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Second-Order Dynamics in the Collective Evolution of Coupled Maps and Automata

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ABSTRACT

We review recent numerical studies and the phenomenology of spatially synchronized collective states in many-body dynamical systems. These states exhibit thermodynamic noise superimposed on the collective, quasiperiodic order parameter evolution with typically one basic irrational frequency. We concentrate on the description of the global temporal properties in terms of second-order difference equations.

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Complex dynamical systems exhibit various types of temporal and spatial evolution ranging from uniform to chaotic. The mechanisms leading to self-organization manifested in constant (fixed-point), limit cycle, persistent-structure, quasiperiodic and other "collective" temporal behaviors have been a subject of extensive investigations.[1−11] Unfortunately, only for the simplest families of one-dimensional cellular automata, significant progress in the understanding of individual rules has been achieved.[3−7,9] A more phenomenological approach is based on classifying possible collective behaviors based on the experience with few-parameter systems such as iterative maps in one or more variables. The usual mean-field type idea is to lump the evolution of the complex system in one or more "collective" order parameters, for instance the total magnetization of an Ising-spin automaton, or generally large-wavelength Fourier components of the spin configuration. This approach, as well as the phenomenological characterization of the spatio-temporal patterns observed, are at the heart of most classifications of cellular automata.[1−10]

Recently, arguments have been advanced[8] suggesting that spatially uniform collective evolutions in synchronous-dynamics complex systems are generically limited to "trivial" temporally constant or period-two cyclic states (at large times). The idea has been that for higher-period temporal evolution, with integer periods $3, 4, \ldots$ in units of the discrete updating time steps of the automaton, the domain walls between regions "out of phase" will have generically nonzero velocity. Thus, randomly generated small droplet regions "out of phase" with the surrounding configuration, are expected to spread irreversibly, destroying the spatial coherence. Similar arguments were advanced for quasiperiodic evolution, with domain walls replaced by regions of phase gradients.[11]

Surprisingly, numerical evidence has been reported indicating the existence of various "nontrivial" self-organized states in higher-dimensional discrete-spin cellular automata. Originally, these collective states were observed in certain $d = 4, 5$ totalistic-rule automata.[10] This discovery was followed by extensive numerical studies[11–17] which both enriched the family of automata models showing these new collective states, and also classified various phenomenological properties to be listed shortly. Moreover, similar collective states were found recently[17–18] in continuous-variable coupled-map lattices in $d = 2$ through 6. Collective states were also observed in $1d$ coupled-map lattices;[19] however, their spatial synchronization pattern and other properties differ from collective states in $d > 1$.

The models studied in Refs. 10–19 showed a great variety of self-organized collective states (see especially Ref. 17). The most intriguing type is characterized by the formation of a stable loop-like structure in the return map, i.e., in the plot of the order parameter $m(t+1)$ vs. $m(t)$. Figure 1 illustrates such a plot for the $d = 5$ totalistic automaton rule[10,17] denoted R_{59}^5, in which a site is updated to the value $\sigma_i(t+1) = 1$ if the neighborhood sum over the von Neumann zone (the site itself and its $2d$ nearest neighbors) is between 5 and 9 at time t, and updated to $\sigma_i(t+1) = 0$ otherwise. The order parameter is defined as

$$m(t) = \sum_j \sigma_j(t) \Big/ \sum_j 1 \ , \tag{1}$$

where the sum is over all the lattice points j, at which the spins $\sigma_j(t) = 1$ or 0 are located. In Figure 1, about 1000 data points are shown after a transient of 5000 time steps has been discarded, for a system of 12^5 sites. The initial condition was a randomly half-filled (with values 1) lattice; and boundary conditions were periodic. The large-time average value, M, was estimated and subtracted out. This quantity can be formally defined as

$$M = \lim_{\tau\to\infty} \left[\tau^{-1} \sum_{t=t_0}^{t_0+\tau} m(t) \right] \ . \tag{2}$$

Bifurcation diagrams were studied as well, for coupled-map lattices for which there is a control parameter.[17−19] Restricting our discussion to the $d > 1$ models, we list several phenomenological observations based largely on numerical studies. Firstly, although noisy integral-period states (for example, period three) were found, noisy quasiperiodic behavior attracted more interest. Here we focus on such quasiperiodic states, typically with quasiperiods close to integral values $3, 4, \ldots$, measured, for example, by the winding numbers[10−11,17] about a point near the origin in plots like Figure 1.

The noise exemplified by the plot in Figure 1, was found to decrease "thermodynamically" with increasing number of spins,[11] according to

$$\text{noise amplitude} \sim 1 \Big/ \sqrt{\sum_j 1} \quad . \tag{3}$$

This important observation indicates that the spatial synchronization mechanism is statistical, short-range in nature, rather than global, i.e., that the synchronization is achieved as a local effect, via correlations of adjacent system parts. Unfortunately, the nature of the spatial self-organization has not been understood theoretically thus far.

Regarding the temporal evolution, numerical evidence based on maximum-entropy analyses,[15] as well as some phenomenological considerations,[16] suggest that the apparently quasiperiodic behavior observed in the new collective states both in $d = 1$ and $d > 1$, has in fact periodic time dependence with a single basic frequency. The quasiperiodicity is thus solely due to the incommensurability of the associated period with the clock period $\Delta t = 1$ of the automaton time steps. This observation can be quantified by the relation

$$m(t \gg 1) = M + \sum_{n=1} a_n \sin\left(\frac{2\pi n t}{T} + \phi_n\right) + \text{noisy contributions} \ . \tag{4}$$

The noisy contributions have no characteristic frequencies associated with them, and they are presumably important only for $d > 1$ models, as mentioned earlier.

All the above properties were obtained for fully deterministic evolution. However, stability of the collective states was checked[11,17] by adding a small stochastic element in the evolution rules and for a range of initial conditions. Empirically, these states disappear when the stochastic element increases, or when the initial conditions significantly depart from the random uniform configuration. In the latter case the system evolves into a trivial, typically dilute, disordered-type state. Configurations with imposed domain walls, and the way the system "heals" such perturbations, were studied as well.[11]

The theoretical situation remains largely unsatisfactory. Indeed, with the accumulated numerical information, it would be of interest to understand both the spatial and temporal self-organization mechanisms of these new collective states. One step in this direction, focusing on phenomenological conclusions one can draw from the numerical observation (4), was the proposed *second-order in time* nature of the discrete-time dynamics of the collective behavior at hand.[16] In the rest of this Short Review we summarize the approach of Ref. 16.

Empirically, one observes that the leading oscillatory term in (4) is usually the dominant contribution:

$$|a_1| \gg |a_{n>1}| \; . \tag{5}$$

Such time evolution for $t \gg 1$ can be described by the linear second-order dynamical rule

$$\mu(t) = 2\cos(c)\mu(t-1) - \mu(t-2) \; , \tag{6}$$

where

$$\mu(t) \equiv m(t) - M \ . \tag{7}$$

Indeed, the solution of (6) is

$$\mu(t) = a_1 \sin\left(\frac{2\pi t}{T} + \phi_1\right) \ , \qquad \text{with} \qquad T = \frac{2\pi}{c} \ . \tag{8}$$

The linear evolution rule (6) only contains one parameter, c, determining the basic frequency. Linear fits of the plots of the discrete second-order derivative, $\mu(t+1) - 2\mu(t) + \mu(t-1)$, vs. $\mu(t)$, for several different models,[16] yielded period values consistent with both the maximum-entropy and winding-number estimates. Figure 2 illustrates such a plot for the data used in Figure 1.

The noisy curve in Figure 2 is not precisely a straight-line, nor even single-valued. However, it is well known that such "Lissajous-like" distortions can be incorporated by allowing for nonlinear terms in the dynamical rule, which we now rewrite as follows,

$$\mu(t+1) - 2\mu(t) + \mu(t-1) = -4\sin^2(c/2)\mu(t) + \sum_{n=2}^{\infty} b_n \mu^n(t) + \text{noisy contributions.} \tag{9}$$

The form of the contributions leading to noise in finite-size samples is not clear at present.

In some cases, the single-harmonic "linear" second-order difference equation (6) works quite well. As an illustration, we present in Figure 3 a plot analogous to Figure 2, but for a $d = 3$ automaton described in Ref. 13: the spin is updated to 1 if the sum over the $3d$ von Neumann zone is 0 or 5, and set to 0 otherwise. However, examples are known of large nonlinearity[16] as well as of "Lissajous-shaped" features obscured by the noise, especially in high dimensions where the system sizes accessible to numerical simulations are small.

A direct maximum-entropy fit of the parameters of the first six terms in (4), i.e., including the terms up to a_6 but excluding the noisy contribution, yields the $\mu(t)$ sequence which generates the continuous curve in Figure 4. If we only keep the leading harmonic, (8), then the elliptical-shaped curve is obtained; see Figure 4. The general equation of such an elliptical-shaped single-harmonic return map is

$$\left[\frac{\mu(t)}{A}\right]^2 + \left[\frac{\mu(t+1)}{A}\right]^2 - 2\cos(c)\left[\frac{\mu(t)}{A}\right]\left[\frac{\mu(t+1)}{A}\right] = \sin^2(c)\ . \tag{10}$$

While these observations shed little light on the nature of the spatial self-organization, they indicate that when the noise can be neglected, i.e., presumably in the infinite-system limit, the collective states essentially follow a "mechanistic" time evolution which in terms of the average order parameter $m(t)$ is fully deterministic and *reversible*. The latter property follows from the second-order (in time) nature of the dynamics. As a result, the droplet-type arguments which are largely irreversible in nature, should be incorrect although the mechanism of their breakdown can be fully illuminated only after the spatial self-organization is described theoretically. Loosely speaking, reversibility implies that for any locally disordering mechanism there should exist a reversed locally ordering mechanism. It seems likely that neither of these processes involves droplets.

In summary, we surveyed a new emerging field of study of collective, self-organized behavior in complex systems. The phenomenological information collected by extensive numerical simulations suggests that these systems offer an interesting challenge of theoretical description of spatially self-organized and synchronized evolution, for which the conventional equilibrium "droplet" stability concepts do not apply. While the spatial mechanisms remain to be explained, the time-dependence at least of the global order parameter, could be quantified phenomenologically by the second-order discrete-time dynamics outlined in this Short Review.

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FIGURE CAPTIONS

Fig. 1: Return map, $m(t+1) - M$ vs. $m(t) - M$, for the cellular automaton rule R^5_{59}, described in the text: about 1000 time steps are shown. A transient of the first 5000 time steps was discarded. System size was 12^5.

Fig. 2: Second-order discrete time derivative, the left-hand side of (9), plotted vs. $\mu(t)$, for the data shown in Figure 1.

Fig. 3: Second-order time derivative plot similar to Figure 2, for the 3-dimensional Hemmingsson model,[13] described in the text.

Fig. 4: The data of Figure 1 are shown together with their "noiseless skeleton" reconstruction obtained by keeping the first six oscillatory terms in (4), i.e., up to a_6. The ellipse corresponds to keeping only the leading oscillatory term, corresponding to relations (6), (8), (10).