Finite volume mixing conditions for lattice spin systems and exponential approach to equilibrium of Glauber dynamics.

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Abstract. We critically review various finite volume conditions in classical statistical mechanics together with their implications both for the Gibbs state and for an associated Glauber type dynamics. Moreover we considerably improve some old results by Holley and Aizenamn and Holley on the relationship between mixing properties of the Gibbs state and fast convergence of the Glauber dynamics. Our results are optimal in the sense that, for example, they show for the first time fast convergence of the dynamics above the critical point for the d-dimensional Ising model with or without an external field.

1. Introduction.

In the recent years a series of works by several authors have been dedicated to the so called "finite size conditions" to analyze the properties of systems of classical statistical mechanics in the pure phase region; they have been introduced in order to explicitely show that, in the pure phase region, a system behaves as if it was weakely coupled provided it is analyzed on the correct, sufficiently large, scale. To simplify the exposition let us consider Ising-like lattice spin systems. Finite size conditions or, in the language of Dobrushin and Shlosman (DS) constructive conditions are mixing properties, involving some parameters, of the Gibbs measure, corresponding to a given interaction, in a finite volume Λ. Let us call, for instance, such a condition $C(\Lambda, k, \gamma)$, where Λ is the volume and k,γ are (in this case two) parameters; for example γ is a rate of exponential decay of truncated correlations, k is a constant in front of the exponential .

The importance of these conditions becomes clear once one is able to prove a statement of the following type:

There exists a finite set of volumes $\Gamma = \{\Lambda_1, \ldots, \Lambda_N\}$, depending on k,γ such that if one supposes true $C(\Lambda, k, \gamma)$ for every $\Lambda \in \Gamma$ then the infinite volume Gibbs measure is unique and enjoys some properties of weak dependence like exponential decay of truncated correlations.

For example DS in their theory of Completely Analytical Interactions prove a theorem like above with Γ given by *all subvolumes* of a cube with suitable edge $L = L(k, \gamma)$. Whith this hypothesis they prove exponential clustering not only for the unique infinite volume Gibbs measure but also for Gibbs measure relative to *arbitrary finite or infinite volumes.* This result is extremely strong since it shows very good mixing properties of the Gibbs state also in strange (pathological) shapes. This point will be discussed in Section 3 in connection with another approach ([12],[13]) giving weaker results but with a wider applicability. The main point of our discussion will be that DS conditions are too strong to be verified near the coexisting curve corresponding to a first order phase transition. This is because these conditions, always assuming uniformity w.r.t. the boundary conditions (b.c.), have to include cases of regions Λ whose shapes are such that the exterior boundary $\partial\Lambda$ " enters ", so to speak, into the bulk of Λ (think to a subset of a 2-D layer in $\mathbf{Z^3}$ or to a cube with a regular array of holes). Then, for some particular b. c. one can produce, in these pathological Λ's, exactly the situation corresponding to values of thermodynamical parameters producing a first order phase transition. In [12], [13] for regular (say Van Hove) regions results similar to the ones of DS are proven starting from weaker conditions involving only "fat" regions (say cubes) and covering also the part of the phase diagram near the coexistence line where DS condition fails.

Finite size conditions play a role also in some dynamical problem.

Consider a Glauber dynamics namely a single spin flip Markov process reversible w.r.t. the Gibbs measure corresponding to a given interaction. Some authors and especially R.Holley and more recently Zegarlinski and Strook and Zegarlinski investigated the connection between mixing properties of the (unique) invariant Gibbs measure and the speed of approach to equilibrium under very general assumptions on the dynamics. We want to quote, for example, the fundamental paper [6] by Holley where the author, in particular, for the case of short range translation invariant attractive (see Section 2) stochastic Ising models introduces a strong finite size condition, referring to the invariant Gibbs measure, that ensures exponential convergence to equilibrium in a strong sense.

Before concluding this short introduction we warn the reader that there are different types of finite volume equilibrium mixing properties and different notions of exponential approach to equilibrium. The first ones can be divided into *weak mixing* and *strong mixing* . Both notions can be expressed as weak dependence, inside Λ, say in $x \in \Lambda$, on the value of a conditioning spin, say in $y \in \partial\Lambda$. We have strong mixing if the influence of what happens in x decays with the distance $|x - y|$ of x from y whereas we speak of weak mixing when the influence decays with the distance of x from the boundary $\partial\Lambda$ and not from y.

There are models, (the so called Czech models) satisfying weak mixing but violating strong mixing also for regular domains. They exhibit absence of phase transition in the bulk but a sort of phase transition with long range order in a layer near to the boundary takes place.

On the other hand we can also consider approach to equilibrium in different possible senses depending on which norm we want to use (L^2 or L^∞) and whether we want results directly for the infinite volume dynamics or for the finite volume dynamics with estimates uniform in the volume and in the boundary conditions.

In the present paper, after some definitions concerning our model (Section 2), we

i) review and discuss the role of the different notions of mixing conditions both at equilibrium and in dynamics (Section 3)
ii) improve previous results by proving, for ferromagnetic systems, exponential convergence in the uniform norm by only assuming weak mixing (Section 4)
iii) prove uniform exponential approach to equilibrium in a very strong sense for *any* (finite or infinite) sufficiently regular volume starting from strong mixing condition in a cube (Section 5)
iv) give some applications (Section 6).

We only sketch the proofs and refer to [10] for more details.

2. Definitions.

2.1. The model.

Let us first describe the Hamiltonian of our spin system. Given a subset Λ of the lattice $\mathbf{Z^d}$ and given a spin configuration τ outside Λ we set for any spin configuration $\sigma \in \Omega_\Lambda \equiv \{-1,1\}^\Lambda$:

$$H^\tau_\Lambda(\sigma) \;=\; -\sum_{X\cap\Lambda\neq 0} J(X)\sigma_X \tag{2.1}$$

where σ_X is the product of the values of the spins at the sites of the set X. It is understood here that the values of the spins at sites x not in Λ are those of the "boundary" configuration $\tau \in \Omega_{\Lambda^c}$. The potential J(X) is assumed to be translation invariant and finite range. Some of our results are proved with an additional hypothesis which in the sequel will be called *attractivity.* The system is said to be *attractive* or *ferromagnetic* if the local field at the origin

$$h(\sigma) \;=\; \sum_{X;\,0\in X} J(X)\sigma_{X\setminus\{0\}} \tag{2.2}$$

is an increasing function of the spins $\sigma_x \quad x \neq 0$

Remark It is easy to check that in the case of only two body interaction attractivity coincides with the requirement $J(x,y) \;\geq\; 0$.

Given H^τ_Λ we will denote by μ^τ_Λ the associated Gibbs state. If there exists a unique Gibbs state in the infinite volume limit $\Lambda \to \mathbf{Z^d}$ independent of the boundary conditions τ then it will be simply denoted by μ. For shortness we will use the notation $\mu^\tau_\Lambda(f)$ to denote the mean of the observable f under the Gibbs state μ^τ_Λ

2.2. The dynamics.

The stochastic dynamics that will be discussed in this paper will be one of the many standard stochastic Ising models for the hamiltonian (2.1) (see [8]). We will need to analyze the stochastic Ising model in finite volume Λ with boundary conditions τ as well as in the whole lattice $\mathbf{Z^d}$. Both cases are defined through their jump rates that will always be denoted by $c_x(\sigma, a)$, $x \in \mathbf{Z^d}$ or $x \in \Lambda$ and $a \in \{-1, +1\}$ whenever confusion does not arise. Then the generator L of the dynamics takes the form :

$$Lf(\sigma) \;=\; \sum_{x,a} c_x(\sigma, a)(f(\sigma^{a,x}) \;-\; f(\sigma)) \tag{2.3}$$

where $\sigma^{a,x}$ is the configuration obtained from σ by putting the spin at x equal to the value a.

In order to simplify the exposition and the computations we decided to take from the beginning a precise form for our jump rates :

$$c_x(\sigma, a) \;=\; \mu^\sigma_{\{x\}}(\eta(x) \;=\; a) \;=\; \frac{1}{1 + \exp(-2a\sum_{X;\, x \in X} J(X)\sigma_{X \setminus \{x\}})} \tag{2.4}$$

where it is understood that if we are in a finite volume Λ the configuration σ agrees with the boundary configuration τ outside Λ. This choice corresponds to what is known as the *heat bath* dynamics. In the finite volume case our expression for the jump rates makes sure that the Markov process generated by the jump rates on $\{-1, 1\}^\Lambda$ is reversible with respect to the Gibbs state μ^τ_Λ. This implies that μ^τ_Λ is the unique invariant measure of the process. This important fact holds also in the infinite volume limit if the Gibbs state is unique . Finally if the interaction is attractive and if in the space of spin configurations we introduce the partial order $\sigma \;\leq\; \eta$ iff $\sigma(x) \;\leq\; \eta(x)$ for all x then there exists a coupling in our probability space such that if σ^η_t denotes the evoluted at time t of the above stochastic Ising model starting from the configuration η then, using the coupling :

$$\sigma^-_t, \leq \; \sigma^\eta_t \;\leq\; \sigma^{\eta'}_t \;\leq\; \sigma^+_t$$

if $\eta' \;\geq\; \eta$, where σ^+_t is the evolution starting from all spins equal to plus one and analogously for σ^-_t.

3. Review and critical analysis.

3.1. Finite volume mixing conditions.

- We first give the definition of *variation distance* between two probability measures P, Q on the finite set Y :

$$Var(P,Q) = \frac{1}{2}\sum_{y\in Y}|P(y)-Q(y)| = \max_{X\subset Y}|P(X)-Q(X)| \tag{3.1}$$

- Given a measure μ_Λ on Ω_Λ we call *relativization* of μ_Λ to Ω_Δ with $\Delta\subset\Lambda$, the measure $\mu_{\Lambda,\Delta}$ on Ω_Δ given by

$$\mu_{\Lambda,\Delta}(\sigma_\Delta) = \sum_{\sigma_{\Lambda\setminus\Delta}}\mu_\Lambda(\sigma_{\Lambda\setminus\Delta},\sigma_\Delta) \tag{3.2}$$

- We say that a Gibbs measure μ_Λ on Ω_Λ satisfies a strong mixing condition with constants C,γ if for every subset $\Delta\subset\Lambda$:

$$\sup_{\tau\in\Omega_{\Lambda^c}} Var(\mu^\tau_{\Lambda,\Delta}\ \ \mu^{\tau^{(y)}}_{\Lambda,\Delta}) \le Ce^{-\gamma\mathrm{dist}(\Delta,y)} \tag{3.3}$$

where $\tau^{(y)}_x = \tau_x$ for $x\neq y$.
We denote this condition by $SM(\Lambda,C,\gamma)$
- We say that a *strong mixing* condition in the sense of *truncated* expectations holds for the measure μ_Λ on Ω_Λ, with constants C,γ if for every cylindrical functions f,g with supports $S_f, S_g\subset\Lambda$

$$|\mu_\Lambda(f,g)| \le\ \ C\parallel f\parallel\ \ \parallel g\parallel\ \ e^{-\gamma\mathrm{dist}(S_f,S_g)} \tag{3.4}$$

and we denote is by $SMT(\Lambda,C,\gamma)$.

Here $\mu_\Lambda(f,g)$ denotes the truncated expectation of f and g:

$$\mu_\Lambda(f,g) = \mu_\Lambda(fg) - \mu_\Lambda(f)\mu_\Lambda(g)$$

It is easy to show that, supposing that $SMT(\Lambda,C,\gamma)$ holds true, then there exists C' such that $SM(\Lambda,C',\gamma)$ holds. It is possible to give some weaker converse statement (see below) so that, in a proper sense, SM and SMT are equivalent.
- We say that a Gibbs measure μ^τ_Λ satisfies a *weak mixing condition* with constants C,γ if for every subset $\Delta\subset\Lambda$

$$\sup_{\tau,\tau'\in\Omega_{\Lambda^c}} Var(\mu^\tau_{\Lambda,\Delta},\mu^{\tau'}_{\Lambda,\Delta}) \le C\sum_{x\in\Delta, y\in\partial^+_r\Lambda}\exp(-\gamma|x-y|) \tag{3.5}$$

where

$$\partial^+_r\Lambda = \{x\notin\Lambda : \mathrm{dist}(x,\Lambda)\le r\}$$

We denote this condition by $WM(\Lambda,C,\gamma)$.
An important notion, already quoted in the Introduction, is what we call *effectiveness*.
- Given two families Γ,Γ' of subsets of $\mathbf{Z^d}$ a strong mixing condition is (Γ,Γ')-effective if C,γ are such that by supposing that $SM(\Lambda,C,\gamma)$ holds for every Λ in the class Γ we have that there exist C',γ' such that $SM(\Lambda',C',\gamma')$ holds for every Λ' in Γ'.

Of course the interesting cases correspond to a *finite* family Γ and an *infinite* Γ' (finite size condition for exponential decay of truncated correlations on arbitrarily large volumes).

3.2. Some known equilibrium results.

The following theorems DS1,DS2,OP hold in the general (finite range, translation invariant, not necessarily attractive) case. Theorem H1 is restricted to the attractive case.
We start from a result by Dobrushin and Shlosman concerning weak mixing [3].

Theorem DS1 (Dobrushin - Shlosmann [3])
Suppose that a suitable finite size condition $DSU(\Lambda_0, \delta)$ on a finite volume Λ_0 is satisfied for some Λ_0 and $\delta < 1$; then $\exists\ C > 0, \gamma > 0$ such that condition $WM(\Lambda, C, \gamma)$ holds *for every* Λ.

In the general case to give condition $DSU(\Lambda_0, \delta)$ we need some more definitions. In the attractive (ferromagnetic) case the condition is :

$$\sup_{\tau \in \Omega_{\Lambda_0^c}} Var(\mu^{\tau}_{\Lambda_0,x} \quad \mu^{\tau^{(y)}}_{\Lambda_0,x}) \leq \alpha_{x,y} \tag{3.6}$$

$$\sum_{x \in \Lambda_0, y \in \partial_r^+ \Lambda_0} \alpha_{x,y} \quad \leq \delta |\Lambda_0| \tag{3.7}$$

In [3],[4] Dobrushin and Shlosman introduced the concept of *complete analytical interactions.*They are those potentials whose corresponding Gibbs measure *for every (finite or infinite)* volume Λ satisfy $SMT(\Lambda, C, \gamma)$ for some $C > 0, \gamma > 0$. DS show that $SMT(\Lambda, C, \gamma)$, when supposed true for every volume Λ is equivalent to many other mixing conditions (in particular $SM(\Lambda, C, \gamma),\ \forall \Lambda$) and to analyticity properties of thermodynamical and correlation functions.
The main result, in the framework of the *constructive description* of completely analytical interactions is the following

Theorem DS2 (Dobrushin-Shlosmann [4])
There exists a function $L = L(C, \gamma)$ such that $SM(\cdot, C, \gamma)$ is (Γ, Γ')-effective with Γ given by the set of all subsets of a cube of edge $L(C, \gamma)$ and $\Gamma' \equiv$ the set of *all* (*finite or infinite*) subsets Λ of $\mathbf{Z^d}$.

As remarked in the Introduction this result is extremely strong (it covers volumes of arbitrary sizes and shapes).However there are situations (especially near a first order coexisting line) to which Theorem DS2 does not apply. In these cases one needs a somehow different approach where only quite regular (but arbitrarily large) volumes appear.

Theorem OP (Olivieri, Picco [12], [13], [10]).

There exists $L = L(C, \gamma)$ such that $SM(\cdot, C, \gamma)$ is (Γ, Γ') effective whith Γ given *only* by the cube Q_L of side L and Γ' contains all "sufficiently fat" regions.

Γ' can be taken as the set of all *multiples* of Q_L. (We say that Λ is a multiple of Q_L if it is partitionable into cubes, with disjoint interior , obtained from Q_L by translating its center by vectors of the form $y = xL$ with $x \in \mathbf{Z^d}$).
We would like to quote, at this point, an example due to R. Schonmann [14]. Consider a ferromagnetic Ising system with nearest neighbours and next to the nearest neighbours interactions at low temperature. It is easy to see that for some particular value ($\neq 0$) of the magnetic field h the corresponding potential is not completely analytical (in Dobrushin-Shlosman's sense) so that no finite volume strong mixing condition can be simultaneously verified *for all* subvolumes of any cube in an *effective* manner. More subtle examples of this phenomenon can be found in a recent paper by A.v.Enter, R.Fernandez and A.Sokal [5]
On the other hand exactly for the same models in the same region of parameters Theorem OP applies so that one obtains uniform exponential decay of truncated correlations for any sufficiently "fat" volume.(see Section 6).
Finally we want to quote a result by Holley (valid for the attractive case) referring only to some particular shapes: the boxes. A box in $\mathbf{Z^d}$ is the cartesian product of d finite intervals. Holley introduces a finite size condition referring to a cube Λ_0 that we call condition $H(\Lambda_0, \delta)$; it can be considered as a stronger version of $DSU(\Lambda, \delta)$ and it is given by:

for every $x \in \Lambda_0$, $y \in \partial_r^+ \Lambda_0$,there exists $\bar{\alpha}_{x,y} > 0$ such that for every $\Lambda \subset \Lambda_0, \Delta \subset \Lambda$:

$$\sup_{\tau \in \Omega_{\Lambda^c}} Var(\mu^{\tau}_{\Lambda,x} \quad \mu^{\tau^{(y)}}_{\Lambda,x}) \leq \bar{\alpha}_{x,y} \tag{3.8}$$

with

$$\sum_{x \in \Lambda_0, y \in \partial_r^+ \Lambda_0} \bar{\alpha}_{x,y} \quad \leq \delta |\Lambda_0| \tag{3.9}$$

Theorem H1 (Holley [6])

The existence of a cube Λ_0 such that $H(\Lambda_0, \delta)$ holds with $\delta < 1$ is equivalent to the existence of $C > 0, \gamma > 0$ such that $SM(\Lambda, C, \gamma)$ holds *for every box* Λ.

Remark Consider a 3-D n.n. ferromagnetic Ising system with $h = 2J$ ($h =$ magnetic field, $J =$ coupling constant) at inverse temperature β larger then the 2-D critical temperature $\beta_c^{(2)}$. Consider a particular class of boxes: the squared two-dimensional layers; for suitable boundary conditions we get zero effective field (-1 b.c. compensate for h in the interior of Λ).So it is impossible to get $SM(\Lambda, C, \gamma)$ for some $C > 0, \gamma > 0$ and for every (even thin) Λ because otherwise we would contraddict the existence of long range order for 2-D Ising system at $h = 0$, β large. Then, in this case,the hypotheses of Theorem H1 cannot be satisfied . On the other hand, as it is easy to verify,the hypotheses of Theorem DS1 are satisfied ($DSU(\Lambda_0, \delta)$ holds with $\delta < 1$ for a sufficiently large *cube* Λ if β is large enough); we conclude that it is the *arbitrariness* of the subsets Λ in Λ_0 to create problems (as it is easy to verify directly by considering, again, two-dimensional boxes $\Lambda \subset \Lambda_0$ and trying to verify (3.8)).

Finally it is easy to see that Theorem OP applies to the above case (the corresponding finite size condition on a cube with sufficiently large edge is satisfied), giving $SMT(\Lambda, C, \gamma)$ for every sufficiently "fat" region Λ ("thin" regions are excluded!).

3.3. Notions of exponential approach to equilibrium.

Let us now analyze the Glauber dynamics associated to our generalized Ising models. We want first to define some different notions of exponential approach to equilibrium (we always suppose that there exists a unique infinite volume Gibbs measure μ).
In what follows T_t will denote the Markov semigroup generated by L.

1) $EC, L^2(d\mu), \mathbf{Z^d}$: exponential convergence in L^2 for the infinite volume dynamics. It means that there exists $\gamma > 0$ such that $\forall f \in D$ (the space of cylindrical functions), $\exists\ C_f > 0$:

$$\|T_t f - \mu(f)\|_{L^2(\mu)} \leq C_f\ e^{-\gamma t}$$

2) $UEC, \mathbf{Z^d}$: Uniform (L^∞) exponential convergence for infinite volume dynamics. It means:

$$\exists \gamma > 0 : \ \forall f \in D\ \exists\ \ C_f > 0 :$$

$$\|T_t f - \mu(f)\|_u \leq C_f\ e^{-\gamma T}$$

namely

$$\sup_\sigma\ |E_\sigma f(\sigma_L) - \mu(f)| \leq C_f\ e^{-\gamma t}$$

where E_σ denotes expectation over the process starting from σ.

3) $EC, L^2(\mu_\Lambda^\tau) \quad \forall \Lambda \in \Gamma$: exponential convergence in L^2 for finite volume dynamics in Λ uniformly in Λ varying in a class Γ and in the b.c. τ; namely:

$$\exists \gamma > 0\ \ : \ \forall f \in D\ \exists\ C_f > 0 : \forall\ \Lambda \in \Gamma, \Lambda \supset S_f,\ \ \forall \tau \in\ \ \Omega_{\Lambda^c}$$

$$\|T_t^{\Lambda,\tau} f - \mu_\Lambda^\tau(f)\|_{L^2(\mu_\Lambda^\tau)} \leq C_f\ e^{-\gamma t}$$

4) $UEC, \quad \forall \Lambda \in \Gamma$: uniform exponential convergence for finite volume dynamics in Λ uniformly in Λ varying in a class Γ and in the b. c. τ; namely :

$$\exists \gamma > 0 \quad : \ \forall f \in D\ \exists C_f > 0 : \forall \Lambda \in \Gamma\ \ : \ \ \Lambda \supset S_f, \forall \tau \in \Omega_{\Lambda^c} :$$

$$\|T_t^{\Lambda,\tau} f - \mu_\Lambda^\tau(f)\|_u \leq C_f\ e^{-\gamma t}$$

Of course $UEC, \quad \forall\Lambda$ in a Van Hove sequence imply $UEC, \mathbf{Z^d}$ and $EC, L^2(\mu_\Lambda^\tau) \quad \forall\Lambda$ in a Van Hove sequence imply $EC, L^2(d\mu), \mathbf{Z^d}$; a less obvious statement, due to Holley, says that, in the attractive case, $EC, L^2(\mu_\Lambda^\tau) \quad \forall$ box Λ implies $UEC, \mathbf{Z^d}$.

Recently some papers have been devoted to the relations between the different notions of exponential approach to equilibrium and between this speed of approach to equilibrium and mixing properties of the invariant Gibbs measure.
In particular the problem has been studied of deducing exponential approach to equilibrium (in the different above senses) from *finite size* conditions.

3.4. Some known results on Glauber dynamics

The following theorems H2, AH hold for the attractive case.

Theorem H2 (Holley [6])
Suppose that there exists a cube Λ_0 such that $H(\Lambda_0, \delta)$, with $\delta < 1$, holds; then $UEC, \mathbf{Z^d}$ holds; moreover $EC, L^2(\mu^\tau_\Lambda)$ holds *for every box* Λ.

Notice that, as previously remarked, the hypotheses of Theorem H2 do not apply to situations (like the previously discussed 3-D Ising system with $h = 2J$) for which, however, the thesis is certainly expected to be true provided that we replace *for every box* Λ with *for every cube* Λ.

Theorem AH (Aizenmann,Holley [2])
If there is a cube Λ_0 such that $DSU(\Lambda_0, \delta)$ is satisfied with $\delta < 1$, then $EC, L^2(d\mu), \mathbf{Z^d}$ holds.

For the general, not necessarily attractive case we want to quote the following Theorem, due to Stook and Zegarlinski, obtained in the framework of the theory making use of logarithmic Sobolev inequalities.

Theorem SZ (Strook, Zegarlinski [15])
The following statements are equivalent

i) There exists a finite region Λ such that $H(\Lambda, \delta)$ holds with $\delta < 1$.
ii) There exist $C > 0, \gamma > 0$ such that $SMT(\Lambda, C, \gamma)$ holds *for every* volume Λ; namely complete analyticity, in the Dobrushin-Shlosman's sense, holds.
iii) UEC *for every* Λ holds.
iv) EC, $L^2(d\mu^\tau_\Lambda)$ for every Λ holds.

4. Exponential convergence under a weak mixing condition

In this section we discuss the first one of our main results, namely that for an attractive stochastic Ising model a weak mixing condition on the Gibbs state implies the ergodicity of the infinite volume Markov process and the exponential convergence in the strong $UEC, \mathbf{Z^d}$ sense of its distribution at time t to the unique invariant measure as $t \to \infty$.

Let us first reformulate our *weak mixing* condition in the context of the attractive systems.

We recall that by Q_L we denote the cube $\{y \in \mathbf{Z^d}\, ; \, |\, y_i| \ \leq \ \frac{L-1}{2} \ \forall\, i = 1...d\}$, for L odd. $Q_L(x)$ will denote the translated by the vector $x \in \mathbf{Z^d}$ of Q_L

Weak mixing $WMA(C, \gamma)$
There exist two positive constants C and γ such for every integer L

$$\mu^{+}_{Q_L}(\sigma(0)) \ - \ \mu^{-}_{Q_L}(\sigma(0)) \ \leq \ C\exp(-\gamma L)$$

It is immediately seen that $WM(Q_L, C, \gamma)$, for every L implies $WMA(C, \gamma)$.

Remark One sees immediately that the above mixing condition implies that there exists a unique Gibbs state in the thermodynamic limit that will be denoted by μ .

As already mentioned in the introduction an important question is what kind of implication has a mixing condition on the Gibbs state for the convergence to equilibrium of the associated stochastic Ising model. A first partial result (by Aizenmann and Holley) is the above quoted Theorem AH where exponential convergence, in the $L^2(d\mu)$ sense, follows from $DSU(\Lambda_0, \delta), \delta < 1$ (recall that, by Theorem DS1, $DSU(\Lambda_0, \delta), \delta < 1$ implies $WM(\Lambda, C, \gamma)$ for some $C > 0, \gamma > 0$ and for every Λ which, in turn, implies $WMA(C, \gamma)$.
Here we will prove a much stronger result:

Theorem 1
Weak mixing implies that there is a positive constant m and for any cylindrical function f there exists a constant C_f such that:

$$\sup_{\sigma} |T_t(f)(\sigma) \ - \ \mu(f)| \ \leq \ C_f \exp(-mt)$$

In other words if $WMA(C, \gamma)$ holds for some $C > 0, \gamma > 0$ then $UEC, \mathbf{Z^d}$ holds.

Sketch of the Proof
Let us define

$$\rho(t) \ = \ \sup_{x} P(\sigma^{+}_t(x) \neq \sigma^{-}_t(x)) \tag{4.1}$$

where σ^{+}_t and σ^{-}_t are the configurations starting from all pluses and all minus respectively.

Given $\rho(t)$ we estimate the quantity appearing in Theorem 1 by:

$$\sup_{\sigma} |T_t(f)(\sigma) \ - \ \mu(f)| \ \leq \ |||f|||\rho(t) \tag{4.2}$$

where $|||f||| \ = \ \Sigma_x \sup_\sigma ||f(\sigma^x) \ - \ f(\sigma)||$ and σ^x is the configuration obtained from σ by flipping the spin at x.

Thus we have to show that $\rho(t)$ decays exponentially to zero. Actually thanks to an important result by Holley [6] (see also [2] for a different derivation of the same result) it is sufficient to show that $\rho(t)$ goes to zero faster than $\frac{1}{t^d}$. The standard way to try to get this input is the following: one increases $\rho(t)$ by imposing extra plus (minus)- boundary conditions on the boundary of a box centered at the origin and of side L for the evolution which starts from all pluses (all minuses). Then one gets very easily that:

$$\rho(t) \;\le\; \mu^+_{Q_L}(\sigma(o) \;=\; +1) \;-\; \mu^-_{Q_L}(\sigma(o) \;=\; +1) \;+$$

$$+ \exp(c_1 L^d \;-\; \mathrm{gap}(Q_L, +)t) \;+\; \exp(c_1 L^d \;-\; \mathrm{gap}(Q_L, -)t) \tag{4.3}$$

where $\mathrm{gap}(Q_L, +)$ is the gap in the spectrum of the (selfadjoint) generator of the stochastic Ising model in the cube Q_L of side L with plus boundary conditions and analogously for $\mathrm{gap}(Q_L, -)$. If we assume *weak mixing* then we have that the first difference in the r.h.s. of (4.3) is smaller than :

$$C\exp(-\gamma L)$$

If we now assume a lower bound on the gap uniform in the volume Q_L, then, by choosing L $= \delta t^{1/d}$ with δ sufficiently small, we get that $\rho(t)$ decays faster than exp(-const $t^{1/d}$) $<<$ $\frac{1}{t^d}$ and therefore, thanks to the Holley's theorem, $\rho(t)$ will decay exponentially fast.

However, as one can easily check, the rather strong input that the gap in the spectrum of the generator of the process in finite volume is bounded away from zero uniformly in the volume implies in some sense that one is able to prove fast convergence to equilibrium (in the $L^2(d\mu_\Lambda)$ sense) not only in the bulk but also close to the boundary. From a static point of view this is equivalent to a *local* weak dependence on the boundary conditions, where local means that the change of one boundary spin does not affect far away spins even if they are located close to the boundary. As it has been discussed in the Introduction and in Section 3 this in general is not implied by the *weak mixing* condition above but requires *strong mixing.*

Under only the weak mixing condition it might very well be that the gap is no longer bounded away from zero uniformly in the volume and one is left with a very rough and rather trivial bound of the form:

$$\mathrm{gap}(\Lambda, +/-) \;\ge\; \exp(-c(J)|\Lambda|) \tag{4.4}$$

where c(J) is a suitable positive constant depending only on the interaction J.

Such a weak bound forces us to choose the side L as : L = L(t) = *const.*$\{log(t)\}^{\frac{1}{d}}$. By plugging L(t) into (4.3) we get :

$$\rho(t) \;\le\; \exp(-\bar{\gamma} log(t)^{\frac{1}{d}}) \tag{4.5}$$

which is certainly not sufficient to apply Holley's theorem.

Thus we need to find a new method that allows us to improve the above very rough bound. The main new technical tool for our analysis is the following recursive inequality satisfied by $\rho(t)$ that for convenience we state as a proposition (see [10] for a proof):

Proposition 1

Under the hypotheses of Theorem 1 there exist two finite positive constants C and γ such that for any integer L :

$$\rho(2t) \leq (2L+1)^d \rho(t)^2 + C\exp(-\gamma L)$$

It is not difficult to see that the above recursive inequality allows us to transform the bound (4.5) into a bound of the form :

$$\rho(t) \leq \exp(-\exp(+\bar{\gamma} log(t)^{\frac{1}{d}})) \tag{4.6}$$

($\bar{\gamma} > 0$) which is clearly faster than the inverse of any power of t (see [10] for details).

5. Exponential convergence in "fat" finite volumes

We discuss in this section the exponential convergence to equilibrium in finite volumes with rates that are estimated uniformly in the volume by assuming a finite volume condition of strong type. All the results of this section can be proved without the assumption of the attractivity of the dynamics (see [11]); some of the proofs are however much simpler in the attractive case. The mathematics involved in the non attractive case becomes much more sophisticated and relies upon the theory of logarithmic Sobolev inequalities applied to Gibbs measures as it has been developed in an important series of papers by Zegarlinski and Zegarlinski and Strook (see, for instance, [16],[15] and references quoted there). An independent proof based on renormalization group ideas of the existence of a finite logarithmic Sobolev constant for the Gibbs state under a finite volume mixing condition (see below) can be found in [11].
In our proof it turns out to be convenient to use a particular finite volume condition which does not contain parameters (like C, γ). In the sequel we will refer to it as L_o-mixing.It is the following one:

L_o-mixing : Let $\Lambda_o \equiv Q_{2L_o+1}$ be the cube of side $2L_o+1$ with sides parallel to the coordinate axes and let for any $V \subset \Lambda_o$ $\mu^\sigma_{\Lambda_o,V}$ be the relativization of $\mu^\sigma_{\Lambda_o}$ to the set V. Then for any k outside Λ_o and any V in Λ_o with dist(V,k) $\geq L_o^{\frac{1}{2}}$ we must have:

$$Var(\mu^\sigma_{\Lambda_o,V}, \mu^{\sigma^{(k)}}_{\Lambda_o,V}) \leq \frac{1}{\text{dist}(k,V)^{2(2d+1)}} \quad \forall \sigma \in \{-1,+1\}^{\Lambda_o^c} \tag{5.1}$$

Notice that L_o-mixing easily follows from $SM(\Lambda_o, C, \gamma), SMT(\Lambda_o, C, \gamma)$ once, given $C > 0, \gamma > 0, L_o$ is taken sufficiently large. We emphasize again that our condition has to hold only in a definite geometric shape , in our case a cube, contrary to what assumed by Aizenman and Holley [2] or Zegarlinski and Strook [15] where the arbitrariness of the geometric shape of the finite volume plays an important role. Of course, as already remarked, in weakening the condition there is a price to pay: we will prove our results only in volumes that are "multiple" of the elementary volume Λ_o. However this has to be the case if we want to apply our condition to systems like the Ising model at low temperature in the presence of a positive external field where it can be proved (see Section 3) that the previous conditions of Aizenman-Holley or Strook-Zegarlinski can fail.

Let us now state our main results (see [10] and [11] for details). In what follows we will call L_o-compatible any subset of the lattice $\mathbf{Z^d}$ which is the union of translates of the cube Λ_o such that their vertices lay on the rescaled lattice $(2L_o + 1)\mathbf{Z^d}$.

Theorem 2 (Effectiveness)
There exists a positive constant $\bar{L} \geq R$ such that if L_o-mixing holds with $L_o \geq \bar{L}$ then there exists positive constants γ and C such that for any L_o-compatible set Λ, any $L \geq L_o^{\frac{1}{2}}$, any σ and any site k outside Λ we have :

$$||\mu_\Lambda^\sigma - \mu_\Lambda^{\sigma^{(k)}}||_{\sigma,k,L} \leq C\exp(-\gamma L) \tag{5.2}$$

where $||\mu_\Lambda^\sigma - \mu_\Lambda^{\sigma^{(k)}}||_{\sigma,k,L}$ is the variation distance between the relativization of the Gibbs states in Λ, with boundary conditions σ and $\sigma^{(k)}$,respectively, to the maximal subset $\bar{\Lambda}(L)$ of Λ which is at distance greater than L from k.

Remark Thus the Theorem says that, provided L_o is large enough, L_o-mixing propagates to all larger scales that are multiple of the basic length scale L_o. In particular it implies the exponential decay of correlations in any L_o-compatible volume uniformly in the volume and thus also of the unique infinite volume Gibbs state.

The content of the above Theorem 2 is similar to the one of Theorem OP. However in [10] a simple dynamical proof of it is provided, avoiding the complicated geometrical constructions and the theory of the cluster expansion that where at the basis of the arguments of proof in [12], [13] where, on the other hand, also analyticity properties where proved.

The next result says that L_o-mixing implies exponential convergence to equilibrium in the strongest possible sense namely in any L_o-compatible finite volume both in the L^2-norm and in the uniform norm.

Theorem 3
There exists a positive constant $\bar{L} \geq R$ such that if L_o-mixing holds with $L_o \geq \bar{L}$ then there exist two positive constants m_o and m such that for any L_o-compatible set Λ, any boundary configuration τ and for any function f in $L^2(d\mu_\Lambda^\tau)$:

i)

$$||T_t^{\Lambda,\tau}(f) - \mu_\Lambda^\tau(f)||_{L^2(d\mu_\Lambda^\tau)} \leq ||f||_{L^2(d\mu_\Lambda^\tau)}\exp(-m_o t) \tag{5.3}$$

ii)

$$\sup_{\sigma} |T_t^{\Lambda,\tau}(f)(\sigma) \; - \; \mu(f)| \; \leq \; |||f||| \exp(-mt) \tag{5.4}$$

where $T_t^{\Lambda,\tau}$ denotes the Markov semigroup of the process evolving in Λ with boundary conditions τ.

Sketch of the proof in the attractive case

Let us fix an L_o-compatible set Λ and a boundary configuration τ and let $\{Q_i\}$ be a covering of the set Λ with the following two properties:

a) Each element of the covering is a a cube of side $2L_o + 1$ with sides parallel to the coordinate axes.
b) If two different cubes Q_i and Q_j overlap then necessarily each one of them is the translated by L_o along at least one coordinate axes of the other.

It is very easy to check that for any L_o-compatible set Λ such a covering always exists.

Next we introduce a new dynamics (Gibbs sampling) on $\{-1,+1\}^{\Lambda}$ by defining its generator L_Q as :

$$L_Q f(\sigma) \; = \; \sum_{\eta,i} c_{Q_i}(\sigma,\eta)(f(\eta) \; - \; f(\sigma)) \tag{5.5}$$

where the new jump rates $c_{Q_i}(\sigma,\eta)$ are a generalization of those of the heat bath dynamics and are given by :

$$c_{Q_i}(\sigma,\eta) \; = \; \mu^{\sigma}_{Q_i}(\eta) \tag{5.6}$$

if η agrees with σ outside the cube Q_i and zero otherwise. It is understood that outside Λ the configurations σ and η agree with τ.

Remark The above version of the Gibbs sampling is different from the one employed by Holley [6] Aizenman and Holley [2] and Strook and Zegarlinski [15]. In these previous works the updating was as follows: each site x is chosen in $\mathbf{Z^d}$ with rate one and then the configuration in $\Lambda_o(x) \cap \Lambda$ is put equal to η with probability $\mu^{\sigma}_{\Lambda_o(x)\cap\Lambda}(\eta)$, where $\Lambda_o(x)$ is the cube of side $2L_o + 1$ centered at x. This dynamics has however the incovenience of updating sometimes regions that are not squares Λ_o but rather boxes (= intersection between two cubes). Contrary to what happen for cubes Λ_o, not only we have no control at all on the mixing properties of the Gibbs states associated to such geometric regions, but there are situations (see Section 3) in which our mixing condition while being true for cubes fails for certain boxes !

It is rather simple to show that the above Gibbs sampling is still reversible with respect to the Gibbs state in Λ with boundary conditions τ ; more important one easily proves (see Lemma 2.3 of [15]) that if gap(L_Q) and gap(L) denote the gap in the spectrum of the generators L_Q and L respectively as operators in $L^2(d\mu^\tau_\Lambda)$, then there exists a positive constant c independent of Λ and τ such that:

$$\mathrm{gap}(L) \;\geq\; \exp(-cL_o^d)\mathrm{gap}(L_Q) \tag{5.7}$$

Thus in order to prove the theorem we need only to estimate from below $\mathrm{gap}(L_Q)$ uniformly in Λ and τ .

For this purpose in [10] we adopt a scheme very similar to the one already used in section 2. We couple the Gibbs sampling dynamics starting from different initial conditions and we define the quantity $\rho(t)$ as :

$$\rho^\tau_\Lambda(t) \;=\; \sup_{\sigma,\eta\; x\in\Lambda} P(\,\sigma_t(x) \;\neq\; \eta_t(x)\,) \tag{5.8}$$

It is easy to check Holley's criterium : if there exists a finite time t_o such that $\rho^\tau_\Lambda(t_o) \;<< \frac{1}{t_o^d}$ then $\rho^\tau_\Lambda(t)$ decays exponentially fast. The idea then is to verify the existence of the basic time scale t_o by just using our L_o-mixing condition. In fact if L_o-*mixing* holds with L_o large enough then the updating of each single cube Q_i of the covering becomes almost independent of the value of the spins in the other cubes since their influence dies out outside a thin layer of width $L_o^{1/2}$ around $\partial Q_i \setminus \partial\Lambda \cap \partial Q_i$. In some sense the Gibbs sampling behaves as a high temperature almost independent stochastic Ising model for which the exponential convergence to equilibrium is a very well established result.

Remark It is absolutely crucial for the whole argument to work that the the influence of the neighboring cubes around Q_i dies out outside a layer only around $\partial Q_i \setminus \partial\Lambda \cap \partial Q_i$ and *not* around the whole boundary ∂Q_i; in other words there cannot be propagation of information along the boundary of Λ.

Part i) of the theorem then follows immediately. For attractive systems part ii) follows from the argument used in Theorem 1 once we know that there is a lower bound on the gap of L^τ_Λ if Λ is L_o-compatible which is uniform in Λ. For non attractive systems part ii) follows by proving a logarithmic Sobolev inequality for the Gibbs state via a decimation procedure which uses in a crucial way the "effectiveness" of the mixing condition.

Remark One may wonder why even for attractive systems we needed in this section a condition like L_o-mixing which is much stronger than the weak mixing condition used in the previous section. The reason is that under only the weak mixing condition we cannot prove in finite volume Λ the result of Proposition 1. In fact if we take a site x close to the boundary of the set Λ and we consider, as we did in Proposition 1 the cube $\Lambda_o(x)$ of side $2L_o+1$ centered at x, then x can be very close to the boundary of $\Lambda_o(x)\cap\Lambda$. If this happens then, by changing the boundary conditions on $\partial\Lambda_o(x)\cap\Lambda$, we may considerably affect the Gibbs state at x (this phenomenon occurs for instance in the Czech models [3] which are however non ferromagnetic); this fact is compatible with the weak mixing condition which requires only a control of the effects of changing the boundary conditions inside the bulk! Of course in the infinite volume case this problem never occurs since x is always in the bulk.

6. Applications

In this section we discuss some applications of our results. In particular we prove the exponential convergence to equilibrium for the infinite volume stochastic Ising model for all temperatures above the critical one and for low temperature and non zero external field.

The model that we will consider is the standard nearest neighbor Ising model in an external non negative field h and at inverse temperature β. If we consider the associated stochastic Ising model discussed in the previous sections, then for $h \neq 0$ or $\beta < \beta_c$ it will be an ergodic Markov process on $\{-1,+1\}^{\mathbf{Z^d}}$ with $\mu^{\beta,\,h}$, the unique infinite volume Gibbs state, as unique invariant measure. In the following theorem we will strenghthen this result. Let us denote by $E^{\beta,\,h}_{\sigma}(f(\sigma_t))$ or by $E^{\Lambda,\,\tau,\,\beta,\,h}_{\sigma}(f(\sigma_t))$ the expected value at time t of the function f with respect to the distribution of the process evolving in the infinite lattice $\mathbf{Z^d}$ or in the finite set Λ with boundary conditions τ. Then we have :

Theorem 4

a) Assume that $\beta < \beta_c$. Then for any $h \geq 0$ there exists a positive constant m and for any cylindrical function f there exists a constant C_f such that:

$$\sup_{\sigma} |E^{\beta,\,h}_{\sigma}(f(\sigma_t)) \;-\; \mu^{\beta,\,h}(f)| \;\leq\; C_f \exp(-mt)$$

b) There exists a positive constant β_o such that for any $\beta \geq \beta_o$ and $h > 0$ there exists a positive constant m and for any cylindrical function f there exists a constant C_f such that:

$$\sup_{\sigma} |E^{\beta,\,h}_{\sigma}(f(\sigma_t)) \;-\; \mu^{\beta,\,h}(f)| \;\leq\; C_f \exp(-mt)$$

c) Given $h > 0$ there exist two positive constants $\beta_o(h)$ and $L_o(h)$ such that for any $\beta \geq \beta_o$ there exists a positive constant m such that for any $L_o(h)$-compatible set Λ and for any function f on $\{-1,+1\}^{\Lambda}$:

$$\sup_{\sigma} |E^{\Lambda,\,\tau,\,\beta,\,h}_{\sigma}(f(\sigma_t)) \;-\; \mu^{\tau,\,\beta,\,h}_{\Lambda}(f)| \;\leq\; |||f||| \exp(-mt)$$

Proof

Thanks to Theorem 1 **a)** and **b)** follow immediately once we are able to verify our *weak mixing* condition. In case **a)** *weak mixing* follows from i) of Theorem 2 of a recent paper by Higuchi [7] which exploits in a crucial way the results by Aizenman, Barski and Fernandez on the absence of the third phase [1]. In case **b)** one uses the fact that for low enough temperature and any positve h if one considers a large enough cube of side L with minus boundary conditions, then within a distance from the boundary smaller than Clog(L) and with very high probability there exists a large contour of plus spins which screens the effect of the negative boundary conditions. Such a result has been proved many years ago by Martyrosian [9].

Part **c)** In this case we verify that for any $h > 0$ there exist $L_o(h)$ such that the configuration identically equal to $+1$ is the unique ground state configuration of the Hamiltonian $H^{\tau}_{\Lambda_{L_o(h)}}(\sigma)$ for any boundary condition τ. This implies, as it is easy to verify, that there exists $\beta_o = \beta_o(h)$ such that if $\beta > \beta_o$ our $L_o(h)$-mixing condition is satisfied.

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