Current Algebra of Classical Non-Linear Sigma Models

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Abstract

The current algebra of classical non-linear sigma models on arbitrary Riemannian manifolds is analyzed. It is found that introducing, in addition to the Noether current j_μ associated with the global symmetry of the theory, a composite scalar field j, the algebra closes under Poisson brackets.

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It is well known that for the classical non-linear sigma models, the Poisson brackets between the Noether currents associated with the global symmetry of the theory involve Schwinger terms which, in general, are field dependent. Explicit expressions have been written down, e.g., for the principal chiral models (sigma models on compact Lie groups) [1, 2] and for the spherical models ($O(N)$-invariant sigma models on spheres) [3, 4], but the structure of the resulting algebra has, to our knowledge, not yet been analyzed in full generality.

In the following, we want to show that, for classical non-linear sigma models on arbitrary Riemannian manifolds, introducing the coefficient of the Schwinger term appearing in the Poisson bracket between the time component and the space component of the Noether current j_μ as a new composite field j leads to an algebra which closes under Poisson brackets: this is what we propose to call the current algebra for these models.

To this end, consider the classical two-dimensional non-linear sigma model on an arbitrary Riemannian manifold M with metric g. The configuration space of the theory consists of (smooth) maps φ from a fixed two-dimensional Lorentz manifold Σ (typically two-dimensional Minkowski space) to M, while the corresponding phase space consists of pairs (φ, π) of fields, π being a smooth section of the pull-back $\varphi^*(T^*M)$ of the cotangent bundle of M to Σ via φ. In terms of local coordinates u^i on M, φ and π are represented by multiplets of ordinary functions φ^i and π_i on Σ; then the action reads

$$S \;=\; \tfrac{1}{2} \int d^2x \; g_{ij}(\varphi) \, \partial^\mu \varphi^i \, \partial_\mu \varphi^j \quad , \tag{1}$$

and the canonical Poisson brackets are

$$\begin{aligned} \{\varphi^i(x)\,,\varphi^j(y)\} = 0 \quad &, \quad \{\pi_i(x), \pi_j(y)\} = 0 \quad , \\ \{\varphi^i(x)\,,\pi_j(y)\} \;&=\; \delta^i_j \, \delta(x-y) \quad . \end{aligned} \tag{2}$$

Denoting the time derivative by a dot and the spatial derivative by a prime, we have

$$\pi_i \;=\; g_{ij}(\varphi) \, \dot{\varphi}^j \quad . \tag{3}$$

Under local coordinate transformations $u^i \rightarrow u'^k$, the component fields φ^i, $\partial_\mu \varphi^i$ and π_i transform according to

$$\varphi^i \;\rightarrow\; \varphi'^k \quad , \quad \partial_\mu \varphi^i \;\rightarrow\; \partial_\mu \varphi'^k = \frac{\partial \varphi'^k}{\partial \varphi^i} \, \partial_\mu \varphi^i \quad , \quad \pi_i \;\rightarrow\; \pi'_k = \frac{\partial \varphi^i}{\partial \varphi'^k} \, \pi_i \quad , \tag{4}$$

from which it can be checked that the action (1) and the canonical commutation relations (2) are invariant.

Next, let G be a (connected) Lie group acting on M by isometries. Then every generator X in the Lie algebra $\boldsymbol{g}$ of G is represented by a fundamental vector field X_M

on M, given by

$$X_M(m) \;=\; \frac{d}{dt}\,(\exp(tX)\cdot m)\Big|_{t=0} \quad , \tag{5}$$

obeying the representation condition

$$[X,Y]_M \;=\; -\,[X_M,Y_M] \quad ;$$

moreover, all these vector fields are Killing vector fields. As usual, G-invariance of the action (1) leads to a conserved Noether current j_μ with values in $\boldsymbol{g}^*$ (the dual of $\boldsymbol{g}$). Explicitly, for $X \in \boldsymbol{g}$,

$$\langle j_\mu, X\rangle \;=\; -\,g_{ij}(\varphi)\,\partial_\mu\varphi^i\,X_M^j(\varphi) \quad . \tag{6}$$

On the other hand, we define a scalar field j with values in the symmetric tensor product of g^* with itself by setting, for $X,Y \in \boldsymbol{g}$,

$$\langle j, X\otimes Y\rangle \;=\; g_{ij}(\varphi)\,X_M^i(\varphi)\,Y_M^j(\varphi) \quad . \tag{7}$$

In terms of an arbitrary basis (T_a) of $\boldsymbol{g}$, with structure constants f_{ab}^c defined by $[T_a,T_b] = f_{ab}^c T_c$, and the corresponding dual basis (T^a) of $\boldsymbol{g}^*$, we write

$$j_\mu \;=\; j_{\mu,a}\,T^a \quad , \quad j \;=\; j_{ab}\,T^a\otimes T^b \quad . \tag{8}$$

Then the current algebra takes the following form:

$$\{j_{0,a}(x), j_{0,b}(y)\} \;=\; -\,f_{ab}^c\,j_{0,c}(x)\,\delta(x-y) \quad , \tag{9}$$

$$\{j_{0,a}(x), j_{1,b}(y)\} \;=\; -\,f_{ab}^c\,j_{1,c}(x)\,\delta(x-y) \;+\; j_{ab}(y)\,\delta'(x-y) \quad , \tag{10}$$

$$\{j_{1,a}(x), j_{1,b}(y)\} \;=\; 0 \quad , \tag{11}$$

$$\{j_{0,a}(x), j_{bc}(y)\} \;=\; -\,\Big(f_{ab}^d\,j_{cd}(x) + f_{ac}^d\,j_{bd}(x)\Big)\,\delta(x-y) \quad , \tag{12}$$

$$\{j_{1,a}(x), j_{bc}(y)\} \;=\; 0 \quad , \tag{13}$$

$$\{j_{ab}(x), j_{cd}(y)\} \;=\; 0 \quad . \tag{14}$$

For the proof, we note first of all that the Poisson brackets (11), (13) and (14) vanish because $j_{1,a}$ and j_{ab} depend only on the φ^i but not on the π_i. The Poisson bracket of two j_0's is:

$$\begin{aligned}
\{\langle j_0(x),X\rangle\,,\,\langle j_0(y),Y\rangle\} \;&=\; \{\pi_i(x)\,X_M^i(\varphi(x))\,,\,\pi_j(y)\,Y_M^j(\varphi(y))\} \\
&=\; \pi_i(x)\,\Big(\partial_j X_M^i(\varphi(x))\,Y_M^j(\varphi(x)) - X_M^j(\varphi(x))\,\partial_j Y_M^i(\varphi(x))\Big)\,\delta(x-y) \\
&=\; \pi_i(x)\,[X,Y]_M^i(\varphi(x))\,\delta(x-y) \\
&=\; -\,\langle j_0(x),[X,Y]\rangle\,\delta(x-y) \quad .
\end{aligned}$$

The Poisson bracket of a j_0 with a j_1, however, is more complicated:

$$\begin{aligned}\{\langle j_0(x),X\rangle\,,\,\langle j_1(y),Y\rangle\} &= \{\pi_i(x)\,X_M^i(\varphi(x))\,,\,g_{jk}(\varphi(y))\,\varphi'^j(y)\,Y_M^k(\varphi(y))\}\\ &= X_M^i(\varphi(x))\,g_{ik}(\varphi(y))\,Y_M^k(\varphi(y))\,\delta'(x-y)\\ &\quad -X_M^i(\varphi(x))\,\partial_i g_{jk}(\varphi(x))\,Y_M^k(\varphi(x))\,\varphi'^j(x)\,\delta(x-y)\\ &\quad -X_M^i(\varphi(x))\,g_{jk}(\varphi(x))\,\partial_i Y_M^k(\varphi(x))\,\varphi'^j(x)\,\delta(x-y)\quad .\end{aligned}$$

Using the identity

$$f(x)\,\delta'(x-y) \;=\; f(y)\,\delta'(x-y)-f'(x)\,\delta(x-y)\quad ,$$

together with the fact that X_M is a Killing field, so

$$\partial_j X_M^i\;g_{ik} \;=\; -\,\partial_k X_M^i\;g_{ij}-\partial_l g_{jk}\,X_M^l\quad ,$$

we get

$$\begin{aligned}\{\langle j_0(x),X\rangle\,,\,\langle j_1(y),Y\rangle\} &= X_M^i(\varphi(y))\,g_{ik}(\varphi(y))\,Y_M^k(\varphi(y))\,\delta'(x-y)\\ &\quad +\partial_k X_M^i(\varphi(x))\,g_{ij}(\varphi(x))\,Y_M^k(\varphi(x))\,\varphi'^j(x)\,\delta(x-y)\\ &\quad -X_M^k(\varphi(x))\,g_{ij}(\varphi(x))\,\partial_k Y_M^i(\varphi(x))\,\varphi'^j(x)\,\delta(x-y)\\ &= X_M^i(\varphi(y))\,g_{ik}(\varphi(y))\,Y_M^k(\varphi(y))\,\delta'(x-y)\\ &\quad -g_{ij}(\varphi(x))\,[X_M,Y_M]^i(\varphi(x))\,\varphi'^j(x)\,\delta(x-y)\\ &= -\,\langle j_1(x),[X,Y]\rangle\,\delta(x-y)\;+\;\langle j(y),X\otimes Y\rangle\,\delta'(x-y)\quad .\end{aligned}$$

Finally, the Poisson bracket of a j_0 with a j reads:

$$\begin{aligned}\{\langle j_0(x),X\rangle\,,\,\langle j(y),Y\otimes Z\rangle\} &= -\,\{\pi_i(x)\,X_M^i(\varphi(x))\,,\,g(Y_M,Z_M)(\varphi(y))\}\\ &= \mathbf{L}_{X_M}(g(Y_M,Z_M))(\varphi(x))\;\delta(x-y)\\ &= -\,\langle\,j(x)\,,\,[X,Y]\otimes Z+Y\otimes[X,Z]\,\rangle\;\delta(x-y)\end{aligned}$$

(since $\mathbf{L}_{X_M}g=0$, $\mathbf{L}$ being the Lie derivative).

The definition of the model and the derivation of the current algebra (9-14) have here been given in the (gauge invariant) local coordinate formulation, which is the only one available for general Riemannian manifolds. For the special case where the target manifold is Riemannian homogeneous space $M=G/H$, however, there is an alternative (gauge dependent) formulation [5-8] in terms of fields g with values in G, determined modulo fields h with values in H by the condition that $\varphi=gH$ (at least locally), and it is instructive to rewrite the definition of the composite fields j_μ and j in this language. The assumption that M is a Riemannian homogeneous space implies that it can be written as the quotient space of some (connected) Lie group G, with Lie algebra $\boldsymbol{g}$, modulo some compact subgroup $H\subset G$, with Lie algebra $\boldsymbol{h}\subset\boldsymbol{g}$, and that it is reductive. This means that there exists an $\mathrm{Ad}(H)$-invariant subspace $\boldsymbol{m}$ of $\boldsymbol{g}$ such that $\boldsymbol{g}$ is the (vector space) direct sum of $\boldsymbol{h}$ and $\boldsymbol{m}$:

$$\boldsymbol{g} = \boldsymbol{h} \oplus \boldsymbol{m} \quad . \tag{15}$$

The corresponding projections from $\boldsymbol{g}$ onto $\boldsymbol{h}$ along $\boldsymbol{m}$ and from $\boldsymbol{g}$ onto $\boldsymbol{m}$ along $\boldsymbol{h}$ will be denoted by $\pi_{\boldsymbol{h}}$ and $\pi_{\boldsymbol{m}}$, respectively. Due to $\mathrm{Ad}(H)$-invariance of the direct decomposition (15), we have the commutation relations

$$[\boldsymbol{h}, \boldsymbol{h}] \subset \boldsymbol{h} \quad , \quad [\boldsymbol{h}, \boldsymbol{m}] \subset \boldsymbol{m} \quad . \tag{16}$$

Moreover, we shall suppose that the $\mathrm{Ad}(H)$-invariant positive definite inner product $(.\,,.)$ on $\boldsymbol{m}$, corresponding to the given G-invariant Riemannian metric on M, is induced from an $\mathrm{Ad}(G)$-invariant non-degenerate inner product $(.\,,.)$ on $\boldsymbol{g}$, corresponding to a G-biinvariant pseudo-Riemannian metric on G, so that the direct decomposition (15) is orthogonal. (This amounts essentially to requiring that M be naturally reductive; we refer to [8] for a detailed discussion.) Now using this scalar product on $\boldsymbol{g}$ to pull up and down Lie algebra indices, we can interpret j_μ as a vector field with values in $\boldsymbol{g}$ and j as a scalar field with values in the space $\mathrm{End}(\boldsymbol{g}) \cong \boldsymbol{g} \otimes \boldsymbol{g}^*$ of endomorphisms of $\boldsymbol{g}$ (i.e., of linear transformations on $\boldsymbol{g}$). Then using that (5) can be rewritten as

$$X_M(gH) \;=\; g\,\pi_{\boldsymbol{m}}(\mathrm{Ad}(g)^{-1}X) \quad , \tag{17}$$

and that the covariant derivative $D_\mu g$ of g is defined by

$$D_\mu g \;=\; g\,\pi_{\boldsymbol{m}}(g^{-1}\,\partial_\mu g) \quad , \tag{18}$$

we get

$$\begin{aligned}
(j_\mu, X) \;&=\; -\,(\partial_\mu\varphi\,,\, X_M(\varphi)) \;=\; -\,(D_\mu g\,,\, g\,\pi_{\boldsymbol{m}}(\mathrm{Ad}(g)^{-1}X)) \\
&=\; -\,(g^{-1}D_\mu g\,,\, \pi_{\boldsymbol{m}}(\mathrm{Ad}(g)^{-1}X)) \;=\; -\,(g^{-1}D_\mu g\,,\, \mathrm{Ad}(g)^{-1}X) \\
&=\; -\,(D_\mu g\, g^{-1}, X) \quad ,
\end{aligned}$$

and

$$\begin{aligned}
(j(X), Y) \;&=\; (X_M(\varphi), Y_M(\varphi)) \;=\; (\pi_{\boldsymbol{m}}(\mathrm{Ad}(g)^{-1}X)\,,\, \pi_{\boldsymbol{m}}(\mathrm{Ad}(g)^{-1}Y)) \\
&=\; ((\pi_{\boldsymbol{m}}\,\mathrm{Ad}(g)^{-1})X\,,\, \mathrm{Ad}(g)^{-1}Y) \;=\; ((\mathrm{Ad}(g)\,\pi_{\boldsymbol{m}}\,\mathrm{Ad}(g)^{-1})X\,,\, Y) \quad ,
\end{aligned}$$

i.e.,

$$j_\mu \;=\; -\,D_\mu g\, g^{-1} \quad , \tag{19}$$

and

$$j \;=\; \mathrm{Ad}(g)\,\pi_{\boldsymbol{m}}\,\mathrm{Ad}(g)^{-1} \quad . \tag{20}$$

The formula for the Noether current j_μ is well known [5-8], while the formula for the new field j simply states that it is conjugate to the projector $\pi_{\boldsymbol{m}}$. It should also be noted that the two fields are not independent. Thus for example, we have the following identity expressing the derivatives of j in terms of j and j_μ:

$$\partial_\mu j \;=\; [\, j\,,\, \mathrm{ad}(j_\mu)\,] \;=\; j\,\mathrm{ad}(j_\mu) \;-\; \mathrm{ad}(j_\mu)\,j \quad . \tag{21}$$

If $M = G/H$ is not only a homogeneous space but even a symmetric one, so that in addition to (16), we have the commutation relation

$$[\boldsymbol{m}\,,\boldsymbol{m}] \subset \boldsymbol{h} \quad , \tag{22}$$

then we get an additional algebraic identity between j and j_μ:

$$\mathrm{ad}(j_\mu) \;=\; j\,\mathrm{ad}(j_\mu) \;+\; \mathrm{ad}(j_\mu)\,j \quad . \tag{23}$$

This case is of particular importance since it is precisely the non-linear sigma models on Riemannian symmetric spaces which, in two dimensions, are integrable field theories. The implications of the current algebra derived above for the canonical structure of these integrable models will be discussed in a separate paper [9].

Incidentally, the derivation of the current algebra (9-14) given above is not restricted to the two-dimensional case but is valid for Lorentz manifolds Σ of arbitrary dimension. Indeed, passing to higher dimensions means that the spatial variables $x, y, \ldots$ become vectors $\vec{x}, \vec{y}, \ldots$, and the only changes required are that the spatial derivative $.'$ is replaced by the gradient $\vec{\nabla}$ and j_1 by $\vec{j}$.

To conclude, we consider two examples which have been discussed in the literature before, namely the principal chiral models [1, 2] and the so-called $O(N)$-models [3, 4].

In the principal chiral models, the target space is a compact Lie group G, equipped with a G-biinvariant metric $(.\,,.)$. The symmetry group of the theory is now the direct product $G_L \times G_R$ of two copies G_L and G_R of G, acting on G by left and right translations, respectively. Then G becomes a Riemannian symmetric space by identifying it with the quotient $G \times G/\Delta G$, ΔG being the diagonal subgroup. Correspondingly, the fields φ and g used before are now denoted by g and (g_1, g_2), respectively, and the formula $\varphi = gH$ becomes $g = g_1 g_2^{-1}$. With this notation, the Noether current j_μ, having values in $\boldsymbol{g}_L \oplus \boldsymbol{g}_R$, splits into two components, $j_\mu = (j_\mu^L, j_\mu^R)$, with

$$j_\mu^L \;=\; -\,\tfrac{1}{2}\,\partial_\mu g\, g^{-1} \quad , \quad j_\mu^R \;=\; +\,\tfrac{1}{2}\, g^{-1}\,\partial_\mu g \quad , \tag{24}$$

cf. [5]. Similarly, the scalar field j, having values in $\mathrm{End}(\boldsymbol{g}_L \oplus \boldsymbol{g}_R)$, can be written as a (2×2)-block matrix:

$$j \;=\; \tfrac{1}{2} \begin{pmatrix} 1 & -\mathrm{Ad}(g) \\ -\mathrm{Ad}(g)^{-1} & 1 \end{pmatrix} \quad . \tag{25}$$

In the so called $O(N)$-models, we have $M = S^{N-1}$, $G = SO(N)$. The Noether current j_μ, having values in $so(N)$, is given by the well-known formula

$$j_\mu \;=\; \varphi\,\partial_\mu\varphi^T \;-\; \partial_\mu\varphi\,\varphi^T \quad . \tag{26}$$

Similarly, the scalar field j, having values in $\mathrm{End}(so(N))$, is found to act on $X \in so(N)$ according to

$$j(X) \;=\; \varphi\varphi^T X \;+\; X\,\varphi\varphi^T \quad . \tag{27}$$

(The proof is most conveniently performed by using that $\pi_{\boldsymbol{m}} = \frac{1}{2}(1-\sigma)$, where $\sigma = \mathrm{Ad}(\theta)$ with $\theta = \mathrm{diag}(1,-1,\ldots,-1)$, so $g\theta g^{-1} = 2\varphi\varphi^T - 1$.)

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