Finite Size Effects and Shock Fluctuations in the Asymmetric Simple Exclusion Process*

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Abstract

We consider a system of particles on a lattice of L sites, set on a circle, evolving according to the asymmetric simple exclusion process, *i.e.* particles jump independently to empty neighboring sites on the right (left) with rate p (rate $1-p$), $1/2 < p \leq 1$. We study the nonequilibrium stationary states of the system when the translation invariance is broken by the insertion of a blockage between (say) sites L and 1; this reduces the rates at which particles jump across the bond by a factor r, $0 < r < 1$. For fixed overall density $\rho_{\rm avg}$ and $r <\approx (1-|2\rho_{\rm avg}-1|)/(1+|2\rho_{\rm avg}-1|)$ this causes the system to segregate into two regions with densities ρ_1 and $\rho_2 = 1-\rho_1$, where the densities depend only on r and p, with the two regions separated by a well-defined sharp interface. This corresponds to the shock front described macroscopically in a uniform system by the Burgers equation. We find that fluctuations of the shock position about its average value grow like $L^{1/2}$ or $L^{1/3}$, depending upon whether particle-hole symmetry exists. This corresponds to the growth in time of $t^{1/2}$ and $t^{1/3}$ of the displacement of a shock front from the position predicted by the solution of the Burgers equation in a system without a blockage and provides a new method for studying such fluctuations.

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1 Introduction

The one-dimensional asymmetric simple exclusion process (ASEP) is a continuous-time stochastic process in which particles jump independently and randomly at unit rate with probability p to a vacant neighboring site on the right and with probability $1-p$ to one on the left, $1/2 < p \leq 1$ [1]. It is one of the simplest of the driven diffusive lattice-gas models in that the only interaction between particles is the hard-core exclusion which prevents any site from being occupied by more than one particle. This makes it one of the simplest models for studying the behavior of systems whose dynamics do not satisfy detailed balance and for the derivation of Euler-like hydrodynamical equations from microscopic dynamics, *e.g.* the Burgers equation [2, 3, 4, 5, 6]. This equation reflects the property of physical fluids in that shock fronts, where the macroscopic density is discontinuous, can form between regions of different density, even if the initial distribution is smooth. Since the macroscopic description breaks down at the shock position we must study the structure of the shock directly at the microscopic level. This has been the focus of much recent research [2, 3, 4, 5, 6].

When a shock forms, or if one is present in the initial state, it will move with an average velocity determined by the densities on either side of the shock via the Burgers equation. One of the interesting phenomena found in studying the simple exclusion process is the dependence of the fluctuations about that average on initial conditions—while there are fluctuations due to the stochastic nature of the evolution, these are usually swamped by fluctuations caused by microscopic variations in the initial state [3, 4]. For example, fluctuations of a shock's position grow like $t^{1/3}$ if one compares independent stochastic evolutions with identical initial conditions, but like $t^{1/2}$ (relative to the deterministic hydrodynamic evolution given by the Burgers equation) if one takes into account the random fluctuations in the initial state [5, 6].

In contrast to the time dependent behavior, the stationary states for the system with translation invariance are not very interesting; the stationary states on the circle give equal weight to all permissible configurations with a given number of particles, and their infinite volume limits are uniform product measures with average density ρ, $0 \leq \rho \leq 1$. We can generate more interesting behavior, however, by breaking the translation invariance and partially blocking one bond so that particles which occupy sites on either end of the bond will have different transition probabilities than particles at other sites. This is analogous to a restriction in a pipe containing a fluid flow. This yields, in addition to its own interest as an example of the generic long range coherence present in stationary nonequilibrium systems with particle conserving dynamics [7] (related to self-organized criticality), a new, more robust method for studying the time dependent case.

By studying only the stationary state we eliminate most of the dependence on the initial conditions (only total particle number remains relevant), but the basic nature of the translation invariant model, which produces macroscopic shocks with microscopic structure, remains. The blockage generally introduces a shock into the stationary state, and by studying its fluctuations we can recover in a numerically more accessible way the exponents describing the growth in time of the fluctuations of an initially sharp shock in the infinite system. Specifically, for a system of size L the generic shock fluctuation scales as $L^{1/2}$, but if the average density is 0.5, placing the shock exactly opposite the blockage, we obtain fluctuations in the shock position that scale like $L^{1/3}$. This can be understood in terms of a cancellation that takes place when there is particle-hole symmetry combined with the time-dependent scaling of the infinite system; we also show that by altering the blocking mechanism we can eliminate the need for a cancellation and construct a model that always has $L^{1/3}$ behavior.

2 The Model

Our system consists of particles on a lattice with unit spacing set on a circle of circumference L, moving (randomly) in one direction, with a limit of one particle per site, *i.e.* we consider the totally asymmetric exclusion process, $p = 1$. This limitation seems to have little impact on the nature of our results, as determined by simulations for both types of systems.

We break the translation invariance of this periodic system by inserting a blockage into the system between sites L and 1, which reduces the probability of a particle traveling between those two sites—a "slow bond" which acts as a traffic jam for the particles. In the language of driven diffusive systems, this is similar to altering the driving field at this one bond [8].

More formally, our process is defined by the generator $\mathcal{L}$ giving the rate of change of any function of the configuration $\eta = \{\eta(1), \eta(2), \ldots, \eta(L)\}$, where $\eta(k) = 0$ or 1 is the occupation number at site k:

$$\mathcal{L}f(\eta) = \sum_{i=1}^{L-1} [f(\eta_{i\to i+1}) - f(\eta)] + r[f(\eta_{L\to 1}) - f(\eta)], \tag{1}$$

where

$$\eta_{i\to j}(k) = \begin{cases} 1, & k = j \text{ and } \eta(i) = 1, \\ 0, & k = i \text{ and } \eta(j) = 0, \\ \eta(k), & \text{otherwise;} \end{cases} \tag{2}$$

the rate r, $0 \le r \le 1$, determines the degree of blocking. For $r = 1$ the model is translation invariant; for $r = 0$ the model is fully blocked; the stationary state has density one behind the blockage and density zero in front of it and there is no current flowing through the system. For $0 < r < 1$ the model has nontrivial behavior, resulting from the requirement that the current J_k through any bond $k \to k+1$, *i.e.* the expected number of particles jumping from k to $k+1$ in unit time, must be independent of k.

As already mentioned, one motivation for this study comes from the fact that this is perhaps the simplest model which shows the dramatic global effects caused by a local perturbation in the dynamics in a system with conservative dynamics that does not satisfy a detailed balance condition [7]. For the homogeneous case $r = 1$, the stationary measure $\mu_{\rho_{\text{avg}}}$ for a given system size L and number of particles $N = \rho_{\text{avg}} L$ is one in which all $\binom{L}{N}$ permissible configurations have equal weight. While this is true for all values of $p \in [0, 1]$, it is only for the symmetric case ($p = 1/2$) that the dynamics satisfy detailed balance with respect to $\mu_{\rho_{\text{avg}}}$. A local blockage then has only a local effect.

We recover detailed balance even for $p \neq 1/2$ if we consider $r = 0$, *i.e.* full blockage, since then the current in the stationary state must be zero. The stationary state is then given by

$$P(\eta) = \frac{1}{Z} \prod_k \left(\frac{p}{1-p} \right)^k \eta(k), \tag{3}$$

where Z is a normalization factor such that the sum of $P(\eta)$ over all allowed configurations is one. When $P = 1$ the stationary state has $\eta(k) = 0$ for $k = 1, \ldots, L - N$ and $\eta(k) = 1$ for $k = L - N + 1, \ldots, L$.

Equivalent Growth Model

The ASEP can also be considered as a simple model for surface growth [9]. The surface height is related to the particle occupancy as follows: the presence of a particle in the ASEP is equivalent to a decrease in the height of the surface (by 1), the absence of a particle to an increase (by 1); thus the relative surface height is essentially the integrated particle number:

$$h(x) - h(0) = \sum_{k=1}^{x} (1 - 2\eta(k)). \tag{4}$$

If the total number of particles is $L/2$ the surface is periodic; otherwise we use helical boundary conditions with $h(L) - h(0) = L(1 - 2\rho_{\text{avg}})$.

Since $\eta(k)$ must be zero or one, $(1 - 2\eta(k))$ is ± 1 and the height difference between neighboring sites must have absolute value 1. A particle followed by a hole (a decrease followed by an increase) forms a local minimum in the surface, and a hole followed by a particle (increase/decrease) forms a local maximum. Of course the dynamics is the same: local minima become local maxima with rate p (deposition) as particles move to vacant sites on the right, and the reverse process (evaporation) occurs with rate $1 - p$ as particles move to the left. Our introduction of a blockage is equivalent to altering the deposition/evaporation rates at a single site [10], *i.e.* we consider deposition that is uniform everywhere except at one site where the rate is reduced; this can have a global effect in the growth of the surface.

3 Results

We simulated our model for a large number of different system sizes, blocking rates and average densities. To carry out a simulation, we choose at each time step one site, say site i, at random. If $i \neq L$ and site i is occupied and site $i+1$ is not, then the particle at site i jumps to site $i+1$, otherwise nothing happens. If we do choose site L, then the jump to site 1 takes place with probability r, provided that site L is filled and site 1 is empty.

Qualitatively there is little difference in the behavior of the system as we vary the parameters, provided that $r \neq 0, 1$ (which would correspond to the two trivial cases mentioned earlier) and (for reasons discussed following eq. (8)) that the average density is not too far from 1/2. A typical (time-averaged) density profile is presented in figure 1. As one would expect,

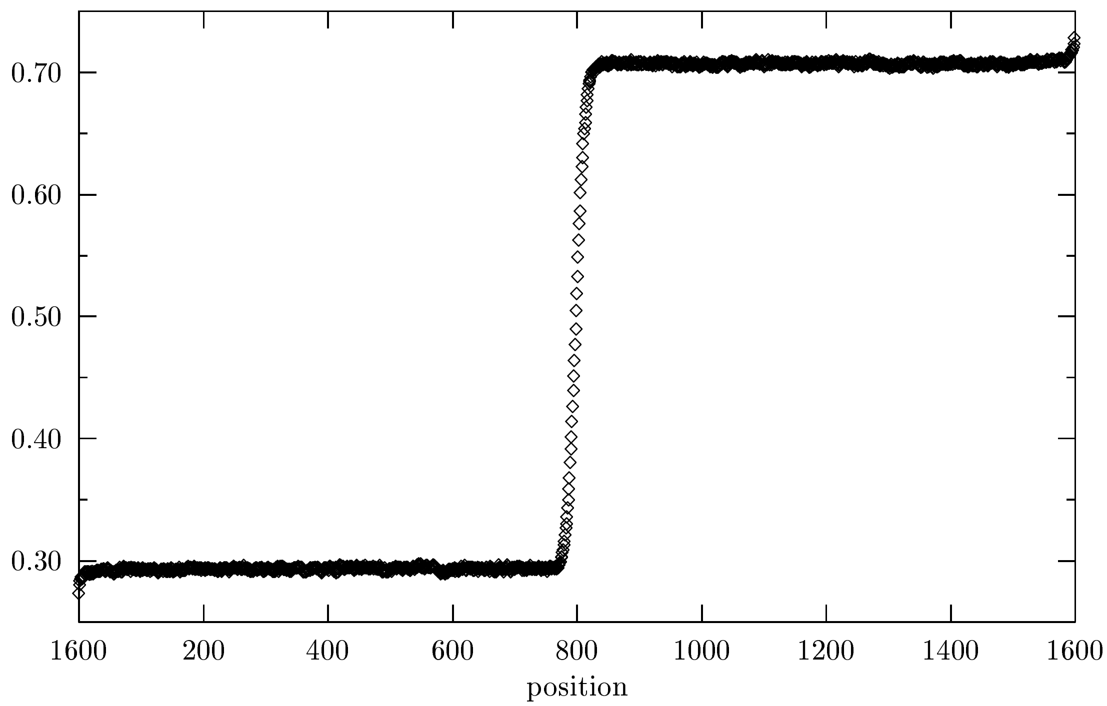

Figure 1: Density profile: 1600 sites, blockage transmission=.35, average density=0.5

particles pile up behind the blockage and are relatively scarce in front of it. Away from the blockage the system appears similar to the infinite model where two uniform phases meet to form a shock [11], while there is a deviation from uniformity near the blockage. Suprisingly, a closer examination of the density near the block indicates that the excess density appears to decay with the distance from the blockage x like $1/|x + \text{const.}|$, as opposed to exponentially. A close-up of the density near one side of the blockage (from a long run with good statistics—10^7 samples) is presented in figure 2. We see the excellent agreement between the data and the function exhibiting the $1/|x|$ behavior.

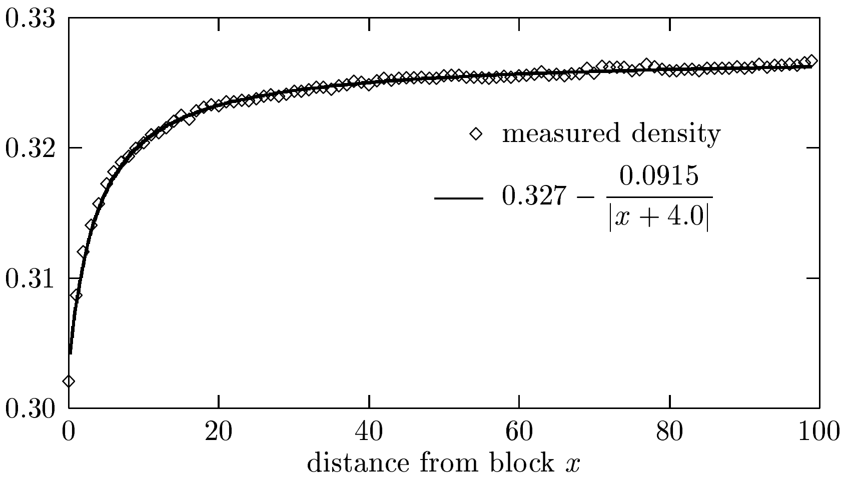

Figure 2: Density profile near the block: 1200 sites, blockage transmission=.40, average density=0.5

Note that the profile is symmetric about density $\rho = 0.5$; the symmetry between the high and low density phases is an important part of the dynamics and the stationary state: instead of moving particles we can just as well consider the situation where when the site i is picked, and it contains a hole (*i.e.* is empty), that hole jumps to the left if the site $i-1$ does not already contain a hole, or jumps with probability r to site L if $i = 1$.

Since the average current through the bond $k \to k+1$ is given by $\langle \eta(k)[1-\eta(k+1)]\rangle$, we see that the current in a state ν_ρ of uniform density ρ without correlations is $\rho(1-\rho)$, so if the measure far from the blockage and the shock is just a product measure, current conservation requires

$$\rho_{\text{low}}(1-\rho_{\text{low}}) = \rho_{\text{high}}(1-\rho_{\text{high}}); \tag{5}$$

equivalently,

$$\rho_{\text{low}} = \rho_{\text{high}} \qquad \text{or} \qquad \rho_{\text{low}} = 1-\rho_{\text{high}}. \tag{6}$$

If the low density current were not equal to the high density current, particle conservation would require that the (infinite system) interface move with velocity $v_{\text{shock}} = 1 - \rho_{\text{low}} - \rho_{\text{high}}$; the requirement that this velocity be zero if the densities are not equal may also be used to derive (6).

Since we have a deviation from uniform product measure near the blockage, the exact values of $\rho_{\text{low}}, \rho_{\text{high}}$ can not be computed directly but must be determined by solving for the stationary state, or lacking that, empirically. We can obtain approximate values for these densities by neglecting correlations between all sites, not just sites far from the blockage. Then, (since the average current must be constant)

$$r\rho_{\mathrm{high}}(1-\rho_{\mathrm{low}}) \approx J_{\mathrm{block}} = J_{\mathrm{bulk}} = \rho_{\mathrm{high}}(1-\rho_{\mathrm{high}}), \tag{7}$$

so that

$$\rho_{\mathrm{high}} \approx \frac{1}{r+1}, \qquad \rho_{\mathrm{low}} \approx \frac{r}{r+1}. \tag{8}$$

For the system of figure 1, this computation yields the result $\rho_{\mathrm{low}} \approx 0.26$; compare it with the actual value $\rho_{\mathrm{low}} = 0.29$. Clearly the system manages to increase the current (and thus ρ_{low}) by making the density non-uniform and introducing correlations near the block. The specifics of how this occurs, *i.e.* the structure of the stationary state near the blocked site, will be the subject of a separate inquiry.

For density different from 1/2, the average shock location is displaced from the center, but the basic structure remains the same—provided that the density is such that the shock would not need to be shifted all the way to (or past) the blockage. In other words, ρ_{low} and ρ_{high} remain the same as before provided that we have a solution to

$$\rho_{\mathrm{avg}} = a_l\rho_{\mathrm{low}} + a_h\rho_{\mathrm{high}}, \qquad a_l + a_h = 1, \tag{9}$$

with $a_l, a_h > 0$. If we use the approximation (8), we must have $|\rho_{\mathrm{avg}} - 1/2| < (1-r)/(2(1+r))$. If this is not the case we no longer have the basic structure of two near-uniform regions meeting at a shock front, with local perturbations near the blockage; instead we have only one approximately uniform phase. Apparently, if the current in the uniform phase is sufficiently small, there is no need for phase segregation to reduce the bulk current to that at the blockage [12]. Similar behavior is observed when we examine an infinite system with one site blocked (see Section 5).

Tracking the Shock

While the average shock location is easy to determine, finding its microscopic position or even defining it precisely at a given time is nontrivial. Instead of determining the shock position directly, we track it through the use of a **second class particle** [13]. The second class particle is an extra particle added to the system, which is treated as a hole in exchanges with particles and as a particle in exchanges with holes. In other words, if we select site i and it is occupied by an ordinary particle, it moves to site $i+1$ either if that site is empty or if it is occupied by the second class particle; in the latter case the second class particle simultaneously jumps (to the left) to site i, *i.e.* there is an exchange between the first and second class particles. If site i is occupied by a second class particle, it moves to site $i+1$ only if site $i+1$ is unoccupied; if site $i+1$ is occupied it stays put. Therefore, if the second class particle is in a high density region (of ordinary particles), it will be forced to the left, while if it is in a low density region it will be able to move to the right.

A shock consists of an abrupt change from a low density region to a high density region. To the left of a shock (a low density region) a second class particle moves predominantly to the right, while on the right side (a high density region) it moves predominantly to the left. The second class particle therefore executes a biased random walk with drift towards the shock position. Simple analysis shows and computer simulations confirm that the inherent fluctuations of this random walk are small compared with the movement of the shock—namely that if we fix the densities on either side of the shock ($\rho_{\text{low}}, \rho_{\text{high}}$) and consider different values of the system size L, the motion of the second class particle about the shock position will have fixed variance while the variance of the shock position itself will grow with system size. Thus analyzing the motion of the second class particle will allow us to indirectly examine the shock motion.

An alternative method of finding the shock is suggested by the surface growth model. Recalling equation (4), we see that surface height increases in low (particle) density regions and decreases in high density regions, so that the shock location should correspond intuitively to the location of the maximum height. Unfortunately this position need not be (microscopically) unique, so we prefer the use of the second class particle. It suggests, however, that in addition to the shock position we study other features from the growth model, such as the difference between the maximum and minimum surface heights, which is equivalent to studying the number of particles on either side of the interface in the ASEP.

As a third possibility we could ignore the microscopic shock position entirely and simply study the width of the time-averaged shock density profile, such as that in figure 1. However, this method obscures the distinction between the instantaneous microscopic width of the shock and the width due to fluctuations in the shock's position (particularly relevant in the extension to two-dimensional models), and the width is difficult to determine accurately because of lattice effects. In any case, we expect that as the system size goes to infinity that rescaling the shock profile by the standard deviation of the shock fluctuation (determined by any of the three methods) should yield a well-defined limiting shape.

We sample the position of the second class particle after allowing the system to reach a steady state. For each system we obtain a distribution (histogram) of shock (second class particle) positions and compute the variance of the obtained distribution; a typical run consisted of approximately 25,000 samples where 15 sweeps ($15 \times L$ monte carlo steps) were made between samples. We then compare the variances obtained from systems with different values of the parameters. We also perform the same set of computations for the number of particles found between site 1 and the second class particle position.

In figure 3 we plot the standard deviation of the interface position *vs.* system size for average density 0.55 and blocking rate $r = 0.35$. The error bars (three symbols are plotted for each measurement: the actual value and that value shifted up or down by the error bound) represent statistical error based on the approximate number of independent samples selected from the steady state in each system. The line drawn through the data points has a slope of $1/2$, so that fluctuations grow like the square root of the system size, indicating standard finite size

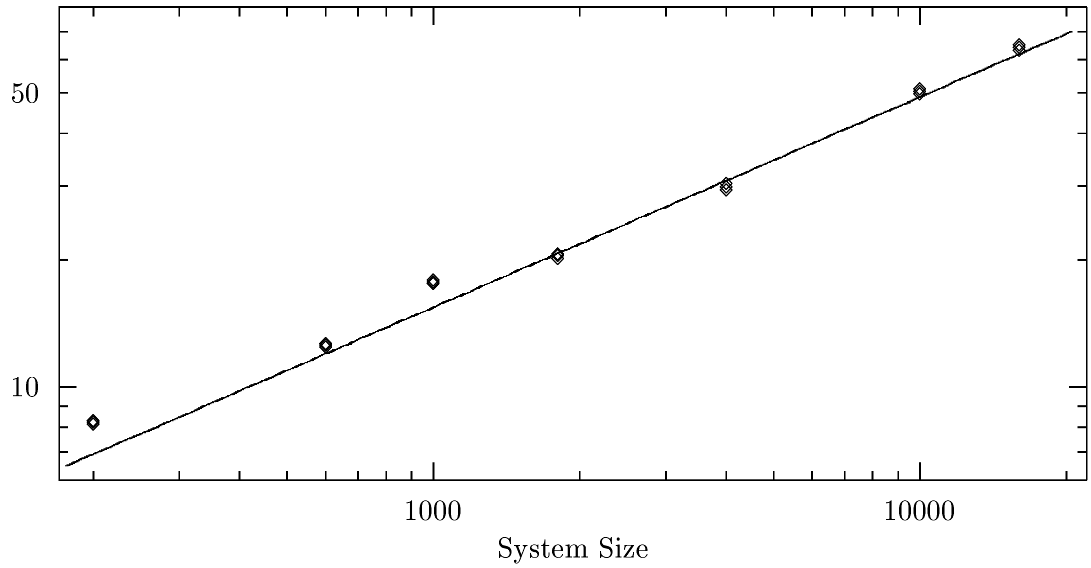

Figure 3: Interface fluctuations with average density 0.55, $r = 0.35$. The solid line has slope 1/2.

behavior.

In figure 4 we plot the same quantities as in figure 3, but the average particle density is 0.5. We examined two sets of systems, with blocking rates $r = 0.35, 0.5$. Here the lines drawn through the data points have slope 1/3, so that fluctuations are suppressed compared with systems whose densities are different from 0.5, yielding growth only like the cube root of the system size. (Note that changing r does not affect this conclusion; the relevant variable is whether or not the density is 0.5.)

The reduction in the fluctuations of the interface when the average density is 1/2 is clearly caused by some type of cancellation since other fluctuations in the system still scale like $L^{1/2}$—in figure 5 we plot the standard deviation of the number of particles found to the left of the interface, *i.e.* between site 1 and the second class particle; this data is consistent with scaling exponent 1/2 (and inconsistent with 1/3). An explanation of this behavior is given in the next section.

4 Analysis

To make sense of our results, it is convenient to decompose the stochastic nature of our system into two pieces: the random flow of particles through the blockage, and the random dynamics everywhere else. We hypothesize that the "blockage randomness" is responsible for the $L^{1/2}$

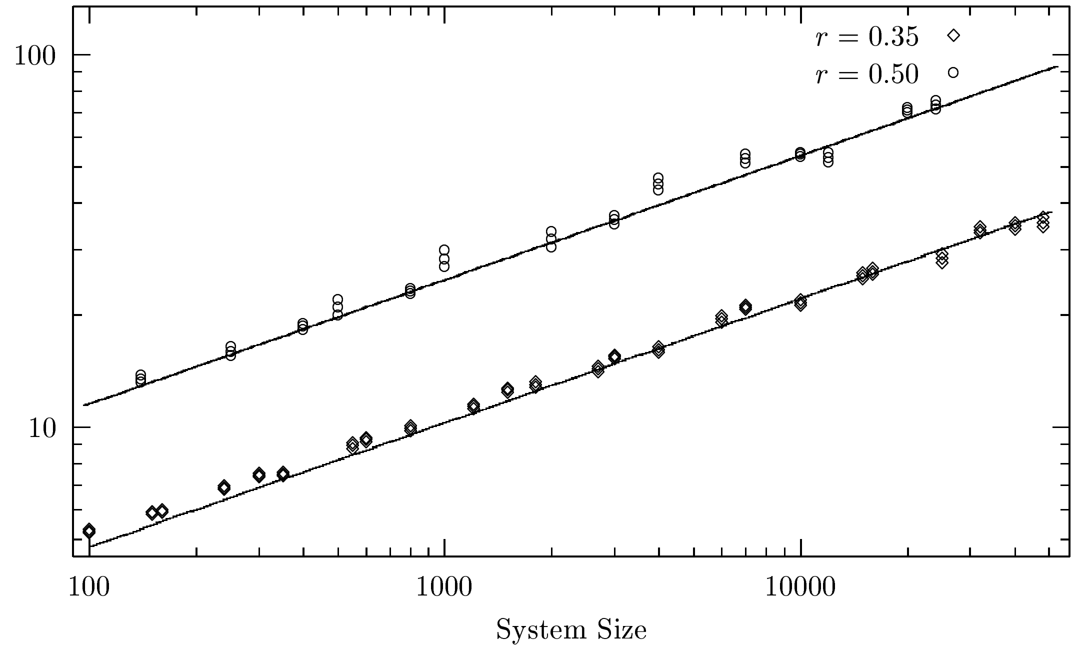

Figure 4: Interface fluctuations with average density 0.5, $r = 0.35, 0.5$. The solid lines have slope 1/3.

behavior and that the "dynamical randomness" is responsible for the $L^{1/3}$ behavior.

A particle traversing the blockage can be thought of as two simultaneous events: a particle being released at site 1 while a hole is released at site L. These two events are of course completely correlated, but the motions of the particle and hole *after* release are nearly independent.

The position of the interface is determined by the *difference* between the number of particles which arrive from the left and the number of holes which come from the right. (It may be convenient for visualization purposes to consider the situation where the density to the right of the shock is very close to one and the density on the left is near zero.) If we neglect the dynamical randomness—*i.e.* if we assume that particles in the low density region move at their average velocity $1 - \rho_{\mathrm{low}}$, and holes in the high density region move at their average velocity $-\rho_{\mathrm{high}} = -(1 - \rho_{\mathrm{low}})$, and consider a shock in the center of the system, then no matter what the nature of the creation of excitations at the blockage, particles and holes will always impinge upon the shock in pairs: when a pair of excitations is created, the hole and particle will travel with equal speeds and opposite directions, meeting (and annihilating) when they have each traveled a distance $L/2$ and reach the shock position. Since the "blockage randomness" plays no role here, a system with density 0.5, and therefore particle-hole symmetry (figure 4), has

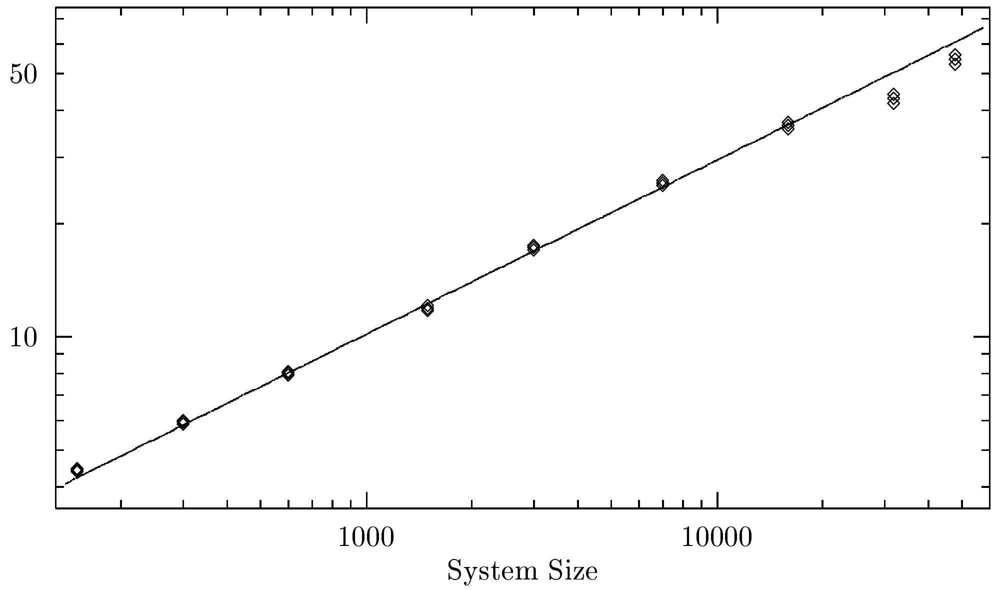

Figure 5: Fluctuations in the number of particles to the left of the interface: average density 0.5, $r = 0.35$. The solid line, which is a best linear fit to the data, has slope 0.46.

fluctuations determined only by the "dynamical randomness"—the deviation of the particle velocities from their average.

On the other hand, if the shock is not in the center of the system, the nature of "blockage randomness" becomes relevant as the times for particles and holes to reach the interface are no longer the same. For example, if the average density is greater than 1/2 the (average) shock position will be less than $L/2$, and particles which start at the blockage and travel to the right will reach the shock before holes (starting at the same time and place) traveling to the left—the arrival times of the particles no longer match those of the holes because the distances they need traverse are different, and the resulting motion of the shock position will depend upon exactly how the excitations are created. So a system with density different from 0.5 (figure 3) and thus with an average shock position shifted from the center will have fluctuations from the "blockage randomness" proportional to the amount of the shift.

The $L^{1/2}$ scaling of the "blockage randomness" results from the rapid decay of correlations in both time and space in the stationary state of the ASEP, as well as the use of an independent random choice for each attempt to cross the blockage. Thus the creation of one excitation is very nearly independent from the creation of another. Therefore the fluctuation in the number of excitations will scale as the square root of the average number. There is a finite density of excitations, so their number is $O(L)$, yielding fluctuations that scale like $O(L^{1/2})$. In fact, if we neglect the dynamical randomness and treat particle flow through the blockage as a Poisson process, we can determine through straightforward analysis the variance of the shock position. Each particle or hole added to the shock moves it by $(\rho_{\rm high}-\rho_{\rm low})^{-1}$. Thus the shock is shifted by an amount $|\rho_{\rm avg}-1/2|L/(\rho_{\rm high}-\rho_{\rm low})$, which is half the difference in path length between particles and holes. Therefore the time differential is $2|\rho_{\rm avg}-1/2|L/\left(\rho_{\rm high}(\rho_{\rm high}-\rho_{\rm low})\right)$. The probability of the formation of an excitation is $\rho_{\rm high}\rho_{\rm low}$ so that the standard deviation of the shock position should be

$$(\rho_{\rm high}-\rho_{\rm low})^{-1}\left(\frac{2|\rho_{\rm avg}-1/2|L}{\rho_{\rm high}(\rho_{\rm high}-\rho_{\rm low})}\rho_{\rm high}\rho_{\rm low}\right)^{1/2} = \left(\frac{2\rho_{\rm low}|\rho_{\rm avg}-1/2|L}{(\rho_{\rm high}-\rho_{\rm low})^3}\right)^{1/2}. \qquad (10)$$

This calculation should be accurate for r near zero but will be too large otherwise, as the actual variance of the particle flow is less than that of a Poisson process since we do have (small) memory effects. For the system of figure 3, the measured behavior is $0.50L^{1/2}$ while eq. (10) yields $0.63L^{1/2}$.

What is the nature of the randomness in the dynamics? Recent work by van Beijeren [6] indicates that density fluctuations grow in time like $t^{1/3}$, while since the distance is of order L and the speed is $O(1)$ the time required for an excitation to reach the interface from the blockage is of order L. Thus an initially nonrandom arrangement (or a random arrangement with number particles = number holes) of excitations will have fluctuations of order $L^{1/3}$ by the time it reaches the interface. We confirm this by examining a system where we alter the behavior near the blockage so that particles pass through the blockage in a near-uniform manner. Of course we can not have a completely uniform current since the occupation numbers of each site are limited to 0 and 1, but we try to make the flow as uniform and non-random as possible: if $n(t)$ is the number of particles already having passed through the blockage at time t, we specify the current exactly by opening the blockage if $n(t) < Jt$ and closing it otherwise. If a particle is immediately to the left of the blockage while it is open and the site to the right is vacant, the particle jumps through the blockage. Thus we attempt to schedule particle jumps at intervals of $1/J$; if a jump is not possible because the appropriate occupancy conditions are not satisfied we "make up" that jump as soon as possible.

By altering the nature of the blockage, we alter the behavior of the shock fluctuations. For this uniform-block model, the lack of "blockage randomness" means that even without particle-hole symmetry, *i.e.* even when the shock is displaced from the center we have fluctuations that scale like $L^{1/3}$ for *both* the interface position *and* the number of particles to the left of the interface—see figure 6. Thus we have (partial) confirmation of our hypothesis.

5 Infinite System with Block

We have been considering particles evolving according to the ($p = 1$) ASEP dynamics on a circle of L sites with a blockage parameterized by r. For each such system with a fixed number

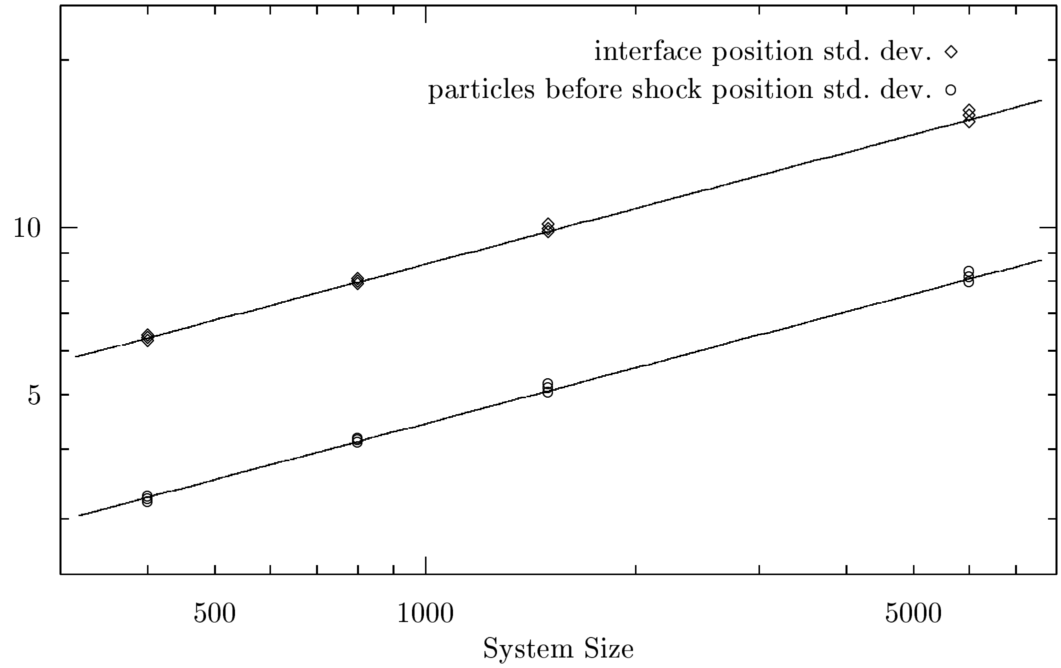

Figure 6: Fluctuations with uniform (non-random) flow through the block. Average density=0.55. Solid lines have slope 1/3.

of particles N (or average density $\rho_{\text{avg}} \equiv N/L$) the stationary state is unique. The question naturally arises, "What happens to these states in the infinite volume limit $L \to \infty$, $N \to \infty$, $N/L \to \rho$ fixed?" To make this question precise we have to specify the position of our domain of observation relative to the blocked site as $L \to \infty$. This is unlike the case in the finite system where except for labeling the state is independent of the position of the blocked bond, since the domain of observation corresponds to all L sites.

Let us label the sites of our system (relative to the observation domain) from $-L/2$ to $L/2$, and put the left end of the blocked bond at K_L. Then if $K_L \to \pm\infty$ as $L \to \infty$ we are in the situation of an infinite uniform system, and the stationary states are superpositions of product states. The product states are of two kinds: for $0 < J \leq 1/4$ they are translation invariant with densities $\rho_{\pm} = \left(1 \pm \sqrt{1-4J}\right)/2$, while for $J = 0$ the density is zero to the left of some site i_0 and one to the right of it.

As $L, K_L \to \infty$, the finite volume current J_L will converge to a limit J and the stationary state μ_s will be a superposition of two product measures ν_{ρ_+} and ν_{ρ_-},

$$\mu_s = \alpha\nu_{\rho_+} + (1-\alpha)\nu_{\rho_-}, \qquad 0 \leq \alpha \leq 1; \tag{11}$$

α depends on the average position of the shock as $L \to \infty$, *i.e.* if the distance to the shock goes to infinity sufficiently fast compared to the fluctuations in the shock position ($L^{1/2}$ or $L^{1/3}$) then α will be zero or one, but it will be non-trivial if the distance between the shock and the origin grows more slowly [11].

The situation is different if K_L remains finite as $L \to \infty$. In this case we can just as well take $K_L = 0$ and the class of stationary states with $J > 0$ will then depend on r (for $J = 0$ we have again the fully blocked states and r is irrelevant).

We believe that it can be shown that for any $J > 0$ the asymptotic distribution to the left and right of the origin is a product measure with densities ρ_{left} and ρ_{right} where either

$$\rho_{\mathrm{left}} = \rho_{\mathrm{right}} = \left(1 \pm \sqrt{1-4J}\right)/2 \tag{12}$$

or

$$\rho_{\mathrm{right}} = \left(1 - \sqrt{1-4J}\right)/2 < \rho_{\mathrm{left}} = \left(1 + \sqrt{1-4J}\right)/2. \tag{13}$$

The problem is then to find the phase diagram in the J–r plane. We have used the results from our finite system studies as well as numerical simulations intended to imitate the infinite situation (we use boundary conditions of fixed density) to obtain the following results:

Away from the blockage the infinite stationary state resembles a product measure, so that the need for the current to be the same on both sides requires (as in equations (5)–(6)) that the densities on either side of the block be equal or be symmetric about 1/2, in full correspondence with (12)–(13).

Let $\rho_{\mathrm{crit}} = \rho_{\mathrm{low}} = 1 - \rho_{\mathrm{high}}$ refer to the lower density of the segregated periodic system with the same blockage transmission r as the infinite system. Then there are three nontrivial classes of extremal stationary states for the infinite system in addition to the zero-current measures (we can also have superposition measures as in (11)):

1. $\rho_{\mathrm{left}} = \rho_{\mathrm{right}} < \rho_{\mathrm{crit}}$,
2. $\rho_{\mathrm{left}} = \rho_{\mathrm{right}} > 1 - \rho_{\mathrm{crit}}$,
3. $1 - \rho_{\mathrm{left}} = \rho_{\mathrm{right}} = \rho_{\mathrm{crit}}$.

We see that we are justified in referring to ρ_{crit} as a critical density; the system cannot be stationary at densities between ρ_{crit} and $1 - \rho_{\mathrm{crit}}$ because the required current is too high. We can illustrate this in figure 7 where the critical line is approximated by the no-correlation formulae (7)–(8).

The first two families of (near) uniform states correspond to the (near) uniform states in the periodic system, when $\rho_{\mathrm{avg}} < \rho_{\mathrm{crit}}$ or $\rho_{\mathrm{avg}} > 1 - \rho_{\mathrm{crit}}$. The nonuniform state corresponds to segregated systems with $\rho_{\mathrm{crit}} < \rho_{\mathrm{avg}} < 1 - \rho_{\mathrm{crit}}$.

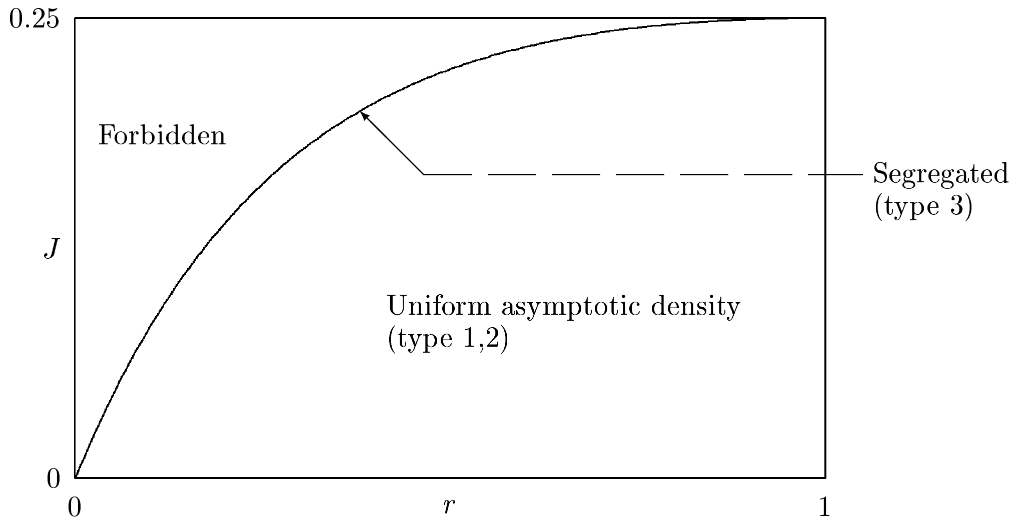

Figure 7: Phase diagram for infinite system with block; current *vs.* blockage rate.

An apparent fourth type of state with $1 - \rho_{\rm right} = \rho_{\rm left} < \rho_{\rm crit}$ is not stationary since the interface is not bound to the block and thus will eventually drift to infinity, even though its average velocity is zero.

Nearly as interesting as the stationary states themselves are their basins of attraction. Choosing as initial conditions $\rho = \rho_{\rm left,0}$ to the left of the blockage and $\rho = \rho_{\rm right,0}$ on the right we found that

- For $\rho_{\rm left,0} < \rho_{\rm crit}$, $\rho_{\rm left,0} < 1 - \rho_{\rm right}$, the system approaches a state from class 1 above with $\rho_{\rm left} = \rho_{\rm right} = \rho_{\rm left,0}$.
- For $\rho_{\rm right,0} > 1 - \rho_{\rm crit}$, $\rho_{\rm right,0} > 1 - \rho_{\rm left}$, the system approaches a state from class 2 above with $\rho_{\rm left} = \rho_{\rm right} = \rho_{\rm right,0}$.
- For $\rho_{\rm left,0} \geq \rho_{\rm crit}$, $\rho_{\rm right,0} \leq 1 - \rho_{\rm crit}$, the system approaches a state from class 3 with $1 - \rho_{\rm left} = \rho_{\rm right} = \rho_{\rm crit}$.

This behavior is presented graphically in figure 8.

In the first two cases the system chooses whichever initial state has the lowest current. In the last case the system forces itself critical whenever the initial current on both sides of the block is greater than that which can be sustained through the block.

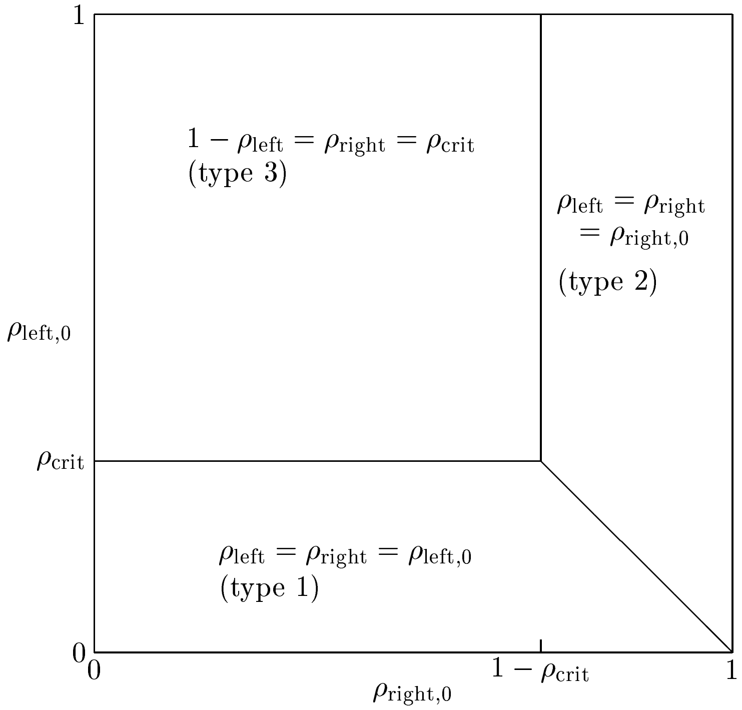

Figure 8: Phase diagram for infinite system with block; the axes represent the initial densities on each side of the block.

6 Conclusion

We have examined a version of the asymmetric simple exclusion process on a finite periodic lattice. Our model has many of the features of the time dependent system but none of the sensitivity to initial conditions, and one can use the presence or absence of particle-hole symmetry to indicate the model's behavior. It thus provides a more convenient platform for the study of the asymmetric simple exclusion process and similar models.

We have used the presence of a "slow bond" as a technique to introduce a shock into the finite system, but the study of such an impurity is of intrinsic interest as well. One can examine the effects of inserting one blockage, or a periodic or random array of blockages, into the infinite system. We have identified some of the gross features of such systems, but much of the determination of the local and global properties remains an interesting open problem.

Our investigations can also be extended to dimension greater than one. Preliminary results [14] indicate the importance of particle-hole symmetry in the two-dimensional system and illustrate the utility of studying interface behavior by forcing the formation of a shock in a stationary state.

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