q–Canonical Commutation Relations and Stability of the Cuntz Algebra

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Abstract. We consider the q-deformed canonical commutation relations $a_i a_j^* - q\, a_j^* a_i = \delta_{ij} \mathbb{1}$, $i, j = 1, \ldots, d$, where d is an integer, and $-1 < q < 1$. We show the existence of a universal solution of these relations, realized in a C*-algebra $\mathcal{E}^q$ with the property that every other realization of the relations by bounded operators is a homomorphic image of the universal one. For $q = 0$ this algebra is the Cuntz algebra extended by an ideal isomorphic to the compact operators, also known as the Cuntz-Toeplitz algebra. We show that for a general class of commutation relations of the form $a_i a_j^* = \Gamma_{ij}(a_1, \ldots, a_d)$ with Γ an invertible matrix the algebra of the universal solution exists and is equal to the Cuntz-Toeplitz algebra. For the particular case of the q-canonical commutation relations this result applies for $|q| < \sqrt{2} - 1$. Hence for these values $\mathcal{E}^q$ is isomorphic to $\mathcal{E}^0$. The example $a_i a_j^* - q\, a_i^* a_j = \delta_{ij} \mathbb{1}$ is also treated in detail.

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1. Introduction

In this paper we study the relations

$$a_i a_j^* - q\, a_j^* a_i = \delta_{ij} \mathbb{1} \qquad i,j = 1, \ldots, d \tag{1}$$

for bounded operators a_i, $i = 1, \ldots, d$, $d < \infty$ on a Hilbert space, and a deformation parameter q satisfying $-1 < q < 1$. For $q = 0$ these relations are known as the Cuntz relations, and it is well-known that in this case the C*-algebra generated by the a_i is essentially unique: it is either the so-called Cuntz-Toeplitz algebra, or the quotient of this algebra by its only closed two sided ideal (generated by $\mathbb{1} - \sum_i a_i^* a_i$), which is known as the Cuntz algebra $\mathcal{O}_d$. Our main result in this paper is that the same statement holds for the relations (1) for small q. The technique we use also applies to more general relations of the form

$$a_i a_j^* = \Gamma_{ij}(a_1, \ldots, a_d) \quad , \tag{2}$$

where Γ is an invertible matrix of functions in the functional calculus of d variables, which satisfy a continuity and a growth condition specified below. We then show that one can decompose the generators as $a_i = v_i \rho$, where the v_i satisfy the Cuntz relations, and ρ satisfies an equation which can be solved by iteration. This iteration can be performed within the algebra generated by the v_i implying that ρ and a_i are in the C*-algebra generated by the v_i, and hence generate a Cuntz-Toeplitz or a Cuntz algebra.

In the recent physics literature there has been much interest in deformed versions of the canonical commutation relations for Bosons, partly as a tool for studying representations of quantum groups [**Bie,Mcf**], but also as a possible generalization of the standard quantum description of indistinguishable particles [**Gre**]. The "q-relations" (1), or "quon q-mutation relations" [**Gre**], are of special interest because they interpolate between the canonical commutation relations for Bosons for $q = 1$ (which do not have any bounded realizations), and the canonical anti-commutation relations for Fermions for $q = -1$. In these cases the relations are usually supplemented by the relations $a_i a_j - q a_j a_i = 0$. However, for $|q| < 1$ such additional relations lead to an overdetermined system.

In the case of the Bose or Fermi relations, uniqueness questions have a long history. In the Bose case, where one usually writes the relations in terms of the exponentials of the $a_i \pm a_i^*$ (Weyl operators), a theorem of von Neumann [**vNe**] asserts uniqueness up to unitary equivalence for $d < \infty$. For infinitely many generators, which is the case of interest in quantum field theory, unitary equivalence breaks down. However, in every representation the C*-algebra generated by the exponentials is C*-isomorphic to the so-called CCR algebra, and is thus independent of the representation. Analogous results hold for the Fermionic case.

We are looking for a similar characterization of the C*-isomorphism classes of representations of the q-relations. We will define first a universal C*-algebra $\mathcal{E}^q(d)$ associated with the q-relations, with the property that any realization of the q-relations by bounded operators generates a homomorphic image of $\mathcal{E}^q(d)$ (Proposition 3). Thus the classes of representations of the q-relations generating the same C*-algebra are in one-to-one correspondence to the closed two-sided ideals of $\mathcal{E}^q(d)$.

There is a special representation of the relations chacacterized by the property that it contains a cyclic vector Ω with $a_i\Omega = 0$ for all i. Since each polynomial in the generators can be brought into a normal ("Wick ordered") form with all starred operators to the left of all unstarred ones by applying the q-relations, it is easy to see that the equation $a_i\Omega = 0$ determines this representation uniquely. In analogy with ordinary Bose or Fermi commutation rules it will be called the *Fock representation* of the q-relations. In order to show that this is indeed a representation in a Hilbert space with positive definite scalar product, one has to show that $a_i\Omega = 0$ implies the positivity of $\langle\Omega, X^*X\Omega\rangle$ for all polynomials X in the variables a_i and a_j^*. This rather non-trivial result has been obtained in [**BS,Fiv**].

Two cases of the q-relations can be studied in complete detail. The first is the case of a single relation ($d = 1$). Most of the physics literature treats either this case, or else commuting operators, each of which satisfies the q-relations separately. It is easy to see that, for $d = 1$, the Fock representation is faithful, and contains only one proper two-sided ideal, which is generated by the one-dimensional projection onto the vacuum vector Ω, and is isomorphic to the compact operators. The algebras $\mathcal{E}^q(1)$ are isomorphic, for all q, to the universal C*-algebra $\mathcal{E}^0(1)$ generated by an isometry. $\mathcal{E}^0(1)$ is the Toeplitz algebra [**Dou,Co1,Co2**]. It is an extension [**BDF**] of $\mathcal{C}(\mathrm{S}^1)$ by the algebra $\mathcal{K}$ of compact operators, i.e. there is an exact sequence $0 \to \mathcal{K} \to \mathcal{E}^0(1) \to \mathcal{C}(\mathrm{S}^1) \to 0$. Under the isomorphisms $\mathcal{E}^0(1) \cong \mathcal{E}^q(1)$ the generators for different q are related by multiplication with a suitable function of the number operator in Fock space, i.e. they are "weighted shifts" [**Con**]. It has been noted by several authors [**Das,OKK**] that such results are not specific to the q-relations, but also hold for more general relations like $aa^* = \Gamma(a^*a)$ for suitable functions Γ.

The second case which has been fully analyzed is the case $q = 0$, $d < \infty$. Each a_i^* is then isometric, and the relations are essentially those studied by [**Cun**]. Again we have that the Fock representation is faithful, and contains a single closed two-sided ideal, which is generated by $\mathbb{I} - \sum_i a_i^* a_i$, and is isomorphic to the compact operators $\mathcal{K}$. The quotient of $\mathcal{E}^0(d)$ by this ideal is a simple C*-algebra known as the Cuntz algebra, and is usually denoted by $\mathcal{O}_d$. Again $0 \to \mathcal{K} \to \mathcal{E}^0(d) \to \mathcal{O}_d \to 0$ is exact. By analogy with the case $d = 1$, we will call $\mathcal{E}^0(d)$ the Cuntz-Toeplitz algebra.

Combining these two well-known cases one is led for general $q \in (-1, 1)$ to the questions (1) whether $\mathcal{E}^q(d)$ is an extension of a simple C*-algebra by the compact operators; (2) whether the Fock representation of $\mathcal{E}^q(d)$ is faithful; (3) whether $\mathcal{E}^q(d)$ and $\mathcal{E}^0(d)$ are isomorphic for all q, and (4) whether such an isomorphism can be obtained by multiplying the generators of $\mathcal{E}^0(d)$ with a fixed q-dependent element of $\mathcal{E}^0(d)$.

Questions (3) and (4) make sense also for the more general "Γ-relations" of the form given in equation (2). For a universal C*-algebra analogous to $\mathcal{E}^q(d)$ to exist, the matrix of functions Γ_{ij} has to satisfy an upper bound guaranteeing a uniform upper bound on any solution of equation (2). Assuming, in addition, a Lipshitz continuity estimate for Γ_{ij}, and, most importantly, a lower bound making the matrix $\Gamma(a_1, \ldots, a_d)$ invertible for all admissible arguments, we show in Theorem 9 that the universal solution of the Γ-relations (2) is of the form $a_i = v_i \rho$, where the v_i are the generators in $\mathcal{E}^0(d)$, and $\rho \in \mathcal{E}^0(d)$ is a positive element computed from Γ. Hence (3) and (4) hold true, and the Cuntz-Toeplitz algebra $\mathcal{E}^0(d)$ is the universal C*-algebra for all relations covered by our Theorem.

We emphasize that our stability result does not fall within the range of analytic deformation theory (see Gerstenhaber's article in [**HG**], and references cited there), since the Γ_{ij} need not be analytic functions. It is also not a case of isomorphism of "close C*-algebras" in the sense of [**Chr**], since we make explicit use of the fact that the "perturbed" algebra is given in terms of generators and relations of a specific type.

For the special case of the q-relations, the bounds needed in Theorem 9 are satisfied for $|q| < \sqrt{2} - 1$. Thus the above conjectures (1) to (4) are valid for these values of q. Moreover, all information that has been accumulated about the Cuntz algebra [**Eva,DR**] becomes immediately relevant to the representations of the q-relations.

Our paper is organized as follows: In section 1 we construct the universal algebra $\mathcal{E}^q(d)$ for the q-relations, and state some of its elementary properties. In section 2 we prove the stability result for the Cuntz-Toeplitz algebra. In section 3 give some applications of this result. Apart from the q-relations, we study in some detail the relations $a_i a_j^* - q a_i^* a_j = \delta_{ij} \mathbb{1}$ which differ from the q-relations in the position of the indices in the term containing q. For these modified relations we also compute the spectrum of $\sum_i a_i^* a_i$ in all possible bounded representations, and show that representations fail to exist for some negative values of q.

2. The C*-algebra $\mathcal{E}^q(\mathcal{H})$

In this section, we use a slightly different version of the q-relations, which makes the basic symmetries of these relations more transparent. Let $\mathcal{H}$ be a vector space with sesquilinear form $\langle\cdot,\cdot\rangle$, and let q be a number with $-1 < q < 1$. We then study $\mathbb{C}$-linear maps $a : \mathcal{H} \to \mathcal{A}$ into a C*-algebra $\mathcal{A}$ with identity such that

$$a(f)a(g)^* - q\,a(g)^*a(f) = \langle g, f\rangle \mathbb{1} \qquad g, f \in \mathcal{H} \quad . \tag{3}$$

Any map a with these properties will be called a representation, or a realization of the q-relations. The connection with the relations described in the introduction is then, that for some basis $\{e_i\} \subset \mathcal{H}$ which is orthonormal with respect to $\langle\cdot,\cdot\rangle$, we can write $a_i = a(e_i)$ and $a(\sum_i f_i e_i) = \sum_i f_i a_i$.

First, we need a result about the case of a single generator $a = a_1$ where $\mathcal{H}$ is one-dimensional. Further details about this case will be given below in Example 2.

Proposition 1. *Let $\mathcal{A}$ be a C^*-algebra with unit, $q \in (-1, 1)$, $c \in \mathbb{R}$, and $a \in \mathcal{A}$ a non-zero element satisfying*

$$aa^* - qa^*a = c\mathbb{1} \quad .$$

Then $c > 0$, and either $aa^* = a^*a = c/(1-q)\mathbb{1}$, or the spectra of aa^* and a^*a are equal to the closure of the sequence

$$c\frac{1-q^n}{1-q} \quad ,$$

where $n \in \mathbb{N}$, and $n \geq 0$ for a^*a and $n \geq 1$ for aa^*.

Proof : We may assume $q \neq 0$. Let $\sigma(x)$ denote the spectrum of an element $x \in \mathcal{A}$. Then $\sigma(aa^*) = c + q\sigma(a^*a)$, and $\sigma(aa^*) \setminus \{0\} = \sigma(a^*a) \setminus \{0\}$. Hence with $f(x) = c + qx$ we have for $x \neq 0$ that $x \in \sigma(aa^*)$ if and only if $f(x) \in \sigma(aa^*)$. The sequence $f^{-n}(x)$, $n \geq 0$ is unbounded unless $x = x_\infty = c(1-q)^{-1}$. Thus if $x_\infty \neq x \in \sigma(aa^*)$, the backwards iteration from x must terminate in 0. It follows that $x = f^n(0)$ for some $n \geq 1$. From the explicit solution of the forward iteration given in the Proposition, one sees that the forward iterates are all non-zero, hence the entire sequence must be in $\sigma(aa^*)$. The exceptional value x_∞ is in the closure of the sequence. Since $aa^* \geq 0$ we get $c = (1-q)x_\infty \geq 0$.

It remains to exclude the possibility $c = 0$. Since a was assumed to be non-zero there must be some non-zero $x \in \sigma(aa^*)$. Hence with $c = 0$ the whole sequence $q^{-n}x$ would be in $\sigma(aa^*)$. This contradicts the boundedness of a.

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Corollary 2. *Suppose $a : \mathcal{H} \to \mathcal{A}$ satisfies the q-relations. Then the sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is positive semidefinite, and*

$$\|a(f)\| = c(q)\langle f, f\rangle^{1/2} \quad , \quad \text{where} \qquad c(q) = \begin{cases} \frac{1}{\sqrt{1-q}} & \text{for } 0 \leq q < 1 \\ 1 & \text{for } -1 < q \leq 0 \end{cases} \quad .$$

It follows that the map a is continuous for the topology on $\mathcal{H}$ given by the seminorm $\|f\| = \langle f, f\rangle^{1/2}$ and the norm topology on $\mathcal{A}$. Therefore it extends by continuity to the completion of $\mathcal{H}$, and this extension again satisfies the q-relations. Hence there is no loss of generality in assuming that $\mathcal{H}$ is a Hilbert space, and we will make this assumption from now on.

An important consequence of Corollary 2 is that the norm bound on $a(f)$ is given by a constant independent of the particular realization of a. Only this fact is needed to construct the universal C*-algebra associated with the q-relations, which is given in the following Proposition.

Proposition 3.

(1) For a given Hilbert space $\mathcal{H}$, and any $q \in (-1, 1)$, there exists a C-algebra, denoted by $\mathcal{E}^q(\mathcal{H})$ with a map $a : \mathcal{H} \to \mathcal{E}^q(\mathcal{H})$ satisfying the q-relations with the following universal property: for any map $\tilde{a} : \mathcal{H} \to \mathcal{A}$ satisfying the q-relations there is a unique *-homomorphism $\tilde{\pi} : \mathcal{E}^q(\mathcal{H}) \to \mathcal{A}$ such that $\tilde{\pi}(a(f)) = \tilde{a}(f)$ for all $f \in \mathcal{H}$.*

(2) $\mathcal{E}^q(\mathcal{H})$ is determined up to C-isomorphism. For any isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces, there is a unique *-homomorphism $\mathcal{E}^q(V) : \mathcal{E}^q(\mathcal{H}_1) \to \mathcal{E}^q(\mathcal{H}_2)$ such that $\mathcal{E}^q(V)\big(a(f)\big) = a(Vf)$. Thus $\mathcal{E}^q$ is a covariant functor from the category of Hilbert spaces with isometries into the category of C*-algebras with *-homomorphisms.*

(3) $V \mapsto \mathcal{E}^q(V)$ is continuous on the unitary group of $\mathcal{H}$ in the sense that strong convergence $V_\alpha \to V$ implies $\mathcal{E}^q(V_\alpha)\big(X\big) \to \mathcal{E}^q(V)\big(X\big)$ in norm for all $X \in \mathcal{E}^q(\mathcal{H})$.

(4) $\mathcal{E}^q(\mathcal{H})$ carries a natural $\mathbb{Z}$-grading

$$\mathcal{E}^q(\mathcal{H}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{E}^q(\mathcal{H})_m \quad ,$$

where $X \in \mathcal{E}^q(\mathcal{H})_m$ iff $\mathcal{E}^q(\eta \mathbb{1})(X) = \eta^m X$ for any $\eta \in \mathbb{C}$ with $|\eta| = 1$. The projections $\mathbb{P}_m : \mathcal{E}^q(\mathcal{H}) \to \mathcal{E}^q(\mathcal{H})_m$ onto the summands are given explicitly by

$$\mathbb{P}_m(X) = \int d\eta \; \eta^{-m} \mathcal{E}^q(\eta \mathbb{1})(X) \quad ,$$

where $d\eta$ denotes the normalized Haar measure on the circle.

Proof : (1),(2) Let $\mathbf{E}^q(\mathcal{H})$ denote the quotient of the *-algebra of non-commuting polynomials in the generators $a(f)$ by the ideal generated by the q-relations and the relations arising from the linearity of a. It is straightforward to check that $\mathbf{E}^q(\mathcal{H})$ satisfies the analogoues of (1) and (2) in the category of *-algebras: if $\tilde{a}$ satisfies the relations, the homomorphism $\tilde{\pi}$ is defined simply by substituting the given $\tilde{a}(f)$ into any polynomial in $\mathbf{E}^q(\mathcal{H})$. In order to get a universal **C*-algebra** with the analogous property we have to define a C*-seminorm $\|\cdot\|$ on $\mathbf{E}^q(\mathcal{H})$ with the property that for any *bounded* representation of the relations the substitution homomorphism is continuous, and hence extends to the completion $\mathcal{E}^q(\mathcal{H})$ of $\mathbf{E}^q(\mathcal{H})$ with respect to that seminorm. This is to say that for any polynomial $X \in \mathbf{E}^q(\mathcal{H})$, and any bounded realization $\tilde{a}$ we must have $\|\tilde{\pi}(X)\| \leq \|X\|$. Hence we want to define

$$\|X\| = \sup_{\tilde{a}} \|\tilde{\pi}(X)\| \quad ,$$

where the sup is over all bounded realizations $\tilde{a}$ of the relations, and $\tilde{\pi}$ denotes the associated substitution homomorphism. There are three problems with this definition. The first is the technical point that the bounded realizations of the relations do not form a set, but a proper class. However, since $\|\tilde{\pi}(X)\|$ may be computed in the C*-algebra generated by the $\tilde{a}(f)$, we may restrict the supremum to realizations in C*-algebras with at most a certain cardinal number of elements depending on the number of generators via the dimension of $\mathcal{H}$. We now pick a Hilbert space of sufficiently high dimension such that all the universal representations of all these algebras can be realized in it. It is then clear that the supremum can be restricted to the set of realizations of the q-relations in bounded operators on this big Hilbert space.

The second problem is whether the supremum is possibly over the empty set, i.e. whether any bounded realizations of the relations exist at all. For the case at hand this question has been settled by **[BS]**, who explicitly construct the so-called Fock representation for all $q \in (-1, 1)$ (See Example 1 below).

The third problem is that the supremum might turn out to be infinite for some X. Since the $X \in \mathbf{E}^q(\mathcal{H})$ are polynomials in $\tilde{a}(f)$ this can be ruled out by proving an upper bound on $\|\tilde{a}(f)\|$, which is uniform with respect to all realizations $\tilde{a}$ of the q-relations. For the case at hand a bound of this kind was established in Corollary 2. Hence $\mathcal{E}^q$ exists and has the universal property (1). The uniqueness and the further properties stated in (2) follow as usual from the universal property.

(3) It is clear from Corollary 2 that $V_\alpha \to V$ strongly implies $\mathcal{E}^q(V_\alpha)\big(a(f)\big) \to \mathcal{E}^q(V)\big(a(f)\big)$ for all $f \in \mathcal{H}$. Hence we also have convergence for all polynomials X in the variables $a(f), a(g)^*$, and by a straigthforward estimate also the convergence for X in the norm closure $\mathcal{E}^q(\mathcal{H})$ of the polynomials.

(4) The integral for $\mathbb{P}_m$ is well-defined and in $\mathcal{E}^q(\mathcal{H})$ because by (3) the integrand is continuous in norm. On monomials in $a(f)$ and $a(g)^*$ we can compute the degree m by counting the number of factors $a(f)$ and $a(f)^*$. Hence on polynomials, the Fourier series decomposition $X = \bigoplus_m \mathbb{P}_m(X)$ is evident, and carries over by continuity to all $X \in \mathcal{E}^q(\mathcal{H})$.

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By (2), $\mathcal{E}^q(\mathcal{H})$ depends only on the dimension of $\mathcal{H}$. Since we will later be mostly interested in the case $\dim\mathcal{H} < \infty$, we will write $\mathcal{E}^q(d)$ for $\mathcal{E}^q(\mathbb{C}^d)$.

Example 1. (The Fock representation): Consider a representation of the q-relations by bounded operators in a Hilbert space $\mathcal{K}$. Let

$$\mathcal{N} = \{\xi \in \mathcal{K} \,|\, \forall_{f\in\mathcal{H}}\ a(f)\xi = 0\} \quad . \tag{4}$$

We call $\mathcal{N}$ the set of Fock vectors in this representation. Note that by applying the q-relations we can arrange any polynomial X in the generators "in standard form", i.e. such that all $a(f)$ are to the right of all $a(f)^*$. In this form we have for all $\xi, \eta \in \mathcal{N}$

$$\langle \xi, X\eta\rangle = \omega(X)\langle \xi, \eta\rangle \quad , \tag{5}$$

where $\omega(X)$ denotes the constant term in the standard form. For each $\xi \in \mathcal{N} \setminus \{0\}$ we obtain a cyclic representation of $\mathcal{E}^q(\mathcal{H})$, which is the GNS representation associated with the Fock state ω. This is called the Fock representation. Since the cyclic subspaces generated from orthogonal Fock vectors are orthogonal, we can choose an orthonormal basis in $\mathcal{N}$, and thus obtain a direct sum of isomorphic copies of the Fock representation. The subspace on which this direct sum lives can be represented naturally as $\mathcal{K}_\omega \otimes \mathcal{N}$, where $\mathcal{K}_\omega$ denotes the GNS-Hilbert space of the Fock representation, and $a(f)\restriction\mathcal{K}_\omega \otimes \mathcal{N} \cong a_\omega(f) \otimes \mathbb{1}$. The orthogonal complement of this invariant subspace is characterized by the property that it contains no non-zero Fock vectors. Note that the argument so far does not imply that there is any representation with $\mathcal{N} \neq \{0\}$. This will be the case if and only if the constant term in the standard form is indeed a positive functional on $\mathbf{E}^q(\mathcal{H})$. Equivalently, one has to check that the unique sesquilinear form on the vector space spanned by vectors of the form $a(f_1)\cdots a(f_n)\Omega$ which is computed by using the q-relations together with the equation $a(f)\Omega \equiv 0$, and the normalization condition $\|\Omega\| = 1$, is positive definite. Bozejko and Speicher [**BS**] have shown this by reducing it to the positive definiteness of the function $\pi \mapsto q^{i(\pi)}$ on the permutation group of n elements, where $i(\pi) = |\{(i,j) \,|\, i < j \text{ and } \pi(i) > \pi(j)\}|$ denotes the number of inversions in π.

Example 2. (The case of a single relation): By Proposition 1 a representation of the relation $aa^* - qa^*a = \mathbb{1}$ containing no Fock vectors has the property that $a = (1-q)^{-1/2}u$, with a unitary u. Equivalently, in such representations we have the equation $aa^* = a^*a$. By Example 1 every representation is a direct sum of such an abelian representation and a multiple of the Fock representation. We can obtain an explicit picture of the Fock representation by taking the polar decomposition $a = v(a^*a)^{1/2}$. Since by Proposition 1 aa^* is boundedly invertible it follows that $v^* = a^*(aa^*)^{-1/2}$ is an isometry, which lies in the C*-algebra $\mathcal{E}^q(\mathbb{C})$. Similarly, the projection $\mathbb{1} - vv^* = \mathrm{supp}(a^*a)$ onto the set $\mathcal{N}$ of Fock vectors lies in $\mathcal{E}^q$. For studying the Fock representation we may thus take this projection to be one-dimensional in the representation space. Denoting by $|0\rangle$ the Fock vector, and $|n\rangle = (v^*)^n|0\rangle$ we get $vt^2v^* = \mathbb{1} + qt^2$. Thus v is a one-sided shift implementing the iteration on the spectrum used in the proof of Proposition 1, and we get $\langle n|t^2|m\rangle = \delta_{nm}(1-q^n)/(1-q)$. The algebra generated by v thus contains t^2, and hence $\mathcal{E}^q(\mathbb{C})$ is generated by the single isometry v^*. The ideal generated by $\mathbb{I} - vv^*$ is isomorphic to the compact operators, and the quotient by this ideal gives back the abelian representations. Thus all irreducible representations of $\mathcal{E}^q(\mathbb{C})$ are quotients of the the Fock representation, which therefore is faithful. Moreover, and the algebras $\mathcal{E}^q(\mathbb{C})$ are isomorphic for all $q \in (-1, 1)$ to the well-known C*-algebra $\mathcal{E}^0(\mathbb{C})$ **[Co1,Co2]** generated by a single isometry.

The following Proposition shows that in a representation with several generators each single generator occurs in the Fock representation.

Proposition 4. *For* $\dim \mathcal{H} \geq 2$, *any bounded representation of the* q*-relations, and every* $0 \neq f \in \mathcal{H}$, 0 *is in the spectrum of* $a(f)^*a(f)$.

Proof : Since $a(f)$ generates a homomorphic image of $\mathcal{E}^q(\mathbb{C})$ we only have to show that this homomorphism is faithful, i.e. that $u = \|f\|^{-1}\sqrt{1-q}\,a(f)$ is not unitary. Suppose it were. Then from the commutation relations we would get for $g \perp f$: $Ua(g)^*U = qa(g)^*$. Hence $\|a(g)\| = |q|\,\|a(g)\|$, Because $|q| < 1$ this implies $a(g) = 0$, which implies $g = 0$. This implies that $\mathcal{H}$ is one-dimensional.

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The following is a typical feature of infinite C*-algebras **[Cun]**, i.e. algebras containing elements with $x^*x = \mathbb{I}$, but $xx^* \neq \mathbb{I}$, of which the Cuntz algebra is a prototype.

Corollary 5. *For* $-1 < q < 1$ *and* $\dim \mathcal{H} \geq 2$, $\mathcal{E}^q(\mathcal{H})$ *has no tracial state.*

Proof : Suppose τ is a tracial state. Then its restriction to the subalgebra generated by $a(f)$ is also tracial. Since there is no finite trace on the compact operators, τ thus annihilates the corresponding ideal in $\mathcal{J} \subset \mathcal{E}^q(\mathbb{C})$. Hence $\tau(x^*zy) = \tau(yx^*z) = 0$ for all $x, y \in \mathcal{E}^q(\mathcal{H})$ and $z \in \mathcal{J}$, which means that $\pi_\tau(z) = 0$ for the cyclic representation associated with τ. This contradicts Proposition 4.

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Our main result regarding the q-relations is the stability of the algebra $\mathcal{E}^0$ under small q-deformations stated in the following Proposition. It will be proven in section 4.1 as a consequence of the stability of the Cuntz relations with respect to a much larger class of perturbations.

Proposition 6. *For* $|q| < \sqrt{2} - 1$ *and all* $d < \infty$, $\mathcal{E}^q(\mathbb{C}^d)$ *is isomorphic to* $\mathcal{E}^0(\mathbb{C}^d)$.

3. Stability of the Cuntz-Toeplitz algebra

In this section we will demonstrate a remarkable stability of the algebra $\mathcal{E}^0$ under practically arbitrary small perturbations. The "Γ-relations" we consider will be of the form

$$a_i a_j^* = \Gamma_{ij}(a_1, \ldots, a_d) \qquad i,j = 1, \ldots, d \quad , \tag{6}$$

where Γ_{ij} are now functions of $a \equiv a_1, \ldots, a_d$ in the non-commutative C*-functional calculus. Before elaborating on this notion let us consider mention a few examples. The Cuntz relations are given by $\Gamma_{ij} = \delta_{ij}\mathbb{I}$. For the q-relations we set $\Gamma_{ij}(a) = \delta_{ij}\mathbb{I} + q a_j^* a_i$. In some detail we will later consider the relations $\Gamma_{ij}(a) = \delta_{ij}\mathbb{I} + q a_i^* a_j$. Further examples are quadratic polynomials whose constant term is an invertible matrix, and whose linear nd second order terms are sufficiently small. We emphasize that the functions Γ_{ij} need not be polynomials, and may contain fairly arbitrary continuous functions of their arguments. The exact conditions will be specified below. A fundamental common feature of all relations which we can treat is that the matrix $a_i a_j^*$ is invertible in the algebra $\mathcal{M}_d(\mathcal{A})$ of $d \times d$-matrices over the algebra $\mathcal{A} \ni a_i$. This property may seem surprising since $a_i a_j^*$ looks like the expression for a rank one matrix. In fact, $a_i a_j^*$ cannot be invertible for a_i in any algebra with finite faithful trace, which underlines the infinite character of the Γ-relations.

By a function f in the C*-functional calculus [**RW,SW**] we mean a collection of functions $f_{\mathcal{A}} : \mathcal{A}^d \to \mathcal{A}$, one for each C*-algebra $\mathcal{A}$, such that for any *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ and any tuple $a_1, \ldots, a_d$ of arguments in $\mathcal{A}$ we have

$$\pi f_{\mathcal{A}}(a_1, \ldots, a_d) = f_{\mathcal{B}}(\pi a_1, \ldots, \pi a_d) \quad . \tag{7}$$

Taking for π the canonical injection from the $C^*(a_1, \ldots, a_d)$ into $\mathcal{A}$, we find that f can always be evaluated in the algebra generated by its arguments. Since f is also natural with respect to isomorphisms it suffices to define $f_{\mathcal{B}(\mathcal{K})}$ for some Hilbert space $\mathcal{K}$ in which any algebra generated by d arguments has a faithful representation. Equation (7) only has to be postulated for all subalgebras of $\mathcal{B}(\mathcal{K})$, and then follows for arbitrary algebras. All polynomials in d non-commuting variables and their adjoints are functions of the functional calculus. Conversely, if one restricts each argument by a uniform norm bound, any function of the calculus can be approximated uniformly by polynomials [**RW**].

In the sequel we we will drop the subscript $\mathcal{A}$ for such functions. Tuples of operators satisfying the Cuntz relations $v_i v_j^* = \delta_{ij}\mathbb{I}$ will usually be denoted by v_i, $i = 1, \ldots, d$. We show first how such operators emerge from representations of the Γ-relations with invertible Γ. For $a, b \in \mathcal{A}^d$ we define $\mathbf{e}(a,b) \in \mathcal{M}_d(\mathcal{A})$ by

$$\mathbf{e}(a,b)_{ij} = a_i b_j^* \quad . \tag{8}$$

Proposition 7. *Let $a_1, \ldots, a_d \in \mathcal{A}$. Suppose that $\mathbf{e}(a,a)$ is boundedly invertible. Then there are elements $v_1, \ldots v_d \in \mathcal{A}$ such that with $\rho = (\sum_i a_i^* a_i)^{1/2}$:*

$$\begin{aligned} v_i v_j^* = \delta_{ij} \mathbb{1} \qquad i,j = 1, \ldots, d \quad , \\ a_i = v_i \rho \quad , \quad \rho \geq 0 \quad , \quad \textit{and} \quad \rho = (\textstyle\sum_i v_i^* v_i) \rho \quad . \end{aligned} \tag{9}$$

Proof : Let $\mathcal{A}$ be faithfully represented on a Hilbert space $\mathcal{K}$, and let $\hat{a} : \mathcal{K} \to \mathcal{K}^d$ denote the operator $(\hat{a}\xi)_i = a_i \xi$, where the index i refers to the i^{th} component in $\mathcal{K}^d$. Then $\mathbf{e}(a,a) = \hat{a}\hat{a}^*$. On the other hand, $\hat{a}^* \hat{a} = \sum_i a_i^* a_i = \rho^2$. Consider the polar decomposition of $\hat{a}$, i.e.

$$\hat{a} = V(\hat{a}^* \hat{a})^{1/2} = (\hat{a}\hat{a}^*)^{1/2} V \quad ,$$

where $V : \mathcal{K} \to \mathcal{K}^d$ is a partial isometry, which we can write as $(V\xi)_i = v_i \xi$. In components the first equation becomes $a_i = v_i \rho$. Since $(\hat{a}\hat{a}^*)^{-1} \in \mathcal{M}_d(\mathcal{A})$ we have $V = (\hat{a}\hat{a}^*)^{-1/2} \hat{a}$, so that $v_i \in \mathcal{A}$. Furthermore, $VV^* = \mathbb{1}_{\mathcal{M}_d(\mathcal{A})}$, which means that $v_i v_j^* = \delta_{ij} \mathbb{1}$. On the other hand, $V^* V = \sum_i v_i^* v_i$ is the support projection of ρ^2, which proves the last equation. ■

Suppose now that some Γ-relations guarantee the invertibility of $\mathbf{e}(a,a)$. Then we know that the algebra generated by the a_i is the same as the algebra generated by the v_i and one additional element ρ. If we can show that ρ is also in the algebra generated by the v_i, the algebras generated by the v_i and by the a_i, respectively, are the same. Our strategy therefore is to fix the operators v_i and to turn the Γ-relations into an equation for ρ. Substituting $a_i = v_i \rho$ (in short, $a = v\rho$) in the Γ-relations we obtain the condition $v_i \rho^2 v_j^* = \Gamma_{ij}(v\rho)$. We multiply this with v_i^* from the left and with v_j from the right, sum over i,j, and use the support condition for ρ to obtain

$$\rho^2 = \sum_{i,j=1}^{d} v_i^* \, \Gamma_{ij}(v\rho) v_j \quad . \tag{10}$$

This equation is equivalent to the Γ-relations for $a = v\rho$, since we can restore the original form by multiplying this with v_i and v_j^*, and using the Cuntz relations for the v_i. We will later solve equation (10) by contractive iteration. Since this involves taking the square root of ρ we need the following Lipshitz estimate for square roots.

Lemma 8. *Let $\mathcal{A}$ be a C^*-algebra with identity, $\varepsilon > 0$, and $x, y \geq \varepsilon \mathbb{1}$. Then*

$$\left\| \sqrt{x} - \sqrt{y} \right\| \leq \frac{1}{2\sqrt{\varepsilon}} \left\| x - y \right\| \quad .$$

Proof : By multiplying x and y with the same factor we may assume that $x = \mathbb{I} + \hat{x}$, $y = \mathbb{I} + \hat{y}$ with $\|\hat{x}\|, \|\hat{y}\| \leq \hat{\varepsilon} = 1 - \varepsilon$. Then

$$\left\|\sqrt{\mathbb{I}+\hat{x}} - \sqrt{\mathbb{I}+\hat{y}}\right\| \leq \sum_{n=0}^{\infty} \left|\binom{1/2}{n}\right| \|\hat{x}^n - \hat{y}^n\| \leq \|\hat{x} - \hat{y}\| \sum_{n=0}^{\infty} \left|\binom{1/2}{n}\right| n\hat{\varepsilon}^{n-1}$$
$$= \frac{1}{2}\|x - y\| \sum_{n=1}^{\infty} \left|\binom{-1/2}{n-1}\right| (-\hat{\varepsilon})^{n-1} = \frac{1}{2}(1-\hat{\varepsilon})^{-1/2} \|x - y\| \quad .$$

■

For stating our main Theorem it is convenient to introduce the norm

$$\|a\|^2 = \left\|\sum_{i=1}^{d} a_i^* a_i\right\| \tag{11}$$

on d-tuples $a = (a_1, \ldots, a_d) \in \mathcal{A}^d$. Note that by the proof of Proposition 7 we have $\|\mathbf{e}(a,a)\| = \|\hat{a}\hat{a}^*\| = \|\hat{a}^*\hat{a}\| = \|a\|^2$.

Theorem 9. *Let Γ_{ij}, $i,j = 1, \ldots, d$ be in the C^*-functional calculus of d variables such that $\Gamma(a) \in \mathcal{M}_d(\mathcal{A})$ is hermitian for all C^*-algebras $\mathcal{A}$ and all $a \in \mathcal{A}^d$. Suppose there are constants $\mu, \varepsilon > 0, \lambda < 2\sqrt{\varepsilon}$ such that*

(1) the Γ-relations $a_i a_j^* = \Gamma_{ij}(a)$ imply $\|a\|^2 \leq \mu$.

(2) For arbitrary $a, b \in \mathcal{A}^d$ with $\|a\|^2, \|b\|^2 \leq \mu$ we have the estimates

$$\|\Gamma(a)\| \leq \mu \tag{a}$$
$$\Gamma(a) \geq \varepsilon \mathbb{I} \tag{b}$$
$$\|\Gamma(a) - \Gamma(b)\| \leq \lambda \|a - b\| \quad . \tag{c}$$

Then there is a unique $\rho \in \mathcal{E}^0(d)$ such that $\rho \geq 0$, $\rho \sum_i v_i^* v_i = \rho$, and that $a_i = v_i \rho$ satisfies the Γ-relations, where the v_i are the generators in $\mathcal{E}^0(d)$.
*Moreover, this solution has the following universal property: if $\tilde{a}_1, \ldots, \tilde{a}_d \in \mathcal{A}$ satisfies the Γ-relations in any C^*-algebra $\mathcal{A}$, there is a unique *-homomorphism $\tilde{\pi} : \mathcal{E}^0(d) \to \mathcal{A}$ such that $\tilde{\pi}(v_i \rho) = \tilde{a}_i$ for all i.*

Proof : Let $\mathcal{A}$ be any C*-algebra containing elements $\tilde{v}_i$, $i = 1, \ldots, d$ satisfying the Cuntz relations. We will iterate

$$f(x) = \Big(\sum_{ij} \tilde{v}_i^* \Gamma_{ij}(\tilde{v}x)\tilde{v}_j\Big)^{\frac{1}{2}}$$

on the set

$$X = \left\{x \in \mathcal{A} \middle| \, 0 \leq x^2 \leq \mu\, p\right\} \quad ,$$

where $(\tilde{v}x)_i = \tilde{v}_i x$, and $p = \sum_i \tilde{v}_i^* \tilde{v}_i$. Then by equation (10) the fixed points of f are precisely those $x \in \mathcal{A}$ with $x \geq 0$ and $xp = x$ such that $\tilde{v}x$ satisfies the Γ-relations. The set X is invariant, because $\|\tilde{v}x\|^2 \leq \|x\|^2$, so $x^2 \leq \mu p$ implies $\|f(x)^2\| \leq \|\Gamma(\tilde{v}x)\| \leq \mu$ by (a), and $f(x)^2 = pf(x)^2 p \leq \mu p$ by the Cuntz relations. By assumption (b) $\Gamma(\tilde{v}x) \geq \varepsilon\mathbb{1}$ for $x \in X$, hence $f(x)^2 \geq \varepsilon p$. The square root is thus well defined, and by Lemma 8 (applied in the algebra $p\mathcal{A}p$ with unit p) we get from (c) the estimate

$$\|f(x) - f(y)\| \leq \frac{1}{2\sqrt{\varepsilon}} \left\|f(x)^2 - f(y)^2\right\| \leq \frac{\lambda}{2\sqrt{\varepsilon}} \|\tilde{v}x - \tilde{v}y\| \leq \frac{\lambda}{2\sqrt{\varepsilon}} \|x - y\| \quad .$$

Thus f is contractive, and has a unique fixed point $\tilde{\rho}$. We apply this firstly in the case $\mathcal{A} = \mathcal{E}^0(d)$ with $\tilde{v} = v$ to obtain $\rho \in \mathcal{E}^0(d)$ as claimed in the Theorem.

Now let $\tilde{a} \in \mathcal{A}^d$ satsify the Γ-relations. Then by assumption (1) of the Theorem, $\|\tilde{a}\|^2 \leq \mu$, which implies that $\mathbf{e}(\tilde{a}, \tilde{a}) = \Gamma(\tilde{a}) \geq \varepsilon\mathbb{1}$ is invertible. Hence, by Proposition 7, $\tilde{a} = \tilde{v}\tilde{\rho}$, and $\tilde{\rho}$ must be the unique fixed point of the iteration in $\mathcal{A}$. On the other hand, by the universal property of $\mathcal{E}^0(d)$ for the Cuntz relations there is a unique homomorphism $\tilde{\pi} : \mathcal{E}^0(d) \to \mathcal{A}$ such that $\tilde{\pi}(v_i) = \tilde{v}_i$. By the homomorphism property (7) of the functional calculus we have

$$\begin{aligned}\Gamma_{ij}\left(\tilde{v}_1\tilde{\pi}(\rho), \ldots, \tilde{v}_d\tilde{\pi}(\rho)\right) = \Gamma_{ij}\left(\tilde{\pi}(v_1\rho), \ldots, \tilde{\pi}(v_d\rho)\right) &= \tilde{\pi}\Big(\Gamma_{ij}(v_1\rho, \ldots, v_d\rho)\Big) \\ &= \tilde{\pi}(v_i\rho^2 v_j^*) = \tilde{v}_i\tilde{\pi}(\rho)^2\tilde{v}_j \quad .\end{aligned}$$

Hence $\tilde{\pi}(\rho)$ is also a fixed point of the iteration in $\mathcal{A}$, and we must have $\tilde{\pi}(\rho) = \tilde{\rho}$, and $\tilde{\pi}(a_i) = \tilde{v}_i\tilde{\rho} = \tilde{a}_i$.

∎

4. Applications

4.1. The q-relations

In equation (8) we have introduced the dyads $\mathbf{e}(a,b)$ for $a,b\in\mathcal{A}^d$. Here we need a second kind of dyad, which is equal to $\mathbf{e}(\cdot,\cdot)$ when a_i and b_i commute. We set

$$\mathbf{f}(a,b)_{ij}=b_j^*a_i \qquad , \qquad \mathbf{e}(a,b)_{ij}=a_ib_j^* \quad . \tag{12}$$

Using these dyads we can write the q-relations in the compact form

$$\mathbf{e}(a,a)=\mathbb{1}+q\,\mathbf{f}(a,a) \quad . \tag{13}$$

The norm bounds needed for the application of Theorem 9 are given in the next Lemma.

Lemma 10. *Let $a,b\in\mathcal{A}^d$. Then $\|\mathbf{e}(a,a)\|=\|\sum a_i^*a_i\|\equiv\|a\|^2$, and*

$$\|\mathbf{f}(a,b)\|\leq\|a\|\,\|b\| \qquad , \qquad \|\mathbf{e}(a,b)\|\leq\|a\|\,\|b\| \quad .$$

Moreover, $\|a^*\|^2\leq d\,\|a\|^2$.

Proof : The equation for $\|\mathbf{e}(a,a)\|$ was already noted before Theorem 9.

$$\big(\mathbf{e}(a,b)^*\mathbf{e}(a,b)\big)_{ij}=\big(\mathbf{e}(b,a)\mathbf{e}(a,b)\big)_{ij}=b_i\big(\sum_k a_k^*a_k\big)b_j^* \quad .$$

The last expression can be written as $\hat b(\sum_k a_k^*a_k)\hat b^*$ with a $d\times 1$-matrix $\hat b$. Estimating the bracket by its norm we get $\mathbf{e}(a,b)^*\mathbf{e}(a,b)\leq\|a\|^2\,\mathbf{e}(b,b)$, and $\|\mathbf{e}(a,b)\|^2\leq\|a\|^2\,\|\mathbf{e}(b,b)\|$.

Similarly, we get

$$\big(\mathbf{f}(a,b)^*\mathbf{f}(a,b)\big)_{ij}=\big(\mathbf{f}(b,a)\mathbf{f}(a,b)\big)_{ij}=\sum_k a_k^*b_ib_j^*a_k=\Phi(b_ib_j^*)$$

with the completely positive map $\Phi(x)=\sum_k a_k^*xa_k$. Hence $\mathbf{f}(a,b)^*\mathbf{f}(a,b)\leq\|\mathbf{e}(b,b)\|\,\Phi(\mathbb{1})$, and $\|\mathbf{f}(a,b)\|^2\leq\|b\|^2\,\|a\|^2$.

Finally we get $\|a^*\|=\|\sum a_ka_k^*\|\leq\sum_{k=1}^d\|a_k\|^2\leq d\sum\|\sum a_k^*a_k\|$.

■

Proof of Proposition 6: With $\Gamma(a)=\mathbb{1}+q\mathbf{f}(a,a)$ we get

$$\|\Gamma(a)\|=1+|q|\,\|\mathbf{f}(a,a)\|\leq 1+|q|\,\|a\|^2 \quad .$$

Hence the q-relations imply $\|a\|^2\leq 1+|q|\,\|a\|^2$, and $\|a\|^2\leq(1-|q|)^{-1}\equiv\mu$. For $\|a\|^2\leq\mu$ we get $\|\Gamma(a)\|\leq\mu$ as required in condition (2a) of Theorem 9. Furthermore,

$$\Gamma(a)\geq\mathbb{1}(1-|q|\,\|\mathbf{f}(a,a)\|\geq\mathbb{1}(1-|q|\mu)=\varepsilon\mathbb{1}$$

with $\varepsilon = (1-2|q|)/(1-|q|)\mathbb{1}$. Hence for $\Gamma(a)$ to be bounded away from zero as required in condition (2b) we need $|q| < 1/2$. Using the bilinearity of $\mathbf{f}(a,b)$ we get the Lipshitz bound

$$\|\Gamma(a) - \Gamma(b)\| = |q|\,\|\mathbf{f}(a,a) - \mathbf{f}(b,b)\| \le |q|\,\|a-b\|\,(\|a\| + \|b\|) \le 2|q|\sqrt{\mu}\,\|a-b\| \quad .$$

With $\lambda = 2|q|\sqrt{\mu}$ the contractivity condition then becomes $|q| < \sqrt{1-2|q|}$, or $|q| < \sqrt{2}-1$.

■

For the extension of Proposition 6 to a larger interval, possibly even to the whole range $q \in (-1,1)$, it is crucial to improve the bounds

$$\frac{1-2|q|}{1-|q|}\mathbb{1} \quad \le \mathbf{e}(a,a) \quad \le \frac{1}{1-|q|}\mathbb{1} \tag{14}$$

for a satisfying the q-relations, which were obtained in the above proof. The upper bound is fairly good. In fact, for $q \ge 0$ the upper bound is an equality since by Proposition 1 each diagonal element in $\mathbf{e}(a,a)$ has norm $(1-q)^{-1}$. As a corollary of this observation we find that in every representation with $d+1$ generators there are many vectors on which $\sum_i^d a_i^* a_i$ is small (namely those on which $a_{d+1}^* a_{d+1}$ nearly attains its maximum). The existence of such "almost Fock" vectors for a subset of generators can be seen as a generalization of Proposition 4, and can be extended also to some negative values of q. For $q < 0$ the upper bound in (14) is the best possible d-independent bound. (The vectors in Fock space arising from (anti-)symmetrization of $a_1^* \cdots a_n^*|0\rangle$ are eigenvectors of T with eigenvalue $(1-(\pm q)^n)/(1 \mp q)$). However, for given finite d the bound can be improved. For example, for $d = 2$ we get

$$\mathbf{e}(a,a) = \begin{pmatrix} a_1 a_1^* & 0 \\ 0 & a_2 a_2^* \end{pmatrix} + q \begin{pmatrix} 0 & a_2^* a_1 \\ a_1^* a_1 & \end{pmatrix}$$

$$\begin{pmatrix} 0 & a_2^* a_1 \\ a_1^* a_1 & \end{pmatrix}^2 = \begin{pmatrix} a_2^* a_1 a_1^* a_2 & 0 \\ 0 & a_1^* a_2 a_2^* a_1 \end{pmatrix} \le \begin{pmatrix} a_2^* a_2 & 0 \\ 0 & a_1^* a_1 \end{pmatrix} \le \mathbb{1} \quad ,$$

where we have used the bound $aa^* \le 1$ from Corollary 2. Hence

$$\|\mathbf{e}(a,a)\| \le 1 + |q| \qquad \text{for } d = 2. \tag{15}$$

We can insert this into the bound $\mathbf{e}(a,a) \ge (1 - |q|\,\|\mathbf{e}(a,a)\|)\mathbb{1}$ to get an improvement of the lower bound in (14):

$$\mathbf{e}(a,a) \ge 1 - |q| - |q|^2 \qquad \text{for } d = 2. \tag{16}$$

Hence $\mathbf{e}(a,a)$ is boundedly invertible for $q > -\phi^{-1} = -\frac{1}{2}(\sqrt{5}-1)$. It is not clear what the best lower bounds on $\mathbf{e}(a,a)$ are. Numerical evidence from Fock space suggests that

$\mathbf{e}(a,a)$ might be bounded away from zero for all $-1 < q < 1$ with bounds going to zero at the endpoints of the interval.

4.2. The relations $a_i a_j^* - q a_i^* a_j = \delta_{ij}\,\mathbb{1}$

These relations are maybe the most natural choice of a quadratic polynomial Γ apart from the q-relations. The only difference between these is the position of the indices in the term containing the parameter q. In terms of the dyads introduced in section 4.1 we can write these relations as

$$\mathbf{e}(a,a) = \mathbb{1} + q\,\mathbf{e}(a^*,a^*) \quad . \tag{17}$$

The two sets of relations are identical if either $d = 1$, or $q = 0$. Thus we can directly take over Proposition 1 and Corollary 2. The existence of any bounded realizations for general q is not immediately clear. But if for any values of (q,d) there is any bounded realization, then there is a universal algebra $\mathcal{D}^q(d)$ satisfying Proposition 3 with one modification: The relations are now not invariant under the unitary group U_d, but only under the orthogonal subgroup O_d, and correspondingly we should take a real Hilbert space $\mathcal{H}$ as the analogue of the complex space $\mathcal{H}$ in Proposition 3. Applying Theorem 9 to these relations yields

Proposition 11. *For $d \in \mathbb{N}$, and $|q| < 1/(d\phi) = (\sqrt{5}-1)/(2d)$ the functions $\Gamma_{ij}(a) = \delta_{ij}\mathbb{1} + a_i^* a_j$ satisfy the conditions of Theorem 9. Consequently, $\mathcal{D}^q(d)$ exists for these values of q and is isomorphic to $\mathcal{E}^0(\mathbb{C}^d)$.*

Proof : Since $\mathbf{e}(a,a) \geq 0$ it pays in this case to distinguish $q > 0$ and $q < 0$. For $q > 0$ we apply Lemma 10 to get $\|\Gamma(a)\| \leq 1 + q\ \|\mathbf{e}(a^*,a^*)\| \leq 1 + qd\,\|a\|^2$, and set set $\mu = (1-qd)^{-1}$ in Theorem 9. Clearly, we can set $\varepsilon = 1$. With $\lambda = 2qd\sqrt{\mu}$ the contractivity estimate is $qd < \sqrt{1-qd}$, or $qd < \phi^{-1}$.

For $q < 0$ we can take $\mu = 1$. Then

$$\Gamma(a) \geq \mathbb{1}(1 - |q|\,\|\mathbf{e}(a^*,a^*)\|) \geq \varepsilon\mathbb{1}$$

with $\varepsilon = (1 - |q|\,d)$. The Lipshitz bound is $\lambda = 2|q|\,d$, so contractivity holds once more for $|q|\,d < \sqrt{1-|q|\,d}$.

■

The fact that the interval for q for which our technique works now depends on d came in through the estimate $\|a^*\|^2 \leq d\,\|a\|^2$ in Lemma 10, which may appear to be exceedingly crude. However, this d-dependence is typical for the relations (17). We will show this now by using the idea of Proposition 1 to obtain rather detailed necessary conditions for such representations. The core of the idea is contained in the following Proposition:

Proposition 12. *Let a_i, $i = 1, \ldots, d$ be bounded operators satisfying the relations (17). Let $T = \sum_{i=1}^{d} a_i^* a_i$, and consider the function $f(x) = 1 + qd + q^2 x$. Then*

$$x \in \sigma(T) \iff f(x) \in \sigma(T) \quad ,$$

provided that both $f(x) \neq 1$ and $f(x) \neq 0$.

Proof : We consider two $d \times 1$-matrices with entries in $\mathcal{A}$, namely $A_i = a_i$, and $\widehat{A}_i = a_i^*$. This somewhat redundant notation is necessary to keep apart the different meaning of adjoints. We can then write

$$\begin{array}{llllll} \mathbf{e}(a,a) & = AA^* & \mathbf{e}(a^*,a^*) & = \widehat{A}\widehat{A}^* & AA^* & = \mathbb{I} + q\,\widehat{A}\widehat{A}^* \\ T := \sum_i a_i^* a_i & = A^* A & \widehat{T} := \sum_i a_i a_i^* & = \widehat{A}^* \widehat{A} & \widehat{T} & = d\mathbb{I} + qT \end{array}$$

Now suppose that x satisfies the two conditions in the Proposition. Then $x \in \sigma(T) \iff d+qx \in \sigma(\widehat{T})$. Since we have assumed $d+qx = (1/q)(f(x)-1) \neq 0$ this is in turn equivalent to $d+qx \in \sigma(\widehat{A}^*\widehat{A})$, and $d+qx \in \sigma(\widehat{A}\widehat{A}^*)$. This is equivalent to $(1+q(d+qx)) = f(x) \in \sigma(AA^*)$. Since we have assumed that $f(x) \neq 0$ this is equivalent to $f(x) \in \sigma(T)$.

■

Proposition 13. *Let $a_1, \ldots, a_d$ with $d > 1$ be bounded operators satisfying the relations (17). Let $T = \sum_{i=1}^{d} a_i^* a_i$, and $x_\infty = (1+qd)/(1-q^2)$. For $n \in \mathbb{N}$, $n \geq 0$ let $\lambda_n = x_\infty(1-q^{2n})$ and $\mu_n = x_\infty + (1-x_\infty)q^{2n}$. Then the spectrum of T is contained in*

$$\{\lambda_n \,|\, n \geq 0\} \cup \{\mu_n \,|\, n \geq 0\} \cup \{x_\infty\} \quad .$$

Moreover, one has

(1) For $q \geq -d^{-1}$, all μ_n are in the spectrum, and either all or no λ_n. Moreover, for $q \neq 0$ $\lambda_n \neq \mu_m$ for all $n, m \geq 0$.

(2) For $q < -d^{-1}$, the spectrum of T is of the form

$$\{\mu_0, \mu_1, \ldots \mu_{N-1}\}$$

with $\mu_N = 0$ for some finite N. Thus representations can exist only for the discrete set of values q for which the equation $\mu_N = 0$, or equivalently

$$q^{2N} = \frac{1+qd}{q(d+q)}$$

has a solution.

Proof : If $x \in \sigma(T)$, $x \neq x_\infty$, the iteration of f^{-1}, with f as in Proposition 12 yields an unbounded sequence. Thus by Proposition 12 the iterates must either hit 0 or 1, which means that either $x = \lambda_n$ or $x = \mu_n$ for some $n \in \mathbb{N}$. Next observe that Proposition 1 and the argument in Proposition 4 show that $0 \in \sigma(a_1^* a_1)$. Thus by Proposition 1 we can choose a non-zero $\xi \in \ker(a_1^* a_1) = \ker(a_1)$. Now the vector $(\xi, 0, \ldots, 0)$ is in $\ker(\widehat{A}\widehat{A}^*)$. This implies $1 \in \sigma(AA^*)$, and $\mu_0 \in \sigma(T)$.

If $q > -1/d$ then x_∞ is positive, and the sequence λ_n is strictly increasing towards x_∞. If in addition $q \leq 0$, then $0 \leq x_\infty \leq 1$, and the sequence μ_n is decreasing towards x_∞, implying that the sequences are disjoint. If $q > 0$, both sequences are increasing. They do not intersect, because $\mu_n > \mu_0 = 1 > 0 = \lambda_0$, and $\lambda_n \geq \lambda_1 = 1 + qd > 1 = \mu_0$ for $n > 0$, and the inverse iteration exists.

If $q < -1/d$, then x_∞ and all $\lambda_n, n > 0$ are strictly negative. Hence $0 \notin \sigma(T)$. μ_n is strictly decreasing. Thus we can only have that $\mu_N = 0$ for some $N \in \mathbb{N}$.

■

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