FULLY DEVELOPED CHAOS

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ABSTRACT

We describe a non linear partial differential equation on a compact domain with solutions regular but with rough behaviour down to a small scale.

1. Introduction

The aim of this paper is to present a simple model exhibiting the development of a multifractal structure through a dynamical evolution. There are many physical systems with an apparent complexity which increases in time like for example the dendritic growth of a solid, the DLA dynamics, the Safmann-Taylor fingers, interfaces etc. One of the most spectacular example being the snowflakes. The phenomenon of spatial intermittency in turbulent flows can also be interpreted as a manifestation of singularities at some scale[1,2], and the discussion below will be partly inspired by this example.

All these systems evolve in time according to some known dynamical equations and develop structures of increasing complexity (sometimes called turbulent structures). During a long transient period, the complexity of theses systems does not seem to be stationary, and the apparent roughness of the solutions seem to invade smaller and smaller scales down to a physical cut-off (diffusion length, capillarity length etc.).

One of the difficulties in understanding theoretically these complex structures is that the dynamical evolution of the different spatial scales are intricately coupled. One would like to understand why these time evolutions which are given by partial differential equations (supplemented in some problems with moving boundaries) produce these apparently irregular structures. Such questions are rarely considered in the Mathematics literature where one finds a lot of information about existence, uniqueness and regularity of solutions or the opposite case of the development of a singularity in finite time.

In order to gain a better understanding of the phenomenon we study a simple model whose time evolution is given by a partial differential equation. The solution is unique, bounded and regular at any finite time. However it develops a spatial roughness which increases in time and invades smaller and smaller scales. Our aim is to discuss this simple model which can be understood without too much technical

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difficulties. As we will see, the behaviour seems to be qualitatively similar to the more realistic time evolutions mentioned above, although the important effect of the coupling between the different scales remains to be understood.

We emphasize that the models discussed below have deterministic time evolutions, and their complexity is coming from an intrinsic stochasticity. This is a different situation from the models with stochastic forcing which have been previously studied (see for example[3]).

The so called turbulent transport of a passive quantity is a phenomenon related to the questions discussed in the present paper[4]. However the complexity is again of an exterior (though deterministic) origin: the chaotic properties of the transporting flow.

2. The model

We now describe our model by giving first the phase space and then the time evolution. We will only describe a very simple version of the idea, many other models can be developed along the same lines.

We consider as physical base space a circle where the points are indexed by an angle θ. This angle will be the space variable of the model. At each spatial location we have a local order parameter which is a three dimensional vector. Therefore, the phase space of our system will be the space of periodic vector valued functions on the circle

$$\theta \to v(\theta) \ .$$

This is to be compared with the case of a fluid motion where the phase space is the set of vector fields (velocity fields) in the container with the appropriate boundary conditions or with the case of a growing closed curve given parametrically. We now introduce the time evolution, first in degenerate form. This time evolution will determine the configuration $v(\theta,t)$ of the system at time t given the configuration $v(\theta,0)$ of the system at time 0. It is given by (a partial differential equation)

$$\partial_t v(\theta,t) = X(v(\theta,t)) \ , \tag{1}$$

where X is the vector field of the Lorenz model with parameter values in the chaotic regime, for example

$$\begin{aligned} X_1(x_1,x_2,x_3) =& 10(x_2 - x_1) \ , \\ X_2(x_1,x_2,x_3) =& 28x_1 - x_2 - x_1x_3 \ , \\ X_3(x_1,x_2,x_3) =& -8x_3/3 + x_1x_2 \ . \end{aligned} \tag{2}$$

To be rigorous, one should take instead a geometric Lorenz flow for which chaos has been proven to occur[5,6]. In order to simplify the exposition we will however continue to use the Lorenz system.

The above evolution equation may look surprising at first sight in particular because it does not contain terms with partial derivatives in θ (although we want to view this equation as a partial differential equation in time and space). This is however an interesting simplification which allows for an easy discussion of the main phenomenon. It is at least formally reminiscent of the Eulerian limit of the Navier-Stokes equation with a vanishing viscosity.

Let us consider an initial condition $v(\theta,0)$ for our system, that is to say a periodic vector valued function. Let us assume that this function is very regular (infinitely differentiable), and not constant. After a period of time t has elapsed, the configuration is a function $v(\theta,t)$ which is still periodic in the angle θ. It is easy to verify that this function is also infinitely differentiable. However down to a certain scale it has in a certain sense very large derivatives. This can be seen more precisely as follows. Fix a small number ϵ and consider the variation of the function on an interval of length ϵ. This is given by

$$v(\theta+\epsilon,t)-v(\theta,t)\sim\epsilon \mathrm{D}\phi_t(v(\theta,0))\partial_\theta v(\theta,0)$$

where ϕ_t is the flow of the Lorenz vector field (2):

$$\partial_t\phi_t(\cdot)=X(\phi_t(\cdot))\ ,$$

and $\mathrm{D}\phi_t$ is the differential of this map. We see from this formula that due to the sensitivity to initial condition (or positive Lyapunov exponent) of the Lorenz system, for a fixed number ϵ, if we wait long enough, the increment of the configuration on an interval of length ϵ will become of order 1. Of course, if on the other hand we fix the time and consider a scale which is much smaller than the expansion factor, we see regular increments since the configuration is always infinitely differentiable. In other words, small variations of the angle θ induce small variations of the vector field v, but these small variations are exponentially amplified during the time evolution.

Note also that the space derivative of v exhibits spatial intermittency. Indeed, we expect that for most values of $v(\theta,0)$, the vector

$$\mathrm{D}\phi_t(v(\theta,0))\partial_\theta v(\theta,0)$$

will be of the order of $e^{\lambda t}$ where λ is the Lyapunov exponent. However this result is in general not true everywhere (except for very uniform systems), and we expect a larger result ($e^{\alpha t}$ with $\alpha>\lambda$) on sets of Hausdorff dimension smaller than 1 in the one dimensional manifolds of $v(\theta,0)$ (or θ) variable.

As mentioned before, this model does not exhibit any coupling between the different angles θ (our space variable). One can add to the evolution equation a coupling in the form of a small diffusive term. The evolution equation is now

$$\partial_t v(\theta,t)=\eta\partial_\theta^2 v(\theta,t)+X(v(\theta,t))\ . \tag{3}$$

This equation is not as simple to discuss rigorously as the one before, but it is not very difficult to prove that regular initial conditions will evolve into uniformly bounded and uniformly regular solutions. This follows by standard energy-like estimates. However, if the diffusion constant η is small, one gets a rather rough function down to a scale $\mathcal{O}(\eta^{1/2})$ if time is large enough.

Another variant of the model is obtained by using a vector field different from the Lorenz system. One can for example use a dynamical system with intermittent properties and this will lead to more space-time intermittency.

3. Quantitative results

As we have seen above, the model described by equation (1) develops roughness at small scale. We now briefly describe the multifractal properties of the solutions by looking at the local variation of one of the component. This local variation can be defined in terms of a local Hölder exponent. One possible definition for a real function $g(\theta)$ is (see Hardy[7] for a different one)

$$\alpha(\theta) = \lim_{\epsilon \to 0} \frac{\log |g(\theta+\epsilon) - g(\theta)|}{\log \epsilon} \ .$$

It is easy to generalize this formula for vector valued functions. The associated $f(\alpha)$ function would measure as usual the Hausdorff dimension of the level sets of the function α.

As it is now well known, a powerful method for dealing with the function $f(\alpha)$ is the thermodynamic formalism. Here we can adapt the analysis to the fact that our order parameter is of dimension 3 by defining the partition function by

$$Z_n(\beta) = \sum_{k=0}^{2^n-1} \|\phi_t(v((k+1)2^{-n})) - \phi_t(v(k2^{-n}))\|^\beta \ .$$

The pressure function $P(\beta)$ is defined by

$$P(\beta) = \lim_{n\to\infty} n^{-1} \log Z_n(\beta) \ ,$$

if the limit exists. Under some technical conditions, the function $f(\alpha)$ is then the Legendre transform of P.

The above expression for Z_n can be simplified by using Taylor's formula. We get

$$Z_n(\beta) \simeq 2^{-n(\beta-1)} \int_0^{2\pi} \|\mathrm{D}\phi_t(v(\theta,0))\partial_\theta v(\theta,0)\|^\beta \, d\theta \ .$$

Strictly speaking, this formula gives a good approximation for $Z_n(\beta)$ only if the remainder in the Taylor formula can be neglected. This is certainly the case if n is large enough. The integral above looks very much like the partition function of the Lyapunov exponent. The measure is not however the invariant measure of the Lorenz system but the Lebegue measure on the curve $v(\cdot,0)$. However, if the curve meets the unstable foliation on a set of positive Lebegue measure, one gets asymptotically for large time the partition function of the Lyapunov exponent of the SRB measure (see Bowen[8] for a definition). We therefore obtain

$$P(\beta) \simeq 1 - \beta + \frac{t}{n} P_L(\beta) \ ,$$

where $P_L(\beta)$ is the pressure of the Lyapunov exponent for the SRB measure. From this formula, we see that if $n \gg t$ we get a trivial result corresponding to the fact that the solution is regular at very small scales. However, if we take a diagonal procedure, letting n diverge with t/n fixed, we get a non trivial result. The formula gives of course a good approximation only if t/n is small, because otherwise the remainder term in the Taylor formula cannot be neglected anymore. In particular, it is easy to see that if t/n is larger than the minimal expansion rate of the system, we get $P(\beta) = 1$.

One would like of course to get a result in the infinite time regime. Such a result could be obtained using an invariant measure for the dynamical system. It turns out that in the case of zero diffusive coupling one can construct easily an invariant measure. It is the continuous product[9] of the Lorenz SRB measures on the circle. The support of this measure seems however to be very large, and it is not known if the pressure is defined for almost every curve with respect to this measure.

The introduction of a diffusive coupling should improve the situation by producing a more reasonable invariant measure. One can then ask if there is a diagonal limit obtained by letting n diverge with $n/\log\eta$ fixed which is almost surely independent of the initial condition chosen at random according to this reasonable measure.

4. References

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