Drift and diffusion in phase space. An application to celestial mechanics.

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Abstract: some results on the theory of Arnold's diffusion and an application to the motion of non spherical heavenly bodies revolving about a point in conic sections.

§1 *Diffusion paths.*

We consider an integrable system with action angle coordinates $(\vec{A},\vec{\alpha})$, with $\vec{A}$ in a sphere $V\subset R^l$, and $\vec{\alpha}\in T^l$ (l-dimensional torus).

Let $h(\vec{A})$ be the hamiltonian. Let h be analytic in V and let $f(\vec{A},\vec{\alpha})$ be a perturbation analytic in $\vec{A},\vec{\alpha}$. Let ρ be the amount by which the actions can be complexified and ξ be the amount by which the angles can be complexified still keeping $(\vec{A},\vec{\alpha})$ in the holomorphy domain of h, f. The hamiltonian will be:

$$H(\vec{A},\vec{\alpha})=h(\vec{A})+\varepsilon f(\vec{A},\vec{\alpha}) \tag{1.1}$$

and the vector $\vec{\omega}(\vec{A})\equiv\partial_{\vec{A}}h(\vec{A})$ will be called the *rotation vector*. We say that h is anisochronous if $\det\partial^2_{\vec{A}^2}h(\vec{A})\neq 0$.

Let $\mathcal{L}$ be a smooth curve on the $\vec{\nu}_0$-resonance: $\mathcal{R}_{\vec{\nu}_0}\equiv\{\vec{A}|\,\vec{\omega}(\vec{A})\cdot\vec{\nu}_0=0\}$, where $\vec{\nu}_0$ is a non zero integer components vector, and on a constant energy surface, $h(\vec{A})=E$, of a anisochronous unperturbed hamiltonian.

We denote $s\to\vec{A}(s)$, $s\in[0,1]$ the parametric equations of the curve and $D=\max_s|\vec{\omega}(\vec{A}(s))|$. Then:

Definiton 1_1: *The curve $\mathcal{L}$, with energy E and on the resonance $\vec{\nu}_0$, is a diffusion path if there exist $K,t,\tau>0$ such that:*

$$\text{measure of}\{s|\,|\vec{\omega}(\vec{A}(s))\cdot\vec{\nu}|^{-1}\le C|\vec{\nu}|^{\tau}\}\ge\big(1-\frac{K}{(DC)^{1/t}}\big) \tag{1.2}$$

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for all $C > 0$ and for all $\vec{\nu}$ not parallel to the resonance numbers $\vec{\nu}_0$.

In many cases one is interested in forced systems rather than in autonomous systems like the above (1.1). Such systems are perturbations of an integrable hamiltonian supposed to be linear in (say) A_1: $h(\vec{A}) = \omega A_1 + h'(\vec{A}')$, with $\vec{A}' = (A_2, \ldots, A_l)$, still supposing $\det \partial^2_{\vec{A}'} h'(\vec{A}') \neq 0$. The perturbation f is supposed to be A_1 independent (so that the variables A_1, α_1 describe the motion of a (perfect) clock). The regularity properties on h and f will be the same as in the autonomous case, and the hamiltonian is:

$$H(\vec{A}, \vec{\alpha}) = \omega A_1 + h'(\vec{A}') + \varepsilon f(\vec{A}', \alpha_1, \vec{\alpha}') \tag{1.3}$$

where $\vec{\alpha}' = (\alpha_2, \ldots, \alpha_l)$.

Let $\mathcal{L}$ be a smooth curve in the $\vec{\nu}_0 \equiv (\nu_1, \vec{\nu}')$-resonance: $\mathcal{R}_{\vec{\nu}_0}\{\vec{A}' \,|\, \omega\nu_1 + \vec{\omega}'(\vec{A}') \cdot \vec{\nu}_0' = 0\}$, where $\vec{\nu}_0$ is a non zero integer components vector. The curve $\mathcal{L}$ will be also thought as a curve in V by defining A_1 so that the energy $h(\vec{A})$ is a given constant E.

Definition 1_2: *The curve $\mathcal{L}$, on the resonance $\vec{\nu}_0$, is a diffusion path if there exist $K, t, \tau > 0$ such that (1.2) holds for all $C > 0$ and for all integer vectors $\vec{\nu}$ not parallel to the resonance numbers $\vec{\nu}_0$.*

Two more classes of systems are interesting: the *a priori* unstable systems (as opposed to the above two systems that will be called *a priori* stable), autonomous or forced. In the new cases we shall call the canonical variables $(\vec{A}, I) \in V$ and $(\vec{\alpha}, \varphi) \in T^l$. In the autonomous case the hamiltonian is:

$$h(\vec{A}) + \frac{I^2}{2J} + Jg^2(\cos\varphi - 1) + \varepsilon f(\vec{A}, I, \vec{\alpha}, \varphi) \tag{1.4}$$

The h, f will be supposed to fulfill the above regularity and non degeneracy conditions.

Let $\mathcal{L}$ be a smooth curve on a constant energy surface $h(\vec{A}) = E$. Then:

Definiton 1_3: The curve $\mathcal{L}$, with energy E is a diffusion path if there exist $K, t, \tau > 0$ such that (1.2) holds for all $C > 0$ and for all integer vectors $\vec{\nu}$.

In some sense in this case the role of the resonance is plaid by the set $(I, \varphi) = (0, 0)$.

Finally the corresponding non autonomous case is defined by a hamiltonian:

$$H(\vec{A}, I, \vec{\alpha}, \varphi) = \omega A_1 + h'(\vec{A}') + \frac{I^2}{2J} + Jg^2(\cos\varphi - 1) + \varepsilon f(\vec{A}', I, \vec{\alpha}, \varphi) \tag{1.5}$$

where $\vec{A} = (A_1, \vec{A}') \in V \subset R^{l-1}$, $\vec{\alpha} = (\varphi_1, \vec{\alpha}')$ and $h'(\vec{A}')$ verifying $\det \partial_{\vec{A}'^2} h'(\vec{A}') \neq 0$. The diffusion path $\mathcal{L}$ is defined to be a curve in the space R^{l-1} such that: the curve $\mathcal{L}$ will be also thought as a curve in $R \times V$ by defining F_1 so that the energy $h(\vec{A})$ is a given constant E.

Definition 1_4: *The curve $\mathcal{L}$ is a diffusion path if there exist $K, t, \tau > 0$ such that (1.2) holds for all $C > 0$ and for all integer vectors $\vec{\nu} \neq \vec{0}$.*

More generally one can consider paths $\mathcal{L}$ on the surfaces of constant $h(\vec{A})$ or $h'(\vec{A}')$ without references to resonances.

§2 *General results.*

The above definitions will permit us to formulate the results known in the theory of drift and diffusion after introducing the following concept:

Definition 2: *A path $\mathcal{L}$: $\{s \to \vec{A}(s)\}$ is open for diffusion by a perturbation f if, given any "tube U around $\mathcal{L}$" (i.e. an open vicinity of $\mathcal{L}$), for all ε small enough, but non zero, one can determine initial data $(\vec{A}_\varepsilon, \vec{\alpha}_\varepsilon)$ (and $(I_\varepsilon, \varphi_\varepsilon)$ as well, in the a priori unstable cases), such that, denoting $(\vec{A}_{\varepsilon,t}, \vec{\alpha}_{\varepsilon,t})$ their evolution under the hamiltonian flow with hamiltonian H_ε, there exists T_ε and:*

$$\begin{aligned} 1: &\qquad \vec{A}_\varepsilon \xrightarrow[\varepsilon\to 0]{} \vec{A}(0) \\ 2: &\qquad \vec{A}_{\varepsilon,t} \in U \quad \text{for} \quad 0 < t < T_\varepsilon \\ 3: &\qquad \vec{A}_{\varepsilon,T_\varepsilon} \xrightarrow[\varepsilon\to 0]{} \vec{A}(1) \end{aligned} \tag{2.1}$$

In the a priori unstable cases one adds the further requirement that:

$$I_{\varepsilon,0}, \varphi_{\varepsilon,0} \xrightarrow[\varepsilon\to 0]{} (0,0), \qquad I_{\varepsilon,T_\varepsilon}, \varphi_{\varepsilon,T_\varepsilon} \xrightarrow[\varepsilon\to 0]{} (0,0) \tag{2.2}$$

Hence *a diffusion path is open under the perturbation f if it is a path that can be closely followed by the actions of a true trajectory of the perturbed system.* The latter trajectories will be called *drift* trajectories, see [CG] for a motivation of the name "diffusion".

It is convenient to give the definition of drift for the paths which are not necessarily diffusion paths: in fact such a requirement would be too restrictive.

Question: can one give sufficient conditions for a path to be open for diffusion?

A consequence of the KAM theory and of the definition of diffusion path is:

Theorem 1: If $l = 2$ no path can be open for diffusion unless it reduces to a point.

This is a trivial consequence of definition 1_3 or 1_4 in the cases of *a priori* unstable systems; but in the case of *a priori* stable systems it follows from the KAM theory.

A first positive result concerns *a priori* unstable systems (see [A], [CG]); let the separatrix motion of the free pendulum, starting in $\varphi = \pi$ at $t = 0$, be $t \to (I(t), \varphi(t))$, then:

Theorem 2: Consider an a priori unstable hamiltonian (forced or not). Suppose that the function:

$$M_s(\vec{\alpha}) \equiv \int_{-\infty}^{\infty} [f(I(t),\varphi(t),\vec{A}_s,\vec{\alpha}+\vec{\omega}t) - f(0,0,\vec{A}_s,\vec{\alpha}+\vec{\omega}t)]\,dt \tag{2.3}$$

is such that the equation $\vec{\partial}_{\vec{\alpha}} M(\vec{\alpha}) = \vec{0}$ admits a smooth solution $s \to \vec{\alpha}_s$ which is non degenerate, i.e. $\delta \equiv \det \vec{\partial}^2 M(\vec{\alpha}_s) \neq 0$, and suppose that $\vec{\nu}\cdot\vec{\omega}(\vec{A}(s) \neq 0$ for all integres vectors $\vec{\nu}$ with $|\vec{\nu}| < N$ for N suitably large (depending on $\mathcal{L}$). Then the perturbation f opens $\mathcal{L}$ for diffusion.

Furthermore there exist three constants $T(\mathcal{L}), b(\mathcal{L}), c(\mathcal{L})$ such that the time T_ε can be taken for ε small enough:

$$T_\varepsilon \le T(\mathcal{L}) e^{b(\mathcal{L})\,\varepsilon^{-2}}, \qquad \text{for} \qquad |\varepsilon| < c(\mathcal{L}) \tag{2.4}$$

which is an upper bound on the drift time.

The *a priori* stable case is harder as one has first to study the motion near a resonance.

It would seem that the theory of normal forms near a resonance, [N, G2, BG], would allow us to reduce the *a priori* stable problem to the *a priori* unstable: this was probably the motivation of Arnold's first example, exhibiting the first case in which the theorem 2) could be proved (apart from the explicit estimate (2.4)).

In fact if $\partial^2_{\vec{\alpha}} h$ is a non degenerate matrix one can find a vicinity of the resonance surface on which $\mathcal{L}$ lies where the system can be described in a new canonical coordinate system, usually called the "fast-slow" coordinates and denoted $(\vec{F}, S) \in R^l$ and $(\vec{\varphi},\sigma) \in T^l$ with $\vec{F} = (F_1,\ldots,F_{l-1})$, $\vec{\varphi} = (\varphi_1,\ldots,\varphi_{l-1})$, in which the hamiltonian takes the form:

$$\frac{h_\varepsilon(\vec{F})}{\sqrt{\varepsilon}} + \Big(\frac{S^2}{2J_{\varepsilon,\vec{F}}} + \sqrt{\varepsilon} v_{\vec{F}}(\sigma) + \varepsilon p_\varepsilon(S,\sigma,\vec{F})\Big) + e^{-b\varepsilon^{-a}} \bar{f}_\varepsilon(\vec{F},S,\vec{\varphi},\sigma) \tag{2.5}$$

for suitable functions $h_\varepsilon, v_{\vec{F}}, J_{\varepsilon,\vec{F}}, p_\varepsilon, \bar{f}_\varepsilon$ regular in their arguments. Furthermore, under the strong additional assumption that $\partial^2_{\vec{\alpha}} h$ is a *positive* matrix, initial data starting close to the resonance (*i.e.* that have small enough S, φ), will stay in the vicinity of the resonance where the coordinates $\vec{F}, S$ are defined for a time of the order of $e^{+b\varepsilon^{-a}}$, [N],[G2],[BG].

The (2.5), apart from the change of the variables name, differs from the previously considered hamiltonians, (1.4):

$$h(\vec{F}) + \frac{S^2}{2J} + Jg^2(\cos\sigma - 1) + \mu f(\vec{F},S,\vec{\varphi},\sigma) \tag{2.6}$$

because:

1) $v_{\vec F}(\sigma)$ is not proportional to $(\cos\sigma-1)$ and $J_{\varepsilon,\vec F}$ is not constant: this is not a problem as the above theorem 2 can be easily generalized to cover such cases ([CG]).
2) the "higher order integrable correction" p_ε is not zero: this is also not a problem as the above theorem 2 can be easily generalized to cover such cases ([CG]).
3) the ratio between the pendulum time scale and the fast angle variables time scales has order $O(\varepsilon)$ rather than $O(1)$: this is not necessarily a problem as the results in theorem 2 can be formulated in terms of the relative orders of magnitude of various quantities (as they have "dimensional nature", see [CG]) and one may still hope that they apply at least to some cases of the form (2.5).
4) the size of the perturbing function *depends* on the time scales ratio ε; while in the (2.6) they are two independent parameters and ε (corresponding to g^2 in (2.6)) is fixed while μ can be taken as small as we please. In (2.5) μ is very small (as $\varepsilon\to0$) but it is a function of ε.

The latter item expresses a deep problem: the quantitative version of theorem 2 could be applied to the case (2.6), and hence to the theory of the *a priori* stable systems, if μ could be taken small of the order $O(\exp -b\varepsilon^{-a})$ with $a>1/2$, see §7 in [CG]. *However the constant a in (2.6) is usually much smaller than* $1/2$ *and only in very special cases it can be estimated to be* $1/2$, [N]: never larger!

It is not impossible that, by imposing suitably many *ad hoc* hypotheses on $\mathcal{L}$ and f, one could treat some *a priori* stable problems. But it seemed to us that a more interesting attitude would be to study a particular case which admittedly has some relevance, mathematical and physical, and which provides a natural selection of special properties that might make the problem soluble. This is discussed in the next section.

§3 Planetary precession. Existence of drift and diffusion.

Imagine a planet $\mathcal{E}$ as a homogeneous rigid body with cylindrical symmetry. The body surface will be described in polar coordinates by $\rho=Rh(\cos\vartheta)$ for some R and some h, $R>0$, $0<h\le1$, *e.g.* for a rotation ellipsoid with equatorial radius R and polar radius $R/(1+2\eta)^{1/2}$ it is $h(z)=(1+2\eta z^2)^{-1/2}$.

We suppose the planet center T to revolve on a keplerian orbit $t\to\vec r_T(t)$: the orbit plane will be called the *ecliptic* plane and $\vec{\vec k}$ will denote its unit normal vector which sees the planet rotating counterclockwise.

The longitude λ_T of $\vec r_T$ on the ecliptic will be reckoned from the major semiaxis of the ellipse; hence $\lambda_T=0$ is the *aphelion* position *i.e.* when $r_T\equiv|\vec r_T|$ is maximal: $r_T(0)=a(1+e)$, a being the major semiaxis of the Keplerian ellipse and e its eccentricity.

With the above conventions, r_T and λ_T are related by the *focal equation* (see, e.g.

[G1], p.304):

$$r_T \equiv |\vec r_T| = \frac{p}{1 - e\cos\lambda_T}, \quad p \equiv a(1-e^2). \tag{3.1}$$

In this section we shall always denote by e the eccentricity of the orbit and to avoid confusion with the Neper's constant we denote the exponential of a number α by $\exp\alpha$*, while* e^α *will denote everywhere the* α*-th power of the eccentricity e.*

Kepler's law, $\dot\lambda_T r_T^2 = const$, and (3.1) imply that if λ is the keplerian *average anomaly*:

$$\lambda \equiv (1-e^2)^{3/2}\int_0^{\lambda_T} \frac{d\beta}{(1-e\cos\beta)^2} = \lambda_T + 2e\sin\lambda_T + \frac{3}{4}e^2\sin 2\lambda_T + \ldots, \tag{3.2}$$

then:

$$\lambda_T = \lambda - 2e\sin\lambda + (5/4)e^2\sin 2\lambda + \ldots, \qquad \frac{a}{r_T} = 1 - e\cos\lambda + e^2\cos 2\lambda + \ldots \tag{3.3}$$

and the motion is $\lambda \to \lambda + \omega_T t$, where $2\pi/\omega_T = 2\pi a^{3/2} g^{-1/2}$ is the year of the planet, $g = k(m_S + m_T)$ if k is Newton's constant and m_T, m_S are the masses of the planet and of its star.

The unit vector $\vec\imath$ pointing from the focus towards the aphelion will be used together with $\vec{\vec k}$ and a third vector $\vec\jmath$ to form an orthonormal triad $(\vec\imath, \vec\jmath, \vec{\vec k})$ of fixed directions in space.

A comoving frame $(T; \vec\imath_1, \vec\imath_2, \vec\imath_3)$ will be attached to the planet with $\vec\imath_3$ axis coinciding with the symmetry axis (*polar axis*) of the planet and $\vec\imath_1$ is arbitrarily chosen on the *equatorial plane*, (*i.e.* the plane orthogonal to $\vec\imath_3$).

The position of $(T; \vec\imath_1, \vec\imath_2, \vec\imath_3)$ referred to $(T; \vec{\vec\imath}, \vec{\vec\jmath}, \vec{\vec k})$ will be determined by the three Euler angles $\bar\vartheta, \bar\varphi, \bar\psi$ with $\bar\vartheta$ being the angle between $\vec{\vec k}$ and $\vec\imath_3$, $\bar\varphi$ being the angle on the ecliptic between $\vec\imath$ and the ecliptic – equator node $\vec{\vec n}$, while $\bar\psi$ is the angle on the equator between $\vec{\vec n}$ and $\vec\imath_1$, (drawings with the above and the following notations can be found in [G1], p.318%321]).

In the coordinates $(\bar\vartheta, \bar\varphi, \bar\psi)$ the motion of the planet $\mathcal{E}$ is described by the Euler-Lagrange equation associated with the lagrangian:

$$\mathcal{L} \equiv \frac{1}{2}J_3(\dot{\overline\varphi}\cos\overline\vartheta + \dot{\overline\psi})^2 + \frac{1}{2}J_1(\dot{\overline\vartheta}^2 + \dot{\overline\varphi}^2\sin^2\bar\vartheta) + \int_{\mathcal{E}} \frac{k m_T m_S}{|\vec r_T + \vec x|}\frac{d\vec x}{|\mathcal{E}|} \tag{3.4}$$

where $J_3, J \equiv J_1 = J_2$ are the inertia moments of $\mathcal{E}$, m_T its mass, $|\mathcal{E}|$ its volume, m_S is the mass of the heavenly body keeping the planet $\mathcal{E}$ on its celestial path, $t \to \vec r_T(t)$, and k is Newton's constant.

Very remarkable is a theorem by Andoyer-Deprit, see [G1], p.318%321], which produces canonically conjugate variables casting the Hamiltonian corresponding to $\mathcal{L}$ in a simple form. To describe such variables we consider the unit vector $\vec{k}$ parallel to the angular momentum $\vec{K}_T \equiv M\vec{k}$, $M = |\vec{K}_T|$ and call *angular momentum plane*, or *spin plane*, the plane orthogonal to $\vec{k}$. We define the angles δ and ϑ between $\vec{\vec{k}}$ and $\vec{k}$ and, respectively, $\vec{k}$ and $\vec{\imath}_3$, so that the components of the angular momentum on $\vec{\vec{k}}$ and on $\vec{\imath}_3$ will be, respectively:

$$K = M\cos\delta, \qquad L = M\cos\vartheta \tag{3.5}$$

We also associate with $\vec{K}_T$ two more remarkable angles: in fact the spin plane has a node $\vec{m}$ on the ecliptic plane and one $\vec{n}$ on the equator plane. We call γ' the angle on the ecliptic between $\vec{m}$ and $\vec{\vec{\imath}}$ and χ the angle on the angular momentum plane between the node $\vec{m}$ and the node $\vec{n}$. Finally we let ψ denote the angle between $\vec{n}$ and $\vec{\imath}_1$.

Deprit's theorem states that the variables (K,γ'), (M,φ), (L,ψ) are canonically conjugated for the hamiltonian H associated with $\mathcal{L}$, and that H in such variables takes the form:

$$H = \frac{M^2}{2J_3} + \frac{J_3 - J_1}{2J_1J_3}(M^2 - L^2) + \omega_T B' + V \tag{3.6}$$

where V is the integral in (3.4) changed in sign, and (B',λ) is a fourth pair of canonical coordinates with λ being the average anomaly of the planet in its revolution about the ellipse focus, see (3.2). The pair (B',λ) has been introduced in order to eliminate the explicit time dependence from the hamiltonian.

In what follows it will be more useful to employ as canonical coordinates the pairs (K,γ), (M,χ), (L,ψ), (B,λ) with $\gamma = \gamma' - \lambda$ and $B = B' + K$.

It is convenient to bear in mind that $\omega_T B$ has a simple physical interpretation: it is the energy stored in the device providing the external force that keeps the heavenly body $\mathcal{E}$ on its (keplerian) celestial path, $t \to \vec{r}_T(t)$.

By symmetry considerations it is clear that V is a function of the angle λ (or of λ_T), and of the angle α between the position vector $\vec{r}_T$ and the axis $\vec{\imath}_3$ of the planet. In fact, it is easy to find out an expression for $V = V(\alpha,\lambda)$. Recalling the relation between the Legendre polynomials $P_l(z)$ and their generating function $(1 + x^2 - 2xz)^{-1/2}$, one finds:

$$\begin{aligned} V =& \frac{-km_S m_T}{|\vec{r}_T|} \int \frac{d\vec{x}}{|\mathcal{E}|} \left(1 + (\frac{|\vec{x}|}{r_T})^2 + 2\frac{|\vec{x}|}{r_T}(\frac{\vec{x}}{|\vec{x}|} \cdot \frac{\vec{r}_T}{r_T})\right)^{-1/2} = \\ =& \frac{-km_S m_T}{|\vec{r}_T|} \sum_{l=0}^{\infty} \int_{\mathcal{E}} \frac{d\vec{x}}{|\mathcal{E}|} (\frac{-|\vec{x}|}{r_T})^l P_l(\frac{\vec{x}}{|\vec{x}|} \cdot \frac{\vec{r}_T}{r_T}) \end{aligned} \tag{3.7}$$

and the above expression can be used to compute the series expansion of the potential energy in the eccentricity.

If we perform the calculation neglecting the terms in (3.7) which come from $l \geq 4$ (which roughly means neglecting $O(R/a)^2$ compared to 1, with R being the planet radius and a being the major semiaxis of its orbit, because the odd orders in l vanish by symmetry), it is well known, (see [L]), that the only properties of the rigid body that matter are the inertia moments. It is also clear that the hamiltonian must be expressible in terms of the physical quantities that establish the orders of magnitude of the problem. Thus we expect that the hamiltonian be a function depending, besides the angles and their conjugated momenta from the daily rotation of the planet ω_D, from the yearly rotation ω_T, from the inertia moment J_3: *the physical periods are introduced into the problem through the initial data around which we want to set up a perturbation theory.* It is a classical calculation to check that in fact the exact form of the hamiltonian is, to order k in the eccentricity and denoting $[\cdot]^{[\leq k]}$ the truncation to power k of a power series in the eccentricity e, the hamiltonian:

$$H = \frac{M^2}{2J_3} + \eta_2 \frac{M^2 - L^2}{2J_3} + \omega_T B + \eta\omega \frac{\overline{M}^2}{\overline{K}} \big[\frac{(1 - e\cos\lambda_T)^3}{(1-e^2)^3} \cos^2\alpha\big]^{[\leq k]} \tag{3.8}$$

with $\eta_2 = (J - I)/I$; it will be called the *D' Alembert precession-nutation* model.

If we define $\omega_D = \overline{M}/J_3$, the rotation velocity, and $2\pi/\omega_T$, the revolution period, (related to the parameters of the system by the Kepler laws), and $\cos\bar{\imath} = \overline{KM}^{-1}$ (the cosine of the planet spin inclination $\bar{\imath}$ over the ecliptic), the value of ω is the celebrated result of D' Alembert:

$$\omega = \omega_T^2 \omega_D^{-1} \cos i_0 \tag{3.9}$$

whose physical meaning is that $-\eta\omega$ is the average angular velocity of *precession of the equinoxes* $\langle\dot{\gamma}'\rangle$, as it appears also from the following analysis (for details see [CG], Appendices 6,7).

Concerning the approximations involved in passing from (3.6) to (3.8) we note that the terms of $O(\eta(R/a)^4)$ are believed to be really negligible for all practical purposes in many astronomy problems while, for the truncation approximation, D' Alembert did not have data on the Moon mass accurate enough to wish to consider orders $k > 0$ in his theory of lunisolar precession, finally providing a theory for the equinoxes precession phenomenon discovered by Hipparcus about two millennia earlier. Here we consider only the case $k = 2$: but it is very likely that what follows does not really require neither the truncation nor neglecting the higher orders in $(R/a)^2$. Considering such more general problems should only lead to some (minor) modifications, except in the case $k = 0$, where the result is simply false (*i.e.* no diffusion can take place) and the case $k = 1$ which cannot be decided by a "lowest order" perturbation analysis as, instead, the cases $k \geq 2$ are, (at least if the initial data are chosen as we are going to do).

To compute the D' Alembert hamiltonian (3.8) we have, of course, to find how $\cos^2\alpha$ depends on the canonically conjugated variables $(K, M, L, B, \gamma, \varphi, \psi, \lambda)$.

Simple spherical trigonometry arguments, (see also [CG] appendix A8), lead to (setting $\tilde\lambda_T \equiv \lambda_T - \lambda$):

$$\begin{aligned}\cos\alpha =& \sin(\tilde\lambda_T-\gamma)\,\big(\cos\varphi\sin\vartheta\cos\delta+\sin\delta\cos\vartheta\big)-\cos(\tilde\lambda_T-\gamma)\sin\vartheta\sin\varphi=\\ &=\sin(\tilde\lambda_T-\gamma)\,\Big((K/M)\,\big(1-(L/M)^2\big)^{1/2}\cos\varphi+(L/M)\,\big(1-(K/M)^2\big)^{1/2}\Big)-\\ &-\big(1-(L/M)^2\big)^{1/2}\sin\varphi\cos(\tilde\lambda_T-\gamma)\equiv s(\kappa\nu c_\varphi+\mu\sigma)-\nu s_\varphi c\end{aligned}\tag{3.10}$$

where:

$$\begin{array}{llll}\mu\equiv L/M, & \nu^2\equiv 1-\mu^2, & s\equiv\sin(\tilde\lambda_T-\gamma), & s_\varphi\equiv\sin\varphi,\\ \kappa\equiv K/M, & \sigma^2\equiv 1-\kappa^2, & c\equiv\cos(\tilde\lambda_T-\gamma), & c_\varphi\equiv\cos\varphi.\end{array}\tag{3.11}$$

Hence we see that (3.8), as well as the full (3.7), does not contain ψ. Therefore L is a constant of motion and it will be regarded as a parameter. It has the physical interpretation that $\overline\nu=(1-L^2/\overline{M}^2)^{1/2}$ is the angle between the spin axis and the symmetry axis and in the theory of nutation it is called the *eulerian nutation constant*, at the initial *epoch*, *i.e.* at a prefixed reference time, when $\overline M, \overline L, \overline K, \overline\varphi, \overline\psi, \overline\gamma$ are the values of the canonical variables.

Therefore setting:

$$V\equiv\Big[\frac{(1-e\cos\lambda_T)^3}{(1-e^2)^3}\cos^2\alpha\Big]^{[\le 2]}\equiv V_0+eV_1+e^2V_2\tag{3.12}$$

and using:

$$\frac{(1-e\cos\lambda_T)^3}{(1-e^2)^3}=1+\frac{3}{2}e^2-3e\cos\lambda+\frac{9}{2}e^2\cos2\lambda+\ldots\tag{3.13}$$

one finds that:

$$V=\sum_{h=0}^{2}e^h\sum_{\substack{r,p,j\\ r,\,p+h=even}}\overline{B}^h_{rpj}\cos(r\gamma+p\lambda+j\chi)\tag{3.14}$$

where $\overline{B}^h_{rpj}$ are suitable coefficients depending on M, K. For instance:

$$\overline{B}^0_{000}\equiv c_0\equiv\frac14[2\sigma^2\mu^2+(1+\kappa^2)\nu^2]\tag{3.15}$$

Thus, setting $\overline{E}\equiv\omega\frac{\overline{M}^2}{\overline{K}}$ and (only for convenience of notation) $\eta_2=0$, the full ("order 2") D' Alembert hamiltonian, in the canonical variables (γ, K), (χ, M), (λ, B), takes the form:

$$\omega_T B+h_0(K,M;\eta)+\eta f(K,M,\gamma,\chi,\lambda;e)\tag{3.16}$$

where:

$$h_0 \equiv -\omega_T K + \frac{M^2}{2J} + \eta \overline{E}\, c_0(K,M), \qquad f \equiv \overline{E}\,[\overline{V}_0 - c_0 + e\overline{V}_1 + e^2\overline{V}_2] \tag{3.17}$$

with $V_i = V_i(K, M, 2\gamma, \chi, \lambda)$, $c_0 = c_0(K, M)$, and $\langle\cdot\rangle$ denotes average over the angles and, more explicitly to give an idea of the result:

$$\begin{aligned}
\overline{V}_0 =& \sum c_j \cos j\chi + d_j \cos(2\gamma + j\chi) \qquad (3.18)\\
\overline{V}_1 =& \sum \Big(-3c_j \cos(\lambda + j\chi) + \tfrac{1}{2} d_j \big(\cos(2\gamma + \lambda + j\chi) - 7\cos(2\gamma - \lambda + j\chi)\big)\Big)\\
\overline{V}_2 =& \sum \Big[c_j \Big(\tfrac{3}{2}\cos j\chi + \tfrac{9}{2}\cos(2\lambda + j\chi)\Big) + d_j \Big(\tfrac{17}{2}\cos(2\gamma - 2\lambda + j\chi) - \tfrac{5}{2}\cos(2\gamma + j\chi)\Big)\Big]
\end{aligned}$$

where the sums run over $j = -2, \ldots, 2$ and the coefficients $\overline{B}^h_{rpj}$ (cfr. (3.14)) vanish unless they belong to the following list where $|j| \le 2$:

$$\begin{aligned}
&\overline{B}^0_{00j} \equiv c_j\ , \quad \overline{B}^0_{20j} \equiv d_j, \quad \overline{B}^1_{01j} \equiv -3c_j\ , \quad \overline{B}^1_{211} \equiv \tfrac{d_j}{2}\ , \quad \overline{B}^1_{2-1j} \equiv -\tfrac{7}{2} d_j\\
&\overline{B}^2_{00j} \equiv \tfrac{3}{2} c_j\ , \quad \overline{B}^2_{02j} \equiv \tfrac{9}{2} c_j\ , \overline{B}^2_{20j} \equiv -\tfrac{5}{2} d_j\ , \overline{B}^2_{2-2j} \equiv \tfrac{17}{2} d_j
\end{aligned} \tag{3.19}$$

and the coefficients c_j, d_j are (complicated, see [CG], appendix A14) functions of M, K and of the parameter L, whose explicit expression will play no role here.

The model (3.17) is a model with *two* parameters η, e and it is a forced system with forcing described by the (B, λ) variables. The parameter η will be regarded as a control parameter while e wil be regarded as a perturbation parameter. We consider a diffusion path by selecting the resonance $\vec{\nu}_0 = (0, 2, 1)$ and the line $\mathcal{L}$:

$$s \in [0,1] \to (s\overline{K} + (1-s)\overline{K}_2, \overline{M}), \quad \text{with} \quad \frac{\overline{M}}{J_3} = \omega_D \tag{3.20}$$

with $\overline{K}_2$ some constant ($< \overline{M}$) (recall that we are fixing the "initial data" of "interest" to be $\overline{M}, \overline{K}$ for M, K). This corresponds to $2\omega_T = \omega_D$, *i.e.* to a resonance $2:1$ between the daily rotation ω_D and the annual rotation ω_T ("two days in one year").

The latter is a resonance generating, in our theory, the simplest calculations; more interesting resonances (*e.g.* $1:1$ or $3:2$ corresponding to $\vec{\nu}_0 = (0, -1, 1)$ and $\vec{\nu}_0 = (0, 3, -2)$) could also be treated but they require more calculations and, perhaps, keeping a few of the terms neglected in our simplified model.

One should also remark that the selected resonance is in fact a double resonance, as also $\vec{\nu}'_0 = (1, -1, 1)$ is a resonance vector. Hence $\bar{\mathcal{L}}$ is not a diffusion path in the sense of definition 1_4 above.

Because of our selection of the resonance, the harmonic $(2\gamma+\chi)$ will be the angle of the pendulum (see (2.5),(2.6)) describing the motion transversal to the resonance. Therefore, we perform the following linear (canonical) change of variables,

$$(K,\gamma),\,(M,\chi),\,(B,\lambda)\;\to(I,\varphi),\,(A,\alpha),\,(B,\lambda) \tag{3.21}$$

where B' denotes here the old B and B becomes the corresponding new variable defined by $B = B' - (A-a)$, with $a \equiv \overline{K} - 2\overline{M}$ and:

$$\begin{aligned} &\gamma = -(\alpha+\lambda+\pi/2) \qquad && K = 2I-(A-a)+4\omega_T J_3 \\ &\chi = 2(\alpha+\lambda)+\varphi && M = I + 2\omega_T J_3 \end{aligned} \tag{3.22}$$

where the shifts have been introduced so that the unstable point of the pendulum is η–close to $(I,\varphi)=(0,0)$ and so that the initial datum $(\overline{K},\overline{M})$ corresponds to $(\overline{I},\overline{A})\equiv(0,0)$. The hamiltonian (3.17), in the canonical variables (I,φ), (A,α), (B,λ), takes the form (up to a neglected constant):

$$H = \omega_T B + \frac{I^2}{2J_3} + h + \eta\Big[v_0 + ev + e^2 v_2\Big] \tag{3.23}$$

where $h = h(A,I)$ is a suitable function.

One can check that, neglecting terms of $O(\nu^2)$, the integrable part of the hamiltonian becomes: $-\eta\omega A - \eta A^2/(8J_3) + \eta O(I) + const$ where $O(I)$ depends on A, I and vanishes for $I=0$, leading to the standard "D' Alembert equinox precession" $\dot\gamma' = -\eta\omega$, (see (3.9) and [CG], appendix A6,A7).

For the purposes of illustration we shall further simplify the analysis by supposing that the term with $O(I)$ is eliminated (a simplified version still giving the D' Alembert precession as well as all the conceptual difficulties that are treated in [CG] without introducing the latter approximation) and that the c_j, d_j coefficients are constants and, furthermore, that most of them vanish. Namely we shall consider the model:

$$H = \omega_T B + \eta\omega A - \frac{\eta A^2}{8J} + \frac{I^2}{2J} + \eta(V_0 + eV_1 + e^2 V_2) \tag{3.24}$$

where $J>0, \overline{K}>0$ and:

$$\begin{aligned} V_0 =& \nu\overline{d}_1\cos(2\gamma+\chi) + \nu c_1 \cos\chi \\ V_1 =& \frac{1}{2}\nu\overline{d}_1 \cos(2\gamma+\chi+\lambda) \\ V_2 =& \overline{d}_2\nu^2\cos(2\gamma+2\lambda) \end{aligned} \tag{3.25}$$

with $\overline{c}_1, \overline{d}_1, \overline{d}_2 > 0$ and with $\overline{\nu} \equiv \nu > 0$ and the angles have to be evaluated in terms of the new α, φ, λ, see (3.22).

The above form of the constants reflects the form that they actually have in the non truncated model (in particular we could absorb ν into the c's and d's), see [CG]: and we imagine fixing the constants to be equal to quantities to which the corresponding non approximated functions are close (see [CG]). For instance $g^2 = \nu \overline{d}_1$ turns out to be: $g^2 = \frac{1}{2}\nu(1+\cos i)\omega_T^2$, if $\cos i$ is $K/\overline{M}$ at the point of coordinate K (see (3.6)).

Our simplified model becomes:

$$H = \omega_T B + \eta\omega A - \frac{\eta A^2}{8J_3} + \frac{I^2}{2J_3} + \eta g^2(\cos\varphi - 1) + \eta\big(\beta v_0 + e v_1 + e^2 v_2\big) \tag{3.26}$$

where $\beta \equiv 1$ is introduced for later reference and:

$$\begin{aligned} g^2 =& \frac{1}{2}\nu(1+\cos i)\omega_T^2 \\ v_0 =& c_1 \cos 2(\alpha+\lambda) + \varphi, \qquad v_1 = 2d_1\nu\cos(\varphi+\lambda), \qquad v_2 = \frac{3}{4}d_2\nu^2\cos 2(\alpha+\varphi) \end{aligned} \tag{3.27}$$

To fix the time scale of the pendulum (I,φ) in (3.27) to be η independent we perform a rescaling of the action variables $(A,I,B) = \eta^{\frac{1}{2}}(A',I',B')$ followed by a time rescaling by η^{-1} and we transform the above hamiltonian into a new hamiltonian given, abolishing the primes over the new variables to avoid introducing too many symbols, by:

$$H = \frac{\omega_T}{\eta^{\frac{1}{2}}} B + \eta^{\frac{1}{2}}\omega A - \frac{\eta A^2}{8J_3} + \frac{I^2}{2J_3} + g^2(\cos\varphi - 1) + \big(v_0 + e v_1 + e^2 v_2\big) \tag{3.28}$$

where the angle variables do not change in the rescaling transformations.

In the new variables the line $\bar{\mathcal{L}}$ becomes simply:

$$\mathcal{L} = \{I = 0;\ -\eta^{-\frac{1}{2}}\overline{A} < A < \eta^{-\frac{1}{2}}\overline{A}\} \tag{3.29}$$

and $\eta^{\frac{1}{2}}A$ proceeds linearly from $-\overline{A}$ to $\overline{A}$ as s increases from 0 to 1. If we regard the terms $v_0 + ev_1 + e^2v_2$ as a perturbationn the it is immediate to check that $\mathcal{L}$ is a diffusion path and one can take the constants K, τ, t to be $O(\eta^{-\frac{1}{2}}), 2, 7$ respectively.

The (3.28) exhibits the difficulties related to the *a priori* stable systems mentioned in §1: but with a few differences due to the fact that the system with $\eta = e = 0$ is quite degenerated in the sense of §1: the h is in fact with zero hessian determinant *even in the non forced variables* K, M: this is reflected in the fact that the fast frequences in the (3.28), which could be considered a normal form of our hamiltonian near the resonance, are not really fast.

In fact there are two frequences $\omega_T \eta^{-\frac{1}{2}}$, the rotation speed of the yearly motion (related by the resonance condition to that of the daily motion), and the $\eta^{\frac{1}{2}}\omega$, the precession speed which is of order $\eta^{\frac{1}{2}}$ rather than $\eta^{-\frac{1}{2}}$. And the latter is much slower.

The theory of §2 cannot be directly applied: for two reasons.

a) the Melnikov integral (2.3) yields a critical point $\vec{\alpha}_s \equiv \vec{0}$, as the parameter s for the $\mathcal{L}$ curve varies in $[0,1]$, at $\alpha = \lambda = 0$ with a hessian matrix determinant $\delta(e,\eta) = \det \vec{\partial}^2_{\vec{\alpha}} M(\vec{0})$ which has order $O(e \exp -\eta^{-\frac{1}{2}}\pi/2g)$. Therefore, unless we are willing to take e extremely small compared to η, the quantitative condition in theorem 1 cannot be met.

b) even if we would be willing to take the step (very unsatisfactory, because in realistic cases e and η are not too different) of assuming that e is much smaller than $\exp -O(\eta^{-\frac{1}{2}})$, we still could not apply the theorem 2 by the cause that the perturbation in (3.28) *is not* small as $e \to 0$ because of the term $v_0 = O(1)$.

In the next section we discuss how the above two difficulties can be bypassed.

§4 *Averaging and large homoclinic splitting.*

To use perturbation theory we must first understand why the perturbation in (3.26) can be in some sense neglected: this is basically due to the fact that if $e = 0$ the angle λ rotates very fast compared to the angle α (the first at velocity $O(\eta^{-\frac{1}{2}})$ and the second at velocity $O(\eta^{\frac{1}{2}})$ relative to the resonance oscillations which are at velocity of $O(1)$ by our rescalings). Thus the qualitative ideas on the *averaging of the fast variables* would suggest that the part v_0 (and the part ev_1 which also depends only on fast variables) couple to the other variables with an *effective coupling of order at least as small as* $O(e)$.

We consider, in general, the hamiltonian system:

$$H = \eta^{-1/2}\overline{\omega}_1 B + \eta^{1/2}\overline{\omega}_2 A + \eta\frac{A^2}{2J} + \frac{I^2}{2J_0} + J_0 g_0^2(\cos\varphi - 1) + \beta(F + \mu f) \tag{4.1}$$

with the quantities J, J_0, g_0 being positive constants and with the f, F being trigonometric polynomials with constant coefficients of degree $\le N$. The function F will be supposed to be "unimodal", *i.e.* to depend on the $\vec{\alpha} = (\lambda, \alpha)$, only via the quantity $\vec{\nu}_0 \cdot \vec{\alpha}$. In other words the F contains only one particular "fast angle" and its multiples.

Then the following *averaging* property holds:

Theorem 3: Suppose that $|\mu| \le \eta^c$, $c > 10$, and fix $x > 0$ and $0 < \sigma < 1/2$. Then there is a holomorphic canonical map casting the hamiltonian H in a form (for η small):

$$\frac{\overline{\omega}_1 B}{\eta^{1/2}} + \eta^{1/2}\overline{\omega}_2 A + \frac{\eta A^2}{2\hat{J}} + g(\eta^{1/2}A, pq)pq + \eta^x \hat{f}(I, A, \varphi, \lambda, \alpha) \tag{4.2}$$

with $\hat{J}, g, \hat{f}$ bounded and holomorphic in a domain $\Omega(\kappa,\rho,\xi,\eta^c)$ defined by:

$$\big\{|p|,|q|<\kappa,\ |\operatorname{Im}A|<\rho\eta^{-1/2},\ |\operatorname{Im}\alpha_j|<\xi,\ |\mu|<\eta^c\big\} \tag{4.3}$$

and for $|\beta|<B^*\eta^{-\sigma}$, with suitable $B^*,\kappa,\rho,\xi>0$ and for η_0 small enough, (one can take $\eta_0^{1/2}\log\eta_0^{-1}<d\cdot x$ for a suitable $d>0$).

One tries to prove the lemma by solving recursively the equations for the generating function of the transformation that conjugates (4.1) to (4.2): the variables p,q exist in the case of the free pendulum as it is well known since Jacobi (see [CG] for an explicit construction of the p,q variables in the case of the pendulum) and in the perturbed case they are shown to still exist and to be close to the unperturbed ones, [CG].

The reason for the validity of (4.2) is then simply that in the free part of (4.1) (*i.e.* for $\beta=0$) no strong resonances occur with $|\vec{\nu}|\le O(\eta^{-1})$, and all denominators appearing in trying to apply perturbation theory, as described, are bounded below by $O(\eta^{1/2})$. Hence we can proceed to perturbation theory of very large order, essentially $O(\eta^{-1/2})$, after taking advantage in the first step of *large* denominators (of order $O(\eta^{-1/2})$, *i.e.* after taking advantage of the averaging phenomenon) to reduce the size of F. At higher order one still gains as the terms involving only F still have large divisors, while the ones involving both f and F are much smaller so that a divisor as big as $\eta^{\frac{1}{2}}$ is not sufficient to make them large.

The method used to deduce (4.2) is very similar to the usual method developed in the Nekhorossev resonance theory, [N], [BG], [G2]. Note however, that in (4.2) the slow modes are in the remainder term (while at first, perhaps, one would expect them to remain of order μ): see [CG] for a proof.

The above theorem can be used, in discussing (3.28), to guarantee that the line $\mathcal{L}$ of two dimensional invariant tori that exist if $\eta,e=0$ and correspond to the $I=\varphi=0$ and $\vec{A}(s)\in\mathcal{L}$ (parameterized by λ,α) still exists after the perturbation, only slightly deformed, togheter with their stable and unstable manifolds, or *whiskers*, with the exception, perhaps, of a few value of s clustered in intervals, "gaps", of width of order $e^{x/t}\eta^{x/t}$, where t is the constant introduced in the definition of diffusion path. In fact the following theorem holds ([CG], §11, lemma 5, and remarks following it):

Theorem 4: consider a hamiltonian which, along a diffusion path $\mathcal{L}$, has a form like (4.1) with μ replacing η^x: then the set of s for which one cannot construct invariant two dimensional tori, and the relative stable and unstable manifolds, which are analytic (as functions of e,η) deformations of the unperturbed invariant tori (and of their stable and unstable manifolds) corresponding to the line $\mathcal{L}$ has measure $O(\mu^{1/t})$ if $\mu<\eta^c$.

On the other hand the condition, in theorem 2, $\varepsilon<c(\mathcal{L})$ is a condition which has a geometrical meaning: it really means that the minimum angle between a tangent to the stable and to the unstable manifolds of two surviving invariant tori at the homoclinic

point $\varphi = \pi, \alpha = \lambda = 0$ (which can be seen to exists by symmetry reasons when the perturbation contains only cosines of the angles) is larger than the gap, measured in terms of the parameter s, between the invariant tori surviving the perturbation, see [CG].

Hence the above theorem tells us that in the precession case the gaps, along a diffusion line $\mathcal{L}$, between the invariant tori surviving the perturbation are smaller than any prefixed power in $\eta + e$ in spite of the fact that the perturbation looks of order $O(1)$.

Thus the presence of a perturbation of order $O(1)$ does not affect the density of the tori: at least *if the perturbation is unimodal on a fast angle in the above sense.*

But we still have to see why the minimum angle between the stable and unstable manifolds of a surviving torus is larger than the gaps, at least in the model for the precession of §3.

This is true if e is not *too small* compared to η: we find that $e = \eta^c$ for some $c > 0$ is "not too small". In fact consider the hamiltonian (4.1) near the segment Δ where $\partial_A h = \eta^{\frac{1}{2}}\omega + \eta A/8J_3$ varies, as $A \in \Delta$, in an interval $\eta^{1/2}[\overline{\omega}, \tilde{\omega}]$ with $\tilde{\omega}, \overline{\omega} > 0$. Then we can prove:

Theorem 5: There is $c > 0$ such that if $\mu = \eta^c$ the hamiltonian system has invariant tori which, if η is small enough, have whiskers with a homoclinic splitting δ of $O((\eta^{3/2}\mu^2)^2)$ as $\eta \to 0$, provided a certain sum (see [CG], §11, lemma 4) does not vanish accidentally. In the case of the model (3.28) we find (see [CG]) up to a factor $(1 + O(\nu))$ and to leading order in η:

$$\sqrt{-\delta} = \frac{9}{2}\,\eta^{3/2}\, e^2\, J_3\omega_T\, \sin\vartheta\, \frac{\sin i \cos^2 i\,(1 - \cos i)}{(1 + \cos i)} \tag{4.4}$$

where i denotes the inclination (i.e. $\cos i = K/\overline{M}$ at a generic point of $\mathcal{L}$, (denoting again by K the original variable of the z-axis spin component); $\cos\vartheta \equiv \overline{L}/\overline{M}$ is the eulerian nutation constant. The result is uniform for i in a closed interval strictly contained in $(0, \pi/2)$.

Hence in the case (3.24) the homoclinic splitting is a power of η, while the gaps are smaller than a prefixed power of η, provided $e = \eta^c$ with $c > 0$ large enough (we find that $c > 92$ is sufficient)

The above discussion shows the ideas that are behind the proof of:

Theorem 6: Consider the precession model and the diffusion line $\mathcal{L}$ of §3. The line $\mathcal{L}$ physically describes a variation of the inclination angle of the spin axis at fixed total angular momentum M and in the resonance 2:1 between the annual motion and the daily motion. Then $\mathcal{L}$ is open for diffusion under the perturbation given by the *gravitational attraction by the centre of the keplerian orbit of the body if* $e = \eta^c$ *for some* c *large enough.*

Note that to obtain theorems 3,4 one makes use, in [CG], of the special structure of the doubly resonant hamiltonian arising in the celestial problem. Note also that one has to take e to be a power of η ($e = \eta^c$, with $c > 92$): the larger the power the smaller η has to be in order that a given piece of $\mathcal{L}$ be open for diffusion; on the other hand c cannot be taken too small as one need it to be large to show the averaging properties in theorem 4. The complete proof can be found in [CG], where also the more general model (3.7) is considered.

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