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Abundance of Translation Invariant Pure States on Quantum Spin Chains

M. Fannes[1,2], B. Nachtergaele[3], and R.F. Werner[4]

fgbda20@blekul11.bitnet   bxn@math.princeton.edu   reinwer@dosuni1.bitnet

Abstract
We construct a set of translation invariant pure states of a quantum spin chain, which is w*-dense in the set of all translation invariant states of the chain. Each of the approximating states has exponential decay of correlations, and is the unique ground state of a finite range Hamiltonian with a spectral gap above the ground state energy.

1 Inst. Theor. Fysica, Universiteit Leuven, B-3001 Leuven, Belgium
2 Bevoegdverklaard Navorser, N.F.W.O. Belgium
3 Dept. of Physics, Princeton University, NJ-08544-708, USA;
on leave from Universiteit Leuven, B-3001 Leuven, Belgium
4 Fachbereich Physik, Universität Osnabrück, Pf. 4469, Osnabrück, Germany

It has long been noted that quantum statistical mechanics is in many ways more difficult than statistical mechanics based on a classical microscopic theory. For example, the thermodynamics of one-dimensional classical spin systems can be solved completely using the transfer matrix technique, whereas in the quantum case even very basic questions about the ground states of systems with nearest neighbour interactions remain unanswered. The main difficulty in the study of quantum spin chains is perhaps constructing translation invariant states with given local expectations. The fundamental problem with such constructions is the fact that families of quantum states defined on overlapping subintervals of the chain fail to have common extensions to the whole chain algebra [**9**], whereas for classical chains common extensions always exist. Consequently, a global state of minimal energy density cannot be found by varying the state on local subalgebras. Similar problems arise in the classical case only for dimensions two and higher, where Bell-type inequalities associated to cycles in the lattice [**8**] constitute obstructions to such state extensions, also known as "frustration". In this note we demonstrate a new property of states on one-dimensional quantum systems, which highlights another aspect of the higher complexity of these systems in comparison with classical ones.

To be specific we shall consider a system of "spins" localized on the sites of the one-dimensional lattice $\mathbb{Z}$, each of which is described by the same observable algebra $\mathcal{A}_i \equiv \mathcal{A}$, which we take as the algebra $\mathcal{M}_d$ of $d\times d$-matrices. For a finite subset $\Lambda \subset \mathbb{Z}$ the observable algebra is $\bigotimes_{i\in\Lambda} \mathcal{A}_i$, and for the entire chain the observable algebra is the inductive limit of these algebras, i.e. the norm closure $\mathcal{A}_{\mathbb{Z}}$ of $\bigcup_N \mathcal{A}_{[-N,N]}$. Clearly, there is an action $(\tau_x)_{x\in\mathbb{Z}}$ of the lattice translations by automorphisms of $\mathcal{A}_{\mathbb{Z}}$. For convenience we will often write τ instead of τ_1. By $\mathcal{S}$, or $\mathcal{S}(\mathcal{A}_{\mathbb{Z}})$ for emphasis, we will denote the set of states on $\mathcal{A}_{\mathbb{Z}}$. This is a compact convex set with the w*-topology. By definition, a net of states $\omega_\alpha \in \mathcal{S}$ converges in this topology to $\omega \in \mathcal{S}$ if $\omega_\alpha(A) \to \omega(A)$ for all $A \in \mathcal{A}_{\mathbb{Z}}$. The translation invariant states will be denoted by $\mathcal{T} = \{\omega \in \mathcal{S} \,|\, \omega\circ\tau = \omega\}$. This is a w*-closed subset of $\mathcal{S}$. For any convex set we will denote by $\partial_e\mathcal{K}$ its extreme boundary, i.e. the points $\omega \in \mathcal{K}$ which cannot be represented as a convex combination $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ with $0 < \lambda < 1$ and $\omega_1 \neq \omega_2$. As usual, the elements of $\partial_e\mathcal{S}$ (resp. $\partial_e\mathcal{T}$) will be referred to as pure (resp. ergodic) states.

It is well-known that $\mathcal{T}$ is a (metrizable) simplex ([**3**], Cor. 4.3.11), i.e. that every $\omega \in \mathcal{T}$ has a unique representation as the barycenter of a measure supported by $\partial_e\mathcal{T}$. It also has the peculiar property, first found by Poulsen in a different example [**7**], that its extreme boundary $\partial_e\mathcal{T}$ is dense in $\mathcal{T}$ (cf. Example 4.3.26 in [**3**]). This property uniquely characterizes $\mathcal{T}$ among metrizable simplices, up to affine homeomorphism [**6**]. Infinite dimensional convex sets with dense extreme boundary are quite common. The state space $\mathcal{S}$ also has this property (cf. Example 4.1.31 in [**3**]). Hence it is clear that every translation invariant state can be approximated by pure states. It is, however, not at all clear whether these approximating states can be chosen to be translation invariant, as well. To show this is the main content of this note.

Theorem. *$\partial_e\mathcal{S}\cap\mathcal{T}$ is w*-dense in $\mathcal{T}$.*

This result concerns not so much the intrinsic geometry of $\mathcal{T}$, but rather the position of $\mathcal{T}$ inside $\mathcal{S}$. In order to get an intuitive picture of its geometrical meaning, note that for finite dimensional convex subsets $\mathcal{T}_0\subset\mathcal{S}$, $\partial_e\mathcal{S}\cap\mathcal{T}_0$ is usually a very small, even empty subset of $\partial_e\mathcal{T}$. If, on the other hand, $\partial_e\mathcal{S}\cap\mathcal{T}_0$ generates $\mathcal{T}_0$, the extreme points of $\mathcal{T}$ must "fit" into suitable "corners" of $\mathcal{S}$. Closer to the system at hand is the case of a classical spin system, i.e. the case of a chain $\mathcal{A}_{\mathbb{Z}}$ where $\mathcal{A}=\mathcal{C}(X)$ is a commutative algebra. The pure states in $\partial_e\mathcal{S}$ are a closed subset of the simplex $\mathcal{S}$, which is homeomorphic to $X^{\mathbb{Z}}$. The points in $X^{\mathbb{Z}}$ are best seen as paths paths $i\in\mathbb{Z}\mapsto x_i\in X$, and the only translation invariant pure states are those for which the path is constant. Hence in the classical case the translation invariant pure states precisely correspond to the single site configurations, i.e. $\partial_e\mathcal{S}\cap\mathcal{T}\cong\partial_e\mathcal{S}(\mathcal{A})$. Clearly, this set is not dense in $\mathcal{T}$. The quantum analogues of the clasical states in $\partial_e\mathcal{S}\cap\mathcal{T}$ are infinite tensor products of a fixed pure state on the one-site algebra $\mathcal{A}$. Again these special states are clearly not dense, and our result shows that $\partial_e\mathcal{S}\cap\mathcal{T}$ contains many non-product states. Such states must have another remarkable property: since their restrictions to local algebras cannot be pure (this would imply the product property), it is possible to give proper convex decompositions of these local restrictions. However, none of these decompositions can be extended to a decomposition of the global state.

The basic construction that we use in this note is the following generalization of the so-called Valence-Bond-Solid states [**2**], which was first given in [**4**] and is based on an earlier proposal for the construction of Quantum Markov States in [**1**]. It is, in fact, the only construction (apart from quasi-free CAR-states) that we know of for obtaining non-product pure translation invariant states.

A state ω of $\mathcal{A}_{\mathbb{Z}}$ is completely determined by its expectation values of observables of the form $\omega(A_m\otimes\cdots A_n)$, $m\le n\in\mathbb{Z}$ and $A_i\in\mathcal{A}$ for $m\le i\le n$. In order to define a translation invariant state of $\mathcal{A}_{\mathbb{Z}}$, i.e. a state ω satisfying $\omega\circ\tau=\omega$, it is sufficient to give a prescription for computing $\omega(A_1\otimes\cdots A_n)$ for all $n\ge 1$, and $A_i\in\mathcal{A}$. Such a prescription must be compatible with the obvious requirement that for a translation invariant state one must have

$$\omega(\mathbb{1}\otimes A_1\otimes\cdots A_n)=\omega(A_1\otimes\cdots A_n\otimes\mathbb{1})=\omega(A_1\otimes\cdots A_n)\tag{1}$$

Conversely, in order to define a translation invariant state on the chain, it is sufficient to specify for each $n\ge 1$ the expectations $\omega(A_1\otimes\cdots A_n)$ of the elementary tensors. These expectations have to satisfy condition (1) and their linear extension to $\mathcal{A}_{[1,n]}$ must extend to a state.

Let $k\ge 1$ be an integer and consider a linear map $V:\mathbb{C}^k\to\mathbb{C}^d\otimes\mathbb{C}^k$, satisfying $V^*V=\mathbb{1}$, i.e. V is an isometry. Then for $A\in\mathcal{M}_d$, define a linear transformation $\mathbb{E}_A^V$:

$\mathcal{M}_k \to \mathcal{M}_k$ by $\mathbb{E}_A^V(B) = V^* A \otimes BV$, for all $B \in \mathcal{M}_k$. Obviously for $A \geq 0$, $\mathbb{E}_A^V$ is completely positive and the fact that V is an isometry translates into the property that $\mathbb{E}_{\mathbb{1}}^V(\mathbb{1}) = \mathbb{1}$. So, $\mathbb{E}_{\mathbb{1}}^V$ is a Markov operator. It follows that there is at least one density matrix $\rho \in \mathcal{M}_k$ such that for all $B \in \mathcal{M}_k$, $\operatorname{tr} \rho \mathbb{E}_{\mathbb{1}}^V(B) = \operatorname{tr} \rho B$, i.e. ρ is an invariant state for $\mathbb{E}_{\mathbb{1}}^V$. It is now straightforward to check that for any such ρ the formula

$$\omega^V(A_1 \otimes \cdots A_n) = \operatorname{tr} \rho \mathbb{E}_{A_1}^V \circ \cdots \mathbb{E}_{A_n}^V(\mathbb{1}) \quad , \tag{2}$$

for $n \geq 1$, and $A_1, \ldots, A_n \in \mathcal{M}_d$, defines a translation invariant state on $\mathcal{A}_{\mathbb{Z}}$. A state ω which can be given in terms of the construction outlined above is called a purely generated C*-finitely correlated state in [**4**]. Such states form a particular subset of the set of finitely correlated states, which are defined by the property that the correlations across any bond can be modeled on a finite dimensional vector space (in this case, the space $\mathcal{M}_k$). Here, for brevity, we shall call them generalized VBS states. In [**4**], Proposition 5.9 we also gave a proof of the following proposition.

Proposition. *If* 1 *is the only eigenvalue of* $\mathbb{E}_{\mathbb{1}}^V$ *with modulus* 1, *and if it is a simple eigenvalue, then* ω^V *is a pure state, i.e.* $\omega^V \in \partial_e \mathcal{S}$.

The set of eigenvalues of modulus 1 of $\mathbb{E}_{\mathbb{1}}^V$ is called the peripheral spectrum of $\mathbb{E}_{\mathbb{1}}^V$, and since $\mathbb{E}_{\mathbb{1}}^V(\mathbb{1}) = \mathbb{1}$ it always contains the eigenvalue 1. We summarize the hypothesis of the Proposition by saying that $\mathbb{E}_{\mathbb{1}}^V$ has "trivial peripheral spectrum". Since our technique for proving the Theorem rests on this Proposition, we are, in fact, proving the slightly stronger result that the special pure states constructed in this fashion (possibly a proper subset of $\partial_e \mathcal{S} \cap \mathcal{T}$) are dense. In particular, we may impose known properties of this class of states [**4**] as further conditions on the approximating states without affecting the density statement. The most interesting such condition is "local exposedness": a state φ on an algebra $\mathcal{A}$ is called exposed, if it is uniquely characterized by the property $\varphi(H) = 0$ for a certain $H \in \mathcal{A}$ with $H \geq 0$. We call $\omega \in \mathcal{S}(\mathcal{A}_{\mathbb{Z}})$ locally exposed, if there is some positive $h \in \mathcal{A}_{[1,n]}$ for some n such that ω is uniquely characterized by the vanishing of $\omega\big(\tau_x(h)\big)$ for all $x \in \mathbb{Z}$. In physical terminology, ω is the unique ground state (in a very strong sense) of the finite range interaction h. We showed in [**4**], Theorem 6.4 that, for any state ω^V satisfying the hypothesis of the Proposition, there is an exposing interaction h such that, in addition, the support of $\omega^V \restriction \mathcal{A}_{[1,m]}$ has constant dimension k^2 for large m, and coincides with the kernel of the local Hamiltonians $\sum_{x=0}^{m-n} \tau_x(h)$. Moreover, the global Hamiltonian in the GNS representation with respect to ω^V has a non-zero spectral gap above the ground state energy. Since the correlation functions of ω^V are determined by the powers of the finite-dimensional operator $\mathbb{E}_{\mathbb{1}}^V$, they decay exponentially. (Of course, when approximating a state with power-law decay of correlations, the correlation length of the approximating pure state must diverge.) Finally, we point to [**5**], where we establish a converse of the above Proposition in the context of C*-finitely correlated states, and give a necessary and sufficient condition for purity of states in terms of their entropy density.

The Proposition provides us with a simple criterion for the purity of the state ω^V, directly in terms of a simple property of $\mathbb{E}^V_{\mathbb{1}}$, i.e. in terms of a property of the isometry V. The following Lemma shows that a "generic" isometry V satisfies the hypothesis of the Proposition. This result, which is of independent interest in the theory of finitely correlated states, is slightly stronger than what we need in the proof of the Theorem, namely the density of $\mathcal{V}_{abc}$ in $\mathcal{V}_a$.

Lemma 2. *Let $d, k \in \mathbb{N}$, $d, k \geq 2$. Let $\mathcal{V}$ denote the set of isometries $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$, and let $\mathbb{E}^V_{\mathbb{1}}(X) = V^*(\mathbb{1} \otimes X)V$ for $V \in \mathcal{V}$. Consider the subset $\mathcal{V}_{abc} = \mathcal{V}_a \cap \mathcal{V}_b \cap \mathcal{V}_c$ of $V \in \mathcal{V}$ such that*

(a) the only fixed points of $\mathbb{E}^V_{\mathbb{1}}$ are in $\mathbb{C}\mathbb{1}$,

(b) the unique $\mathbb{E}^V_{\mathbb{1}}$-invariant state is faithful, and

(c) apart from 1, $\mathbb{E}^V_{\mathbb{1}}$ has no eigenvalues of modulus 1.

Then $\mathcal{V}_{abc}$ is open and dense in $\mathcal{V}$.

Proof : The map $V \mapsto \mathbb{E}^V_{\mathbb{1}}$ is continuous. Hence by analytic functional calculus the sets $\mathcal{V}_a \subset \mathcal{V}$ and $\mathcal{V}_c \subset \mathcal{V}$ satisfying (a) and (c), respectively, are open. In a neighbourhood of $V \in \mathcal{V}_a$ the eigenprojection P^V_1 of $\mathbb{E}^V_{\mathbb{1}}$ for eigenvalue 1 can be computed by a Cauchy integral in the functional calculus, hence depends continuously on V. Applying the adjoint of P^V_1 to an arbitrary state we see that the invariant state also depends continuously on V. Hence $\mathcal{V}_b \subset \mathcal{V}_a$ is also open and consequently so is $\mathcal{V}_{abc}$.

We show next that $\mathcal{V}_a \cap \mathcal{V}_b$ is non-empty. Let $(\varphi_n) \in \mathbb{C}^k$ be an orthonormal basis, and let $\chi_n \in \mathbb{C}^d$, $n = 1, \ldots, k$ be distinct unit vectors. We define an isometry $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ by $V\varphi_n = \chi_n \otimes \varphi_{n+1}$, where indices are to be read mod$k$. Then

$$\langle \varphi_m, \mathbb{E}^V_{\mathbb{1}}(X)\varphi_n\rangle = \langle \chi_m, \chi_n\rangle \langle \varphi_{m+1}, X\varphi_{n+1}\rangle \quad .$$

Since $|\langle \chi_m, \chi_n\rangle| < 1$ for $n \neq m$, multiples of $\mathbb{1}$ are the only fixed vectors of $\mathbb{E}^V_{\mathbb{1}}$, and one easily checks that its unique invariant state is the normalized trace. Its peripheral eigenvalues are exactly $\exp(2\pi i l/k)$, $l = 0, \ldots, k-1$.

We show the density of $\mathcal{V}_a$ by an analyticity argument. Let $V_1 \in \mathcal{V}_a$, and $V \in \mathcal{V} \setminus \mathcal{V}_a$. Then we may find a hermitian $\Theta \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^k)$ such that $V_1 = \exp(i\Theta)V$, and set $V_\lambda = \exp(i\lambda\Theta)V$. For real λ we therefore have $V_\lambda \in \mathcal{V}$, and $\lambda \mapsto \mathbb{E}^{V_\lambda}_{\mathbb{1}}(X) = V^* \exp(-i\lambda\Theta)X \exp(i\lambda\Theta)V$ extends to an entire analytic function for all X. Let

$$f(\lambda, z) = (z-1)^{-1} \det\big(\mathbb{E}^{V_\lambda}_{\mathbb{1}} - z\,\mathrm{id}\big) \quad .$$

Then f is a polynomial in z, because all $\mathbb{E}^V_{\mathbb{1}}$ share the fixed point $\mathbb{1}$. Moreover, $V_\lambda \in \mathcal{V}_a$ if and only if $f(\lambda, 1) \neq 0$. Clearly, $\lambda \mapsto f(\lambda, 1)$ is entire and not constant, because $f(1,1) \neq 0 = f(0,1)$. Hence $\lambda = 0$ is an isolated zero of $f(\cdot, 1)$, and $V_\lambda \in \mathcal{V}_a$ for small λ. Consequently, $\mathcal{V}_a \subset \mathcal{V}$ is dense.

To show the density of $\mathcal{V}_b$ we show that the complement of this set is contained in a finite union of submanifolds $\mathcal{V}_s \subset \mathcal{V}$ of strictly smaller dimension. Here we consider $\mathcal{V}$ as a homogeneous space of the unitary group $\mathrm{U}(kd)$ of $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^k)$: if $V_1 : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ is any isometry, we can obtain all others by multiplying V_1 from the left by a unitary in $\mathrm{U}(kd)$. The fixed subgroup of V_1 consists of those unitaries, which are the identity on the range of V_1 and arbitrary on its orthogonal complement. Hence $\mathcal{V} \cong \mathrm{U}(kd)/\mathrm{U}(kd-k)$ and the real manifold dimension of $\mathcal{V}$ is $\dim \mathcal{V} = (kd)^2 - (k(d-1))^2 = k^2(2d-1)$. Suppose now that $\mathbb{E}^V_{\mathbb{1}}$ has an invariant state ρ with support projection $S < \mathbb{1}$. Then $\mathrm{tr}\big(\rho V^*(\mathbb{1} \otimes (\mathbb{1} - S))V\big) = \mathrm{tr}(\rho(\mathbb{1} - S)) = 0$, which implies $(\mathbb{1} \otimes (\mathbb{1} - S))VS = 0$. We will now parametrize the set $\mathcal{V}_s$ of isometries V, for which an s-dimensional projection S with this property exists. The set of s-dimensional projections on $\mathbb{C}^k$ can be considered as the homogeneous space $\mathrm{U}(k)/(\mathrm{U}(s) \times \mathrm{U}(k-s))$, which has dimension $k^2 - s^2 - (k-s)^2 = 2s(k-s)$. For each such projection we have to find an isometry $VS : \mathbb{C}^s \to \mathbb{C}^d \otimes \mathbb{C}^s$, i.e. a point in a $s^2(2d-1)$-dimensional space. Then we have to pick the isometry $V(\mathbb{1} - S) : \mathbb{C}^{k-s} \to \mathbb{C}^{kd-s}$ into the complement of the range of VS, which is parametrized by $(kd-s)^2 - (kd-k)^2$ dimensions. Hence we get

$$\dim \mathcal{V} - \dim \mathcal{V}_s = k^2(2d-1) - 2s(k-s) - s^2(2d-1) - (kd-s)^2 + (kd-k)^2 = 2s(k-s)(d-1) \quad .$$

For $1 \leq s < k$ this is strictly positive. Hence the isometries which allow a non-faithful invariant state are contained in the image of a lower dimensional map, and the set $\mathcal{V}_b$ is dense in $\mathcal{V}$.

Finally, let $V \in \mathcal{V}_a \cap \mathcal{V}_b$. Then we know from Proposition 2.3 of **[5]** that the peripheral eigenvalues of $\mathbb{E}^V_{\mathbb{1}}$ form a group of the form ζ^n with $\zeta = \exp(2\pi i/p)$, $p \leq k$. By analytic perturbation theory, these are also the only possible peripheral eigenvalues for V' near V. Let $p > 1$, and let V_1 be any isometry such that ζ is not an eigenvalue of $\mathbb{E}^{V_1}_{\mathbb{1}}$, and consider the function f defined above. Then $f(0, \zeta) = 0$, and $f(\zeta, 1) \neq 0$, so by analyticity we conclude that $f(\lambda, \zeta) \neq 0$ for small λ. Hence $\mathcal{V}_c \cap \mathcal{V}_b \cap \mathcal{V}_a$ is dense in $\mathcal{V}$.

■

Proof of the Theorem: Our strategy of proof is the following: we argue in step 1 that any translation invariant ω can be approximated by pure product states with respect to a sufficiently coarse partitioning of $\mathbb{Z}$ into equal intervals. Since we want to approximate by *translation invariant* states, we take the averages over such states with respect to translations, and show that such averages still approximate ω.

As the degree of approximation is improved the averaging makes the approximating states convex combinations of more and more states, hence less and less pure. The main problem is thus to replace these impure states by w*-close pure ones.

In step 3 we show that the averaged states are of the form ω^V. In step 5 we then use the Lemma to deform the V obtained in step 3 into V' such that $\omega^{V'}$ becomes pure by virtue of the Proposition. This concludes the proof, but in order to conclude in step 5 that $\omega^{V'}$ is close to ω^V we need the continuity of ω^V with respect to V, which holds only for $V \in \mathcal{V}_a$. We show in step 4 that the V from step 3 is in $\mathcal{V}_a$, provided that a non-degeneracy property of the averaged states holds, which we establish without loss of generality in step 2.

Step 1:
For $\ell \in \mathbb{N}$ and $\psi \in (\mathbb{C}^d)^{\otimes \ell}$ a unit vector, let ω_ψ denote the product state on the regrouped chain $(\mathcal{A}^{\otimes \ell})_{\mathbb{Z}}$ formed from the pure state on $\mathcal{A}^{\otimes \ell}$ given by ψ. Then we consider the states

$$\omega_{\ell,\psi} = \frac{1}{\ell} \sum_{p=0}^{\ell-1} \omega_\psi \circ \tau_p \quad . \tag{3}$$

We claim that states of this special form are w*-dense in $\mathcal{T}$. To see this, let ω be translation invariant. For any even $\ell \equiv 2n$, we can find a unit vector $\psi \in (\mathbb{C}^d)^{\otimes \ell}$ with the property that $\omega(A^{(n)}) = \langle \psi \mid \mathbb{1}^{\otimes n} \otimes A^{(n)} \psi \rangle = \langle \psi \mid A^{(n)} \otimes \mathbb{1}^{\otimes n} \psi \rangle$ for all $A^{(n)} \in \mathcal{A}_{[1,n]} \cong \mathcal{M}_d^{\otimes n}$. This follows from the GNS-construction for the state $\omega \restriction \mathcal{A}_{[1,n]}$. For any $A \in \mathcal{A}_{[1,m]}$ one immediately gets from the definition of $\omega_{\ell,\psi}$ that

$$\begin{aligned} \omega_{\ell,\psi}(A) &= \frac{1}{\ell} \sum_{p=0}^{\ell-1} \omega_\psi(\tau_p(A)) \\ &= \frac{1}{2n} \left\{ 2(n-m+1)\omega(A) + \sum_{p=n-m+1}^{n-1} \omega_\psi \circ \tau_p(A) + \sum_{p=\ell-m+1}^{\ell-1} \omega_\psi \circ \tau_p(A) \right\} \end{aligned}$$

and hence

$$|\omega(A) - \omega_{\ell,\psi}(A)| \leq \frac{2(m-1)}{n} \|A\| \quad .$$

Hence for each fixed A we have $\lim_{n\to\infty} \omega_{2n,\psi}(A) = \omega(A)$, and the states given by equation (3) are w*-dense in $\mathcal{T}$.

Step 2:
We claim that the subset of states $\omega_{\ell,\psi}$ such that ψ is not of the form $\psi_p \otimes \psi_{\ell-p}$ for any p with $0 < p < \ell$ is still w*-dense $\mathcal{T}$. By Step 1, we need only show that this subset is dense in the set of all states given by equation (3). The map $\psi \mapsto \omega_{\ell,\psi}$ is clearly w*-continuous for fixed ℓ (an explicit estimate is $|\omega_{\ell,\varphi}(A) - \omega_{\ell,\psi}(A)| \leq 2\big((\ell/m)+1\big) \|\varphi - \psi\| \|A\|$ for $A \in \mathcal{A}_{[1,m]}$). Hence the claim follows from the density of non-factorizable vectors ψ in the unit sphere of $(\mathbb{C}^d)^{\otimes \ell}$.

Step 3:
We now show that $\omega_{\ell,\psi}$ is a purely generated C*-finitely correlated state by explicitly constructing an isometry V such that $\omega_{\ell,\psi} = \omega^V$.

This construction is analogous to the proof of Proposition 2.6 in [**4**], where the density of (not necessarily purely generated) C*-finitely correlated states is shown. We take

(i) $\mathbb{C}^k = \bigoplus_{p=0}^{\ell-1} (\mathbb{C}^d)^{\otimes p}$ with the convention $(\mathbb{C}^d)^{\otimes 0} \cong \mathbb{C}$.

(ii) $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ is defined by:
$V(\bigoplus_{p=0}^{\ell-1} \xi_p) = \bigoplus_{p=1}^{\ell-1} \xi_p \oplus \xi_0 \psi$.

(iii) the state ρ on $\mathcal{M}_k$ is block-diagonal and is given on block diagonal elements $B \in \bigoplus_{p=0}^{\ell-1}(\mathcal{M}_d)^{\otimes p}$, by $\rho(\bigoplus_{p=0}^{\ell-1} B_p) = \frac{1}{\ell}\sum_{p=0}^{\ell-1}\langle\psi \mid \mathbb{1}^{\otimes(\ell-p)} \otimes B_p\psi\rangle$.

Because $\|\psi\| = 1$, V is an isometry. We omit the straightforward verification that ρ is invariant under $\mathbb{E}_{\mathbb{1}}^V$. For any $A \in \mathcal{M}_d$, the operator $\mathbb{E}_A^V$ then leaves the block-diagonal algebra $\bigoplus_{p=0}^{\ell-1}(\mathcal{M}_d)^{\otimes p}$ invariant and it is given by:

$$\begin{aligned}\mathbb{E}_A^V(\bigoplus_{p=0}^{\ell-1} B_p) &= V^* A \otimes \bigoplus_{p=0}^{\ell-1} B_p V \\ &= \langle\psi \mid A \otimes B_{\ell-1}\psi\rangle \oplus \bigoplus_{p=0}^{\ell-2} A \otimes B_p \quad .\end{aligned}$$

Iterating this equation ℓ times we find for $A_1, \dots A_\ell \in \mathcal{M}_d$:

$$(\mathbb{E}_{A_1}^V \cdots \mathbb{E}_{A_\ell}^V)(\bigoplus_{p=0}^{\ell-1} B_p) = \bigoplus_{p=0}^{\ell-1} A_1 \otimes \cdots A_p\langle\psi, A_{p+1} \otimes \cdots A_\ell \otimes B_p\psi\rangle \quad .$$

¿From this equation it can be checked easily that $\omega^V = \omega_{\ell,\psi}$.

Step 4:
We now show that, provided ψ is not factorizable as $\psi_p \otimes \psi_{\ell-p}$, the V constructed in the previous step satisfies condition (a) of the Lemma.

Any operator $B \in \mathcal{M}_k$ can be written in block matrix form with respect to the decomposition $\mathbb{C}^k = \bigoplus_{p=0}^{\ell-1}(\mathbb{C}^d)^{\otimes p}$. Thus the entry B_{ij} of B is an operator from $(\mathbb{C}^d)^{\otimes i}$ to $(\mathbb{C}^d)^{\otimes j}$, or a $d^i \times d^j$-matrix. $\mathbb{E}_{\mathbb{1}}^V$ then acts like

$$\mathbb{E}_{\mathbb{1}}^V(B) = \mathbb{E}_{\mathbb{1}}^V \begin{pmatrix} B_{00} & B_{01} & \cdots & B_{0,\ell-1} \\ \vdots & & & \vdots \\ B_{\ell-1,0} & B_{\ell-1,1} & \cdots & B_{\ell-1,\ell-1} \end{pmatrix}$$

$$= \begin{pmatrix} \langle\psi \mid \mathbb{1} \otimes B_{\ell-1,\ell-1}\psi\rangle & \langle\psi \mid \mathbb{1} \otimes B_{\ell-1,0} & \cdots & \langle\psi \mid \mathbb{1} \otimes B_{\ell-1,\ell-2} \\ \mathbb{1} \otimes B_{0,\ell-1} \mid \psi\rangle & \mathbb{1} \otimes B_{00} & \mathbb{1} \otimes B_{01} & \cdots \\ \vdots & & & \vdots \\ \mathbb{1} \otimes B_{\ell-2,\ell-1} \mid \psi\rangle & \mathbb{1} \otimes B_{\ell-2,0} & \cdots & \mathbb{1} \otimes B_{\ell-2,\ell-2} \end{pmatrix} \quad .$$

Thus the (i,j)-entry of $(\mathbb{E}_{\mathbb{1}}^V)^n(B)$ depends only on $B_{i'j'}$ with $i - i' \equiv j - j' \equiv n \bmod \ell$. A fixed point B is uniquely determined by the entries $B_{m0}, m = 0, \dots \ell - 1$, and the fixed point condition can be evaluated separately for each m. For $m = 0$, B_{00} is just a scalar. Thus the fixed point $B = \mathbb{1}$ is the unique one for $m = 0$, up to a factor.

Now suppose $B_{m0} \neq 0$ for some fixed point B. Since $B_{m0} : \mathbb{C} \to (\mathbb{C}^d)^{\otimes m}$, this entry is given uniquely in terms of the vector $\varphi_m = B_{m0}1$. Iterating the condition $\mathbb{E}_{\mathbb{1}}^V(B) = B$ we obtain

$$
\begin{aligned}
B_{\ell-1,\ell-1-m}\xi_{\ell-1-m} &= \xi_{\ell-1-m}\otimes\varphi_m \\
B_{0,\ell-m}\xi_{\ell-m} &= \langle\psi,\xi_{\ell-m}\otimes\varphi_m\rangle \\
\langle\xi_{m-1},B_{m-1,\ell-1}\xi_{\ell-1}\rangle &= \langle\xi_{m-1}\otimes\psi,\xi_{\ell-1}\otimes\varphi_m\rangle \\
\langle\xi_m,B_{m0}\xi_0\rangle &= \langle\xi_m\otimes\psi,\psi\otimes\varphi_m\rangle\xi_0 \\
&= \langle\xi_m,\varphi_m\rangle\xi_0 \quad ,
\end{aligned}
$$

where in every equation ξ_j denotes an arbitrary vector in $(\mathbb{C}^d)^{\otimes j}$. Hence with $\|\psi\|=1$ we get the condition $\langle\varphi_m\otimes\psi,\psi\otimes\varphi_m\rangle = \|\varphi_m\|^2\,\|\psi\|^2$, which by the Cauchy-Schwartz inequality implies

$$
\varphi_m\otimes\psi = \psi\otimes\varphi_m \quad .
$$

Unless $\varphi_m = B_{m0}1 = 0$, this implies a factorization of ψ [**9**] of the type we have excluded by assumption. Hence the fixed point $\mathbb{1}$ is unique up to a factor.

Step 5:
Combining the steps so far, we have shown that the set of states ω^V with V satisfying the hypothesis (a) of the Lemma (i.e. $V\in\mathcal{V}_a$) is w*-dense in $\mathcal{T}$. It is evident that $\mathbb{E}^V_{\mathbb{1}}$ depends continuously on V. By analytic functional calculus the fixed point ρ depends continuously on $\mathbb{E}^V_{\mathbb{1}}$, where it is unique. Hence $\rho=\rho^V$ is a continuous function of V for $V\in\mathcal{V}_a$. For any finite n, and any local observable $A=A_1\otimes\cdots A_m\in\mathcal{A}_{[1,m]}$ the function $V\mapsto\omega^V(A)=\rho^V\circ\mathbb{E}^V_{A_1}\cdots\mathbb{E}^V_{A_m}(\mathbb{1})$ is continuous. By taking linear combinations and norm limits, this result carries over to arbitrary $A\in\mathcal{A}_{\mathbb{Z}}$. Thus $V\mapsto\omega^V$ is w*-continuous on $\mathcal{V}_a$. By the Lemma we may approximate any $V\in\mathcal{V}_a$ by elements $V'\in\mathcal{V}_{abc}$, and by the Proposition the states $\omega^{V'}$ are pure. Hence the pure states of this form are w*-dense in $\mathcal{T}$.

■

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