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On the inverse problem of Lagrangian supermechanics

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Abstract

The inverse problem for Lagrangian supermechanics is investigated. A set of necessary and sufficient conditions for a system of second order differential equations in superspace to derive from (a possibly non regular) superlagrangian function are given. The harmonic superoscillator is revisited and a family of even and odd alternative superlagrangians are constructed for it. Finally, we comment on the existence of recursion operators.

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1 Introduction

The inverse problem of the calculus of variations has attracted a lot of attention over the years (for a recent review see [Mo90]). An elegant description of necessary and sufficient conditions for a second order differential equation to derive from a Lagrangian function were given in [Ba80] [He82] [Cr81]. These conditions encode in a neat geometrical way the well-known Helmhotz conditions for the existence of a local Lagrangian function. A similar programme can be carried out in the setting of supermechanics (or pseudomechanics). One of the difficulties encountered in trying to formulate precisely the inverse problem in this context, is to establish the correct notions of second order differential equations and an intrinsic derivation of Euler-Lagrange equations from a superlagrangian in such a way that the ordinary conditions can be easily superized. Such a geometrical foundation has been established recently [Ib92] and in this geometrical setting the inverse problem of Lagrangian supermechanics acquires a structure similar to the inverse problem in ordinary Lagrangian mechanics. One of the main sources of interest for addressing the inverse problem lies in the already well-known link between (super) integrability and the existence of alternative Lagrangians [Cr83]. New features emerge in the context of supermechanics [La92] as we will discuss later and the existence of alternative superlagrangians is not always related to a recursion operator. The simple example of the harmonic superoscillator is revisited. Its linear supersymmetries and a whole family of alternative superlagrangians, both even and odd, are described.

2 The Inverse Problem of Lagrangian Supermechanics

Let L be a superlagrangian defined on the tangent supermanifold of a superspace with local supercoordinates (q^a, θ^α), $a = 1, \cdots, n$ and $\alpha = 1, \cdots, m$.

The configuration superspace with local supercoordinates (q^a, θ^α) can be thought to be a $n|m$ dimensional $\mathcal{G}$-supermanifold with superalgebra B_L, the real Grassmann algebra with L generators [Ba91]. The supermanifold is defined by $\mathcal{G}$-smooth transition superfunctions between superdomains

$$\begin{aligned} q'^a &= \Phi^a(q,\theta) \\ \theta'^\alpha &= \Psi^\alpha(q,\theta) \end{aligned} \tag{1}$$

and the tangent supermanifold is the $\mathcal{G}$-supermanifold with local supercoordinates $(q^a, \dot{q}^a, \theta^\alpha, \dot{\theta}^\alpha)$ transforming by means of the tangent cocycle

$$\begin{aligned} q'^a &= \Phi^a(q,\theta); \quad \dot{q}'^a = \frac{\partial \Phi^a}{\partial q^b}\dot{q}^b + \frac{\partial \Phi^a}{\partial \theta^\alpha}\dot{\theta}^\alpha \\ \theta'^\alpha &= \Psi^\alpha(q,\theta); \quad \dot{\theta}'^\alpha = \frac{\partial \Psi^\alpha}{\partial q^a}\dot{q}^a + \frac{\partial \Psi^\alpha}{\partial \theta^\beta}\dot{\theta}^\beta . \end{aligned} \tag{2}$$

In many particular instances it is enough to consider configuration superspaces which are graded manifolds $(Q, \mathcal{A}_Q)$ in the sense of Kostant [Ko77]. In such case the cocycle (1) defining the supermanifold can be drastically simplified [Ba79] and takes the simpler form

$$\begin{aligned} q'^a &= \phi^a(q) \\ \theta'^\alpha &= \psi^\alpha_\beta(q)\theta^\beta \end{aligned} \tag{3}$$

and the tangent supermanifold $(TQ, T\mathcal{A}_Q)$ is again a graded supermanifold defined by the tangent cocycle [Ib92]:

$$\begin{aligned} q'^a &= \phi^a(q); \quad \dot{q}'^a = \frac{\partial \phi^a}{\partial q^b}\dot{q}^b \\ \theta'^\alpha &= \psi^\alpha_\beta(q)\theta^\beta; \quad \dot{\theta}'^\alpha = \psi^\alpha_\beta(q)\dot{\theta}^\beta + \frac{\partial \psi^\alpha_\beta}{\partial q^a}\dot{q}^a\theta^\beta . \end{aligned} \tag{4}$$

It is a simple, but tedious, computation to check that the $(1,1)$-supertensor field defined in local supercoordinates by

$$S = dq^a \otimes \frac{\partial}{\partial \dot{q}^a} + d\theta^\alpha \otimes \frac{\partial}{\partial \dot{\theta}^\alpha} \tag{5}$$

is well-defined. We can use it to define the analogous to the Cartan 1-superform as

$$\Theta_L = S \circ dL = dq^a \frac{\partial L}{\partial \dot{q}^a} + d\theta^\alpha \frac{\partial L}{\partial \dot{\theta}^\alpha} = \frac{\partial L}{\partial \dot{q}^a} dq^a + (-1)^{|L|} \frac{\partial L}{\partial \dot{\theta}^\alpha} d\theta^\alpha , \tag{6}$$

and the Cartan 2-superform

$$\Omega_L = -d\Theta_L \tag{7}$$

The superenergy E_L is defined as the superfunction

$$E_L = \Delta(L) - L = \dot{q}^a \frac{\partial L}{\partial \dot{q}^a} + \dot{\theta}^\alpha \frac{\partial L}{\partial \dot{\theta}^\alpha} - L \tag{8}$$

where Δ denotes the Liouville superfield defined by $\Delta = \dot{q}^a \frac{\partial}{\partial \dot{q}^a} + \dot{\theta}^\alpha \frac{\partial}{\partial \dot{\theta}^\alpha}$.

The superlagrangian L is said to be nondegenerate or regular if the supermatrix defined by the superform Ω_L in the basis $(dq^a, d\dot{q}^a, d\theta^\alpha, d\dot{\theta}^\alpha)$ is invertible. This is equivalent to say that the weak kernel of Ω_L, defined as the set of supervector fields Z such that $\epsilon(\Omega_L(Z,X)) = 0$ for any X, is zero. By ϵ we mean that we take the ordinary part of the superfunction $\Omega_L(Z,X)$. If L is regular, it is simple to show that the dynamical equation

$$i_\Gamma \Omega_L = dE_L \tag{9}$$

has a unique solution Γ which happens to be a super second order differential equation (superSODE) [Ib92], i.e. $S(\Gamma) = \Delta$, or locally

$$\Gamma = \dot{q}^a \frac{\partial}{\partial q^a} + \dot{\theta}^\alpha \frac{\partial}{\partial \theta^\alpha} + f^a \frac{\partial}{\partial \dot{q}^a} + f^\alpha \frac{\partial}{\partial \dot{\theta}^\alpha} \tag{10}$$

where f^a, f^α, even and odd superfunctions respectively, are the generalized superforces. Locally the flow defined by Γ can be written as the set of Newton's like equations

$$\ddot{q}^a = f^a(q, \dot{q}, \theta, \dot{\theta}); \qquad \ddot{\theta}^\alpha = f^\alpha(q, \dot{q}, \theta, \dot{\theta}). \tag{11}$$

It is an immediate consequence that Γ satisfies the equation

$$\mathcal{L}_\Gamma \Theta_L = dL, \tag{12}$$

and then, the dynamical equations above are equivalent to the Euler-Lagrange superequations for L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \frac{\partial L}{\partial q^a}\,; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^\alpha}\right) = \frac{\partial L}{\partial \theta^\alpha}. \tag{13}$$

Statement of the inverse problem.

Given a superSODE Γ, under which conditions does there exists a superlagrangian function L such that Γ is a solution of the dynamical equation defined by L, namely $i_\Gamma \Omega_L = dE_L$?

Proposition 1.

A necessary and sufficient set of conditions assuring the existence of a (local) superlagrangian for the superSODE Γ are:

i) There exists a closed 2-superform Ω.

ii) $\mathcal{L}_\Gamma \Omega = 0$.

iii) $i_{V_1 \wedge V_2} \Omega = 0$ for all V_1, V_2 vertical supervector fields.

A vertical supervector field is a supervector field V that is written locally as $V = k^a \frac{\partial}{\partial \dot{q}^a} + k^\alpha \frac{\partial}{\partial \dot{\theta}^\alpha}$.

The necessity of conditions i) and ii) is obvious from the construction of Ω_L. Computing explicitely Ω_L in local supercoordinates it is also a simple exercise to check condition iii).

Let us check now directly the sufficiency of conditions i), ii), iii) obtaining a local superlagrangian function L for Γ. Because of condition i) and applying the graded Poincare's lemma [Ko77] there exists a 1-superform Φ such that

$$\Omega = d\Phi.$$

Writting down Φ in local supercoordinates we have:

$$\Phi = dq^a M_a(q,\dot{q},\theta,\dot{\theta}) + d\dot{q}^a M_{\dot{a}}(q,\dot{q},\theta,\dot{\theta}) + d\theta^\alpha N_\alpha(q,\dot{q},\theta,\dot{\theta}) + d\dot{\theta}^\alpha N_{\dot{\alpha}}(q,\dot{q},\theta,\dot{\theta}).$$

Denoting by $\dot{\Phi} = d\dot{q}^a M_{\dot{a}} + d\dot{\theta}^\alpha N_{\dot{\alpha}}$, we have that because of iii),

$$i_{V_1 \wedge V_2} d\dot{\Phi} = i_{V_1 \wedge V_2} d\Phi = i_{V_1 \wedge V_2} \Omega = 0$$

for all supervertical V_1, V_2. Then, there exists a superfunction $f(q,\dot{q},\theta,\dot{\theta})$ such that

$$i_V df = i_V \dot{\Phi}$$

for all V supervertical, or locally,

$$\frac{\partial f}{\partial \dot{q}^a} = N_{\dot{a}}; \qquad \frac{\partial f}{\partial \dot{\theta}^\alpha} = M_{\dot{\alpha}}.$$

Then defining the 1-superform

$$\Theta = \Phi - df$$

it is clear that $i_V\Theta = 0$ for all V supervertical, hence, in local supercoordinates Θ has the expression

$$\Theta = dq^a A_a + d\theta^\alpha B_\alpha$$

where A_a is an even superfunction and B_α is odd if Ω is even and conversely if Ω is odd. It is clear that $d\Theta = \Omega$.

Using now the invariance condition ii) we obtain

$$\mathcal{L}_\Gamma d\Theta = d\mathcal{L}_\Gamma\Theta = 0$$

or, in other words, $\mathcal{L}_\Gamma\Theta$ is a closed 1-superform. Using again Poincare's lemma, we can affirm that there exists a superfunction L such that

$$\mathcal{L}_\Gamma\Theta = dL$$

and computing the previous Lie derivative in local supercoordinates we obtain the following identities

$$\begin{aligned}
A_a &= \frac{\partial L}{\partial \dot{q}^a}; \qquad B_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} \\
\dot{q}^a \frac{\partial A_b}{\partial q^a} &+ \dot{\theta}^\alpha \frac{\partial A_b}{\partial \theta^\alpha} + f^a \frac{\partial A_b}{\partial \dot{q}^a} + f^\alpha \frac{\partial A_b}{\partial \dot{\theta}^\alpha} = \frac{\partial L}{\partial q^b} \\
\dot{q}^a \frac{\partial B_\beta}{\partial q^a} &+ \dot{\theta}^\alpha \frac{\partial B_\beta}{\partial \theta^\alpha} + f^a \frac{\partial B_\beta}{\partial \dot{q}^a} + f^\alpha \frac{\partial B_\beta}{\partial \dot{\theta}^\alpha} = \frac{\partial L}{\partial \theta^\beta}.
\end{aligned}$$

Because of eqn. (6) the first two equations clearly imply that $\Theta = \Theta_L$ and in consequence (7) $\Omega = \Omega_L$. Expanding again the invariance condition we get $i_\Gamma d\Theta_L + di_\Gamma\Theta_L = dL$, this is $i_\Gamma\Omega_L = d(i_\Gamma\Theta_L - L)$, but $i_\Gamma\Theta_L = i_\Gamma(S \circ dL) = i_\Delta(dL) = \Delta(L)$, hence $i_\Gamma\Omega_L = dE_L$ because of (8).

Remarks.

1) The 2-superform Ω can be even or odd. In each situation we get an even or odd superlagrangian respectively.

2) It is important to realize that with respect to the ordinary assumptions in [Ba80] [Cr81],[He82], Ω does not have to be regular, i.e. L needs not to be a regular superlagrangian. If this were not the case the weak kernel of Ω would impose algebraic conditions on Γ, and the strong kernel of Ω (defined as the set of supervector fields Z such that $\Omega(Z, X) = 0$, $\forall X$) defines an invariant Lie superalgebra, the gauge superalgebra of Γ and L. Under such circumstances, the tangent supermanifold and the dynamical equation can be reduced quotienting out the gauge degrees of freedom defined by the strong kernel of Ω_L.

3) The existence of alternative superlagrangians allows to construct a mixed $(1,1)$ tensor field which could play the role of recursion operator. In fact, if the superlagrangians have opposite parity this is not the case and they cannot be used in the analysis of complete integrability. Their existence will rather imply the existence of a supersymmetry.

4) A similar inverse problem can be formulated for superequations of motion which are first order in the velocities. This is usually the situation of superparticles and Supersymmetric (Quantum) Classical Mechanics [Ba83]. The inverse problem for such class of systems has been discussed in the setting of ordinary geometry in [Ib91] and the results there can be easily translated into the graded realm following the ideas above.

3 The Superoscillator Revisited

We will discuss in this section a family of alternative superlagrangians for the superoscillator. Let $\mathbf{R}^{n|n}$ be the configuration superspace with local supercoordinates (q^a, θ^a), $a = 1, \ldots, n$. The tangent supermanifold is easily seen to be $\mathbf{R}^{2n|2n}$ with local supercoordinates $(q^a, \dot{q}^a, \theta^a, \dot{\theta}^a)$. The equations of motion of the superoscillator are simply

$$\ddot{q}^a = -q^a \ ; \quad \ddot{\theta}^a = -\theta^a, \tag{14}$$

corresponding to the superSODE

$$\Gamma = \dot{q}^a \frac{\partial}{\partial q^a} - q^a \frac{\partial}{\partial \dot{q}^a} + \dot{\theta}^a \frac{\partial}{\partial \theta^a} - \theta^a \frac{\partial}{\partial \dot{\theta}^a}. \tag{15}$$

The linear supersymmetry algebra of the superoscillator.

Let $M(m|n, B_L)$ be the Lie superalgebra of supermatrices with entries in the superalgebra B_L. A supermatrix A will have the block structure

$$A = \left(\begin{array}{c|c} A_{++} & A_{+-} \\ \hline A_{-+} & A_{--} \end{array} \right).$$

Any $A \in M(m|n, B_L)$, has associated a linear supervector field X_A on $\mathbf{R}^{m|n}$. If we use collective notations for the coordinates, namely $z^i = (x^a, \theta^b)$, the supervector field X_A will be given by

$$X_A = z^j A_j{}^i \frac{\partial}{\partial z^i}. \tag{16}$$

Clearly, $[X_A, X_B] = X_{[A,B]}$, where $[A, B]$ denotes the supercommutator in the Lie superalgebra $M(m|n, B_L)$.

Similarly, if F is another supermatrix, we can define a 2-form Ω_F as

$$\Omega_F = dz^i \wedge dz^j F_{ji}. \tag{17}$$

In order to have a 2-form the matrix F should be skewsymmetric, i.e., $F^t = -F$ or, in components, $F_{ij} = -(-1)^{|i||j|} F_{ji}$ (to be precise, in the previous expression, will survive only the part of F with these properties), irrespective of the degree of F. The Ω_F will be even or odd if F is even or odd respectively.

Since F is a constant matrix, Ω_F is closed and the action of the supervector field X_A is easly obtained. It turns out to be

$$\begin{aligned} \mathcal{L}_{X_A}\Omega_F &= (-1)^{|A||i|} dz^i \wedge dz^j \{(AF)_{ji} - (-1)^{|i||j|}(AF)_{ij}\} \\ &= (-1)^{|A||i|} dz^i \wedge dz^j \{(AF) - (AF)^t\}_{ji} \,. \end{aligned} \tag{18}$$

In particular, $\mathcal{L}_{X_A}\Omega_F$ can be associated to a supermatrix iff A is even. If this is the case, then

$$\mathcal{L}_{X_A}\Omega_F = 2\Omega_{(AF)^{sa}}\ , \qquad (AF)^{sa} = \frac{1}{2}((AF) - (AF)^t)\ . \tag{19}$$

For the analysis of the harmonic superoscillator we shall consider the configuration superspace $\mathbf{R}^{n|n}$ as a graded supermanifold. This assumption will also implies that the only constant numbers are ordinary (i.e. real or complex) numbers. Consider then the supermatrix J in $M(2n|2n,\mathbf{R})$ given by

$$J = \left(\begin{array}{c|c} J_0 & 0 \\ \hline 0 & J_0 \end{array}\right)$$

where J_0 is the $2n \times 2n$ symplectic matrix

$$J_0 = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)\ .$$

Clearly the superSODE Γ defining the harmonic superoscillator dynamics, eq. (15), is simply

$$\Gamma = X_J \tag{20}$$

and then, a linear supervector field X_A is a symmetry of Γ iff

$$0 = \mathcal{L}_{X_A}\Gamma = [X_A, X_J] = X_{[A,J]}$$

i.e., iff $[A,J] = 0$. In other words, the linear symmetry supergroup for Γ is the supergroup of linear transformations $Gl(2n|2n,\mathbf{R})$.

Linear Inverse problem.

Let Ω_F be a 2-superform associated with the skewsymmetric supermatrix F in $M(2n|2n,\mathbf{R})$. The first condition in the inverse problem, namely that Ω_F is closed, is automatically satisfied. ¿From (19), the invariance condition $\mathcal{L}_\Gamma\Omega_F = 0$ implies that $(JF)^{sa} = 0$ and this is equivalent to F being in the Lie superalgebra of the Lie supergroup $Gl(2n|2n,\mathbf{R}) \cap Osp(2n|2n)$, and having the block structure

$$F = \left(\begin{array}{cc|cc} F^d_{++} & F^a_{++} & F^d_{+-} & F^a_{+-} \\ \hline -F^a_{++} & F^d_{++} & -F^a_{+-} & F^d_{+-} \\ \hline -(F^d_{+-})^t & (F^a_{+-})^t & F^d_{--} & F^a_{--} \\ \hline -(F^a_{+-})^t & (F^d_{+-})^t & -F^a_{--} & -F^d_{--} \end{array}\right)$$

with F^a_{++} a symmetric matrix of ordinary numbers and F^a_{--} a skwesymmetric matrix of ordinary numbers.

The vertical condition iii) in proposition 1, implies that the coefficients of $d\dot{q} \wedge d\dot{q}$, $d\dot{q} \wedge d\dot{\theta}$ and $d\dot{\theta} \wedge d\dot{\theta}$ in Ω_F must vanish, thus F has the form

$$F = \left(\begin{array}{cc|cc} 0 & F^a_{++} & 0 & F^a_{+-} \\ \hline -F^a_{++} & 0 & -F^a_{+-} & 0 \\ \hline 0 & (F^a_{+-})^t & 0 & F^a_{--} \\ \hline -(F^a_{+-})^t & 0 & -F^a_{--} & 0 \end{array}\right) = F_{even} + F_{odd} \ ,$$

with

$$F_{even} = \left(\begin{array}{cc|cc} 0 & F^a_{++} & & \\ -F^a_{++} & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & F^a_{--} \\ \multicolumn{2}{c|}{0} & -F^a_{--} & 0 \end{array}\right)$$

and

$$F_{odd} = \left(\begin{array}{cc|cc} & & 0 & F^a_{+-} \\ \multicolumn{2}{c|}{0} & -F^a_{+-} & 0 \\ \hline 0 & (F^a_{+-})^t & & \\ -(F^a_{+-})^t & 0 & \multicolumn{2}{c}{0} \end{array}\right) \ .$$

Alternative superlagrangians.

In both cases what we get is that the product FJ defines a supermetric G on the superspace $\mathbf{R}^{2n|2n}$, even or odd if F is even or odd respectively and the superlagrangians from which Ω_F derives are given by

$$L_{even} = \frac{1}{2}((F^a_{++})_{ab}\dot{q}^a\dot{q}^b + (F^a_{--})_{ab}\dot{\theta}^a\dot{\theta}^b - (F^a_{++})_{ab}q^aq^b - (F^a_{--})_{ab}\theta^a\theta^b) \quad (21)$$

or

$$L_{odd} = (F^a_{+-})_{ab}\dot{q}^a\dot{\theta}^b - (F^a_{+-})_{ab}q^a\theta^b \tag{22}$$

Any choice of an admissible Lagrangian will break the group $Gl(2n|2n, \mathbf{R})$ to a subgroup that depends on L_F. This implies that the association of symmetries and constants of the motion depends on L_F and moreover, upon quantization, we get different quantum mechanical systems depending on the Lagrangian we choose. This fact is evident without the need of heavy computations. Indeed, according to which Lagrangian we use, we can select a subgroup of symmetries that contains a compact or a non compact even part and upon quantization we shall have a discrete or a continuous spectrum respectively.

4 Alternative Lagrangians and recursion operators

What we would like to show now is that the existence of alternative superlagrangians of opposite parity for the same superSODE Γ does not allow to construct recursion operators and cannot be used for the complete integrability of Γ. If the alternative Lagrangians are of the same parity one has the usual kind of analysis.

Before we do that we rewiev the analysis done with ordinary (i.e. non graded) Lagrangian dynamical systems [Cr83] by stressing the conditions that should (and in same cases cannot) be superized.

So we work on an ordinary tangent bundle TQ; the following objects can (and some have already been [Ib92]) superized. There are two natural lifting procedure for vector fields from Q to TQ, namely the tangent and the vertical lifting. If $X = X^a(q)\frac{\partial}{\partial q^a} \in \mathcal{X}(Q)$, its *tangent lift* X^T and its *vertical lift* X^V are the elements in $\mathcal{X}(TQ)$ given by

$$X^T = X^a(q)\frac{\partial}{\partial q^a} + \dot{q}^b \frac{\partial X^a}{\partial \dot{q}^b}\frac{\partial}{\partial \dot{q}^a} \ , \quad X^V = X^a(q)\frac{\partial}{\partial \dot{q}^a} \ . \tag{23}$$

A third lifting procedure is associated with any SODE Γ: the *horizontal* *lift* of $X \in \mathcal{X}(Q)$ is the vector field $X^H \in \mathcal{X}(TQ)$ given by

$$X^H = X^a H_a \ , \quad H_a =: (\frac{\partial}{\partial q^a})^H = \frac{\partial}{\partial q^a} + \frac{1}{2}(\frac{\partial \Gamma^b}{\partial \dot{q}^a})\frac{\partial}{\partial \dot{q}^b} \ , \quad a \in \{1, \dots, n\}. \tag{24}$$

The horizontal lift provides a connection on TQ and, for any point $(q, \dot{q})$ of TQ, the set of horizontal lifts of vectors from Q forms a subspace of $T_{(q,\dot{q})}(TQ)$ which is complementary to the subspace spanned by vertical lifts. This decomposition of the tangent spaces to TQ allows also to define an almost complex structure $\mathcal{J}$ on TQ by setting

$$\mathcal{J}(\xi^H) = \xi^V \ , \qquad \mathcal{J}(\xi^V) = -\xi^H \ , \qquad \forall \ \xi \in T_{(q,\dot{q})}(TQ) \ . \tag{25}$$

If L is a regular Lagrangian on TQ, its Cartan 2-form Ω_L besides being invariant under Γ has additional properties related to the horizontal distribution generated by Γ. The most relevant one is the possibility of defining a pseudo-hermitian metric g on TQ by

$$g(X, Y) = \Omega_L(X, \mathcal{J}Y) \ , \quad \forall \ X, Y \in \mathcal{X}(TQ) \ . \tag{26}$$

Then, horizontal and vertical subspaces are ortogonal with respect to g and $g(\xi^V, \eta^V) = g(\xi^H, \eta^H)$.

Let us suppose now that the field Γ is the common Euler-Lagrange field associated with two regular Lagrangians L_1 and L_2 which are not trivially related by addition of a total time derivative or by multiplication by a constant. The associated Cartan 2-forms Ω_{L_1} and Ω_{L_2}, which both have the properties described before, can be used to define a type $(1,1)$ tensor field T on TQ by the property $i_X \Omega_{L_2} = i_{T(X)} \Omega_{L_1}$, namely

$$T = \Omega_{L_2} \circ \Omega_{L_1}^{-1} \ , \tag{27}$$

where we use the same symbols for the 2-forms and the associated endomorphisms of $\mathcal{X}(TQ)$ and $\mathcal{X}(TQ)^*$.

The tensor T is compatible with $(\Omega_L, \mathcal{J}, g)$ and as a consequence it preserves the direct sum decomposition of the tangent spaces to TQ, and maps vertical vectors to vertical vectors and horizontal ones to horizontal ones. In addition, T is the direct sum of two linear non-singular transformations of the vertical and the horizontal subspaces, which are identical in the sense that they are given by the same matrix in a chosen basis of vertical and horizontal vectors. Since T is simmetrical with respect to the metric g, its eigenvalues at each points are real, and, acting identically on the vertical and horizontal subspaces, each eigenvalue is at least doubly degenerate with a vertical and a horizontal eigenvector. If the eigenvalues of T are as little degenerate as possible, namely doubly degenerate, they determine functions λ_a , $a \in \{1, \ldots, n\}$, on TQ, whose values at each point are just the values of the eigenvalues of T at that point. These functions are constants of the motion for Γ as follows from the invariance of T under Γ. They will give its complete integrability if there are in involution. In order to show the latter, one needs to impose an additional condition on T, namely that its *Nijenhuis tensor* $\mathbf{N}_T$, defined for all $X, Y \in \mathcal{X}(TQ)$ by

$$\mathbf{N}_T(X,Y) = T^2[X,Y] + [TX,TY] - T[TX,Y] - T[X,TY] \ , \tag{28}$$

vanishes. This is equivalent to the fact that the two Poisson structures associated with Ω_{L_1} and Ω_{L_2} are compatible; their sum is still a Poisson structure. An immediate consequence of the condition $\mathbf{N}_T = 0$ is the fact that, if X_a is any eigenvector field corresponding to the eigenfunction λ_a, then $\mathcal{L}_{X_b}\lambda_a = 0$, whenever $a \neq b$; this also expresses the vanishing of the Poisson bracket of any two eigenfunctions λ_a, λ_b.

An equivalent way of using a (1-1) tensor field T with vanishing Nijenhuis torsion is to generate a sequence of constants of motion in involution by successive application of T. This is (locally) the content of the

Proposition 2.

If the Nijenhuis tensor $\mathbf{N}_T$ vanishes and $d(TdF) = 0$ then $d(T^k dF) = 0$ for all $k > 1$.

Proof. Let α be any 1-form. Some simple algebra gives

$$i_{X\wedge Y} d(T^2\alpha) = i_{\{X\wedge TY + TX\wedge Y\}} d(T\alpha) - i_{\{TX\wedge TY\}} d\alpha - i_{\mathbf{N}_T(X,Y)}\alpha \ . \tag{29}$$

If both α and $T\alpha$ are closed, than $T^2\alpha$ is closed if and only if $\mathbf{N}_T = 0$.

With T as in (27), one has $TdE_{L_1} = dE_{L_2}$ and $T^k dE_L$ is a function of the eigenfunctions of T.

Let us analize now the graded situation. It turns out that one cannot define a (super) Nijenhuis torsion. Let us suppose T is a graded $(1,1)$ tensor field which is homogeneous of parity $|T|$. Then, if α is any 1-form, the closest possible expression to (29) reads

$$\begin{aligned} i_{X\wedge Y} d(T^2\alpha) &= i_{\{(-1)^{|T||Y|}X\wedge TY + (-1)^{|T|(|X|+|Y|)}TX\wedge Y\}} d(T\alpha) \\ &\quad -(-1)^{|T|(|X|+|T|)} i_{TX\wedge TY} d\alpha - (-1)^{|T|}\ i_{{}^G\mathbf{N}_T(X,Y)}\alpha \\ &\quad +(-1)^{|T||X|}[1-(-1)^{|T|}]\mathcal{L}_{TX}(i_{TY}\alpha)\ . \end{aligned} \tag{30}$$

Where ${}^G\mathbf{N}_T$ is defined by

$$\begin{aligned} {}^G\mathbf{N}_T(X,Y) &=: T^2[X,Y] + (-1)^{|T||X|}[TX,TY] - T[TX,Y] \\ &\quad -(-1)^{|T||X|}T[X,TY]\ . \end{aligned} \tag{31}$$

We see from (30), that for an $(1,1)$ odd tensor a $(2,1)$ tensor corresponding to its torsion (super-Nijenhuis torsion) can be defined only when $|T| = 0$. This is a consequence of the fact that the map ${}^G\mathbf{N}_T$ defined in (31) is linear over superfunctions (and graded antisymmetric) if and only if $|T| = 0$.

The Lagrangians (21) and (22) for the harmonic oscillator provide examples of numerical $(1,1)$ tensors that cannot be recursion operators. As an explicit example we consider a harmonic oscillator on $\mathbf{R}^{4|4}$ with an even and an odd Lagrangian given respectively by

$$L_1 = \frac{1}{2}[(\dot{q}^1)^2 + (\dot{q}^2)^2 - (q^1)^2 - (q^2)^2] + i(\dot{\theta}^1\dot{\theta}^2 - \theta^1\theta^2)\ , \tag{32}$$
$$L_2 = \dot{q}^1\dot{\theta}^1 + \dot{q}^2\dot{\theta}^2 - q^1\theta^1 - q^2\theta^2\ . \tag{33}$$

The Cartan 2-forms and the energy functions are respectively

$$\Omega_1 = dq^1\wedge d\dot{q}^1 dq^2\wedge d\dot{q}^2 + id\theta^1\wedge d\dot{\theta}^1 - id\theta^2\wedge d\dot{\theta}^2\ , \tag{34}$$
$$\Omega_2 = dq^1\wedge d\dot{\theta}^1 dq^1\wedge d\dot{\theta}^1 + d\theta^1\wedge d\dot{q}^1 + d\theta^2\wedge d\dot{q}^2\ ; \tag{35}$$
$$E_1 = \frac{1}{2}[(\dot{q}^1)^2 + (\dot{q}^2)^2 + (q^1)^2 + (q^2)^2] + i(\dot{\theta}^1\dot{\theta}^2 + \theta^1\theta^2)\ , \tag{36}$$
$$E_2 = \dot{q}^1\dot{\theta}^1 + \dot{q}^2\dot{\theta}^2 + q^1\theta^1 + q^2\theta^2\ . \tag{37}$$

The associated $(1,1)$ odd tensor field is given by

$$T = \Omega_2 \circ (\Omega_1)^{-1} = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{matrix} \\ \hline \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} & 0 \end{array} \right) , \tag{38}$$

and, is spite of the fact that T is numerical, one finds that

$$\begin{aligned} &T dE_1 = dE_2 \\ &T^2 dE_1 = i((dq^1)q^2 - (dq^2)q^1 + (d\dot{q}^1)\dot{q}^2 + (d\dot{q}^2)\dot{q}^1) \\ &\qquad\quad +(d\theta^1)\theta^1 + (d\theta^2)\theta^2 + (d\dot{\theta}^1)\dot{\theta}^1 + (d\dot{\theta}^2)\dot{\theta}^2 \ , \\ &dT^2 dE_1 \neq 0 \ . \end{aligned} \tag{39}$$

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