FOUNDATIONS OF SUPERMANIFOLD THEORY: THE AXIOMATIC APPROACH †

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Abstract. We discuss an axiomatic approach to supermanifolds valid for arbitrary ground graded-commutative Banach algebras B. Rothstein's axiomatics is revisited and completed by a further requirement which calls for the completeness of the rings of sections of the structure sheaves, and allows one to dispose of some undesirable features of Rothstein supermanifolds. The ensuing system of axioms determines a category of supermanifolds which coincides with graded manifolds when $B = \mathbb{R}$, and with G-supermanifolds when B is a finite-dimensional exterior algebra. This category is studied in detail. The case of holomorphic supermanifolds is also outlined.

Introduction

Supergeometry has been developed along two different guidelines: Berezin, Leĭtes and Kostant introduced the so-called *graded manifolds* via algebro-geometric techniques (cf. [**10,17,22,23,8**]), while DeWitt and Rogers's treatment ([**16,31,32**]; cf. also [**20,37**]) relies on more intuitive local models expressed in the language of differential geometry. As a matter of fact, this pretended dichotomy has no *raison d'être*, for at least two motivations. First of all, it is our opinion that the relative formulation of graded manifold

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theory [**25**] in some sense includes supermanifolds *à la* DeWitt-Rogers; secondly, and more concretely, in order to provide a sound mathematical basis to the DeWitt-Rogers theory, one need use sheaf theory as well [**33,8**], at least when the ground algebra is finite-dimensional. Anyway, the precise relationship between the two models is still unclear.

In his paper [**33**], Rothstein devised a set of four axioms which any sensible category of supermanifolds should verify; however, it turns out that the category of supermanifolds singled out by his axiomatics (that we call R-supermanifolds) is too large, in the sense that, contrary to what is asserted in [**33**], it is neither true that if the ground algebra is commutative the axiomatics reduces to Berezin-Leĭtes-Kostant's graded manifold theory (see Example 3.2 of this paper), nor that when the ground algebra is a finite-dimensional exterior algebra, the axiomatics singles out the category of supermanifolds that are extensions of Rogers's G^∞ supermanifolds.

The purpose of the present work is to analyze Rothstein's axiomatics, discussing the interdependence among the axioms and singling out the additional axiom necessary to characterize those Rothstein supermanifolds which are free from the aforementioned drawbacks. The new axiom calls for the completeness of the rings of sections of the 'structure sheaf' with respect to a certain natural topology.

The ensuing system of five axioms can be reorganized into four statements, defining a category of supermanifolds, called R^∞-supermanifolds, that coincide with graded manifolds when the ground algebra is either $\mathbb{R}$ or $\mathbb{C}$, and provide the most natural generalization of differentiable or complex manifolds. When the ground algebra is a finite-dimensional exterior algebra, the resulting category of supermanifolds is equivalent to the category of G-supermanifolds that some of the authors have independently introduced and discussed elsewhere [**2-8,14,15**]. This means that G-supermanifolds (in the case of a finite-dimensional ground algebra) are the unique concrete model for supermanifolds fulfilling the extended axiomatics, or alternatively, that they can be defined through that axiomatics, thus stressing their relevance in supergeometry. This also means that G-supermanifolds are exactly those Rothstein supermanifolds that extend Rogers's G^∞ supermanifolds in the sense of [**33**].

Other results that we present in this paper are the following: any R-supermanifold morphism is continuous as a morphism between the rings of sections of the relevant structure sheaves; any R^∞-supermanifold morphism is also G^∞; any R-supermanifold can be in one sense completed to yield an R^∞-supermanifold.

Finally, in the last section the case of complex analytic supermanifolds is discussed.

Many of the results contained in this paper have already been presented in [**8**] in the case of a finite-dimensional ground algebra B.

We briefly recall the basic definitions and facts we shall need. We consider $\mathbb{Z}_2$-graded (for brevity, simply 'graded') algebraic objects; any morphism of graded objects is assumed to be homogeneous. (For details, the reader may consult [**22-24,8**]). Let B denote a graded-commutative Banach algebra with unit; so B_0 and B_1 are, respectively, the even and odd part of B. With the exception of Section 6, we consider the case of a real B. The analysis of the properties of supermanifolds is greatly simplified when B is local and, moreover, satisfies a very natural additional property that we discuss in Section 3: that of being a Banach algebra of Grassmann origin. Some of our results are true only under this additional assumption, which however does not seem to be truly restrictive, in that all examples of graded-commutative Banach algebras that have been used as ground algebras for supermanifolds are actually Banach algebras of Grassmann origin.

We define the (m,n) dimensional 'superspace' $B^{m,n}$ as $B_0^m \times B_1^n$ with the product topology.

By *graded ringed B-space* we mean a pair $(X,\mathcal{A})$, where X is a topological space and $\mathcal{A}$ is a sheaf of graded-commutative B-algebras on X. A graded ringed space is said to be *local*, as it occurs in the most interesting examples, if the stalks $\mathcal{A}_z$ are local graded rings for any $z \in M$ (a graded ring is said to be local if it has a unique maximal graded ideal). The *sheaf $\mathcal{D}er\,\mathcal{A}$ of derivations* of $\mathcal{A}$ is by definition the completion of the presheaf of $\mathcal{A}$-modules $U \mapsto \{\text{graded derivations of } \mathcal{A}_{|U}\}$, where a graded derivation of $\mathcal{A}_{|U}$ is an endomorphism of sheaves of graded B-algebras $D\colon \mathcal{A}_{|U} \to \mathcal{A}_{|U}$ which fulfills the graded Leibniz rule, sc. $D(a\cdot b) = D(a)\cdot b + (-1)^{|a||D|}a\cdot D(b)$.

Furthermore, $\mathcal{D}er^*\mathcal{A}$ denotes the dual sheaf to $\mathcal{D}er\,\mathcal{A}$, i.e. $\mathcal{D}er^*\mathcal{A} = \mathcal{H}om_{\mathcal{A}}(\mathcal{D}er\,\mathcal{A}, \mathcal{A})$. A morphism of sheaves of graded B-modules $d\colon \mathcal{A} \to \mathcal{D}er^*\mathcal{A}$ — called the *exterior differential* — is defined by letting $df(D) = (-1)^{|f||D|}D(f)$ for all homogeneous $f \in \mathcal{A}(U)$, $D \in \mathcal{D}er\,\mathcal{A}(U)$ and all open $U \subset M$.

Rothstein's axiomatics revisited

In order to state Rothstein's axioms for supermanifolds, we consider triples $(M,\mathcal{A}, ev)$, where $(M,\mathcal{A})$ is a graded ringed space over a graded-commutative Banach algebra B, the space M is assumed to be (Hausdorff) paracompact, and $ev\colon \mathcal{A} \to \mathcal{C}_M$ is a morphism of sheaves of graded B-algebras, called the 'evaluation morphism;' here $\mathcal{C}_M$ is the sheaf of continuous B-valued functions on M. Such a triple will be called an *R-superspace*. We shall denote by a tilde the action of ev, i.e. $\tilde{f} = ev(f)$. A morphism of R-superspaces is a pair $(f, f^\sharp)\colon (M,\mathcal{A}, ev^M) \to (N,\mathcal{B}, ev^N)$, where $f\colon M \to N$ is a continuous map and $f^\sharp\colon \mathcal{B} \to f_*\mathcal{A}$ is a morphism of sheaves of graded B-algebras, such that $ev^M \circ f^\sharp = f^* \circ ev^N$.

After fixing a pair (m,n) of nonnegative integers, one says that an R-superspace $(M,\mathcal{A}, ev)$ is an (m,n) dimensional R-supermanifold if and only if the following four axioms are satisfied.

Axiom 1. *$\mathcal{D}er^*\mathcal{A}$ is a locally free $\mathcal{A}$-module of rank (m,n). Any $z \in M$ has an open neighbourhood U with sections $x^1,\dots,x^m \in \mathcal{A}(U)_0$, $y^1,\dots,y^n \in \mathcal{A}(U)_1$ such that $\{dx^1,\dots,dx^m, dy^1,\dots,dy^n\}$ is a graded basis of $\mathcal{D}er^*\mathcal{A}(U)$.*

The collection $(U,(x^1,\dots,x^m,y^1,\dots,y^n))$ is called a *coordinate chart* for the supermanifold. This axiom implies evidently that $\mathcal{D}er\,\mathcal{A}$ is locally free of rank (m,n), and is locally generated by the derivations $\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^\alpha}$ defined by duality with the dx^i's and dy^α's.

Axiom 2. *If $(U,(x^1,\dots,x^m,y^1,\dots,y^n))$ is a coordinate chart, the mapping*

$$\begin{aligned}\psi\colon U &\to B^{m,n}\\ z &\mapsto (\tilde{x}^1(z),\dots,\tilde{x}^m(z),\tilde{y}^1(z),\dots,\tilde{y}^n(z))\end{aligned}$$

is a homeomorphism onto an open subset in $B^{m,n}$.

Axiom 3. (Existence of Taylor expansion) *Let $(U,(x^1,\dots,x^m,y^1,\dots,y^n))$ be a coordinate chart. For any $z\in U$ and any germ $f\in\mathcal{A}_z$ there are germs $g_1,\dots,g_m,h_1,\dots,h_n\in\mathcal{A}_z$ such that*

$$f=\tilde{f}(z)+\sum_{i=1}^m g_i\,(x^i-\tilde{x}^i(z))+\sum_{\alpha=1}^n h_\alpha\,(y^\alpha-\tilde{y}^\alpha(z))\,.$$

Axiom 4. *Let $\mathcal{D}(\mathcal{A})$ denote the sheaf of differential operators over $\mathcal{A}$, i.e., the graded $\mathcal{A}$-module generated multiplicatively by $\mathcal{D}er\,\mathcal{A}$ over $\mathcal{A}$, and let $f\in\mathcal{A}_z$, with $z\in M$. If $\widetilde{L(f)}=0$ for all $L\in\mathcal{D}(\mathcal{A})_z$, then $f=0$.*

The sections of $\mathcal{A}$ will be called *superfunctions.* Morphisms of R-supermanifolds are just R-superspace morphisms.

It is convenient to restate this axiomatics in a slight different manner, more suitable for dealing with the topological completeness of the rings of sections of $\mathcal{A}$. Let us consider, as before, an R-superspace $(M,\mathcal{A},ev)$. For any $z\in M$ define a graded ideal $\mathfrak{L}_z$ of $\mathcal{A}_z$ by letting

$$\mathfrak{L}_z=\{f\in\mathcal{A}_z \mid \tilde{f}(z)=0\}.$$

Axiom 3 can be obviously reformulated as follows:

Let $(U,(x^1,\dots,x^m,y^1,\dots,y^n))$ be a coordinate chart. For any $z\in U$ the ideal $\mathfrak{L}_z$ is generated by $\{x^1-\tilde{x}^1(z),\dots,x^m-\tilde{x}^m(z),y^1-\tilde{y}^1(z),\dots,y^n-\tilde{y}^n(z)\}$.

Axiom 1 allows one to replace this axiom by a weaker requirement; to this aim we need some preliminary discussion.

Lemma. *There is an isomorphism of $\mathcal{A}_z/\mathfrak{L}_z$-modules*

$$\begin{aligned}\mathfrak{L}_z/\mathfrak{L}_z^2 &\to \mathcal{D}er^*\mathcal{A}_z\otimes_{\mathcal{A}_z}\mathcal{A}_z/\mathfrak{L}_z\\ \bar{f} &\mapsto df\otimes 1\end{aligned}$$

where a bar denotes the class in the quotient.

Proof. It can be easily shown that $df\otimes\bar{g}\mapsto\overline{(f-\tilde{f}(z))g}$ defines a morphism $\mathcal{D}er^*\mathcal{A}_z\otimes_{\mathcal{A}_z}\mathcal{A}_z/\mathfrak{L}_z\to\mathfrak{L}_z/\mathfrak{L}_z^2$ which inverts the previous one. $\square$

If we denote by $d_z f$ the class of the element $f - \tilde{f}(z) \in \mathfrak{L}_z$ in $\mathfrak{L}_z/\mathfrak{L}_z^2$, then Axiom 1 for $(M, \mathcal{A}, ev)$ implies that — given a coordinate chart $(U, (x^1, \dots, x^m, y^1, \dots, y^n))$ — the elements $\{d_z x^i,\, d_z y^\alpha\}$ are a basis for the $\mathcal{A}_z/\mathfrak{L}_z$-module $\mathfrak{L}_z/\mathfrak{L}_z^2$.

Let us suppose until the end of this Section that $(M, \mathcal{A})$ is a *graded locally ringed space*. Since in that case any graded ideal of $\mathcal{A}_z$ is contained in its radical, one can apply a graded version of Nakayama's lemma (cf. [**8**]). Thus we obtain

Lemma. *Assume that $\mathfrak{L}_z$ is finitely generated. Then the elements $\{x^i - \tilde{x}^i(z),\, y^\alpha - \tilde{y}^\alpha(z)\}$ are generators for $\mathfrak{L}_z$ if and only if their classes $\{d_z x^i,\, d_z y^\alpha\}$ generate the $\mathcal{A}_z/\mathfrak{L}_z$-module $\mathfrak{L}_z/\mathfrak{L}_z^2$.* □

Thus, we have proved the following result.

Proposition. *If the graded rings $\mathcal{A}_z$ are local, and $\mathfrak{L}_z$ is finitely generated, then Axiom 1 implies Axiom 3.* □

We are therefore led to consider the apparently weaker axiom

Axiom 3′. *For every $z \in M$ the ideal $\mathfrak{L}_z$ is finitely generated.*

It is an important fact that Axiom 3′ does not depend on the choice of a coordinate chart. So, while in order to check Axiom 3 one has to prove the existence of a Taylor expansion for any coordinate chart, if $(M, \mathcal{A})$ is a graded locally ringed space it is sufficient to show that there is one coordinate chart for which a Taylor expansion does exist.

We can summarize this discussion as follows.

Proposition. *If an R-supermanifold is also a graded locally ringed space, we can replace Axiom 3 by Axiom 3′.* □

Example Here we show that Rothstein's Axiom 3 is independent of Axioms 1, 2, and 4. Let $B = B_0 = \mathbb{R}$, $M = B^{1,0} = \mathbb{R}$. Let us fix a continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that for every open and non-empty $U \subset \mathbb{R}$ the restriction $\phi_{|U}$ is neither constant nor one-to-one; an example of such a function is Weierstrass' nowhere differentiable continuous function [**34**]. We denote $\mathcal{F} = \phi^{-1}\mathcal{C}_\mathbb{R}$ and by $i_\mathcal{F} : \mathcal{F} \hookrightarrow \mathcal{C}_\mathbb{R}$ the canonical injection. Let $i_\mathcal{P}$ be the embedding of the sheaf $\mathcal{P}$ of germs of real polynomial functions on $\mathbb{R}$ into $\mathcal{C}_\mathbb{R}$, and let $\mathcal{A} = \mathcal{F} \otimes_\mathbb{R} \mathcal{P}$. Implicit function arguments enable one to show that the morphism $ev := i_\mathcal{F} \otimes i_\mathcal{P} \colon \mathcal{A} \to \mathcal{C}_\mathbb{R}$ is injective; thus, the R-superspace $(\mathbb{R}, \mathcal{A}, ev)$ satisfies Axiom 4. Let $\mathfrak{K}_x = \mathfrak{L}_x \cap (\mathcal{F}_x \otimes 1)$; since for each $x \in \mathbb{R}$ one has $\mathfrak{K}_x^2 = \mathfrak{K}_x$, then for any open and non-empty $U \subset \mathbb{R}$, each derivation of the algebra $\mathcal{A}(U)$ is trivial on $\mathcal{F}_U$. Thus, the sheaves of derivations $\mathcal{D}er\,\mathcal{A}$ and $\mathcal{D}er\,\mathcal{P}$ are canonically isomorphic, there is a global coordinate system $\{x\}$ on M, and Axioms 1 and 2 are satisfied. Now, let us suppose that ϕ admits a decomposition as in Axiom 3; then ϕ is C^1, and since it is not constant, there are points of local injectivity for ϕ, contrary to the assumed properties of ϕ. Thus, Axiom 3 is violated.

We conclude this Section by noticing that morphisms of R-supermanifolds can behave in a rather unsatisfactory way, as the following Example shows. Example Consider the R-supermanifolds $(M, \mathcal{P}, \mathrm{Id})$ and $(M, \mathcal{C}, \mathrm{Id})$, where $M = \mathbb{R}$, $\mathcal{P}$ is the sheaf of polynomials on $\mathbb{R}$, and $\mathcal{C}$ is the sheaf of smooth functions on $\mathbb{R}$. The only R-supermanifold morphisms $(f, f^\sharp) \colon (M, \mathcal{P}) \to (M, \mathcal{C})$ are given by *constant* maps $f \colon \mathbb{R} \to \mathbb{R}$ with $f^\sharp = f^*$, as one can check directly.

G^∞ supermanifolds and Z-expansion

We wish now to introduce the notion of G^∞ function [**27,13,36,16,20**]. Let $U \subset B^{m,0}$ be an open set; a C^∞ map $f \colon U \to B$ is said to be G^∞ if its Fréchet differential is B_0-linear; the resulting sheaf of functions on $B^{m,0}$ will be denoted by $\hat{\mathcal{G}}^\infty$. A G^∞ function $f(x, y)$ on $B^{m,n}$ is a smooth map that can be written in the form $f(x, y) = \sum_{\mu \in \Xi_n} f_\mu(x)\, y^\mu$ for some (in general not uniquely defined) G^∞ functions $f_\mu(x)$. Here Ξ_n is the set of sequences $\mu = \{\mu(1), \ldots, \mu(r)\}$ of integers such that $1 \le \mu(1) < \cdots < \mu(r) \le n$, including the empty sequence μ_0, and we let $y^\mu = y^{\mu(1)} \cdot \ldots \cdot y^{\mu(r)}$. The sheaf of G^∞ functions on $B^{m,n}$ will be denoted by $\mathcal{G}^\infty$.

Definition. *An (m, n) dimensional G^∞ supermanifold is a graded ringed space $(M, \mathcal{A}^\infty)$ locally isomorphic with $(B^{m,n}, \mathcal{G}^\infty)$, with M (Hausdorff) paracompact.* ∎

One should notice that, generally speaking, a G^∞ supermanifold is not an R-supermanifold [**13,33,8**], in that Axiom 1 may be violated.

It is natural to ask whether, given an R-supermanifold $(M, \mathcal{A}, ev)$, the pair $(M, \mathcal{A}^\infty)$, where $\mathcal{A}^\infty = \mathrm{Im}\, ev$, is a G^∞ supermanifold; contrary to what asserted in [**33**], this question in general has a negative answer. Indeed, the sheaf $\mathcal{A}$ may not be topologically complete with respect to the even coordinates; the following Example should clarify what we mean.

Example Let us take $B = \mathbb{R}$, $n = 0$ and $M = \mathbb{R}^m$. If we consider the sheaf $\mathcal{A} = \mathbb{R}[x^1, \ldots, x^m]$ of polynomial functions on $\mathbb{R}^m$ and the trivial evaluation morphism $ev \colon \mathcal{A} \hookrightarrow \mathcal{C}_\mathbb{R}$, $ev(f) = f$, then $(M, \mathcal{A}, ev)$ is an R-supermanifold of dimension $(m, 0)$. But $(M, ev(\mathcal{A})) = (M, \mathbb{R}[x^1, \ldots, x^m])$ is certainly not an $(m, 0)$-dimensional G^∞ supermanifold, which in this case would be an m-dimensional smooth manifold.

Thus, there are R-supermanifolds which do not satisfy Rothstein's *structural definition* of supermanifolds [**33**]. In order to characterize those R-supermanifolds which fulfill that definition, a further axiom must be imposed. This will be discussed in next Section.

In the rest of this Section we discuss a method that, to a large extent, enables one to reduce the study of G^∞ functions to that of B-valued functions on Euclidean space, namely, the so-called *Z-expansion* [**7,8,18,20,26,31,32**]. We show that the Z-expansion is applicable to a larger class of graded-commutative Banach algebras than it was known earlier.

Theorem. *Let B be a graded-commutative Banach algebra. The following conditions are equivalent:*

(1) *B is local and the linear span of products of odd elements is dense in the radical $\mathfrak{Rad}B$ of B;*
(2) *Any closed unital subalgebra of B containing B_1 coincides with B;*
(3) *The reflection of B in the category of purely even Banach algebras is $\mathbb{R}$. (In other terms, for any graded Banach algebra morphism h from B to a purely even Banach algebra, the image $h(B)$ is isomorphic to $\mathbb{R}$.)*
(4) *For an appropriate cardinal number η, there exists a submultiplicative seminorm p on a Grassmann algebra B_η with η anticommuting generators such that B is isomorphic to the Banach algebra associated with (B_η, p). (That is, B is isomorphic to the completion of the quotient normed algebra of B_η by the ideal $\{x \in B_\eta | p(x) = 0\}$.)*

Proof. (1) $\Leftrightarrow$ (2): obvious. (2) $\Leftrightarrow$ (3): it follows from the fact that any graded Banach algebra morphism h from a graded Banach algebra B to any purely even Banach algebra can be factored through the quotient algebra of B by the closed ideal generated by the odd part B_1; now, the quotient algebra is $\mathbb{R}$ if and only if (2) is true. (3) $\Rightarrow$ (4): let η be the cardinality of B_1. Denote by π the graded algebra morphism from B_η to B such that the image under π of the set of generators coincides with B_1, and for all $x \in B_\eta$ set $p(x) = \|\pi(x)\|_B$. (4) $\Rightarrow$ (3): Let $\pi\colon B_\eta \to B$ be the projection, and let h be any morphism from B to a purely even Banach algebra. Then the composite morphism $h \circ \pi$ is a graded algebra morphism from a Grassmann algebra B_η to an even algebra; clearly, the image of $h \circ \pi$ is $\mathbb{R}$, and at the same time it is dense in the image of h. □

Jadczyk and Pilch were the first to consider the above property (in their paper [**20**] this feature, in the form (1), was one of the two conditions determining the class of Banach-Grassmann algebras). One of the authors of the present paper has studied the algebras satisfying this property under the name of 'supernumber algebras'[**26,27,29,30**]. Here we propose to call the graded-commutative Banach algebras B satisfying one of the equivalent conditions (1)-(4) *Banach algebras of Grassmann origin* because of (4); we shall shorten this into 'BGO-algebras.' Seemingly, these algebras form the most important class of local graded-commutative Banach algebras; as a matter of fact, all ground algebras for supermanifolds that have been so far introduced are BGO-algebras. So are indeed the finite-dimensional Grassmann algebras (in this paper we denote them by B_L, L being the number of generators) and Rogers's infinite-dimensional B_∞ algebra [**31**] (that in particular is a Banach-Grassmann algebra). A large number of new examples of Banach-Grassmann algebras is described in [**29,30**]. The so-called Grassmann-Banach algebras [**18**] are also BGO-algebras. Moreover, any algebra of superholomorphic functions on a purely even graded Banach space [**35**] can be made into a BGO-algebra.

Let B be a local Banach algebra. We will denote by σ_B or simply σ the augmentation morphism (that is, the unique character) $\sigma\colon B \to \mathbb{R}$, and by $s\colon B \to \mathfrak{Rad}B$ the complementary mapping, $s+\sigma = Id_B$. The mappings $\sigma^{m,n}$ (body map) and $s^{m,n}$ (soul map) from $B^{m,n}$ to $\mathbb{R}^m$ and $(\mathfrak{Rad} B)^{m,n}$, respectively, are defined as direct sums of copies of the former two mappings. For a subset $X \subset \mathbb{R}^m$, we denote $X^\sim = (\sigma^{m,0})^{-1}(X)$, and call *DeWitt open* sets the open subsets of $B^{m,n}$ of the form $U^\sim$, $U \subset \mathbb{R}^m$ [**16,8**].

For any $U \subset \mathbb{R}^m$, the Z-expansion is the morphism of graded algebras

$$Z\colon \mathcal{F} \to \mathcal{C}^\infty((\sigma^{m,0})^{-1}(U)),$$

(where $\mathcal{F}$ is a dense subalgebra of the graded algebra $\mathcal{C}^\infty(U)$ of B-valued C^∞ functions on U) defined by the formula

$$Z(h)(x) = \sum_{j=0}^{\infty} \frac{1}{j!} D^{(j)} h_{\sigma^{m,0}(x)}(s^{m,0}(x))$$

for $h \in \mathcal{C}^\infty_{L'}(U)$ and all $x \in U^\sim$; here the j-th Fréchet differential $D^{(j)}h_{\sigma^{m,0}(x)}$ of h at the point $\sigma^{m,0}(x)$ acts on $B^{m,0} \times \cdots \times B^{m,0}$ (j times) simply by extending by B_0-linearity its action on $\mathbb{R}^m \times \cdots \times \mathbb{R}^m$. When B is finite-dimensional one can take $\mathcal{F} = \mathcal{C}^\infty(U)$. The Z-expansion can be written in another form by using partial derivatives:

$$Z(h)(x) = \sum_{|J|=0}^{\infty} \frac{1}{J!} \left(\frac{\partial^{|J|} h}{\partial x^J}\right)_{\sigma^{m,0}(x)} (s^{m,0}(x))^J ,$$

where J is a multiindex.

The proof of the following result is the same as in [**20**] where B is a Banach-Grassmann algebra; actually, in that proof only the property of being a BGO-algebra is used.

Theorem. *Let B be a BGO-algebra, let m be a positive integer and V be an open subset of $B^{m,0}$. An arbitrary G^∞ function f on V admits a unique extension to a G^∞ function over the DeWitt open set $(\sigma^{m,0}(V))^\sim$.* □

We study now the convergence of the Z-expansion.

Theorem. *Let B be a Banach algebra of Grassmann origin, let m be a positive integer and U be an open subset of $\mathbb{R}^m$. For an arbitrary G^∞ function f on $U^\sim$, the Z-expansion of the restriction $f_|$ of f to U converges to f. The convergence is uniform on compacta lying in any 'soul fibre' $\{x\}^\sim$, with $x \in U$. For any $i = 1, \ldots, m$ the following holds: $\partial f/\partial x^i = Z(\partial f_|/\partial x^i)$.*

Proof. Denote by $\hat{B}$ a unital subalgebra of B generated by the odd part B_1; $\hat{B}$ is local and dense in B. Taylor formula for a G^∞ function f [**36**] shows that the Z-expansion of $f_|$ converges to $f(z)$ at any $z \in \hat{B}^{m,0}$ since the remainder of the series vanishes for $|J|$ large enough. Now, fix $x \in U$; since the Z-expansion converges pointwise on the 'nilpotent fibre' $\{x\} + (\mathfrak{Rad}\hat{B})^{m,0}$ over x to a continuous function, and the terms of the Z-expansion restricted to this space are polynomials on a normed linear space, the convergence is *normal at zero* [**12**]. This means that for some neighbourhood U of zero in $\{x\} + (\mathfrak{Rad}\hat{B})^{m,0}$ the convergence of the Z-expansion to f is uniform on U. As a consequence, the Z-expansion converges uniformly to f on the closure of U in the 'quasinilpotent fibre' $\{x\}^{\sim}$, that is, it converges to f normally at zero in the Banach space $\{x\}^{\sim}$ as well. (Remark that $\{x\} + (\mathfrak{Rad}\hat{B})^{m,0}$ is dense in $\{x\}^{\sim} = \{x\} + (\mathfrak{Rad}B)^{m,0}$.) Due to quasinilpotency of $\mathfrak{Rad}B$, this implies the pointwise convergence of the Z-expansion to f on any fibre $\{x\}^{\sim}$ and thus on the whole of $U^{\sim}$. The first statement is thus proved. On the other hand, this implies that for any $x \in U$ the restriction $f|\{x\}^{\sim}$ is an entire function [**12**], hence is analytic (*ibid.*, Prop. 8.2.3.) Since the Taylor series of an analytic function converges to it uniformly on compacta lying in the interior of the domain of convergence, the second statement follows as well. Finally, the claim regarding partial derivatives follows from the fact that restriction of $\partial f/\partial x^i$ to U coincides with $\partial f_|/\partial x^i$. □

A Banach space-valued function f on an open subset U of $\mathbb{R}^m$ is said to be *Pringsheim regular* if its Taylor series converges in a neighbourhood of every point $x \in U$ (not necessarily to f itself). One can show [**28**] that for $B = B_\infty$ all G^∞ functions are obtained by Z-expansion of Pringsheim regular functions, and that whenever a C^∞ function has a convergent Z-expansion then its sum is a G^∞ function. Thus, for $B = B_\infty$ the algebra $\mathcal{F}$ is formed by all Pringsheim regular B-valued mappings.

R^∞-SUPERMANIFOLDS

Let $(M, \mathcal{A}, ev)$ be an R-superspace over a graded-commutative Banach algebra B, that in this and the next Sections is assumed to be *real*, and let $\| \;\; \|$ denote the norm in B; the rings of sections $\mathcal{A}(U)$ of $\mathcal{A}$ on every open subset $U \subset M$ can be topologized by means of the seminorms $p_{L,K}\colon \mathcal{A}(U) \to \mathbb{R}$ defined by

$$p_{L,K}(f) = \max_{z \in K} \left\| \widetilde{L(f)}(z) \right\|, \qquad ()$$

where L runs over the differential operators of $\mathcal{A}$ on U, and $K \subset U$ is compact (cf. [**17,22**]). The resulting topology in $\mathcal{A}(U)$, that we call the R^∞ topology, endows it with a structure of locally convex graded B-algebra (possibly non-Hausdorff). In the case where $(M, \mathcal{A}, ev)$ is an R-supermanifold, one obtains as a consequence of the axioms that the R^∞-topology is alternatively defined by the family of seminorms

$$p_K^I(f) = \max_{\substack{z \in K \\ |J| \le I,\, \mu \in \Xi_n}} \left\| ev\left(\left(\frac{\partial}{\partial x}\right)^J \left(\frac{\partial}{\partial y}\right)_\mu f \right)(z) \right\| ,$$

where K runs over the compact subsets of a coordinate neighbourhood W with coordinates $(x^1, \ldots, x^m, y^1, \ldots, y^n)$ (as a matter of fact, in this case Axiom 4 means that $\mathcal{A}(U)$ is Hausdorff).

The R^∞-topology on an algebra $\mathcal{A}(U)$ of superfunctions can also be described as the coarsest topology with the properties:

(i) the evaluation map ev_U from $\mathcal{A}(U)$ to the space $\mathcal{C}_M(U)$ of all continuous B-valued functions on U endowed with the topology of compact convergence is continuous;

(ii) all the differential operators $L \in \mathcal{D}er\mathcal{A}(U)$ are continuous.

Theorem. *Let $(f, f^\sharp)$ be an R-superspace morphism between two R-supermanifolds $(M, \mathcal{A}, ev^M)$ and $(N, \mathcal{B}, ev^N)$. Then $f^\sharp_V \colon \mathcal{B}(V) \to \mathcal{A}(f^{-1}(V))$ is continuous for every open subset $V \subset N$.*

Proof. It suffices to verify the property for the case where V is a coordinate neighbourhood. Fix a coordinate system $\varphi = (x^1, \ldots, x^m, y^1, \ldots, y^n)$ on V. Let L be an arbitrary differential operator over $f^{-1}(V)$ of order $k \geq 0$ and let K be a compact subset of $f^{-1}(V)$. For multi-indices $J \in \mathbb{N}^m$ and $\mu \in \Xi_n$ such that the total length does not exceed k, i.e., $|J| + |\mu| \leq k$, we let $C_{J,\mu} := \max_{z \in K} \| ev^M\big(L(f^\#(x^J y^\mu))\big)(z)\|_B$; because of the continuity of the map $x \mapsto \|ev^M\big(L(f^\#(x^J y^\mu))\big)(p)\|_B$, the nonnegative real numbers $C_{J,\mu}$ are well defined. We will now prove that if a superfunction $g \in \mathcal{B}(V)$ is such that for every J and μ with $|J| + |\mu| \leq k$ one has

$$\max_{z \in K} \|C_{J,\mu}\, ev^N\big(\partial^{J,\mu} g\big)(f(z))\| \leq 1,$$

where

$$\partial^{J,\mu} = \frac{\partial^{J_1}}{\partial(x^1)^{J_1}} \cdots \frac{\partial^{J_m}}{\partial(x^1)^{J_1 m}} \cdots \frac{\partial}{\partial y^{\mu_1}} \cdots \frac{\partial}{\partial y^{\mu_n}}$$

with $J = (J_1, \ldots, J_m)$ and $\mu = \mu_1, \ldots, \mu_n$; then for all $z \in K$ one also has

$$\max_{z \in K} \|ev^M\big(L(f^\#(g))\big)(z)\| \leq \exp(m+n),$$

which observation will obviously complete the proof. To prove this, let $z \in K$ be fixed. By repeated application of Axiom 3 we represent g in a small neighbourhood of $f(z)$ as follows:

$$\begin{aligned} g = &\sum_{|J|+|\mu|<k} \frac{1}{J!} ev^N\big(\partial^{J,\mu} g\big)(f(z))\, (x - f(z))^J (y - f(z))^\mu \\ &+ \sum_{|J|+|\mu|=k} \nu_{J,\mu}\, (x - f(z))^J (y - f(z))^\mu, \end{aligned}$$

where the $\nu_{J,\mu}$'s are some superfunctions whose evaluations vanish at the point $f(z)$; one can verify that $ev^M\big(f^\# \nu_{J,\mu}\big)(z) = ev^N\big(\nu_{J,\mu}\big)(f(z)) = (1/J!)\, ev^N\big(\partial^{J,\mu} g\big)(f(z))$. Thus, for any $g \in \mathcal{B}(V)$ with the above properties the following holds:

$$\begin{aligned}
&\| ev^M\big(L(f^\#(g))\big)(z)\|_B \\
&= \Big\| \sum_{|J|+|\mu|<k} \frac{1}{J!} ev^N\big(\partial^{J,\mu} g\big)(f(z))\, ev^M\big(L(f^\#((x-f(z))^J(y-f(z))^\mu))\big)(z) \\
&\qquad + \sum_{|J|+|\mu|=k} ev^M\big(L(f^\#(\nu_{J,\mu} f^\#((x-f(z))^J(y-f(z))^\mu)))\big)(z) \Big\|_B \\
&\le \sum_{|J|+|\mu|<k} \frac{1}{J!} C_{J,\mu} \\
&\qquad + \sum_{|J|+|\mu|=k} \| ev^M\big(f^\#(\nu_{J,\mu}\big)(z)\, ev^M\big(L(f^\#((x-f(z))^J(y-f(z))^\mu))\big)(x)\|_B \\
&\le \sum_{|J|+|\mu|\le k} \frac{1}{J!} < \exp(m+n)\,.
\end{aligned}$$

□

Let $(M, \mathcal{A}, ev)$ be an (m,n) dimensional R-supermanifold, and let (U,φ) be a coordinate chart on it with $\varphi = (x^1, \dots, x^m, y^1, \dots, y^n)$. Define $\hat{\mathcal{A}}_\varphi$ as the subsheaf of $\mathcal{A}_{|U}$ whose sections 'do not depend on the odd variables,' in the sense that

$$\hat{\mathcal{A}}_\varphi(V) = \big\{ f \in A(V) \;\big|\; \frac{\partial f}{\partial y^\alpha} = 0, \quad \alpha = 1, \dots, n \big\}\,,$$

for every open subset $V \subset U$. We have the following canonical isomorphism (cf. [**33**]):

$$\hat{\mathcal{A}}_\varphi \otimes_{\mathbb{R}} \textstyle\bigwedge_{\mathbb{R}} \mathbb{R}^n \to \mathcal{A}_{|U} \qquad ()$$

having identified $\bigwedge_{\mathbb{R}} \mathbb{R}^n$ with the Grassmann algebra generated by the y's. Moreover, the restriction of ev to $\hat{\mathcal{A}}_\varphi$ is injective.

Lemma. *The isomorphism (4.2), $\mathcal{A}(V) \xrightarrow{\sim} \hat{\mathcal{A}}_\varphi(V) \otimes_{\mathbb{R}} \bigwedge_{\mathbb{R}} \mathbb{R}^n$, is a topological isomorphism for every open subset $V \subset U$.*

Proof. Since the tensor product on the right hand side can be identified with the topological linear space $\hat{\mathcal{A}}_\varphi(V)^{2^n}$ with the usual product topology, in order to check that the algebraic isomorphism is also a homeomorphism, it remains to verify that all the projection maps (under the above identification) $\mathcal{A}(V) \to \hat{\mathcal{A}}_\varphi(V)$ which are labelled by multiindices μ and given by $y^\mu f(x) \mapsto f(x)$ are continuous. But this follows from the very definition of the R^∞-topology because the projection maps are represented as compositions of evaluation maps with differential operators. □

We wish now to investigate the question of the topological completeness of the rings of sections of the structure sheaf of an R-supermanifold. The discussion of the previous Section leads us to introduce the following supplementary axiom.

Axiom 5. (Completeness) *For every open subset $U \subset M$, the topological algebra $\mathcal{A}(U)$ is complete.*

Axioms 4 and 5, taken together, are equivalent to still another axiom:

Axiom 6. *For every open subset $U \subset M$, the topological algebra $\mathcal{A}(U)$ is complete Hausdorff.*

Thus, it turns out that in order to determine a class of supermanifolds whose rings of sections are topologically complete, it is enough to replace Axiom 4 by Axiom 6. We therefore consider the following axiomatic characterization of supermanifolds.

Definition. *An R^∞-supermanifold over B is an R-supermanifold $(M, \mathcal{A}, ev)$ over B satisfying additionally Axiom 5; or, equivalently, it is an R-superspace fulfilling Axioms 1, 2, 3, and 6.*

We have shown in the previous Section that Axiom 3 can be replaced by the simpler Axiom $3'$ provided that $(M, \mathcal{A})$ is a graded locally ringed space. As a matter of fact, in the case of R^∞-supermanifolds a simpler assumption, that of locality of the ground algebra B, can be made.

Theorem. *Let B be a local graded-commutative Banach algebra. An R-superspace $(M, \mathcal{A}, ev)$ over B satisfying axioms 1, 2, $3'$ and 6 is an R^∞-supermanifold.*

Proof. One needs to show that $(M, \mathcal{A})$ is a graded *locally* ringed space. Let $p \in M$; we shall prove that the ideal

$$\mathcal{J}_p := \{g \in \mathcal{A}_p : \tilde{g}(p) \in \mathfrak{Rad}B\}$$

is the only maximal ideal in $\mathcal{A}_p$; it suffices to show that any $g \notin \mathcal{J}_p$ has a multiplicative inverse. Pick a representative $g' \in \mathcal{A}(U)$ of g, where U is a suitable coordinate neighbourhood of p. Since the map $q \mapsto \tilde{g}'(q)$ from U to B is continuous, and since the invertible elements of a Banach algebra B are exactly those not belonging to the radical $\mathfrak{Rad}B$, then one can assume that $\tilde{g}'(p) = 1$ and that for all $q \in U$ one has $\|\tilde{g}'(q) - 1\|_B < 1$ (in particular, $\tilde{g}'(q)$ is invertible in B). We shall show that the series $\sum_{j=0}^{\infty} h^j$, where $h = 1 - g'$, converges in the R^∞-topology on the algebra $\mathcal{A}(U)$; the germ of the sum of this series will be a multiplicative inverse to g.

Let $K \subset U$ be compact and L be a differential operator of order k on U. It suffices to prove that the series with nonnegative real terms $\sum_{j=0}^{\infty} \max_{q \in K} \|L(h^j)\|_B$ converges.

By using local coordinates one can write

$$L = \sum_{i_1 + \cdots + i_{m+n} = k} L_1^{i_1} \dots L_{m+n}^{i_{m+n}} ;$$

here $(m, n) = \dim(M, \mathcal{A}, ev)$, and each L_i is a first-order differential operator. Then one has

$$L(h^j) = \sum_{r=1}^{k} \sum_{|J_1| + \cdots + |J_r| = k} P_{J_1, \dots, J_r}(j)\, h^{j-r} L^{J_1}(h) \dots L^{J_r}(h) ;$$

here the J's are multiindices, the number of summands depends on k (and hence on L) only, and $P_{J_1,\dots,J_r}(j)$ are integer polynomials in j (and in k, but k is fixed) of combinatorial origin. Notation is such that $L^J = L_1^{j_1}\dots L_N^{j_N}$ if $J=(j_1,\dots,j_N)$. Let

$$C_{J_1,\dots,J_r} = \max_{q\in K} \|\widetilde{L^{J_1}(h)}(q)\dots\widetilde{L^{J_r}(h)}(q)\|_B\,.$$

Since $\max_{q\in K}\|\tilde h(q)\|_B = t<1$, one has:

$$\begin{aligned}\sum_{j=0}^\infty \max_{q\in K}\|L(h^j)\|_B &\le \sum_{j=0}^\infty\sum_{r=1}^k \sum_{|J_1|+\dots+|J_r|=k} P_{J_1,\dots,J_r}(j)\, C_{J_1,\dots,J_r}\, t^{j-r}\\ &= \sum_{|J_1|+\dots+|J_r|=k}\sum_{r=1}^k C_{J_1,\dots,J_r}\sum_{j=0}^\infty P_{J_1,\dots,J_r}(j)\, t^{j-r}\,,\end{aligned}$$

the last series being convergent. □

We wish now to check that R^∞-supermanifolds can be defined by means of a local condition. This implies that Rothstein's structural definition [**33**] singles out the category of R^∞ supermanifolds, rather than the wider category of R-supermanifolds. In other terms, R^∞ supermanifolds coincide with Rothstein's $C^\infty(B)$-manifolds. Another consequence is that only for R^∞ supermanifolds it is true that the pair $(M, ev(\mathcal{A}))$ is a G^∞ supermanifold in the sense of Rogers.

During the proof we shall need to assume that B is a BGO-algebra. We start by stating the completeness axiom in an alternative way. The following result is proved straightforwardly.

Proposition. *An R-supermanifold is an R^∞-supermanifold if and only if every point is contained in a coordinate chart (U,φ) such that the rings $\hat{\mathcal{A}}_\varphi(V)$ are complete in the R^∞-topology.* □

We define the *standard R^∞-supermanifold* over $B^{m,n}$ as the graded ringed space $(B^{m,n},\mathcal{G})$, where $\mathcal{G} = p^{-1}\hat{\mathcal{G}}^\infty\otimes_{\mathbb{R}}\bigwedge\mathbb{R}^n$; here p is the projection $B^{m,n}\to B^{m,0}$. The evaluation morphism is given by $ev(f\otimes a) = fa$. One proves that $(B^{m,n},\mathcal{G}, ev)$ is an R^∞ supermanifold; the only nontrivial thing to be checked (when B is infinite-dimensional) is the following.

Lemma. *The algebra $\mathcal{G}(U)$ is complete in the R^∞ topology for every open $U\subset B^{m,n}$.*

Proof. In view of the isomorphism (4.2) we may consider only the case $n=0$, so that we may identify $\mathcal{G}(U)$ with an algebra of G^∞ functions of even variables. Let $\overline{\mathcal{G}(U)}$ be the completion of $\mathcal{G}(U)$ in the R^∞ topology; all differential operators on $\mathcal{G}(U)$ extend to $\overline{\mathcal{G}(U)}$. Being a metric space, U is a k-space [**21**] and therefore $\overline{\mathcal{G}(U)}$ may be regarded as a subalgebra of $\mathcal{C}_M(U)$, the algebra of B-valued continuous functions on U. One needs to check that any function $f \in \overline{\mathcal{G}(U)}$ at any point $p \in U$ is Fréchet differentiable and that its differential is given by multiplicative action of the partial derivatives of f with respect to the x's, formally extended by continuity from $\mathcal{G}(U)$. Since the locally convex space $B^{m,n}$ is Banach, 'Fréchet' can be replaced by 'Gâteaux,' that is, one can restrict to an arbitrary 1-dimensional subspace K of U passing through p. The space of C^∞ B-valued functions on K is complete with respect to its standard topology and therefore $f|_K$ is in this space. This means that the Gâteaux differential $d_p f$ of f at p exists (*a priori* not necessarily bounded). Pick a net (f_α) of functions $f_\alpha \in \mathcal{G}(U)$ converging to f in the R^∞ topology. Clearly $f_\alpha|_K \to f|_K$ in the C^∞ topology over K. Let $K = \{p + at : t \in \mathbb{R}\}$, $a = (a_1, \dots, a_m) \in B^{m,0}$. For all α, due to the usual chain rule for G^∞ functions, one has

$$d_p(f_\alpha|_K)(a) = \sum a_i \left(\frac{\partial f_\alpha}{\partial xi} \right)_p .$$

As $f_\alpha \to f$, the above equality turns by continuity into the following:

$$d_p(f|_K)(a) = \sum a_i \left(\frac{\partial f}{\partial x^i} \right)_p ,$$

which implies that for an arbitrary $h \in B^{m,0}$ the desired property holds:

$$d_p f(h) = \sum h_i \left(\frac{\partial f}{\partial x^i} \right)_p$$

□

Quite evidently, any R-superspace $(M, \mathcal{A}, ev)$ which is locally isomorphic to the standard R^∞-supermanifold over $B^{m,n}$ is an (m,n) dimensional R^∞-supermanifold. By means of Proposition 4.6 we may prove the converse:

Proposition. *Any (m,n) dimensional R^∞-supermanifold $(M, \mathcal{A}, ev)$ over a BGO-algebra B is locally isomorphic to the standard R^∞-supermanifold over $B^{m,n}$.*

To prove this result we need a preliminary Lemma, which can be proved essentially as in [**29**] (cf. also [**8**]), and a result on the density of polynomials in the rings of superfunctions.

Lemma. *Let $(M, \mathcal{A}, ev)$ be an (m,n) dimensional R-supermanifold, and let (U, φ) be a local chart for it. For all $f \in \mathcal{A}(U)$, the composition $\tilde{f} \circ \tilde{\varphi}^{-1}$ is a G^∞ function on $\tilde{\varphi}(U) \subset B_L^{m,n}$.* □

Let $(M, \mathcal{A}, ev)$ be an R-supermanifold, and let, for a fixed coordinate system $\varphi = (x^1, \dots, x^m, y^1, \dots, y^n)$ in U, $\mathcal{P}_\varphi(U)$ be the graded B-subalgebra of $\mathcal{A}(U)$ generated by the coordinates. The following result may be considered as a graded analogue of the Weierstrass approximation theorem. We do not know whether it remains true when B is an arbitrary graded-commutative Banach algebra.

Theorem. *Let B be a BGO-algebra. Then $\mathcal{P}_\varphi(U)$ is dense in $\mathcal{A}(U)$.*

Proof. The demonstration of this result is very lengthy and has been postponed to an Appendix. □

Proof of Proposition 4.7. Let (U, φ) be a coordinate chart for $(M, \mathcal{A}, ev)$, with $\varphi = (x^1, \ldots, x^m, y^1, \ldots, y^n)$. In view of the isomorphism (4.2) one can define an injection

$$\hat{T}_\varphi \colon \hat{\mathcal{A}}_\varphi \hookrightarrow \tilde{\varphi}^{-1}\hat{\mathcal{G}}_{|\tilde{\varphi}(U)}$$

by letting $\hat{T}_\varphi(f) = \tilde{f} \circ \tilde{\varphi}^{-1}$; by Lemma 4.8 $\hat{T}_\varphi(f)$ is a G^∞ function and therefore is a section of $\tilde{\varphi}^{-1}\hat{\mathcal{G}}_{|\tilde{\varphi}(U)}$. Furthermore, $\hat{T}_\varphi$ is a topological isomorphism with its image, so that $\hat{T}_\varphi(\hat{\mathcal{A}}_\varphi)$ is complete. Since this space contains the G^∞ functions that are polynomials in the even coordinates, it contains all the G^∞ functions by virtue of Theorem 4.9; that is, $\hat{T}_\varphi$ is an isomorphism. The morphism $\hat{T}_\varphi$ determines a topological isomorphism

$$T_\varphi \colon \mathcal{A}_{|U} \to \tilde{\varphi}^{-1}\mathcal{G}_{|\tilde{\varphi}(U)}$$

simply by letting $T_\varphi(\sum f_\mu \otimes y^\mu) = \sum \hat{T}_\varphi(f_\mu) \otimes y^\mu$. Now, the commutative diagram

$$\begin{tikzcd}
\mathcal{A}_{|U} \arrow[r, "\sim"] \arrow[d, "ev^U"'] & \tilde{\varphi}^{-1}\mathcal{G}_{|\tilde{\varphi}(U)} \arrow[d, "ev"] \\
\mathcal{A}^\infty{}_{|U} \arrow[r, "\sim"] \arrow[d] & \tilde{\varphi}^{-1}\mathcal{G}^\infty_{|\tilde{\varphi}(U)} \arrow[d] \\
0 & 0
\end{tikzcd}$$

proves the thesis. □

Corollary. *If $(M, \mathcal{A}, ev)$ is an R^∞-supermanifold over a BGO-algebra, then $(M, ev(\mathcal{A}))$ is a G^∞ supermanifold.*

Proof. This result holds evidently for the standard R^∞-supermanifold over $B^{m,n}$, and therefore, by local isomorphism, also for an arbitrary R^∞-supermanifold. □

Finally, we consider the coordinate description of morphisms. What follows generalizes results already known for graded manifolds [**23**] and for finite-dimensional ground algebras [**33,8**]. Let $(M, \mathcal{A}, ev^M)$ be an R^∞-supermanifold over a BGO-algebra B, let U be an open set in $B^{m,n}$, and denote by $(U, \mathcal{G}, ev)$ the restriction to U of the standard R^∞-supermanifold over $B^{m,n}$.

Lemma. *Let B be a BGO-algebra. If $(f, \phi) \colon (M, \mathcal{A}, ev^M) \to (U, \mathcal{G}, ev)$ and $(f, \psi) \colon (M, \mathcal{A}, ev) \to (U, \mathcal{G}, ev)$ are R^∞-supermanifolds morphisms, and $\phi(x^i) = \psi(x^i)$ for $i = 1, \ldots, m$, $\phi(y^\alpha) = \psi(y^\alpha)$ for $\alpha = 1, \ldots, n$, then $\phi = \psi$.*

Proof. ϕ and ψ coincide over the sheaf of polynomials in the coordinates, and therefore by continuity they also coincide over its completion $\mathcal{G}$. □

Proposition. *Let B be a BGO-algebra, and let $U \subset B^{m,n}$ be an open subset.*

(1) *A family of sections $(u^1, \dots, u^m, v^1, \dots, v^n)$ of $\mathcal{G}$ on U is a coordinate system for $(U, \mathcal{G}_{|U}, ev)$ as an R-supermanifold if and only the evaluations $(\tilde{u}^1, \dots, \tilde{u}^m, \tilde{v}^1, \dots, \tilde{v}^n)$ yield a G^∞ coordinate system.*

(2) *Let $(u^1, \dots, u^m, v^1, \dots, v^n)$ be a coordinate system for $(U, \mathcal{G}, ev)$, let $f: U \to W \subset B_L^{m,n}$ be the homeomorphism $z \mapsto (\tilde{u}^1(z), \dots, \tilde{u}^m(z), \tilde{v}^1(z), \dots, \tilde{v}^n(z))$, and let $(x^1, \dots, x^m, y^1, \dots, y^n)$ be a coordinate system on W. There exists a unique isomorphism of R^∞-supermanifolds $(f, \phi): (U, \mathcal{G}_{|U}, \delta) \to (W, \mathcal{G}_{|W}, \delta)$ such that $\phi(x^i) = u^i$ for $i = 1, \dots, m$, and $\phi(y^\alpha) = v^\alpha$ for $\alpha = 1, \dots, n$.*

(3) *Every isomorphism $g : U \to V \subset B^{m,n}$ can be extended (in many ways) to an isomorphism of R^∞-supermanifolds $(g, \phi): (U, \mathcal{G}_{|U}) \xrightarrow{\sim} (V, \mathcal{G}_{|V})$. Here 'extension' means that the diagram*

$$\begin{tikzcd}
\mathcal{G}_{|V} \arrow[r, "\phi"] \arrow[d, "ev"'] & g_*\mathcal{G}_{|U} \arrow[d, "ev"] \\
\mathcal{G}^\infty{}_{|V} \arrow[r, "g^*"'] & g_*\mathcal{G}^\infty{}_{|U}
\end{tikzcd}$$

commutes.

Proof. (1) Since Ker ev is nilpotent, a matrix of sections of $\mathcal{G}$ is invertible if and only if its evaluation is invertible as well, thus proving the statement.

(2) One can define a ring morphism $\phi: \mathcal{P} \to g_*\mathcal{G}$, where $\mathcal{P}$ is the sheaf of polynomials in x and y, by imposing that $\phi(x^i) = u^i$, $\phi(y^\alpha) = v^\alpha$ for $i = 1, \dots, m$, $\alpha = 1, \dots, n$. Since the topology of $\mathcal{G}$ can be described by the seminorms associated with any coordinate chart, ϕ is continuous and therefore induces a morphism between the completions, $\phi: \mathcal{G} \to g_*\mathcal{G}$. To see that (g, ϕ) is an isomorphism, we can construct, by the same procedure, an 'inverse' morphism (g', ψ); then, we have two morphisms of R^∞-supermanifolds $(\mathrm{Id}, \mathrm{Id}), (\mathrm{Id}, \psi \circ \phi): (U, \mathcal{G}_{|U}, ev) \to (U, G_{|U}, ev)$ that coincide on a coordinate system, thus finishing the proof by the previous Lemma.

(3) follows from (1) and (2) since a G^∞ isomorphism transforms G^∞ coordinate systems into G^∞ coordinate systems. □

If $B = B_L$, then R^∞ supermanifolds reduce to the G-supermanifolds introduced by some of the authors [**2**]; they have been extensively studied in [**8**]. This on the one hand shows the relevance of G-supermanifolds, in that they are the unique examples of supermanifolds over B_L satisfying the extended axiomatics, and, on the other hand, demonstrates that that axiomatics admits concrete models.

From R-supermanifolds to R^∞ supermanifolds

In this section we show that with any R-supermanifold one can associate an R^∞-supermanifold in a functorial way. We assume that the ground algebra B is a BGO-algebra.

Let $(M, \mathcal{A}, ev)$ be an R-supermanifold; for any open set $U \subset M$, let $\mathcal{Q}(U)$ be the completion of $\mathcal{A}(U)$ in the R^∞-topology. This defines a presheaf $\mathcal{Q}$; let us denote by $\bar{\mathcal{A}}$ the associated sheaf. Let W be a coordinate neighbourhood, with coordinates $\varphi = (x^1, \ldots, x^m, y^1, \ldots, y^n)$; since the polynomials are dense in $\mathcal{A}$ (Theorem 4.9), there is a presheaf isomorphism $\tilde{\varphi}^{-1}\mathcal{G}_{|\tilde{\varphi}(W)} \simeq \mathcal{Q}_{|W}$. This means that $\mathcal{Q}_{|W}$ is isomorphic with its associated sheaf $\bar{\mathcal{A}}_{|W}$ for each coordinate neighbourhood W, so that $\bar{\mathcal{A}}$ can be endowed with a structure of a sheaf of *complete* Hausdorff locally convex graded B-algebras. The evaluation morphism ev, being continuous, induces a morphism $ev\colon \bar{\mathcal{A}} \to \mathcal{C}_M$, so that $(M, \bar{\mathcal{A}}, ev)$ is an R-superspace over B, which is locally isomorphic with the standard R^∞-supermanifold over $B^{m,n}$. Hence, by Proposition 4.7, we obtain the following result.

Theorem. *The triple $(M, \bar{\mathcal{A}}, ev)$ is an R^∞ supermanifold.* □

Quite obviously, there is a canonical R-superspace morphism $(f, f^\sharp)\colon (M, \bar{\mathcal{A}}, ev) \to (M, \mathcal{A}, ev)$, with $f = \mathrm{Id}$. Moreover, in view of Theorem 4.1, this correspondence between the two categories of supermanifolds is functorial.

In accordance with Corollary 4.10 and with the previous Theorem, any R-supermanifold determines an 'underlying' G^∞ supermanifold; thus, one can prove the following result.

Proposition. *Let $(f, f^\sharp)\colon (M, \mathcal{A}, ev^M) \to (N, \mathcal{B}, ev^N)$ be an R-supermanifold morphism. Then $f\colon M \to N$ is a G^∞ map.*

Proof. One can assume that M and N are coordinate neighbourhoods, in which case the result is proved by Lemma 4.8. □

Holomorphic supermanifolds

Let $(M, \mathcal{A}, ev)$ be a complex R-superspace, that is, an R-superspace over a *complex* graded commutative Banach algebra B. We introduce a topology on the algebra $\mathcal{A}(U)$ for every open $U \subset M$, which we call the *R^ω-topology*, as the coarsest topology with the properties:

(i) the evaluation map ev_U from $\mathcal{A}(U)$ to the space $\mathcal{C}_M(U)$ of all continuous B-valued functions on U endowed with the topology of compact convergence is continuous;

(ii) all odd differential operators $L \in \mathcal{D}er\, A(U)$ are continuous.

One can describe this topology by means of seminorms as it was done for the R^∞-topology. The R^ω-topology makes $\mathcal{A}(U)$ into a locally convex complex topological B-algebra. It can be easily seen that in the non-graded case ($B_1 = 0$), and when $(M, \mathcal{A}, ev)$ is an R-supermanifold, this topology coincides with the customary compact-open topology.

We say that a complex R-supermanifold $(M, \mathcal{A}, ev)$ is an *R^ω-supermanifold* if it fulfills Axioms 1 to 4 and the following Axiom.

Axiom $5_{\mathbb{C}}$. *For every open subset $U \subset M$, the topological algebra $\mathcal{A}(U)$ is complete Hausdorff in the R^ω-topology.*

Arguing as in the case of R^∞-supermanifolds, and appealing to results on holomorphic maps between complex Banach spaces, (see, e.g., [**12**]) one can reformulate in this context all the results of Sections 3 and 4.

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Appendix

Proof of Theorem 4.9. By virtue of the isomorphism (4.2) it is sufficient to consider the case $n = 0$ only. Let f be a G^∞ function defined over an open subset of $B^{m,0}$; by force of Theorem 3.4 this set may be taken of the form $U^\sim$, $U \subset \mathbb{R}^m$ with no loss of generality. Let $K \subset U^\sim$ be a compact set; one may assume that it is of the form $I \times C$, I being an m-cube in $\mathbb{R}^m$ and C a compact set in $\mathfrak{Rad}B$.

Let $\epsilon > 0$. By virtue of Theorem 3.5, we can pick for any $x \in U$ a number N_x such that for all $y \in K$ with $\sigma(y) = x$ one has $\|f(y) - \sum_{|J|=i}^{N(x)} \frac{1}{J!} D^{(J)}(f)(x)(s^{m,0}(y))^J\|_B < \epsilon$. Denote by $p_x(y)$ the polynomial in y of the form $\sum_{|J|=i}^{N(x)} \frac{1}{J!} D^{(J)}(f)(x)(s^{m,0}(y))^J$. The set $U_x = \{y \in U^\sim : \|f(y) - p_x(y)\|_B < \epsilon$ is a neighbourhood of a compact set $\{x\} \times C$, and hence it contains a 'rectangular' neighbourhood of the form $V_x \times W_x$, $x \in V_x \subset U$, $C \subset W_x \subset \mathfrak{Rad}B$ (see [**21**]). Pick a finite subcover $V_{x_1}, \ldots, V_{x_k}$ of the open cover $\{V_x : x \in I\}$ of I. There is a partition of unity $\{h_i\}_{i=1}^k$ subordinated to the cover $V_{x_1}, \ldots, V_{x_k}$. Since all the functions h_i may be chosen to be Pringsheim regular (for example, so are the usual 'bell' functions), the Z-expansions $Z(h_i)$ converge to G^∞ functions (see [**18**] where this result was proved for Grassmann-Banach algebras; however, the proof is true *verbatim* for BGO-algebras). The collection $\{Z(f_i)\}_{i=1}^k$ of G^∞ functions forms a partition of unity for the family of DeWitt open sets $V_{x_1}{}^\sim, \ldots, V_{x_k}{}^\sim$. The function $g = \sum_{i=1}^k Z(f_i) p_{x^i}$ is G^∞ and ϵ-approximates f on K. The totality of C^∞ functions on U such that for some $\alpha > 0$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \alpha^n \sum_{|J|=n} \max_{x \in I} \|D^{(J)}(f)(x)\| < +\infty$$

forms an algebra which we denote by $\mathcal{UP}^\infty(U)$; it contains polynomials and 'bell' functions. Thus, we can assume that $g \in \mathcal{UP}^\infty(U)$.

Turning back to the hypothesis of the first paragraph of our proof, we may assume now that $f|U \in \mathcal{UP}^\infty(U)$. In this case the Z-expansion converges to f *uniformly* on K. Indeed, taking into account the quasinilpotency of elements of $\mathfrak{Rad}B$ and compactness of C, one can prove that for each α with $0 < \alpha < 1$, there exists a constant $M_\alpha > 0$ such that for every $\theta \in C$, where $\theta = (\theta_1, \ldots, \theta_m)$, every $i = 1, \ldots, m$, and every $n \in \mathbb{N}$ the inequality $\|\theta_i^n\| < M_\alpha \cdot \alpha^n$ holds.

Given an $\epsilon > 0$ and a natural number k, we can find a natural number N and a polynomial $p(x)$ on $\mathbb{R}^m$ with coefficients in B_0 such that for all $x \in K$ and all J' with $|J'| \leq k$ one has:

$$\sum_{J=0}^{N+1} \frac{1}{J!} D^{(J+J')}(f-p)(\sigma^{m,0}(x))(s^{m,0}(x))^J < e \cdot \epsilon$$

$$\sum_{J=N+1}^{\infty} \frac{1}{J!} D^{(J+J')} p(\sigma^{m,0}(x))(s^{m,0}(x))^J < \epsilon$$

$$\sum_{J=N+1}^{\infty} \frac{1}{J!} D^{(J+J')} f(\sigma^{m,0}(x))(s^{m,0}(x))^J < \epsilon$$

Because of the uniform convergence of the Z-expansion on $K = I \times C$, the last inequality is true for all J' with $|J'| \leq k$ as soon as $N > N_0$ for some N_0 large enough.

In order to choose a polynomial p, we resort to the classical proof of the Weierstrass approximation theorem [**19**], going back to Weierstrass himself. Usually that proof is applied to real-valued functions, but the case of Banach-valued functions defined on subsets of $\mathbb{R}^m$ makes no difference at all.

A careful analysis of the proof [**19**] shows that for any finitely supported continuous function f in $\mathbb{R}^m$ taking values in a Banach space and any compact set $I \subset \mathbb{R}^m$ there exist real positive constants C_1, C_2, C_3 (which do not depend on f but rather on I) and a sequence of polynomials $p_n(f)$, $n \in \mathbb{N}$ on $\mathbb{R}^m$ with the properties:

1. For each $\epsilon > 0$, if n is such that

$$\frac{C_1 n^{\frac{m}{2}} ((\sum_i \|\partial f/\partial x^i\|_I)^2 - \epsilon^2)^n (\|f\|_I + C_2)}{(\sum_i \|\partial f/\partial x^i\|_I)^{2n}} < \epsilon \,,$$

where

$$\|\partial f/\partial x^i\|_I = \max_{x \in I} \|f(x)\| \,,$$

then

$$\|f - p_n(f)\|_I < \epsilon \,.$$

2. The degree of $p_n(f)$ is n, and for any multiindex J with $|J| \leq n$ one has

$$\frac{\partial^{|J|} p_n(f)}{\partial x^J} = p_n\Big(\frac{\partial^{|J|} f}{\partial x^J}\Big)$$

and

$$\|p_n(f)\|_I \leq C_3 \|f\|_I \,.$$

As a corollary of 2), for all $N > N_0'$ the third inequality is fulfilled for all J' with $|J'| \leq k$ as soon as $N > N_0'$ for some N_0' large enough, if one substitutes $p_n(f)$ for p (this number N_0' does not depend on n). Put $N = \max\{N_0, N_0'\}$. Set

$$n = \epsilon^{-3}\big[(C_0 + C_1 + C_2)^4 \sum_{|J| \leq N+k+1} \big\| \frac{\partial^{|J|} f}{\partial x^J} \big\|_I\big] \,,$$

where the square brackets stand for the integer part of a number. Applying 1), one can show that for all J with $|J| \leq N + k$ one has

$$\|f^{(J)} - (p_n(f))^{(J)}\|_I < \epsilon$$

This implies the first inequality with $p = p_n(f)$.

Since p is a polynomial function on $\mathbb{R}^m$, $m \in \mathbb{N}$ taking values in B, then $Z(p)$ is a polynomial function on $B^{m,0}$ (with the same coefficients) and thus belongs to $\mathcal{P}_\varphi(U)$. The three inequalities above imply that for all $x \in K$ and every J' with $|J'| \leq k$ one has $\|(f - Z(p))^{(J')}(x)\| \leq (2 + e)\epsilon$. This proves that $\mathcal{P}_\varphi(U)$ is dense in $\mathcal{A}(U)$ in the R^∞ topology. $\square$

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