Low Temperature Stability of Nonperiodic Structures

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Abstract. Complex ground states appearing in classical lattice gas models and their behaviour in the presence of thermal motions are discussed. Our examples consist of a nearest-neighbour model based on nonperiodic tilings of the plane and an Ising type model with a fractal ground state.

One of the important goals of nonlinear dynamics is to discuss complex structures arising from simple deterministic rules. Here we are concerned with an equilibrium or static version of this problem. Namely, the main question is as follows: How complicated structures can one get as a result of standard minimization procedures of equilibrium statistical mechanics? Our models consist of systems of many particles interacting by short-range deterministic translation-invariant potentials. The structures we are interested in are ground states minimizing the energy density of a system or equilibrium phases minimizing its free energy density.

To be more precise, we will consider here classical lattice gas models in which every site of a lattice can be occupied by one of several different particles. Particles interact through two-body (many-body) translation-invariant potentials $f(x-y)$. Configurations of particles on an infinite lattice minimizing the potential energy density of a Hamiltonian $H=\sum_{(x,y)} f(x-y)$ are called ground state configurations. It is an old and still unsolved problem, the so called crystal problem, to deduce periodicity of low temperature matter, i.e., periodicity of ground state configurations of systems of many interacting particles [1, 2]. It is a converse of the crystal problem which is discussed here: How complicated ground state configurations can short-range, deterministic (as opposed to random interactions appearing for example in spin glasses) translation-invariant interactions produce? We restrict ourselves to systems in which, although all ground state configurations are nonperiodic, there is a unique translation invariant measure (called a ground state) supported by them. This is in analogy with the Ising antiferromagnet, where there are two checkerboard ground state configurations but only one ground state measure which is just their average. The unique ground state is then inevitably a zero temperature limit of an infinite-volume grand canonical probability distribution. The uniqueness of a ground state is also a signature of nondegeneracy and genericity of a system.

One of our models is based on Robinson's tiles [3, 4]. There is a family of 56 square-like tiles such that using an infinite number of copies of each of them one can tile the plane only in a nonperiodic fashion. This can be translated into a lattice gas model in the following way first introduced by Radin [5, 6, 7, 8]. Every site of the square lattice can be occupied by one of the 56 different particles-tiles. Two nearest neighbour particles which do not "match" contribute positive energy; otherwise the energy is zero. Such a nearest-neighbour translation-invariant lattice gas model obviously does not have any periodic ground state configurations. In fact, there is a one-to-one correspondence between its ground state configurations and Robinson's nonperiodic tilings. The model has a unique ground state which is highly ordered (arbitrarily distant regions are correlated). Let us emphasize that the nonperiodic global order is forced here by local translation-invariant interactions.

In another class of models we have constructed a one-dimensional Ising spin model [9] with $\sigma_i = \pm$ at each lattice site i and a translation-invariant four-body exponentially decaying interaction:

$$H = \sum_{i=-\infty}^{\infty} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} exp[-(r+p^2](\sigma_i + \sigma_{i+2^r})^2(\sigma_{i+(2p+1)2^r} + \sigma_{i+(2p+2)2})^2$$

A unique ground state of this model has two unusual properties: it has a continuous spectrum and perfect fractal symmetry. The ground state is supported by nonperiodic Thue-Morse sequences. A Thue-Morse sequence can be constructed by a repetitive application of the following substitution rules: $1 \rightarrow 1\ -1$ and $-1 \rightarrow -1\ 1$

At positive temperatures, in the presence of particles thermal motions (an analog of a noise in dynamical systems), to construct stable equilibrium phases of a system of many interacting particles one has to minimize not the energy but the free energy of a system taking into account disruptions due to entropy.The main question here is: Will complex structure present in ground states be preserved at positive temperatures or it will be smeared out by chaotic motions?

Low temperature behaviour of the above described nearest neighbour model was investigated in [10, 11, 12]. The existence of a decreasing sequence of temperatures T_n was proved, such that if $T < T_n$ then there exists an equilibrium state with the period at least 2^n. We believe that the sequence of temperatures accumulates at zero temperature giving rise to an infinite number of phase transitions: period doubling of equilibrium phases. We are currently investigating a three-dimensional version of this model which uses Penrose tiles [4]. We conjecture that in this model the sequence accumulates at some positive temperature giving rise to a quasiperiodic equilibrium phase. This is especially interesting with the connection of recently discovered quasicrystals.

Concerning our second model described here: There exists a summable interaction (an interaction energy of any particle with all other particles is finite) with the Thue-Morse measure as its only ground state and an equilibrium state with only nonperiodic Gibbs states in its extremal decomposition [13].

In conclusion let us mention some open problems. It is important to investigate how generic models without periodic ground states are? Does forced nonperiodicity occur only at isolated points or it can be present in an open ball in some space of interactions? See however [14, 15] that generically (countable intersection of dense open sets), summable interactions produce nonperiodic but ordered ground states. Another problem is to construct a classical lattice gas model with a finite range translation-invariant interaction without periodic ground state configurations and without periodic extremal

Gibbs states - a microscopic model of a quasicrystal.

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