Recent Progress in Classical Mechanics

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The goal of this lecture is to review several developments in classical mechanics that have taken place in the last years, that will fit in the time of the talk and that I have become aware of. Unfortunately, the latter is a constraint more severe than what I would like and I apologize to the authors and the audience for many things that have been left out. In particular, I have left out topics such as "twist mappings", or "geometric phases" and "quantum chaos" that are generating a great deal ofactivity in the literature.

Geometric theory of integrable systems

One of the central problems of mechanics has been to integrate Hamilton's equations of motion, or at least decide if such an integration is impossible.

The most geometrically natural notion of integrability is that the system should have as many conserved quantities with vanishing Poisson brackets as degrees of freedom.

It has long been known that the fact that a system is integrable severely restricts the topology of the phase space and of the energy surface. For example, if the system admits action-angle variables — which is strictly stronger than being integrable in the above sense — the phase space should be $\mathbf{R}^n \times \mathbf{T}^n$ so that the topology is determined.

The first foothold in the geometric theory of integrable systems is the Liouville/Arnol'd theorem that says that the system can be decomposed in pieces $\mathbf{R}^{n_i} \times \mathbf{T}^{m_i}$, but the n_i, m_i can change. For example in the Kepler system we have bounded trajectories along ellipses and unbounded ones along hyperbolas. These two sets get "glued" in the intermediate set of parabolic trajectories.

An important realization was that the gluing of the different pieces has to be done in very precise ways so as to preserve the very rigid structure imposed by integrability. Hence, the phase space of an integrable system can be considered as pieces of $\mathbf{R}^{n_i} \times \mathbf{T}^{m_i}$ glued in very precise ways. This turns out to impose severe restrictions on the phase spaces and energy surfaces of integrable systems. (Notice that having standard pieces that get glued in well defined ways according to the singular submanifolds of a vector field is somewhat reminiscent of Morse Theory.)

This beautiful theory, which one may start to learn in the paper [Fo] written by the indefatigable leader of the method, has many spin-offs.

Since one of the best known methods to generate integrable systems has been to create systems in manifolds with lots of symmetries — e.g., groups or quotients of groups — it is possible to obtain many results about topological consequences of algebraic structures. The machinery of cutting and pasting manifolds while preserving a rich geometric structure can be used to generate examples and counterexamples in low dimensional topology. One of the methods of choice to attempt the classification of low dimensional manifolds has been to show that on them one can define canonical geometric structures that produce invariants, and, one hopes, provide with methods to show they are equivalent. Once one has an integrable system on a manifold, it is very easy to generate other types of structures. Hence, these methods have tested the boundaries of what geometric structures determine.

Analytically integrable systems

On the other side of the spectrum one can try to decide which systems can be integrated using only algebraic or rational transformations.

For example, if one takes particles on a line which are very repulsive, the resulting system will be integrable [Hub] [Gu]— essentially, the integration is achieved by the scattering operator — nevertheless, the Calogero potential is special because it can be integrated by algebraic methods.

There are several criteria to show that a system cannot be integrated by algebraic methods. The most time honored is the one used by S. Kowaleska to narrow down the search of algebraically integrable cases for the rigid body to a few particular ones. (Basically, we observe that an integrated system has no singularities and that an algebraic integration can only introduce algebraic singularities. The possible singularities of the system under study can be ascertained by solving the differential equation.)

A more recent method for complex integrability has been introduced by Ziglin, based on the observation that if a system is integrable, the periodic orbits can be deformed, along complex paths. This imposes contraints on the algebraic properties of the variational equations.

To my knowledge, there are no definitive results about converses of these negative criteria. One would like to know if an algebraic system satisfying Kowaleska criterion, and maybe some other global condition, is algebraically integrable.

A much more severe obstruction to the possibility of integrability is the existence of homoclinic points. The verification of this hypothesis usually is done by perturbation theory. We mention that in several cases, this is quite difficult since the perturbation expansions vanishes to all orders.

A very easy to read review of the developments in the physical literature is [RGB]. See also the corresponding chapters in [AANS], [Koz], [Mor].

One could wonder why one should worry about the difference between algebraic integrability versus, say C^{50}. Let me mention two reasons. One is that there are several structures that only manifest themselves when we consider complex extensions. (We will discuss one of those when we discuss perturbations beyond all orders.) Another is that there are many natural systems that are algebraic, for example, one step of algorithms acting on matrices can be considered as an algebraic dynamical system. It has been known for a long time that there were important similarities between the Jacobi algorithm to diagonalize matrices and the scattering of particles on a line. This has been used to analyze in quite detailed fashion the Jacobi algorithm and in turn this has lead to implementations that soon will improve in many respects the now standard algorithms [DLT]. ¿From the mathematical point of view, the theory of algebraically integrable systems has important connections with algebraic geometry. For example, Hilbert's 21^{st} problem is related to issues raised by Ziglin's method.

Asymptotics beyond all orders

There are several interesting quantities in classical mechanics that, do not vanish even if a perturbation expansion yields identically zero.

It has been known for a long time these perturbation expansions (physicists called them "divergent" even if they are the sum of zero term) into upper bounds. In a typical situation, by truncating the series to order N we obtain $|Q| \le \varepsilon^N N^N$. Then, if we take $N = (\varepsilon e)^{-1}$ we obtain $|Q| \le e^{-1/\varepsilon e}$.

Such upper bounds appear very frequently. See for example [Nek] for transport, [Ne2], [FS] for splitting of separatrices. Getting lower bounds seems much harder. Nevertheless, there is a trick that has yielded results in several cases. If we consider complex extensions of radii $O(1/\varepsilon)$ for the objects under consideration, perturbation theory will produce results $Q(x) = A(x)\varepsilon + O(\varepsilon^2)$. Then, using Cauchy estimates, it is possible to prove lower bounds of the right order of magnitude.

Needless to say, this broad sketch does not do justice to the difficulty of concrete situations. Some recent examples are the treatment of the Landau-Zener formula for the corrections to the adiabatic limit [Ha], [JaSe] or the crossing of separatrices in two dimensional twist mappings, [An], [ACKR] a problem that has plagued the literature for several years.

Let us point out that the upper bounds in [Nek] do not have a corresponding set of lower bounds. This is the famous problem of Arnol'd diffusion and it involves, besides analytical lower bounds, a good understanding of several geometric structures. It is known that this phenomenon occurs in examples [Ar1] and there are now more systematic constructions [Do] that show it is generic. Nevertheless, this is a far cry from being able to decide whether a concrete system presents it or not or calculate its magnitude.

Let us point out that in this problem of Nekhorosev bounds/Arnol'd diffusion there has been significant recent progress. For upper bounds, the paper [Nek] was significantly clarified in [BGG] and a radically different proof has appeared [Lo]. For Arnol'd diffusion, there has been progress in the computation of one of the ingredients, the whiskered tori. There are two different perturbative calculations of whiskered tori that yield disjoint sets of tori [LW], [Tr].

Non-collision singularities in the N-body problem

The problem of N-bodies moving classically under their mutual gravitational attraction is perhaps the oldest in mathematical physics.

The first question to ask is whether solutions are defined for all time. A first glance to the differential equations reveals that they become singular if two bodies collide. Nevertheless, a more detailed analysis reveal that the singularity when only two bodies collide is only apparent. If we modify the conditions so that the bodies miss by a small amount, we get a well defined limit as this modification goes to zero (a back of the envelope argument: the conservation of momentum and energy determines what happens). Unfortunately, for triple collisions, this argument does not work and indeed, it is possible to show that if the bodies miss by small amounts many things can happen and one does not get a well defined limit (exercise: show that the same happens with billiard balls). Very detailed information is nevertheless available (see [SM], ch.I for the the classical work of Sundmann, [De]) for the three body collision and glimpses for more bodies.

One natural question to ask is how pervasive these singularities are and whether there are any others. Important results were obtained at the beginning of the century and completed in the early 70's (See [SM]) that showed that if a solution cannot be continued, either there is a triple collision or the moment of inertia of the whole system has to become unbounded.

In [MMcG] it was shown that, indeed there are singularities different from collisions. Their example consists of 4 particles in a line. Two of them are oscillating and a third one is far apart. A "messenger particle" collides with the oscillating pair and, since this is almost a triple collision, leaves at an enormous speed obtained from the potential energy of the pair. It catches the other particle, gives momentum to it and bounces back, just in time to catch the other two in an almost triple collision, in precisely the conditions that will make it bounce back even more violently and so on. The net result is that the fourth particle escapes to infinity in finite time.

Unfortunately, it was quite difficult to generalize their method to higher dimension so as to avoid collisions completely. (This was included in the list of problems in mathematical physics by B. Simon.)

Recently, however, there have been two remarkable papers [Xi], [Ge] in which examples with collisionless singularities are produced. Both are based on having a messenger particle going back and forth between others managing to always arrive near a triple collision (but not colliding). The configuration in [Xi] has two pairs rotating on parallel planes directly above each other and the messenger particle moving perpendicularly. The configuration of [Ge] consists of pairs orbiting around — roughly the vertices of an equilateral polygon and the several messenger particles going around them.

These two papers entail quite involved estimates and are somewhat difficult to read (I have not checked them in detail myself) but it is clear that they are important.

Let me point out that it is not known whether such singularities occupy a set of positive measure.

The most obvious quantum version of the problem of showing that the dynamics is well defined for all times for almost all trajectories is to show that the quantum Hamiltonian is self-adjoint, which is much easier than the classical one. (See, nevertheless, [RS] for some more detailed physical discussion of the analogy) It is amusing to note that the effect of a messenger particle oscillating wildly between two channels is, however, the enemy to beat in the proofs of quantum asymptotic completeness.

Examples of classical ergodic systems

The main argument to prove that a system is ergodic originated in the work of [Hed], [Ho] (the latter published in Leipzig!) which established ergodicity of geodesic flows.

Basically the main ingredients are a geometric study of the trajectories, which establishes that trajectories that converge in the future or the past form a manifold and that by moving alternatively along these stable/unstable manifolds we can go from every point to every point. Secondly, an abstract theorem due to Birkhoff that shows that for almost all trajectories, the statistical behaviour in the future is the same as that in the past. (Unfortunately, due to the "almost all" rather than all in Birkhoff theorem we need an extra technical condition on the foliations.)

The argument was generalized and streamlined in [A]. Further generalizations were due to [Si] who introduced allowing singularities and [Pe] who allowed for non-uniformity of the approaches in the future and the past but constructed the stable and unstable manifolds and concluded that, if they are sufficiently long, the system is indeed ergodic, if they are not, there are counterexamples [Pe2], [W].

Nevertheless, one can say that, in spite of its beauty, the theory was too abstract and the the only concrete examples with physical appeal were the dispersing billiards [Si], joined later by the celebrated stadium [Bu], a non-dispersing ergodic billiard.

Recently, the situation has changed, manageable conditions to verify that concrete systems satisfy the abstract hypothesis of [Pe] were introduced in [W]. This unified the examples of dispersing and non-dispersing billiards and examples of ergodic geodesic flows and flows on scattering potentials were constructed [Don], [DoL].

More or less at the same time, there appeared another elaboration of the basic strategy that can produce ergodicity even in the case that the manifolds are short [SCh] or that there are singularities The basic idea is perhaps the concept of local ergodicity. The methods of this paper have been extended in [KSS].

Let us mention that all the above verifications of ergodicity, almost automatically yield ergodic properties such as K-property, positive entropy, Bernouilli.

Maybe I should mention that presumably proving ergodicity is not quite the only problem one wants to tackle for physical applications. For Hamiltonian systems, it is very easy to create little islands that, even if they will be negligible for practical applications, would destroy ergodicity. A very interesting problem that I learned from T. Spencer is to prove that the well known system $T_\varepsilon(A,\theta) = (A + \varepsilon \sin\theta, \theta + A + \varepsilon \sin\theta)$ has an ergodic component of positive measure for some large ε's. Let me mention that if one takes in place of sin a piecewise linear version

$$f(x) = \begin{cases} x & \text{if } x \in [0, \pi/2] \\ \pi - x & \text{if } x \in [\pi/2,\, 3\pi/2] \\ x - 2\pi & \text{if } x \in [3\pi/2,\, 2\pi] \end{cases}$$

It is reasonably easy to verify hypothesis that imply that the piecewise linear version of T is ergodic for $\varepsilon > 100$. Can one make a prove the same result for some values of ε when we round the corners of the picewise liner version of T?.

Rigidity of dynamical systems

An object is called rigid if, whenever there is another object equivalent to it in a certain sense, it is also equivalent in another stronger sense. For example, a triangle is rigid because any polygon equivalent to it in the sense of having sides of the same length is equivalent in the much stronger sense of being isometric. A somewhat weaker version of rigidity is rigidity under deformations. We say that a system is rigid under deformations if all the deformations that preserve some structure are trivial. Typically when one starts working with equivalences one tries to attach invariants and it is usual to also call a rigidity theorem a result that shows that some set of invariants determines the object up to trivial changes.

One of the reasons why rigidity theorems are amusing to study is that they cross category lines. We investigate measure theoretic consequences of a Riemannian hypothesis and so on.

One set of objects for which many rigidity theorems are known is negatively curved Riemannian manifolds. A very interesting line of development was started in the papers [GK1][GK2] which showed that, for negatively curved manifolds (either two dimensional or satisfying pinching conditions), isospectral deformations of the metric are isometries (this is a version of the famous "can you hear the shape of a drum?" question). The proof of the theorem consists of two steps. One, showing that if we deform metrics of negative curvature keeping the spectrum of the laplacian constant the lengths of closed geodesics are kept constant, and then showing that this implies that the deformations are isometric. The later is purely a problem in Riemannian geometry.

Another investigation of the consequences of keeping the lengths constant was undertaken in [CEG] who showed that, for surfaces of constant negative curvature, any Hamiltonian deformations that kept some action invariants had to be smooth canonical transformations. This result was generalized in [LMM] who showed the same result is true for Hamiltonian Anosov flows in any dimension. The methods of this paper were extended in [L1] [LM] [L2] to show that, for two dimensional Anosov diffeomorphisms or three dimensional Anosov flows, the eigenvalues at periodic orbits — which are obviously invariants of C^1equivalence — form a complete set of invariants for C^∞ or C^ω equivalence. A different proof using the theory of SRB measures was constructed in [Po] and generalized in [L3] to the case of partially hyperbolic systems. So it seems that the fact that there is complete set of local invariants of peridic orbits for smooth conjugacy of Anosov systems is false in higher dimensions. Nevertheless,it is true that in some cases the eigenvalues at periodic orbits are complete sets of invariants for $C^{\tilde{k}}$ conjugacy and that C^{k^*} conjugacies are C^∞ or C^ω. Unfortunately, $\tilde{k} < k^*$ so that there is a range of regularities without counterexamples or theorems.

Related to the geodesic flows, the horocycle flows — which have no periodic orbits — have been shown to be extremely rigid. In that case, measure theoretic equivalence can be bootstrapped to differentiable equivalence and, by the previous results to smooth equivalence. See for example [FO] and the references there to previous work of a more algrbraic nature.

Another set of problems which I believe is strongly related to the previous ones is to show that Cantor sets generated by dynamical processes are differentiably equivalent if they are topologically equivalent. For the Feigenbaum Cantor set, there are several results already [Ra] [Pa][Su] that show that the topology of the Cantor sets and the fact that they are dynamically generated forces them to be C^1 equivalent.

Other problems with a similar flavor are the study of diffeomorphisms that only commute only with their powers –i.e., diffeomorphisms without any differentiable symmetry.– (See [PY1],[PY2] for theorems that show diffeomorphisms with this property and [PM] for examples of germs of diffeomorphisms with non trivial centralizer) or the study of hyperbolic diffeomorphisms with differentiable stable and unstable foliations. (See [HK], [Ghy].) It is sometimes possible to characterize the systems for which inequalities among dynamical quantities or characteristic classes are saturated [HK] [KKW].

An interesting generalization of dynamical systems are actions of groups on manifolds. On the one hand, they can be considered as non-linear generalizations of representations and on the other hand as dynamical systems with a more complicated time — a point of view emphasized by the thermodynamic formalism approach to dynamical systems. There has been very active program (e.g. [Z] and references there) based on the first point of view to show that actions of *"large"* groups are rigid in the sense that, by a smooth change of variables they can be reduced to a canonical one. The dynamical systems point of view has been exploited in [Hu] to obtain results of rigidity of $SL(n, \mathbf{Z})$ on T^n.

Coming back to the original problems posed by the papers [GK1] [GK2] there has been considerable progress made in two papers that are infuriatingly devoid of heavy machinery and depend only on stark cleverness. In [Sn] there is an extremely simple machinery to produce manifolds which are isospectral but not isometric. In [O] there is a very clever proof that knowledge of the lengths of closed geodesics and their homotopy classes determines the manifold up to isometry. For all that we know [Sn] provides the only mechanism to produce isospectral but not isometric manifolds and for a while it was discussed whether it should be possible to show that isospectral two dimensional manifolds are related as in [Sn].

I would be embarrassed not to mention the new developments in symplectic geometry that are related to symplectic capacities. Symplectic capacities are invariants of symplectic maps. The most transparent constructions of these invariants is uses Floer cohomology. Nevertheless, since Floer cohomology is defined through the gradient flows of a variational principle, one can also give a variational definition. There are also rather simple axiomatic characterizations.

Symplectic capacities have many applications. A tentative one that jumps into mind is that, for the study of pseudodifferential operators via the uncertainty principle, one has to pack some regions of phase space with symplectic images of rectangles. The theory of symplectic capacities places severe constraints on what these images could be. Can this be used to prove bounds for pseudodifferential operators?.

More firmly, the theory of symplectic capacities has produced a proof of the celebrated result that the group of C^∞ symplectic diffeomorphisms is C^0 closed in the group of diffeomorphisms. Very readable surveys with references are the papers by Viterbo and Hofer in [Mi].

K. A. M. Theory

Use of frequencies as parameters.

The first theorem of K. A. M. theory had two very different proofs. The first one, proposed by Kolmogorov (see e.g [Ba] for a nodern exposition of the method) consisted in selecting one frequency and producing a torus though several canonical transformations chosen to exhibit that tori of this frequency indeed existed. The other strategy, proposed and carried out in great detail in [Ar2] consisted in studying the resonance regions and performing the transformations suggested by first order perturbation theory in the regions where this transformations were well defined.

The first method of proof is simpler and became the dominant method in the mathematical literature. Especially since it was discovered that it allowed smoothing schemes that were much sharper than the simple truncating of Fourier series required by Arnol'd's method and hence yielded superior results with respect to the differentiability assumptions required or the differentiability of the conclusions. The method could also be abstracted into implicit function theorems, which could be applied in a variety of situations.

One of the consequences of this point of view is that, since the formulation of implicit function theorems requires one to keep the frequency fixed and each step of the perturbation changes the frequencies, it becomes necessary to have extra parameters to adjust so that the frequency is restored. Hence, it became customary, when trying to apply a theorem to try to count free parameters parameters to see if one had enough to generate counterterms that allowed the application of thw implicit function theorem.

In the recent mathematical physics literature, the situation was a little bit more divided due, in good part to the very nice exposition of Arnol'd's method [CG]. Also, there were theorems such as [FSW] in which the role of adjusting extra parameters had to be played by performing probability estimates in a measure space. An important motivation for overcoming the constraints imposed by parameter counting is applications to infinite dimensions, that we will discuss in the next paragraph.

A very direct attack on a problem that parameter counting said was impossible was [El], who studied the problem of the preservation of lower dimensional tori. The method used was to try to change the frequency considered at every step of the iteration. Soon afterwards it was remarked that similar results could be understood as applying Arnold's method [Pö]. Then, there has been a flurry of activity using this type of ideas. Many problems that had been put on the backburner because the parameter counting said they were impossible became accessible. One application that I particularly like is the existence of two dimensional invariant tori in three dimensional volume preserving flows. Flows such as those to be considered in the theorem can be produced in fluid mechanics experiments and the existence of the tori has important physical meaning such as the lack of mixing. Moreover the tori can be seen by injecting dye! Such experiments have been performed in the Center for Nonlinear Dynamics at U. Texas. Before that, there were numerical experiments [Su] confirmed and extended in [FKP]. Proofs of the theorem were finally obtained in [ChSu], [DL]. Other theorems in which the frequencies play the role of extra parameters is [JS]. I certainly expect that there will be many more as Arnol'd's method of proof becomes better known.

Incidentally, another advantage of using Arnol'd's method is that the non-degeneracy conditions one obtains are much better than those required using an implicit function theorem type of proof. Unfortunately, besides making more difficult to obtain sharp results with respect to differentiability it also makes it more difficult to obtain quantitative versions of the result or to use it to validate numerical computations.

K.A.M. theory for infinite systems.

For several years, there has been an active program to study the thermodynamic limit of K.A.M. theorem. One aspect of this program was to show that a system of N particles, each of them with d degrees of freedom and interacting though short range forces has invariant tori of positive measure for values of the perturbation that decrease like $O(N^{-\alpha})$ [W1], [V]. Moreover in [W3], [W4], there is an analogue of Nekhoroshev upper bound when the number of degrees of freedom increases.

In this program, one has to use the short range of the interaction to show that the system is effectively almost finite dimensional. This gives a precise meaning to the physicists intuition that degrees of freedom are frozen and that the interactions do not count.

Perhaps the frist result for infinitely many degrees of freedomn is [FSW]. Nevertheless, the system is such that the amplitude of the oscillations in the degrees of freedom decreases very fast with the distance to a center. In effect, the oscillations are localized – in the sense of the word in solid state physics. An abstract version of this result is in [Pö2].

It is interesting to realize the technical similarities of this with the series of papers [BS] [PS] in which the opposite is attempted, namely, to develop a hyperbolic perturbation theory for hyperbolic systems each of which is hyperbolic but which are coupled through a short range interaction.

Another interesting development is the application of methods of K.A.M. theory to partial differential equations. This is not just a case of taking the thermodynamic limit. If one did that, one would obtain tori of as many dimensions as the number of degrees of freedom i.e., infinite. What does it mean to have a function with infinitely many frequencies? It is not a very interesting result. There are two interesting papers. One is the paper [W2] in which the method used is very similar to Arnol'd's method applied by defining a measure in the set of potentials entering in the definition of the problem. In another set of papers, [K], [K2] performs an iteration in which the frequency changes at every stage, but one obtains the control by direct methods.

Quantitative versions of the K.A.M. theorem.

From the beginning of K.A.M. theory it has been a point of friction with the physicists that the values for which the theory applies are quite hard to ascertain rigorously [Mo]. While the mathematicians were using as the main figure of merit the differentiability properties, physicists were emphasizing numbers. In some particular examples, it was possible to obtain reasonable values [L4] [He] — incidentally, if one chases though the constants in

[Br], one gets better numbers than those in [L4] as was pointed out to me by M. Herman. Nevertheless, it is clear that the methods are quite specific.

One different method was afforded by the use of computer assisted proofs. The method consists in getting computers to perform calculations that verify the conditions of a theorem. The calculations should be performed taking care to prevent the computers from using approximations as they do since they were abandoned by the mathematicians to the engineers. (See [La] for a review.)

There are two basic strategies. One [Ce], [CC1], [CC2], [CCF] considers perturbation expansions using methods similar to the Linstedt series and estimates the remainders and applies K.A.M. theory to them. This has several advantages. First, it produces results which are valid in open ranges. Second, the analysis is somewhat easier since the analytical theorem to be proved are proved for perturbations of the identity.

The other method, [R], [LR] consists in proving an implicit function theorem that shows that, if one has an approximate solution that verifies some extra hypothesis, then, there is a true one nearby. Secondly, one has to verify the hypothesis, again with a computer. This method has the advantage that it requires less calculations and is insensitive to singularities in the complex domain that, even if they do not affect the true results, could affect the convergence of expansions. Indeed, in several cases, there are proofs that these methods — up to the limits of numerical analysis — will not miss any solution.

Perhaps the main inconvenience of the method as applied so far is that the programs developed require severe modifications in order to be adapted to other problems. (Nevertheless, the programs in [R] have been used in four different projects.) It is to be hoped that, with the new developments in software tools, soon there will be packages that not only meet a new version of Salam's criterion for renormalization theories but which become routinely used.

The method of computer assisted proofs can also be used to produce upper bounds of values for which the theorem cannot be true. The first such paper is [MP]. [Ju] produces a very clever method which is not only easy to implement but also gets very close to the presumed value. In [Mu] [MMS] there is a generalization of the method of [MP] to higher dimensions.

The results of K.A.M. theory are nicely counterpointed with the mathematical results of Aubry-Mather thoery and with the very delicate numerical work based on renormalization group ideas. Both these subjects would merit more detailed discussion.

Other mathematical papers that construct systematically examples which fail to exhibit the conclusions of K.A.M. theorems are [He2] – whose methods were used in [NMS]–, [Yo1], [Yo2]

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