Preprint n.

Dipartimento di Fisica
Università di Roma "La Sapienza"
I.N.F.N. - Sezione di Roma

FLUCTUATIONS AND THERMODYNAMIC VARIABLES IN MEAN FIELD SPIN GLASS MODELS‡

by

Francesco Guerra
Dipartimento di Fisica, Università di Roma "La Sapienza",
and Istituto Nazionale di Fisica Nucleare, Sezione di Roma,
Piazzale Aldo Moro, 2, I-00185 Roma, Italy.
e-mail guerra@roma1.infn.it

April 1992

ABSTRACT.

We present two rigorous results on the Sherrington-Kirkpatrick mean field model for spin glasses, proven by elementary methods, based on properties of fluctuations, with respect to the external quenched noise, of the thermodynamic variables and order parameters. The first result gives the *uniform* convergence of the quenched average of the free energy in the thermodynamic limit to its annealed approximation, in the high temperature regime, *including* the assumed critical point ($\beta = 1$ in our notations). The second result shows that the free energy can be expressed through a functional order parameter, of the type introduced by Parisi in the frame of the replica symmetry breaking method. The functional order parameter is implicitely given in terms of fluctuations of thermodynamic variables.

‡ Research supported in part by MURST (Italian Minister of University and Scientific and Technological Research) and INFN (Italian National Institute for Nuclear Physics).

1. INTRODUCTION.

The Sherrington-Kirkpatrick mean field model [1] for spin glasses is defined through the Hamiltonian

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N-1}} \sum_{(i,j)} J_{ij}\sigma_i\sigma_j, \tag{1.1}$$

where $\sigma_1, \dots, \sigma_N$ are Ising spins, with values ± 1, coupled in a highly disordered way through the quenched variables J_{ij}, assumed to be independent random variables with identical unit Gaussian distribution (for the sake of simplicity). The sum runs over all the $N(N-1)/2$ couples (i,j) of spins.

Due to its relevant physical interest as a prototype for disordered complex systems, this model has been the subject of intensive investigation in the last years [2]. By now, its general structure is well understood, at least from a qualitative point of view, and ingeniously described by the Parisi solution [3], originally obtained through the replica symmetry breaking method, and further confirmed by the cavity method [4]. According to the accepted view, the model is trivial in the high temperature regime ($\beta \leq 1$), where the thermodynamic variables coincide with their annealed approximation, in the thermodynamic limit $N \to \infty$. On the other hand, at low temperatures ($\beta > 1$), the model exhibits an extremely rich structure of approximate equilibrium states, separated by high barriers, and organized in a hierarchical system. As a consequence, the thermodynamic variables are expressed through a functional order parameter, introduced by Parisi [5], replacing the numerical order parameter of the approximate Sherrington-Kirkpatrick solution [1], obtained without replica symmetry breaking and violating positivity of the entropy.

Rigorous results, by Aizenman, Lebowitz and Ruelle [6], give a very detailed description of the fluctuations of thermodynamic variables and their limiting behavior, as $N \to \infty$, in the high temperature regime ($0 \leq \beta < 1$) (see also [7]), and interesting estimates on the free energy at low temperature.

On the other hand, Pastur and Scherbina [8] have given a rigorous proof of the damping of the mean square fluctuations of the free energy, in the thermodynamic limit, at *all* fixed temperatures. They have also shown that the approximate Sherrington-Kirkpatrick solution, surely defective at high β, follows necessarily from the assumption that a suitably defined order parameter, connected with the induced magnetization in an external random field, has vanishing mean square fluctuations in the thermodynamic limit (see also [9]). This result strongly supports Parisi theory.

The study of fluctuations is clearly of great importance for this model. In this paper, we give a rigorous proof of two results, based on elementary properties of fluctuations with respect to the external quenched noise J. The first result gives the *uniform* convergence of the quenched average of the free energy to its annealed approximation, in the high temperature regime, *including* the point $\beta = 1$, which is the critical point, according to the accepted picture. The second result is based on the evaluation of the corrections necessary to go from the quenched average to the annealed average, where the variables J participate to the thermodynamic equilibrium. These corrections are expressed through fluctuations of appropriate thermodynamic variables. A simple consequence is the emergency of the functional order parameter and the antiparabolic martingale equation of Parisi theory [3]. Therefore, we show that the free energy can be expressed through the functional order parameter, exactly as in Parisi theory. For further developments along this line we refer to [10].

The paper is organized as follows. In Section 2, we give a detailed description of the model and some elementary properties of the thermodynamic variables. The results on the high temperature behavior are given in Section 3, where we exploit only simple convexity properties and positivity of fluctuations. In Section 4 we introduce the marginal martingale method, equivalent to the cavity method. It is a very powerful technique for the study of the thermodynamic limit in mean field models.

The functional order parameter and the antiparabolic martingale equation are introduced in Section 5. They allow a very simple expression of the thermodynamic variables of the system. The functional order parameter can be easily written in terms of fluctuations, which take into account the corrections necessary to go from quenched averages to annealed averages.

Finally, Section 6 is devoted to conclusions and outlook for future developments.

2. THE MODEL. ELEMENTARY PROPERTIES.

We consider N sites, $i = 1, \ldots, N$. In the thermodynamic limit we will let $N \to \infty$. The configurations of the system are given by

$$\sigma: \quad \{1, 2, \ldots, N\} \ni i \to \sigma_i \in Z_2 = \{-1, 1\}, \tag{2.1}$$

where σ_i are Ising spins. For each of the $N(N-1)$ couples of sites (i, j), $i \neq j$, we introduce independent random variables $J_{ij} = J_{ji}$, $i \neq j$, identically distributed , called quenched variables. The σ's are mesoscopic random variables subject to thermodynamic equilibrium. The J's do not participate to thermodynamic equilibrium, but act as a kind of random environment on the σ's. For the sake of simplicity, we assume that the J's have unit Gaussian distribution with

$$E(J_{ij}) = 0, \quad E(J_{ij}^2) = 1, \tag{2.2}$$

where E denotes averages with respect to the J variables.

The Hamiltonian of the model is given by

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N-1}} \sum_{(i,j)} J_{ij} \sigma_i \sigma_j, \tag{2.3}$$

where the sum extends over all the $N(N-1)$ distinct couples. The square root is necessary in order to assure the right thermodynamic behavior for the extensive variables, as it will be clear in the following. We write it in the form $\sqrt{N-1}$, equivalent to the customary one given by $\sqrt{N}$. Introducing the inverse temperature β (in proper units), we can write the *Boltzmannfaktor* as $\exp(-\beta H_N(\sigma, J))$, and define the partition function $Z_N(\beta, J)$ and the free energy $F_N(\beta, J)$ through

$$Z_N(\beta, J) = \sum_{\sigma_1 \ldots \sigma_N} \exp(-\beta H_N(\sigma, J)) = \exp(-\beta F_N(\beta, J)). \tag{2.4}$$

The associated Boltzmann state $\omega_{N,\beta,J}$ is given by

$$\omega_{N,\beta,J}(A) = Z_N(\beta,J)^{-1} \sum_{\sigma_1 \dots \sigma_N} A \exp(-\beta H_N(\sigma,J)), \tag{2.5}$$

where A is a generic function of the σ's. Its density is

$$\rho_N(\sigma;\beta,J) = Z_N(\beta,J)^{-1} \exp(-\beta H_N(\sigma,J)). \tag{2.6}$$

The internal energy $U_N(\beta,J)$ is

$$U_N(\beta,J) = \omega_{N,\beta,J}(H_N(\sigma,J)) = -\partial_\beta \log Z_N(\beta,J) = \partial_\beta\big(\beta F_N(\beta,J)\big). \tag{2.7}$$

The entropy $S_N(\beta,J)$ is

$$S_N(\beta,J) = -\sum_{\sigma_1 \dots \sigma_N} (\rho_N \log \rho_N)(\sigma;\beta,J), \tag{2.8}$$

with the usual bounds

$$0 \le S_N(\beta,J) \le N \log 2. \tag{2.9}$$

These variables are connected through the second principle of thermodynamics

$$F_N(\beta,J) = U_N(\beta,J) - \beta^{-1} S_N(\beta,J). \tag{2.10}$$

All thermodynamic variables are random with respect to the external noise J. However, we can also introduce the free energy per site $f_N(\beta,J)$, and correspondently the internal energy per site $u_N(\beta,J)$, and the entropy per site $s_N(\beta,J)$, such that

$$\begin{gathered} f_N(\beta,J) = N^{-1}F_N(\beta,J), \quad u_N(\beta,J) = N^{-1}U_N(\beta,J), \quad s_N(\beta,J) = N^{-1}S_N(\beta,J), \\ 0 \le s_N(\beta,J) \le \log 2, \quad f_N(\beta,J) = u_N(\beta,J) - \beta^{-1}s_N(\beta,J). \end{gathered} \tag{2.11}$$

On physical grounds, it is expected that these variables have a good thermodynamic behavior in the limit when $N \to \infty$, in the sense that they should converge almost surely, with respect to the external J noise, *i.e.* with the possible exclusion of a set of J samples of zero measure in the J probability space,

$$f_N(\beta,J) \to f_N(\beta), \quad u_N(\beta,J) \to u_N(\beta), \quad s_N(\beta,J) \to s_N(\beta). \tag{2.12}$$

Alternatively, one can consider also convergence of the quenched averages with respect the external noise

$$E\big(f_N(\beta,J)\big) \to f_N(\beta), \quad E\big(u_N(\beta,J)\big) \to u_N(\beta), \quad E\big(s_N(\beta,J)\big) \to s_N(\beta). \tag{2.13}$$

For the sake of convenience, let us define the quenched average

$$\alpha_N(\beta) = N^{-1}E\big(\log Z_N(\beta,J)\big) = -\beta E\big(f_N(\beta,J)\big). \tag{2.14}$$

Clearly, $\alpha_N(\beta)$ is convex and increasing in β, for $\beta \geq 0$. Then we have the following elementary estimates (see for example [6]).

Proposition 1. For the previously defined $\alpha_N(\beta)$ the following bounds hold, uniformly in N,

$$\begin{aligned} \log 2 \leq \alpha_N(\beta) &\leq \log 2 + \beta^2/4, \quad \text{for } \beta \leq \bar{\beta} = 2\sqrt{\log 2} = 1.665\ldots, \\ &\leq \beta\sqrt{\log 2}, \qquad \text{for } \beta \geq \bar{\beta}. \end{aligned} \tag{2.15}$$

The proof (see for example [6]) is based on the following simple, but important, inequality, relating quenched averages with annealed averages

$$E\big(\log Z_N(\beta, J)\big) \leq \log E\big(Z_N(\beta, J)\big). \tag{2.16}$$

In the annealed average the J's participate to the thermodynamic equilibrium. A simple calculation shows

$$\begin{aligned} E(Z_N(\beta, J)) &= \sum_{\sigma_1\ldots\sigma_N} \prod_{(i,j)} E\big(\exp(\frac{\beta}{\sqrt{N-1}} J_{ij}\sigma_i\sigma_j)\big) \\ &= 2^N (\exp \frac{\beta^2}{2(N-1)})^{\frac{N(N-1)}{2}} = (2\exp\frac{\beta^2}{4})^N. \end{aligned} \tag{2.17}$$

Therefore, for any β

$$\alpha_N(\beta) \leq \log 2 + \beta^2/4. \tag{2.18}$$

On the other hand, the lower bound in (2.15) is trivial, since convexity gives

$$Z_N(\beta, J) = \sum_{\sigma_1\ldots\sigma_N} \exp(-\beta H_N(\sigma, J)) \geq 2^N \exp(-\frac{\beta}{2^N} \sum_{\sigma_1\ldots\sigma_N} H_N(\sigma, J)) = 2^N. \tag{2.19}$$

As shown in [6], a simple argument, based on thermodynamic stability and the bound (2.18), gives the upper bound in (2.15) for $\beta \geq \bar{\beta}$. In fact, introduce the function

$$\phi(\beta) = \log 2 + \beta^2/4, \tag{2.20}$$

and define $\bar{\beta}$ such that, with $\phi'(\beta) = (\partial_\beta \phi)(\beta)$,

$$\phi(\bar{\beta}) - \bar{\beta}\phi'(\bar{\beta}). \tag{2.21}$$

Therefore

$$\bar{\beta} = 2\sqrt{\log 2}, \quad \phi(\bar{\beta}) = 2\log 2, \quad \phi'(\bar{\beta}) = \sqrt{\log 2}. \tag{2.22}$$

Assume by absurd that there exists a point $\beta > \bar{\beta}$ where the upper bound is violated, *i.e.*

$$\alpha_N(\beta) > \beta\sqrt{\log 2}, \quad \beta > \bar{\beta}. \tag{2.23}$$

Notice that by convexity of α_N we would have at this point

$$\alpha'_N(\beta) \geq (\beta-\bar\beta)^{-1}\big(\alpha_N(\beta)-\alpha_N(\bar\beta)\big) > (\beta-\bar\beta)^{-1}(\beta\sqrt{\log 2}-2\log 2) = \sqrt{\log 2}. \quad (2.24)$$

On the other hand, the average entropy is given by

$$E\big(s_N(\beta,J)\big) = \alpha_N(\beta) - \beta\alpha'_N(\beta), \quad (2.25)$$

as a consequence of (2.7,10,11,14). Therefore, by exploiting again convexity and the bounds (2.23) in β and (2.18) in $\bar\beta$, we would get

$$\begin{aligned} E\big(s_N(\beta,J)\big) &\leq \alpha_N(\beta) - \beta(\beta-\bar\beta)^{-1}\big(\alpha_N(\beta)-\alpha_N(\bar\beta)\big) \\ &= -\bar\beta(\beta-\bar\beta)^{-1}\alpha_N(\beta) + \beta(\beta-\bar\beta)^{-1}\alpha_N(\bar\beta) \\ &< -\bar\beta(\beta-\bar\beta)^{-1}\beta\sqrt{\log 2} + \beta(\beta-\bar\beta)^{-1}2\log 2 = 0, \end{aligned} \quad (2.26)$$

which is impossible.

Of course, the bound given by Proposition 1 extends to the limit $N\to\infty$.

Proposition 2. In the thermodynamic limit we have

$$\begin{aligned} \limsup_{N\to\infty}\alpha_N(\beta) &\leq \log 2 + \beta^2/4, \quad \text{for } \beta \leq \bar\beta = 2\sqrt{\log 2} = 1.665\ldots, \\ &\leq \beta\sqrt{\log 2}, \qquad \text{for } \beta \geq \bar\beta. \end{aligned} \quad (2.27)$$

As remarked in [6], these bounds show a phase transition for some $\beta_c \leq \bar\beta = 2\sqrt{\log 2}$, in the sense that annealing of the noise J gives wrong results in the limit $N\to\infty$. On the other hand, next Theorem 3 shows that annealing is correct in the thermodynamic limit for any $\beta \leq 1$. Therefore, the transition must occur at some critical point β_c, such that $1 \leq \beta_c \leq 2\sqrt{\log 2}$. On physical grounds (see for example [2]), it is expected that $\beta_c = 1$. This is also confirmed by the buildig up of strong fluctuations in the $N\to\infty$ limit of global thermodynamic variables, as $\beta \uparrow 1$, shown in the beautyful and detailed analysis of Aizenman, Lebowitz and Ruelle [6], based on graph expansions.

3. THERMODYNAMIC LIMIT IN THE HIGH TEMPERATURE REGIME.

Let us introduce the important order parameter $M_N^2(\beta)$ (see for example [2],[6],[8])

$$M_N^2(\beta) = \frac{2}{N(N-1)} \sum_{(i,j)} E\big(\omega_{N,\beta,J}^2(\sigma_i\sigma_j)\big), \quad 0 \le M_N^2(\beta) \le 1, \tag{3.1}$$

and write the average internal energy as

$$-E\big(u_N(\beta,J)\big) = \alpha_N'(\beta) = \frac{\beta}{2}\big(1 - M_N^2(\beta)\big). \tag{3.2}$$

This comes from a simple integration by parts on the external Gaussian noise, expressed in the form, reminiscent of the Wick theorem in quantum field theory,

$$E\big(J_{ij}F(J)\big) = E\big(\frac{\partial}{\partial J_{ij}}F(J)\big), \tag{3.3}$$

for any smooth function F of the noise.

In fact, we can write, with obvious shorthand notations,

$$\begin{aligned}
-E\big(u_N(\beta,J)\big) &= -N^{-1}E\big(\omega_N(H_N)\big)\\
&= (N\sqrt{N-1})^{-1}\sum_{(i,j)} E\big(J_{ij}\omega_N(\sigma_i\sigma_j)\big)\\
&= \beta N^{-1}(N-1)^{-1}\sum_{(i,j)} E\big(\omega_N(\sigma_i\sigma_j,\sigma_i\sigma_j)\big)\\
&= \beta N^{-1}(N-1)^{-1}\sum_{(i,j)} \Big(1 - E\big(\omega_N^2(\sigma_i\sigma_j)\big)\Big).
\end{aligned} \tag{3.4}$$

We have exploited the general expression

$$\frac{\partial}{\partial J_{ij}}\omega_N(A) = (\beta/\sqrt{N-1})\omega_N(A,\sigma_i\sigma_j), \tag{3.5}$$

where ω_N is the Boltzmann state $\omega_{N,\beta,J}$, and $\omega(A,B)$ denotes truncation, *i.e.* $\omega(A,B) = \omega(AB) - \omega(A)\omega(B)$.

Then we can state the following

Theorem 3. With the definitions (2.14) and (3.1) we have

$$\lim_{N\to\infty} \alpha_N(\beta) = \log 2 + \beta^2/4, \tag{3.6}$$

$$\lim_{N\to\infty} M_N^2(\beta) = 0, \tag{3.7}$$

uniformly for $0 \le \beta \le 1$.

Here we give a streamlined proof, which exploits only elementary thermodynamic properties and positivity of fluctuations. We find convenient to split the proof in a long series of simple statements. It can be also understood as an elementary application of the cavity method.

Let us define

$$\alpha_N(\beta,t) = N^{-1}E\big(\log \sum_{\sigma_1\ldots\sigma_N} \exp(\frac{\beta}{\sqrt{N-1}}\sum_{(i,j)} J_{ij}\sigma_i\sigma_j)\exp(\frac{t}{\sqrt{N}}\sum_i J_i\sigma_i)\big), \tag{3.8}$$

where we have introduced an additional parameter t, $t \geq 0$, and an additional fresh noise J_i, $i = 1, 2, \ldots, N$, with the same properties of the noise J_{ij}. Notice that $\alpha_N(\beta, 0) = \alpha_N(\beta)$, as defined in (2.14). Clearly, $\alpha_N(\beta,t)$ is convex and increasing in β and t, separately. For the sake of simplicity, we call ω_t the state associated to the new *Boltzmannfaktor*. When $t = 0$, ω_t reduces to the even Boltzmann state ω_N.

Let us introduce the order parameters

$$M_N^2(\beta,t) = \frac{2}{N(N-1)}\sum_{(i,j)} E\big(\omega_t^2(\sigma_i\sigma_j)\big), \quad 0 \leq M_N^2(\beta,t) \leq 1,$$
$$\overline{M}_N^2(\beta,t) = N^{-1}\sum_i E\big(\omega_t^2(\sigma_i)\big), \quad 0 \leq \overline{M}_N^2(\beta,t) \leq 1, \tag{3.9}$$

such that

$$\partial_\beta \alpha_N(\beta,t) = \frac{\beta}{2}\big(1 - M_N^2(\beta,t)\big),$$
$$\partial_t \alpha_N(\beta,t) = \frac{t}{N}\big(1 - \overline{M}_N^2(\beta,t)\big). \tag{3.10}$$

Notice that $M_N^2(\beta,0) = M_N^2(\beta)$ as in (3.1), while $\overline{M}_N^2(\beta) = 0$, since ω_N is even. In a sense, the parameter t gives a smooth interpolation between a system with N sites, at $t = 0$, and a system with $N+1$ sites, at $t = \sqrt{N/(N-1)}$, with a small change in the temperature. The parameter t denotes the strength of the coupling between the original N spins and an additional $(N+1)$th spin added, with J_i interpreted as $J_{i,N+1}$, $i = 1, 2, \ldots, N$. A precise statement is given by

Lemma 4. The following equalities hold

$$M_N^2(\beta\sqrt{(N-1)/N},\beta) = \overline{M}_N^2(\beta\sqrt{(N-1)/N},\beta) = M_{N+1}^2(\beta),$$
$$(N+1)\alpha_{N+1}(\beta) = \log 2 + N\alpha_N(\beta\sqrt{(N-1)/N},\beta). \tag{3.11}$$

For the proof let us notice that we have $M_N^2(\beta,t) = E\big(\omega_t^2(\sigma_1\sigma_2)\big)$, $\overline{M}_N^2(\beta,t) = E\big(\omega_t^2(\sigma_1)\big)$, $M_{N+1}^2(\beta) = E\big(\omega_{N+1}^2(\sigma_1\sigma_{N+1})\big) = E\big(\omega_{N+1}^2(\sigma_1\sigma_2)\big)$, due to the complete symmetry among the J's. In fact, for each of the alternatives in $(\ldots,\ldots,\ldots)$, we can write

$$\sum_{\sigma_1\ldots\sigma_N}(1,\sigma_1,\sigma_1\sigma_2)\exp(\frac{\beta}{\sqrt{N}}\sum_{(i,j)}^{1\ldots N} J_{ij}\sigma_i\sigma_j)\exp(\frac{\beta}{\sqrt{N}}\sum_i^N J_i\sigma_i) =$$

$$= \frac{1}{2} \sum_{\sigma_1 \dots \sigma_N \sigma_{N+1}} \dots = \frac{1}{2} \sum_{\sigma_1 \dots \sigma_{N+1}} (1, \sigma_1 \sigma_{N+1}, \sigma_1 \sigma_2) \exp\left(\frac{\beta}{\sqrt{N}} \sum_{(i,j)}^{1 \dots N+1} J_{ij} \sigma_i \sigma_j\right), \tag{3.12}$$

where we have added the dummy summation variable σ_{N+1} in the first equality, and performed the change of variables $\sigma_i \to \sigma_i \sigma_{N+1}$, $i = 1, 2, \dots, N$, in the second equality.

In the following theorem we exploit the mild dependence of $\alpha_N(\beta, t)$ on t, as a consequence of the $1/N$ factor in the t derivative in (3.10), and simple convexity estimates, in order to have bounds on $M_N^2(\beta, t)$, uniform in t.

Theorem 5. Assume $\beta > \Delta\beta > 0$, then we have, uniformly for $0 \le t \le \tilde{\beta}$,

$$M_N^2(\beta + \Delta\beta) - \frac{\Delta\beta}{\beta} - \frac{\tilde{\beta}^2}{N\beta\Delta\beta} \le M_N^2(\beta, t) \le M_N^2(\beta - \Delta\beta) + \frac{\Delta\beta}{\beta} + \frac{\tilde{\beta}^2}{N\beta\Delta\beta}. \tag{3.13}$$

For the proof, let us notice that (3.10) gives

$$0 \le \partial_t \alpha_N(\beta, t) \le \frac{t}{N}, \quad 0 \le \alpha_N(\beta, t) - \alpha_N(\beta) \le \frac{t^2}{2N}. \tag{3.14}$$

Therefore, by convexity we can write

$$\begin{aligned} \Delta\beta\ \alpha'_N(\beta, t) &\ge \alpha_N(\beta, t) - \alpha_N(\beta - \Delta\beta, t) \\ &\ge \alpha_N(\beta) - \alpha_N(\beta - \Delta\beta) - \frac{t^2}{2N} \\ &\ge \Delta\beta\ \alpha'_N(\beta - \Delta\beta) - \frac{\tilde{\beta}^2}{2N}. \end{aligned} \tag{3.15}$$

¿From this and (3.10) we have

$$\alpha'_N(\beta, t) \ge \alpha'_N(\beta - \Delta\beta) - \frac{\tilde{\beta}^2}{2N\Delta\beta},$$

$$\frac{\beta}{2}\big(1 - M_N^2(\beta, t)\big) \ge \frac{\beta - \Delta\beta}{2}\big(1 - M_N^2(\beta - \Delta\beta)\big) - \frac{\tilde{\beta}^2}{2N\Delta\beta}. \tag{3.16}$$

We conclude

$$M_N^2(\beta, t) \le \left(1 - \frac{\Delta\beta}{\beta}\right) M_N^2(\beta - \Delta\beta) + \frac{\Delta\beta}{\beta} + \frac{\tilde{\beta}^2}{N\beta\Delta\beta}. \tag{3.17}$$

This gives the upper bound in the theorem, and the lower bound is proven analogously.

Through a straigthforward calculation and integration by parts, we can now prove the following basic theorem, which gives the t derivative of the order parameter $\overline{M}_N^2(\beta, t)$ in terms of the order parameter $M_N^2(\beta, t)$ and some fluctuations of variables of the theory (see also [8]).

Theorem 6. The following equality holds

$$\partial_{t^2}\Big(N^{-1}\sum_i E\big(\omega_t^2(\sigma_i)\big)\Big) = N^{-2}\sum_{i,j} E\big(\omega_t^2(\sigma_i\sigma_j)\big)$$
$$-E\Big(\big(N^{-1}\sum_i \omega_t^2(\sigma_i)\big)^2\Big) - 4E\big(\omega_t(A,A)\big), \tag{3.18}$$

where A is the variable

$$A = N^{-1}\sum_i \sigma_i\omega_t(\sigma_i). \tag{3.19}$$

For the proof, let us evaluate

$$\partial_t\Big(N^{-1}\sum_i E\big(\omega_t^2(\sigma_i)\big)\Big) = 2N^{-1}\sum_i E\big(\omega_t(\sigma_i)\partial_t\omega_t(\sigma_i)\big) =$$

$$= 2(N\sqrt{N})^{-1}\sum_{i,j} E\big(J_j\omega_t(\sigma_i)\omega_t(\sigma_i,\sigma_j)\big) =$$

$$= 2tN^{-2}\sum_{i,j} E\big(\omega_t(\sigma_i,\sigma_j) + \omega_t(\sigma_i)\omega_t(\sigma_i,\sigma_j,\sigma_j)\big), \tag{3.20}$$

where we have exploited

$$\partial_t\omega_t(\sigma_i) = \frac{1}{\sqrt{N}}\sum_j J_j\omega_t(\sigma_i,\sigma_j), \quad \frac{\partial}{\partial J_j}\omega_t(\sigma_i) = \frac{t}{\sqrt{N}}\omega_t(\sigma_i,\sigma_j),$$

$$\frac{\partial}{\partial J_j}\omega_t(\sigma_i,\sigma_j) = \frac{t}{\sqrt{N}}\omega_t(\sigma_i,\sigma_j,\sigma_j). \tag{3.21}$$

Now we consider the identities

$$\omega_t(\sigma_i,\sigma_j) = \omega_t(\sigma_i\sigma_j) - \omega_t(\sigma_i)\omega_t(\sigma_j),$$
$$\omega_t(\sigma_i,\sigma_j,\sigma_j) = -2\omega_t(\sigma_i,\sigma_j)\omega_t(\sigma_j). \tag{3.22}$$

By collecting all terms, we can write (3.20) in the form

$$\partial_{t^2}\Big(N^{-1}\sum_i E\big(\omega_t^2(\sigma_i)\big)\Big) = N^{-2}\sum_{i,j} E\big(\omega_t^2(\sigma_i\sigma_j)\omega_t^2(\sigma_i)\omega_t^2(\sigma_j) - 4\omega_t(\sigma_i)\omega_t(\sigma_j)\omega_t(\sigma_i,\sigma_j)\big), \tag{3.23}$$

and the theorem is proven.

Now we exploit the obvious positivity inequalities

$$\omega_t(A,A)\geq 0, \quad E\Big(\big(N^{-1}\sum_i \omega_t^2(\sigma_i)\big)^2\Big) \geq E\Big(\big(N^{-1}\sum_i \omega_t^2(\sigma_i)\big)\Big)^2, \tag{3.24}$$

and the inequality

$$N^{-2}\sum_{i,j} E\big(\omega_t^2(\sigma_i\sigma_j)\big) = N(N-1)N^{-2}E\big(\omega_t^2(\sigma_1\sigma_2)\big) + \frac{1}{N}$$
$$= (1-\frac{1}{N})M_N^2(\beta,t) + \frac{1}{N} \leq M_N^2(\beta,t) + \frac{1}{N}, \tag{3.25}$$

together with the definitions (3.9), in order to derive

Theorem 7. The following inequalities hold

$$\partial_{t^2}\overline{M}_N^2(\beta,t) + \overline{M}_N^4(\beta,t) \leq M_N^2(\beta,t) + \frac{1}{N}, \quad \text{for any } t,$$
$$\leq \widetilde{M}_N^2(\beta,\tilde{\beta}), \quad \text{uniformly for } 0 \leq t \leq \tilde{\beta}, \tag{3.26}$$

where

$$\widetilde{M}_N^2(\beta,\tilde{\beta}) = \max_{0\leq t\leq\tilde{\beta}} M_N^2(\beta,t) + \frac{1}{N}, \tag{3.27}$$

$$\overline{M}_N^2(\beta,t) \leq \widetilde{M}_N(\beta,\tilde{\beta}) \tanh\big(t^2\widetilde{M}_N(\beta,\tilde{\beta})\big), \quad \text{uniformly for } 0 \leq t \leq \tilde{\beta}. \tag{3.28}$$

In fact, (3.26) follows from (3.18), (3.24), (3.25) and the definition (3.27), while (3.28) follows from a simple integration.

Let us now put $\beta\sqrt{(N-1)/N}$ in place of β in (3.28), and then $\tilde{\beta} = \beta$, and $t = \beta$. We get

$$M_N^2(\beta\sqrt{(N-1)/N},\beta) = \overline{M}_N^2(\beta\sqrt{(N-1)/N},\beta)$$
$$\leq \widetilde{M}_N(\beta\sqrt{(N-1)/N},\beta) \tanh\big(t^2\widetilde{M}_N(\beta\sqrt{(N-1)/N},\beta)\big). \tag{3.29}$$

Now we are ready for the proof of the following key theorem.

Theorem 8. Define

$$M_N^2 = \max_{0\leq\beta\leq 1} M_N^2(\beta), \quad M_N^2 \leq 1, \tag{3.30}$$

then for large N

$$M_N^4 \leq 3\left(\frac{e^2+1}{e^2-1}\right)^2 (\frac{4}{\sqrt{N-1}} + \frac{1}{N}), \tag{3.31}$$

hence

$$\lim_{N\to\infty} M_N^2 = 0. \tag{3.32}$$

For the proof, let us notice that the definitions (3.27) and (3.30), and the uniform upper bound (3.13) of Theorem 5 give for $\beta \leq 1$

$$\widetilde{M}_N^2(\beta\sqrt{(N-1)/N},\beta) \leq M_N^2(\beta\sqrt{(N-1)/N} - \Delta\beta) + \frac{\Delta\beta}{\beta}\frac{\sqrt{N}}{\sqrt{N-1}} + \frac{\beta}{\Delta\beta}\frac{1}{N}\frac{\sqrt{N}}{\sqrt{N-1}} + \frac{1}{N}$$
$$\leq M_N^2 + \frac{1}{\sqrt{N-1}}(\frac{\Delta\beta\sqrt{N}}{\beta} + \frac{\beta}{\Delta\beta\sqrt{N}}) + \frac{1}{N}. \tag{3.33}$$

It is convenient to put $\Delta\beta = \beta/\sqrt{N}$, so that

$$\widetilde{M}_N^2(\beta\sqrt{(N-1)/N}, \beta) \leq M_N^2 + \frac{2}{\sqrt{N-1}} + \frac{1}{N}. \tag{3.34}$$

But in the same conditions we derive from the lower bound (3.13)

$$\widetilde{M}_N^2(\beta\sqrt{(N-1)/N}, \beta) \geq M_N^2\big(\beta(\sqrt{(N-1)/N} + \frac{1}{\sqrt{N}})\big) - \frac{2}{\sqrt{N-1}}. \tag{3.35}$$

Therefore, from (3.29), (3.34) and (3.35) we derive

$$\begin{aligned} M_N^2\big(\frac{\beta}{\sqrt{N}}(1+\sqrt{N-1})\big) &\leq \frac{2}{\sqrt{N-1}} + M'_N \tanh(\beta^2 M'_N), \\ M'_N &= \sqrt{M_N^2 + \frac{2}{\sqrt{N-1}} + \frac{1}{N}}. \end{aligned} \tag{3.36}$$

Notice that $(1+\sqrt{N-1})/\sqrt{N} > 1$, therefore, if we take the maximum in (3.36), for $0 \leq \beta \leq 1$, we immediately have the following important bound

$$M'^2_N \leq \frac{4}{\sqrt{N-1}} + \frac{1}{N} + M'_N \tanh(\beta^2 M'_N), \tag{3.37}$$

where we added $2/\sqrt{N-1} + 1/N$ to both members for convenience. Since the function $x(x - \tanh x)$ is strictly increasing for $x \geq 0$, and $M_N \leq M'_N$, we have also,

$$M_N(M_N - \tanh M_N) \leq \frac{4}{\sqrt{N-1}} + \frac{1}{N}, \tag{3.38}$$

which shows that $M_N \to 0$ as $N \to \infty$.

In order to have an estimate on the rate of convergence, we exploit the following inequality, holding for $0 \leq x \leq 1$,

$$x(x - \tanh x) \geq x^4 (\tanh 1)^2/3, \tag{3.39}$$

and the theorem follows.

Finally, we can prove Theorem 3. In fact, from $M_N^2(\beta) \leq M_N$, for $0 \leq \beta \leq 1$, we have the uniform convergence in (3.7). On the other hand, we also have

$$\begin{aligned} &\alpha'_N(\beta) = \frac{\beta}{2}\big(1 - M_N^2(\beta)\big) \geq \frac{\beta}{2}(1 - M_N^2), \\ &\alpha_N(\beta) \geq \log 2 + \beta^2(1 - M_N^2)/4, \end{aligned} \tag{3.40}$$

and the uniform convergence in (3.6) follows.

In conclusion, we see that the validity of the annealed approximation holds uniformly in $0 \leq \beta \leq 1$, in the infinite volume limit, as a simple consequence of thermodynamic stability and positivity of the mean square fluctuations.

MARGINAL MARTINGALE AND CAVITY METHODS.

Let ω be a generic even state on the Ising variables $\sigma_1, \dots, \sigma_N$, possibly depending on a stale noise J_{ij}. Introduce the marginal martingale function $\psi_N(\omega, t)$ defined by

$$\psi_N(\omega, t) = E \log \omega(\exp \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i), \tag{4.1}$$

where J_i are a fresh noise, as in (3.8). Since ω is even, we can substitute the cosh in place of the exp in (4.1).

We have the bounds given by

Theorem 9. For any t, the following bounds hold uniformly in ω

$$\int \log\cosh(tz)\ d\mu(z) \le \psi_N(\omega, t) \le N \int \log\cosh(tz/\sqrt{N})\ d\mu(z), \tag{4.2}$$

where $d\mu(z) = \exp(-z^2/2)\ dz/\sqrt{2\pi}$ is the unit Gaussian distribution. The upper bound is realized if ω in (4.1) is taken as the symmetric product state $\omega^{(0)}$, where all configurations of the σ's have the same probability 2^{-N}. This is the case if ω is the Boltzmann state ω_N at $\beta = 0$. The lower bound is realized if ω in (4.1) is a state $\overline{\omega}$ which gives equal weigths $\frac{1}{2}$ to some fixed configurations $\bar{\sigma_1}, \dots, \bar{\sigma_N}$ and its inverted one $-\bar{\sigma}_1, \dots, -\bar{\sigma}_N$, and zero to all other configurations.

Proof. First of all let us show that the bounds are in fact equalities for some states. For the upper bound let ω in (4.1) be the symmetric product state $\omega^{(0)}$. In this case we have

$$\log \omega^{(0)}(\exp \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i) = \sum_i \log \omega^{(0)}(\exp \frac{t}{\sqrt{N}} J_i \sigma_i) = \sum_i \log\cosh(\frac{t}{\sqrt{N}} J_i). \tag{4.3}$$

By taking the average E and exploiting the symmetry in the J's, we have the upper bound in (4.2) as an equality. For the lower bound, let ω be the state $\overline{\omega}$ which gives probability $\frac{1}{2}$ to each of the two configurations $\sigma_i = 1$, and $\sigma_i = -1$ for all i's, for example. In this case we have

$$\overline{\omega}(\exp \frac{t}{\sqrt{N}} J_i \sigma_i) = \cosh(\frac{t}{\sqrt{N}} J_i). \tag{4.4}$$

By taking the E average we get the lower bound in (4.2) as an equality.

Let us now prove the bounds for a generic state. Introduce the additional Ising variables $\epsilon_1, \dots, \epsilon_N$, with symmetric distribution, and let E' denote the related average. Since the J's have symmetric distributions, by annealing in E', we have

$$\begin{aligned} E \log \omega(\exp \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i) &= E E' \log \omega(\exp \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i \epsilon_i) \\ &\le E \log \omega E'(\exp \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i \epsilon_i) \\ &= \sum_i \log\cosh(\frac{t}{\sqrt{N}} J_i), \end{aligned} \tag{4.5}$$

where we have freely exchanged E' and ω. Therefore, we are reduced to (4.3), and the upper bound follows. Obviously, we could anneal completely the J variables and get the slightly weaker bound uniform in N

$$E \log \omega(\exp \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i) \leq \frac{1}{2} t^2. \tag{4.6}$$

For the lower bound, it is enough to write

$$E \log \omega(\cosh \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i) \geq \omega(E \log \cosh \frac{t}{\sqrt{N}} \sum_i J_i \sigma_i), \tag{4.7}$$

by quenching also the variables σ, in some sense. In the E average the variables σ do not play any role, and the lower bound follows.

Let us remark that the upper bound in (4.2) is realized for the state $\omega^{(0)}$ of maximum entropy $N \log 2$, while the lower bound corresponds to states $\overline{\omega}$ of minimum entropy $\log 2$, compatible with eveness. It would be interesting to find the appropriate bounds if ω has some given fixed entropy.

The interest in the introduction of the marginal martingale relies on the possibility to derive information on the thermodynamic variables from information on $\psi_N(\omega, t)$, for some properly chosen state. Let us remark that ψ_N and related quantities play also a central role in the deep study of fluctuations made by Pastur and Scherbina [8,9]. We refer to [10] for an extensive treatment of the marginal martingale. Here we give only some typical results.

Let us define the particular marginal martingale $\psi_N(\beta)$, associated to the mean field spin glass model, by replacing the generic state ω in (4.1) with the Boltzmann state ω'_N at temperature $\beta\sqrt{(N-1)/N}$, and putting $t = \beta$,

$$\psi_N(\beta) = \psi_N(\omega'_N, t) = N\big(\alpha_{N+1}(\beta\sqrt{\frac{N-1}{N}}, \beta) - \alpha_N(\beta\sqrt{\frac{N-1}{N}})\big). \tag{4.8}$$

¿From (3.11) we have

$$(N+1)\alpha_{N+1}(\beta) = \log 2 + N\alpha_N(\beta\sqrt{\frac{N-1}{N}}) + \psi_N(\beta). \tag{4.9}$$

We see that $\psi_N(\beta)$ connects the free energy for a system with $N+1$ particles with the free energy for a system with N particles, with a small change in the temperature. This is a typical feature of the cavity method (see for example [4]).

The following theorem shows how information on the limiting behavior of $\psi_N(\beta)$ translates into information on the behavior of the thermodynamic variables.

Theorem 10. Assume that a lower bound of the type

$$\psi_N(\beta) \geq \overline{\psi}_N(\beta) \tag{4.10}$$

holds uniformly for $0 \le \beta \le \tilde\beta$, $N \ge K$. Then we have

$$\liminf_{N\to\infty} \alpha_N(\beta) \ge \log 2 + \int_0^1 \overline{\psi}(\beta\sqrt{1-q})\ dq, \tag{4.11}$$

for $0 \le \beta \le \tilde\beta$. An analogous statement holds for an upper bound.
Assume that the following limit exists

$$\lim_{N\to\infty} \psi_N(\beta) = \psi(\beta), \tag{4.12}$$

uniformly on a compact region $0 \le \beta \le \tilde\beta$, with $\psi(\beta)$ continuous in β, as a consequence. Let us define

$$\begin{aligned} \alpha(\beta) &= \log 2 + \int_0^1 \psi(\beta\sqrt{1-q})\ dq \\ &= \log 2 + \beta^{-2} \int_0^{\beta^2} \psi(\beta')\ d\beta'^2, \end{aligned} \tag{4.13}$$

so that the β derivative $\alpha'(\beta)$ exists and the following holds

$$\alpha(\beta) + \beta\alpha'(\beta)/2 = \log 2 + \psi(\beta). \tag{4.14}$$

Then we have, for $0 \le \beta \le \tilde\beta$,

$$\lim_{N\to\infty} \alpha_N(\beta) = \alpha(\beta),\ \lim_{N\to\infty} \alpha'_N(\beta) = \alpha'(\beta),\ \lim_{N\to\infty} \big((N+1)\alpha_{N+1}(\beta) - N\alpha_N(\beta)\big) = \alpha(\beta). \tag{4.15}$$

Moreover, $\psi(\beta)$ and $\psi(\beta)/\beta$ are increasing in β, and

$$\psi(\beta) = \beta^2/2, \tag{4.16}$$

in the fluid region $0 \le \beta \le 1$.

We give only a sketch of the proof, by referring to [10] for complete details. Let us iterate (4.9) $N-K$ times

$$\begin{aligned} (N+1)\alpha_{N+1}(\beta) = (N-K+1)\log 2 + K\alpha_K(\beta\sqrt{\frac{K-1}{N}}) \\ + \psi_N(\beta) + \psi_{N-1}(\beta\sqrt{\frac{N-1}{N}}) + \ldots + \psi_K(\beta\sqrt{\frac{K}{N}}). \end{aligned} \tag{4.17}$$

Therefore, by assuming the uniform bound (4.10), we have

$$\alpha_{N+1}(\beta) \ge \frac{N-K+1}{N+1} + \frac{K}{N+1}\alpha_K(\beta\sqrt{\frac{K-1}{N}}) + \frac{1}{N+1}\sum_{K'=K}^{N} \overline{\psi}(\beta\sqrt{\frac{K'}{N}}), \tag{4.18}$$

and derive (4.11) in the limit $N \to \infty$, where the discrete variable K'/N becomes the continuous $1-q$. The other statements follow by standard arguments.

A simple application can be obtained by combining the lower bound in (4.2) with (4.11).

Theorem 11. The following lower bound holds

$$\liminf_{N\to\infty} \alpha_N(\beta) \geq \log 2 + \int_0^1 dq \int \log\cosh(\beta z\sqrt{1-q})\ d\mu(z). \tag{4.19}$$

This bound is a mild improvement over the analogous one found in [6] through a clever choice of trial states in the variational principle for the free energy. In fact, since $\log\cosh x$ is an even convex function of x, we have

$$\int \log\cosh(\beta z\sqrt{1-q})\ d\mu(z) \geq \log\cosh(\beta\sqrt{1-q}\int |z|\ d\mu(z)) = \log\cosh(\beta\sqrt{1-q}\sqrt{2/\pi}), \tag{4.20}$$

and the bound in [6] follows from (4.19).

5. THE FUNCTIONAL ORDER PARAMETER.

While the approximate solution of the model, given by Kirkpatrick and Sherrington in [1], involves a numerical order parameter, function of the temperature, it was soon realized by Parisi that the complete solution [3,5] must be expressed through a functional order parameter x, which for a given temperature depends on an auxiliary variable q, both x and q taking values on the interval $[0,1]$. This was suggested by numerical computations and a deep physical intuition about the rich structure of phases of the system, in the infinite volume limit, for $\beta > 1$.

Here we prove that any marginal martingale can be expressed through a functional order parameter of Parisi type, also in the infinite volume limit $N \to \infty$. Moreover, we show that there is a very large set, in the convex space of functional order parameters, which gives equivalent results in the expression of the marginal martingale. This kind of gauge freedom in the choice of the functional order parameter raises the problem of picking the most convenient representation, according to some criterion. Parisi choice [2,3,5], fully discussed in [10] in the general frame introduced here, is very natural, because it gives simple physical interpretations for the parameters x and q, and very useful for the applications. But, in principle, other choices are also possible.

The main result of this Section is given by the following

Theorem 12. For a given even state ω, on the Ising variables $\sigma_1,\ldots,\sigma_N$, possibly depending on some stale noise, let us introduce, as in (4.1), the marginal martingale

$$\psi(\beta) = E\log\omega(\cosh\frac{\beta}{\sqrt{N}}\sum_i J_i\sigma_i). \tag{5.1}$$

Then, for each ω and β, there exists a functional order parameter

$$x: \quad [0,1] \ni q \to x(q) \in [0,1], \tag{5.2}$$

such that

$$\psi(\beta) = f(0,0), \tag{5.3}$$

where $f(q,y)$, $0 \le q \le 1$, $y \in R$, is the solution of the nonlinear antiparabolic equation

$$(\partial_q f)(q,y) + \frac{1}{2}\big(f''(q,y) + x(q){f'}^2(q,y)\big) = 0, \tag{5.4}$$

with final condition

$$f(1,y) = \log\cosh(\beta y). \tag{5.5}$$

In (5.4), $f' = \partial_y f$ and $f'' = \partial_y^2 f$. A possible expression of x is implicitely given by

$$\big(1 - x(q)\big)E\big(\widetilde{\omega}(\tilde{f}'^2)\big) = N^{-1}\sum_i E\big(\widetilde{\omega}^2(\sigma_i \tilde{f}')\big), \tag{5.6}$$

where

$$\tilde{f} = f(q, \sqrt{q}\frac{1}{\sqrt{N}}\sum_i J_i\sigma_i), \quad \tilde{f}' = f'(q, \sqrt{q}\frac{1}{\sqrt{N}}\sum_i J_i\sigma_i), \tag{5.7}$$

and $\widetilde{\omega}$ is the state

$$\widetilde{\omega}(A) = \omega(A\exp\tilde{f})/\omega(\exp\tilde{f}). \tag{5.8}$$

The proof is very simple, and can be understood as a consistent correction to successive annealing of the J variables [10]. It also shows how naturally the equation (5.4) arises from integration by parts. For a generic $f(q,y)$ let us introduce the function ϕ defined by

$$\begin{aligned}\phi: \quad [0,1] \ni q \to \phi(q) &= E\log\omega\big(\exp f(q, \sqrt{q}\frac{1}{\sqrt{N}}\sum_i J_i\sigma_i)\big)\\ &= E\log\omega(\exp\tilde{f}).\end{aligned} \tag{5.9}$$

Notice the typical Brownian scaling $\sqrt{q}$ in the y variable.

Assume for f the boundary condition (5.5). Then we have

$$\phi(1) = \psi(\beta), \quad \phi(0) = f(0,0). \tag{5.10}$$

Let us now calculate the derivative of $\phi(q)$. We have

$$\frac{d}{dq}\exp\tilde{f} = (\partial_q\tilde{f})\exp\tilde{f} + \frac{1}{2\sqrt{q}}\frac{1}{\sqrt{N}}\sum_i J_i\sigma_i\tilde{f}'\exp\tilde{f}. \tag{5.11}$$

Therefore

$$\frac{d}{dq}\phi(q) = E\big(\widetilde{\omega}(\partial_q\tilde{f})\big) + \frac{1}{2\sqrt{q}}\frac{1}{\sqrt{N}}\sum_i E\big(J_i\widetilde{\omega}(\sigma_i\tilde{f}')\big). \tag{5.12}$$

Now we integrate by parts on the fresh noise J_i, by using

$$\frac{\partial}{\partial J_i}\exp\tilde{f} = \sqrt{\frac{q}{N}}\sigma_i\tilde{f}'\exp\tilde{f}, \quad \frac{\partial}{\partial J_i}\tilde{f}' = \sqrt{\frac{q}{N}}\sigma_i\tilde{f}''. \tag{5.13}$$

By collecting all terms, we have

$$\frac{d}{dq}\phi(q) = E\big(\widetilde{\omega}(\partial_q \tilde{f}) + \frac{1}{2}\tilde{f}'' + \frac{1}{2}\tilde{f}'^2)\big) - \frac{1}{2}\frac{1}{N}\sum_i E\big(\widetilde{\omega}^2(\sigma_i \tilde{f}')\big). \tag{5.14}$$

Therefore, if f is chosen so to satisfy (5.4) with x given by (5.6), we see immediately that $\phi(q)$ does not depend on q, and (5.3) follows from (5.10). Finally, let us notice that the inequalities

$$0 \le \widetilde{\omega}^2(\sigma_i \tilde{f}') \le \widetilde{\omega}(\tilde{f}'^2) \tag{5.15}$$

imply

$$0 \le x(q) \le 1, \tag{5.16}$$

and the theorem is fully proven.

We refer to [10] for a detailed discussion about the properties of the solution $f(q, y; x)$ of the antiparabolic equation (5.4), with final condition (5.5), as a functional of a generic given x, as in (5.2). Here we only state the following

Theorem 13. The function f is monotone in x, in the sense that $x(q) \le \bar{x}(q)$, for all $0 \le q \le 1$, implies $f(q, y; x) \le f(q, y; \bar{x})$, for any $0 \le q \le 1$, $y \in R$. Moreover f is continuous in the $L^1(dq)$ norm. In fact, for generic x, $\bar{x}$, we have

$$|f(q, y; x) - f(q, y; \bar{x})| \le \frac{\beta^2}{2}\int_q^1 |x(q') - \bar{x}(q')|\, dq'. \tag{5.17}$$

Let us now introduce the extremal order parameters $x_0(q) \equiv 0$ and $x_1(q) \equiv 1$, such that for any x we have $x_0(q) \le x \le x_1(q)$. It is simple to solve (5.4), (5.5) in these cases in the form

$$f_0(q, y) = \int \log\cosh\big(\beta(y + z\sqrt{1-q})\big)\, d\mu(z), \quad f_1(q, y) = \log\cosh(\beta y) + \beta^2(1-q)/2. \tag{5.18}$$

Therefore, we have for the f associated to a generic x the bounds

$$f_0(q, y) \le f(q, y) \le f_1(q, y), \tag{5.19}$$

and in particular at the point $q = 0$, $y = 0$,

$$f_0(0, 0) \equiv \int \log\cosh(\beta z)\, d\mu(z) \le f(0, 0) \le (\beta^2/2) \equiv f_1(0, 0). \tag{5.20}$$

We recognize that these bounds are the same as found in (4.2) and (4.6), written for $t = \beta$. This remark leads to far reaching consequences.

Theorem 14 Let $\psi(\beta)$ be a marginal martingale for N particles, or a limit for $N \to \infty$, therefore satisfying the bounds

$$\int \log\cosh(\beta z)\, d\mu(z) \le \psi(\beta) \le \beta^2/2. \tag{5.21}$$

Let x_ϵ be a generic family of functional order parameters depending continuously in the L^1 norm on the variable ϵ, $0 \leq \epsilon \leq 1$, with $x_0 \equiv 0$, and $x_1 \equiv 1$, and nondecreasing in ϵ. Then, there exists an $\epsilon(\beta)$ such that

$$\psi(\beta) = f(0,0;x_{\epsilon(\beta)}), \tag{5.22}$$

where f is defined by (5.4), (5.5) with x replaced by $x_{\epsilon(\beta)}$.

The proof follows easily from the *a priori* bounds and continuity. An easy way to construct families of x_ϵ is the following. Let $\bar{x}$ be some fixed functional order parameter. Introduce x_ϵ, for $0 \leq \epsilon \leq 1$, defined by

$$x_\epsilon(q) = [\bar{x}(q) - 1 + 2\epsilon], \tag{5.23}$$

where $[\ldots]$ denotes truncation outside of the interval $[0,1]$, *i.e.* $[y]$ takes the value 0 if $y \leq 0$, y if $0 \leq y \leq 1$, and 1 if $y \geq 1$.

In conclusion, we have

Theorem 15. If $\psi(\beta)$ is a marginal martingale, then for each β there exists a nonempty hypersurface Σ_β of functional order parameters such that

$$\psi(\beta) = f(0,0;x), \qquad \text{for any} \quad x \in \Sigma_\beta. \tag{5.24}$$

6. CONCLUSIONS AND OUTLOOK.

We have seen that many important properties of the mean field spin glass model can be derived by elementary methods, based on the fluctuations with respect to the external noise.

In particular, we have shown that the positivity of the mean square deviations alone forces the annealed approximation, for the free energy, to be correct in the thermodynamic limit, in the high temperature fluid region, up to the critical point.

In general, the corrections to annealing can be easily expressed through a functional order parameter, of the type introduced by Parisi in his ingenious discoveries about the physical properties of the spin glass system.

The methods exploited in this paper can be easily extended to other mean field disordered theories, with variables more general that the Ising variables, for example continuous ones.

Finally, we refer to [10] for a more complete description of the martingale method and its applications.

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