed nonlinearly nondegenerate, the frequency map is performed first, and only then does the analysis of the invariant sets take place.

An outline of the iteration is as follows. We choose:

(1) A sequence of length scales $L_n = L_0 2^n$ which define the approximating domains $B_n = \{|x| \leq L_n\}$ which exhaust $\mathbf{Z}^+ \times \mathbf{Z}$.

(2) A sequence of tolerances for small divisors (small eigenvalues) $\delta_n = L_n^{-\alpha}$, for a suitable $\alpha > 0$.

(3) A sequence of lengths $\ell_n = L_n^\beta$ over which linear resonances are decoupled.

(4) A sequence $\gamma_n = c_0/(n+1)^2$ which governs loss of exponential decay of the approximate solutions throughout the the iteration.

(5) And a rapidly convergent sequence $\epsilon_n = \epsilon_0^{\kappa^n}$, for $1 < \kappa < 2$, which will bound the error terms during the iteration.

The size of the error is dominated by a rapidly convergent sequence as the iteration scheme has quadratic errors; this is the usual phenomenon with the Nash-Moser technique.

The major issue to contend with is the invertibility of relevant linearized operators. Let $B \subseteq \mathbf{Z}^+ \times \mathbf{Z}$ be a subdomain of the lattice. We define the **Hamiltonian operator** on $\ell^2(B)$ by

$$H_B(p, \Omega; u) = \big(V(\Omega) + D_u W(\varphi(p) + u)\big)_B \quad .$$

The subscript B denotes the restriction of the operators to $\ell^2(B)$. Invertibility depends crucially upon the small spectra of the operator $V_B(\Omega)$, as the following result demonstrates.

Lemma 4.1. *Let $A \subseteq \mathbf{Z}^+ \times \mathbf{Z}$ be a domain such that $|V(\Omega)(x,x)| > d_0$ for all $x \in A$. Then for $r_0^{m-1}/d_0 << 1$ the Green's function*

$$G_A(x,y) = \big(V(\Omega) + D_u W(\varphi(p) + u)\big)_A^{-1}(x,y)$$

satisfies the estimate

$$\|G_A\|_{\sigma_0} \leq \frac{C}{d_0} \quad .$$

We call a lattice site $s \in \mathbf{Z}^+ \times \mathbf{Z}$ **singular** if $|V(\Omega)(s,s)| < d_0$, and regular otherwise. Connected regions of singular sites are called singular regions. The wave equation has singular regions consisting of either isolated sites, or else two adjacent sites, $S = \{s_1, s_2\}$, with $s_1 = (j,k)$, $s_2 = (j+1,k)$. We will consider local Hamiltonians defined on neighborhoods of singular regions. Let $S \subseteq B_{n+1}\backslash B_n$ and $C_{\ell_{n+1}}(S) = \{x; \mathrm{dist}\,(x,S) \leq \ell_{n+1}\}$. We will be concerned with the operators $H_S(p,\Omega;u_n)$ and $H_{C_{\ell_{n+1}}(S)}(p,\Omega;u_n)$.

The proof of Theorem 3.3 is by induction on the following statements.

$(n.1)$ There is a sequence $u_n(x;p,\Omega) = u_0(x;p,\Omega) + \sum_{j=0}^n v_j(x;p,\Omega)$ in $\ell^2(B_{n+1})$, which is C^∞ on $\mathcal{N}_0$, analytic in a $\delta_{n+1}r_0/L_n^2$ complex neighborhood of $\mathcal{N}_{n+1}$ such that

$$\|P(V(\Omega)u_n + W(\varphi(p)+u_n))\|_{\sigma_n} \leq \|p\|^M \epsilon_n$$
$$\|v_n\|_{\sigma_n-\gamma_n} \leq \frac{C_G^n \epsilon_n}{\delta_{n+1}\gamma_n^s}\|p\|^M$$

for some fixed constant s.

$(n.2)$ There exists a closed domain $\mathcal{N}_{n+1} \subseteq \mathcal{N}_n \subseteq \cdots \mathcal{N}_0$ with the following properties.

(i) If $(p,\Omega) \in \mathcal{N}_{n+1}$, and $S_1, S_2 \subseteq B_n^c$ are any two singular regions, then

$$\mathrm{dist}\,(S_1,S_2) > 2\ell_{n+1} \quad .$$

(ii) If $(p,\Omega) \in \mathcal{N}_{n+1}$, and S is a singular region in $B_{n+1}\backslash B_n$, then

$$\mathrm{dist}\,(\mathrm{spec}\,(H_S(p,\Omega;u_n)),0) > \delta_{n+1}$$
$$\mathrm{dist}\,(\mathrm{spec}\,(H_{C_{\ell n+1}(S)}(p,\Omega;u_n)),0) > \delta_{n+1} \quad .$$

(iii) Any C^∞, T_ξ–invariant surface $\Omega(p) = \omega + \lambda\|p\|^m(1+o(\|p\|))$, with $|\lambda| > L_0^{-\nu}$ intersects $\mathcal{N}_{n+1}$ with nonzero measure;

$$\mathrm{meas}\,(\{r \in [0,r_0); \|p\| = r, (p,\Omega(p)) \in \mathcal{N}_{n+1}\}) \geq r_0(1 - Cr_0^\mu) \quad .$$

A consequence of $(n.2)(i)(ii)$ is that the Green's function for any $E \subseteq B_n\backslash N$ is controlled on the parameter region $\mathcal{N}_{n+1}$.

Lemma 4.2. *Let A be a nonsingular region, and $E \subseteq (B_{n+1}\backslash N) \cup A$. The Green's function satisfies*

$$\|G_E(p,\Omega;u_n)\|_{\sigma_n} \leq \frac{C_G^n}{\delta_{n+1}\gamma_n^s} \quad ,$$

and under perturbations of u_n of size $\|u-u_n\|_{\sigma_n-\gamma_n} \leq \|p\|^M\epsilon_n/\delta_{n+1}\gamma_n^s$,

$$\|G_E(p,\Omega;u)\|_{\sigma_n-2\gamma_n} \leq \frac{2C_G^n}{\delta_{n+1}\gamma_n^s} \quad .$$

Proof. The proof is the same as in [CW], Section 5. The arguments involve the decoupling of the local Hamiltonians at singular regions of $B_{n+1}\backslash N$. As long as the spectra of the local Hamiltonians are controlled, and the singular regions are sufficiently separated, resolvant expansions can be employed to recover the full Green's function. □

Induction step $(n.1)$ will follow from $((n-1).2)(i)(ii)$ and Lemma 4.2. Indeed the Newton iteration step is

$$v_{n-1} = -\,G_{B_n\backslash N}(p,\Omega;u_{n-1})\big(V(\Omega)u_{n-1} + W(\varphi(p)+u_{n-1})\big)_{B_n\backslash N} \quad ,$$
$$u_n = u_{n-1} + v_{n-1} \quad .$$

With this definition of u_n the Taylor remainder theorem will exhibit a quadratic error, and the error due to domain truncation will be exponentially small if some decay is sacrificed. We again refer to [CW] for details of the convergence proof.

The remaining task is to realize a large set of parameters $\mathcal{N}_{n+1} \subseteq \mathcal{N}_n$ such that $(n.2)$ is satisfied. Conditions (i) and (ii) decrease the size of $\mathcal{N}_n$, while condition (iii) requires that it be sufficiently large, and further satisfy certain geometrical properties related to the order of contact of the nonlinear degeneracy. Central to the verification of this induction step is a lemma on eigenvalue perturbation theory for the local Hamiltonians. For the case M even we introduce an additional hypothesis on the lattice nonlinearity W, a restriction on the self-interaction of the system within a singular region. It will always be satisfied for the nonlinear wave equation. The case M odd has no such requirement.

Hypothesis: If $z, w \in S$ a singular region, then for $M \leq J < R$,

$$\langle \delta(z) | D_u^J W(0)[(\partial_p \varphi(p))^{J-1}, \delta(w)] \rangle = 0 \quad .$$

Consider a self adjoint operator $H(a)$ depending upon a parameter a, and suppose an eigenvector–eigenvalue pair $(\psi(a), e(a))$ of $H(a)$ is smooth. Then

$$\partial_a e(a) = \langle \psi(a) | \partial_a H(a) | \psi(a) \rangle \quad , \tag{4.1}$$

which is known as the Feynman–Hellman formula.

Lemma 4.3. *Let $(\psi(p,\Omega), e(p,\Omega))$ be an eigenvector–eigenvalue pair for $H_{C(S)}(p,\Omega)$. Then*

$$|\langle \psi(p,\Omega) | \partial_\Omega H_{C(S)}(p,\Omega) | \psi(p,\Omega) \rangle| \geq C_1 L_n^2 \quad . \tag{4.2}$$

For (p,Ω) satisfying $(n+1.2)(i)$,

$$|\langle \psi(p,\Omega) | \partial_p H_{C(S)}(p,\Omega) | \psi(p,\Omega) \rangle| \leq C_2 \|p\|^{m-1}. \tag{4.3}$$

Let $e(p,\Omega)$ be an eigenvalue of a local Hamiltonian (labeled by ordering), and Z be the set in $\mathcal{N}_0$ on which $e(p,\Omega)$ vanishes. Z is given by a graph $(p, \Omega_Z(p))$, and if $(p_1, \Omega_Z(p_1)), (p_2, \Omega_Z(p_2))$ are nearby points satisfying $(n.2)(i)$, then

$$|\Omega_Z(p_2) - \Omega_Z(p_1)| \leq \frac{C_3}{L_n^2} \big| \|p_2\|^m - \|p_1\|^m \big| \tag{4.4}$$

The proof of this is similar to Lemma 4.14 of [CW].

This result allows us to control the excisions of parameters in order to satisfy $(n.2)(iii)$. Consider a T_ξ invariant surface $(p, \Omega(p))$, with $\Omega(p) = \omega + \lambda \|p\|^m (1 + o(\|p\|))$, and $|\lambda| > L_0^{-\nu}$. Let $S \subseteq B_{n+1} \backslash B_n$ be a singular region, and $e(p,\Omega)$ an eigenvalue of a local Hamiltonian $H_{C(S)}$. Suppose that for some p_1, $e(p_1, \Omega(p_1)) = 0$, and that $(p_1, \Omega(p_1))$ satisfies $(n.2)(i)$. We are concerned with nearby points on the surface $(p, \Omega(p))$. In order to inductively construct the next set $\mathcal{N}_{n+1}$ a δ_{n+1}/L_n^2–neighborhood of Z is excised from $\mathcal{N}_n$. If the point $(p, \Omega(p))$ is excised in this process, then

$$\begin{aligned}\frac{\delta_{n+1}}{L_n^2} &\geq |\Omega(p) - \Omega_Z(p)| \\ &\geq |\Omega(p) - \Omega(p_1)| - |\Omega_Z(p_1) - \Omega_Z(p)| \\ &\geq (|\lambda/2| - C_3/L_n^2)\big|\|p_2\|^m - \|p_1\|^m\big| \quad .\end{aligned}$$

Hence any p such that $\|p - p_1\|^m > (4/|\lambda|)(\delta_{n+1}/L_n^2)$ is not excised, and $|e(p, \Omega(p))| > \delta_{n+1}$. Lemma 4.3 provides the main result needed to verify the induction statement $(n.2)(iii)$, and with some patience the convergence proof will follow.

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