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Critical Percolation in Finite Geometries

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ABSTRACT: The methods of conformal field theory are used to compute the crossing probabilities between segments of the boundary of a compact two-dimensional region at the percolation threshold. These probabilities are shown to be invariant not only under changes of scale, but also under mappings of the region which are conformal in the interior and continuous on the boundary. This is a larger invariance than that expected for generic critical systems. Specific predictions are presented for the crossing probability between opposite sides of a rectangle, and are compared with recent numerical work. The agreement is excellent.

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Conformal field theory has been very successful in determining universal quantities associated with two-dimensional isotropic systems at their critical points [1,2]. The range of predictions which can be made appears to be bounded by the enthusiasm and industriousness of the theorist rather than by any intrinsic limitations of the theory. However, the underlying assumptions of conformal field theory, and their appropriateness for describing the scaling limit of critical lattice systems, are not rigorously founded, and it remains important to perform precise numerical tests of the theory whenever possible.

Recently [3], extensive numerical work has been carried out to estimate crossing probabilities in rectangular geometries for critical percolation in very large but finite lattices, with the principal aim of establishing their universality between different models. Percolation provides an important test of the ideas of conformal field theory because large-scale numerical simulations are more readily performed. In this letter we consider the general problem of crossing probabilities in the language of conformal field theory, and derive exact expressions which may be compared with the numerical work.

The most familiar way to think about percolation as a critical phenomenon is through the $q \to 1$ limit of the q-state Potts model [4]. In that model, spins $s(r)$ at the sites of the lattice are allowed to be in one of q possible states $(\alpha, \beta, \ldots)$, and the partition function is the trace of a product over links of the form

$$Z = \prod_{(r,r')} \left(1 + x\delta_{s(r),s(r')}\right) \tag{1}$$

The terms in the expansion in powers of Z in powers of x are in 1-1 correspondence with configurations of bonds appearing in the bond percolation problem, and in the limit $q \to 1$ they are weighted appropriately if $x = p/(1-p)$. Two sites in the same cluster are necessarily in the same state of the Potts model. Consider now two disjoint segments S_1 and S_2 of the piecewise differentiable boundary of a simply connected compact region. Let $Z_{\alpha\beta}$ be the partition function of the q-state Potts model with the constraint that all spins at lattice sites on S_1 are fixed in the state α, and all the spins on S_2 are fixed in the state β. The rest of the boundary spins are unrestricted. Then the crossing probability between S_1 and S_2 is given by

$$\pi(S_1, S_2) = \lim_{q\to 1} \left(Z_{\alpha\alpha} - Z_{\alpha\beta}\right) \tag{2}$$

where, in the second term $\alpha \neq \beta$. In fact, in the limit when $q = 1$, the first term is unity.

The interior of the compact region may be mapped conformally to the upper half plane, so that the boundary is mapped onto the real axis. If there are corners on the boundary, the map will be singular but continuous at these points. Conformal field theory relates the partition functions in the two geometries, in a manner to be described later. Thus, if the images of S_1 and S_2 are the intervals (x_1, x_2) and (x_3, x_4) respectively (where we may assume that the x_i are placed in increasing order), the problem reduces to that of finding the respective partition functions $Z_{\alpha\alpha}$ and $Z_{\alpha\beta}$ in this geometry.

The study of boundary conditions in conformal field theory [5,6] shows that, for a particular theory, there is a given set of boundary conditions consistent with the conformal symmetry of the theory. In general they correspond to the possible fixed points of the renormalization group in the semi-infinite system: thus a generic boundary condition becomes equivalent in the continuum limit to one of those allowed by conformal symmetry. In addition, points on the boundary at which the boundary condition changes may be identified [6] with points of insertion of boundary operators, that is, scaling operators of the theory corresponding to highest weight states of the Virasoro algebra [7,5]. Situations where more than one change of boundary condition occurs then correspond to correlation functions of these boundary operators. In the case in question, let us denote the boundary condition where the spins are free by (f), and those where they are fixed in a given state by (α). Denote the boundary operator corresponding to a switch from boundary condition (i) to (j) at the point x by $\phi_{(i|j)}(x)$. Then the partition functions we need are given in terms of correlators by

$$\begin{aligned} Z_{\alpha\alpha} &= Z_f \,\langle \phi_{(f|\alpha)}(x_1)\,\phi_{(\alpha|f)}(x_2)\,\phi_{(f|\alpha)}(x_3)\,\phi_{(\alpha|f)}(x_4)\rangle \\ Z_{\alpha\beta} &= Z_f \,\langle \phi_{(f|\alpha)}(x_1)\,\phi_{(\alpha|f)}(x_2)\,\phi_{(f|\beta)}(x_3)\,\phi_{(\beta|f)}(x_4)\rangle \end{aligned} \tag{3}$$

where Z_f is partition function with free boundary conditions all along the real axis. Note that, in the upper half plane, all three partition functions in general diverge in the infinite volume limit, and Eq. (3) strictly should be interpreted as being valid only for a large but finite lattice. However, when $q = 1$, $Z_f = 1$ identically, and this problem does not arise.

In order to compute the above correlation functions using the methods of conformal field theory, we need to understand to which representations of the Virasoro algebra the boundary operators belong. It has been known for some time [8,9] that the critical q-state Potts model corresponds, in the continuum limit, to a conformal field theory with conformal anomaly number [7] $c = 1 - 6/m(m+1)$, where $q = 4\cos^2(\pi/m+1)$, with $m \geq 1$. Thus percolation has $c = 0$. This is consistent with the fact that c is related to the finite-size corrections to the free energy [10] in certain geometries, and the free energy vanishes identically when $q = 1$. However, the problem of boundary operator assignment has not been addressed so far, except for the cases $q = 2$ and $q = 3$ [11,6,12]. However, it is not difficult to determine the assignment for the operators $\phi_{(f|\alpha)}$. For minimal conformal field theories, all the scaling operators have the property that their corresponding representations contain null states [7]. This has the consequence that the allowed values of their scaling dimensions are given by the Kac formula

$$h = h_{r,s} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)} \tag{4}$$

where r and s are positive integer are positive integers. In addition, the correlators involving these operators obey differential equations of order at most rs. For unitary models, for example those with positive definite Boltzmann weights, all allowed operators must be of this type. Although this condition is not applicable to the q-state Potts model for general q, the fact that it does apply for $q = 2$ and $q = 3$ suggests that those operators whose position (r, s) in the Kac table does not appear to change as a function of $c(q)$ do correspond to representations with null states even in the non-unitary case. Indeed, it was conjectured in Ref. 5 that the spin operator of the Potts model, when inserted at a boundary with free boundary conditions, corresponds to $(r, s) = (1, 3)$. This agrees with known results for $q = 2, 3$ [11,6,12], and is also consistent with the known assignment of operators in the bulk. It gives a prediction for the case $q = 1$ which agrees with numerical estimates to within their accuracy [13]. There are $(q - 1)$ independent such spin operators.

The continuum limit of duality symmetry [14] for the critical q-state Potts model maps the free boundary condition (f) onto a fixed boundary condition (α). Exactly which state α is chosen is arbitrary, since just one spin on the boundary has to be assigned a given value in order to make the duality mapping 1-1. An insertion of the spin operator at the point x on the free boundary is mapped into an insertion of the disorder operator $\phi_{(\alpha|\beta)}(x)$ where $\beta \neq \alpha$. There are just $(q - 1)$ such operators, for a fixed α. This duality symmetry implies that that the correlators of $\phi(\alpha|\beta)$ are simply related to those of the boundary spin operator with free boundary conditions, and hence that it also corresponds to $(r, s) = (1, 3)$

in the Kac table. However, we are interested in the operators $\phi_{(\alpha|f)}$. Consider the insertion of two such operators $\phi_{(\alpha|f)}(x)\phi_{(f|\beta)}(x')$ as the points x, x' approach each other. This is given by the operator product expansion, which symbolically must have the structure

$$\phi_{(\alpha|f)} \cdot \phi_{(f|\beta)} \sim \delta_{\alpha\beta}\,\mathbf{1} + \phi_{(\alpha|\beta)} + \cdots \tag{5}$$

where $\mathbf{1}$ is the identity operator (no change in boundary condition). According to the fusion rules of conformal field theory [7], there is one such operator in the Kac table which has such a simple operator product expansion with itself, namely $(r,s) = (1,2)$. We therefore conjecture that this is the correct assignment for the operators $\phi_{(f|\alpha)}$, for general q. This agrees with the known results for $q = 2$ and $q = 3$ [11,6,12]. It implies that the correlators involving these operators satisfy second order differential equations. From the Kac formula Eq. (4), we see that, in the limit $q \to 1$, their scaling dimensions are given by

$$h = h_{1,2}(0) = 0 \quad . \tag{6}$$

This vanishing of the scaling dimension will turn out to have remarkable consequences when the result is transformed back into the original geometry. In fact, it has a natural explanation. Consider a compact region on whose boundary there is a single segment S_1 on which the Potts spins are fixed into the state α. On the remainder of the boundary, the spins are free. In the limit $q \to 1$, the partition function in this geometry is equal to unity, and equal to Z_f, since in either case any spin can only be in a single state. But the ratio of these partition functions is equal to the correlation function $\langle\phi_{(f|\alpha)}\phi_{(\alpha|f)}\rangle$, which in general will scale like distance to the power $2h$. This is only consistent if $h = 0$. However, the form of the four-point functions, although simplified by this result, is nontrivial. Consider the half plane geometry, when the points lie along the real axis. Conformal invariance implies [7] that they are of the form $F(\eta)$, depending only on the invariant cross-ratio $\eta = (x_4 - x_3)(x_2 - x_1)/(x_3 - x_1)(x_4 - x_2)$. The absence of other prefactors multiplying F is a consequence of $h = 0$. The fact that the correlators satisfy second order differential equations implies that $F(\eta)$ satisfies a Riemann equation, whose general solution is [7,5]

$$F = P \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & -4h_{1,2} & 0 & \eta \\ h_{1,3} & -4h_{1,2} + h_{1,3} & h_{1,3} & \end{array} \right\} \tag{7}$$

Which solution is chosen depends on whether we calculate $Z_{\alpha\alpha}$ or $Z_{\alpha\beta}$. Although is straightforward to solve this problem for arbitrary q, we restrict ourselves to $q = 1$ for simplicity. In that limit, one of the solutions of the Riemann equation reduces to a constant, and the second solution is proportional to $\eta^{1/3}{}_2F_1(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \eta)$. The combination corresponding to $Z_{\alpha\beta}$ is determined by the requirement that as $(x_3 - x_2) \to 0$, that is $\eta \to 1$, the operator product expansion Eq. (5) requires that the solution vanish like $(1-\eta)^{1/3}$. In addition, in the opposite limit $\eta \to 0$, we expect that $Z_{\alpha\beta} \to Z_{\alpha\alpha} = 1$. Using simple identities on hypergeometric functions, we then find for the crossing probability

$$\begin{aligned} \pi((x_1, x_2), (x_3, x_4)) &= \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} \eta^{1/3}{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; \eta) \\ &= 1 - \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} (1-\eta)^{1/3}{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; 1-\eta) \end{aligned} \quad . \tag{8}$$

Now consider the transformation of the upper half plane onto the interior of a simply connected compact region by a conformal mapping $z \to w$. If the boundary of the region is a differentiable curve, this mapping may be taken to be conformal also on the boundary. In that case, correlation functions of operators on the boundary transform in the standard manner summarized by the formula

$$\langle \phi_1(w_1)\phi_2(w_2)\ldots\rangle = \prod_i |w'(z_i)|^{-h_i} \langle \phi_1(z_1)\phi_2(z_2)\ldots\rangle \tag{9}$$

where the correlation functions on the left and right hand sides refer to the new and the old geometry respectively, and h_i is the scaling dimension of ϕ_i. In our case, however, since $h = 0$, no such prefactors arise, and the correlation function is truly invariant. For a general critical system, the partition function for a compact region (without any operator insertions) is not itself scale invariant, but picks up a factor $(L/L_0)^{ac}$ where L has the dimensions of length and gives the overall size of the region, L_0 is some non-universal microscopic scale (*e.g.* the lattice spacing), and a is geometry dependent [10]. However, for the case of percolation, $c = 0$ and $Z = 1$, so such effects are absent. In general, there is a further complication when the boundary of the compact region is only piecewise differentiable, and boundary operators happen to sit at the corners. In this case Eq. (9) does not apply. Instead there appear additional non-scale invariant factors of the form $(L/L_0)^{-(\pi/\gamma)h}$, where γ is the interior angle at the corner. Such factors have been treated explicitly for the Ising model with various boundary conditions [15]. However, once again, since $h = 0$ for the problem at hand, such factors are absent. We conclude that crossing probabilities are indeed invariant under mappings which are conformal in the interior and are piecewise conformal on the boundary, but that this is not generic for all critical systems, for example when $q \neq 1$.

As an example, consider the case treated in Ref. 3 of the crossing probability between opposite sides of a rectangle of aspect ratio r. This is the image of the upper half plane under a Schwartz-Christoffel transformation. Taking the points x_j to be at $(-k^{-1}, -1, 1, k^{-1})$, the aspect ratio of the rectangle is given by $r = K(1-k^2)/2K(k^2)$, where $K(u)$ is the complete elliptic integral of the first kind.

The prediction is then that the crossing probability is given by Eq. (8), with $\eta = ((1-k)/(1+k))^2$. The results of this are illustrated in Figs. 1,2, and compared with the numerical data obtained in Ref. 3 for bond percolation on square lattices with approximately $4 \cdot 10^4$ sites. It is seen from Fig. 2 that the deviations between the numerical experiment and the theory are consistent with the internal scatter of the data, although there appears also to be a systematic difference which may be due to finite-size effects.

It is possible to generalize the above methods to treat the case of correlations between different crossing events. As long as the segments involved are not adjacent, such probabilities may always be related to correlation functions of $\phi_{(f|\alpha)}$ operators, and they should have the same invariance properties as the simple crossing probabilities considered here. The fact that they enjoy these properties, which are not expected to hold for analogous quantities in generic critical systems, suggests that some of the ideas of conformal invariance might usefully be reformulated for the percolation problem without invoking the mapping to the Potts model.

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Figure Captions

1) Theory *vs.* the numerical data of Ref. 3 for the horizontal crossing probabilities $\pi_h(r)$ for rectangles of aspect ratio r. In the figure, $\mathrm{Ln}\,((1-\pi_h)/\pi_h)$ is plotted against $\mathrm{Ln}\,r$. The numerical data is represented by by points, and the solid curve is the theoretical prediction.

2) Deviation between numerical estimates and theoretical predictions of crossing probabilities $\pi_h(r)$ and $1-\pi_v(r)$.

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