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MODEL OF CLUSTER GROWTH AND PHASE SEPARATION: EXACT RESULTS IN ONE DIMENSION

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ABSTRACT

We present exact results for a lattice model of cluster growth in $1D$. The growth mechanism involves interface hopping and pairwise annihilation supplemented by spontaneous creation of the stable-phase, $+1$, regions by overturning the unstable-phase, -1, spins with probability p. For cluster coarsening at phase coexistence, $p = 0$, the conventional structure-factor scaling applies. In this limit our model falls in the class of diffusion-limited reactions A+A→inert. The $+1$ cluster size grows diffusively, $\sim \sqrt{t}$, and the two-point correlation function obeys scaling. However, for $p > 0$, i.e., for the dynamics of formation of stable phase from unstable phase, we find that structure-factor scaling breaks down; the length scale associated with the size of the growing $+1$ clusters reflects only the short-distance properties of the two-point correlations.

1. INTRODUCTION

Lattice cellular automaton-type models with local tendency for ordering, termed voter models, can be used to study phase segregation and cluster coarsening reminiscent of spinodal decomposition,(1,2) at least in low dimensions. Both the cluster-size(3) and structure-factor(4) scaling at phase separation have been subjects of numerous investigations. However, most of the available results for realistic $2D$ and $3D$ dynamical models are numerical. We distinguish between the two "scaling" terms as follows. By cluster-size scaling we mean scaling properties of the cluster size distribution. The term structure-factor scaling is reserved for the scaling properties of the two-point order parameter correlation function. The latter is accessible to scattering experiments.

The symmetric voter-type models are related also to the diffusion-limited chemical reactions involving particle annihilation, A+A$\to$inert.[5] There are several exact results available mostly in $1D$ which essentially translate to various average and asymptotic properties of the cluster size distribution in the phase-separation nomenclature.[5−14] Recent works also yielded exact results for the two-point correlations.[13,14] Results for the chemical reaction models in $D > 1$ are more limited: see Ref. 10 and literature cited therein. Furthermore, the relation of chemical reaction systems to voter models is less straightforward.[1,15−17]

The purpose of the present work is to introduce a lattice model that incorporates voter-type cluster coarsening by interface diffusion in $1D$, as well as the process of spontaneous formation of stable-phase regions from those of the unstable phase. We derive exact results for the two-point correlations. Our main finding is that structure-factor scaling ideas cannot be extended from cluster coarsening (of both phases) <u>at coexistence</u> to stable-phase cluster growth <u>off coexistence</u>. While the general-D formulation is outlined, the present study is focused on the $1D$ case.

The model is defined in Section 2. The generating function solution of the discrete-time and discrete-space dynamics is presented in Section 3. Section 4 is devoted to the discussion of some special limits including the symmetric case. Our results are consistent with previous studies; the general framework of our formulation is close to the zero-temperature kinetic Ising model studies of $1D$ chemical reactions.[8,13,14] Detailed results are obtained in the appropriately defined continuum limit of the discrete dynamics (Section 5). These results are analyzed (Section 6) with emphasis on the length scales associated with the two-point correlation function. The structure-factor scaling at coexistence and its breakdown off coexistence are elucidated.

2. DEFINITION OF THE MODEL

In this section we define the model in $1D$. We also describe the extension to $D > 1$. However, the emphasis in this work is on the $1D$ case, and the notation is introduced correspondingly. Thus, we consider spin variables $\sigma_i(t) = \pm 1$. Time evolves in unit steps: $t = 0, 1, 2, \ldots$. It is convenient to put the spin variables only at even lattice sites, $i = 0, \pm 2, \pm 4, \ldots$ for even times $t = 0, 2, \ldots$. Similarly, for odd times $t = 1, 3, \ldots$ we put spins at odd lattice sites $i = \pm 1, \pm 3, \ldots$.

The values $\sigma_i(t+1)$ for $(t+1) > 0$ will be determined stochastically by the dynamical rule incorporating interface diffusion and pairwise annihilation leading to cluster coarsening, and also spontaneous formation of the stable, +, phase from the unstable, −, phase thus attempting to model cluster growth in nucleation processes. If the "parent" spins, $\sigma_{i-1}(t)$ and $\sigma_{i+1}(t)$ are both +1 or both −1, then the "offspring" is first set to +1 or −1, respectively. However, the −1 value is then overturned with probability p. If the "parent" spin values are opposite, the "offspring" is first set to one of them randomly. However, the −1 value is again overturned spontaneously with probability p.

The first updating step (corresponding to setting $p = 0$) describes a symmetric "voter model" type dynamics.[1,2,6,12] Indeed, the ordered, all + or all −, regions are unaltered. However, each interface between the neighboring + and − spin pairs hops one lattice spacing to the left or to the right with equal probability. On encounter, interfaces annihilate pairwise leading to cluster coarsening. Note that interfaces can be viewed as located at the odd sublattice at even times and at the even sublattice at odd times. An important property of symmetric voter models and related particle (here, interface) diffusion-with-annihilation models is the decoupling of the hierarchy of recursion relations for the correlation functions,[1,6] as well as the diffusive nature of the resulting equations in the continuum limit, which have allowed derivation of several exact results.[5−14]

The second step of updating, i.e., the spontaneous spin-flips $-1 \to +1$, is introduced here as the means to break the $\pm$ symmetry and model formation of the $+$ phase by growth from the $-$ phase. Thus, we will be interested in the results for $p \ll 1$. The key observation (not limited to $D = 1$) is that the correlation function hierarchy can be set up in such a way that decoupling properties reminiscent of the symmetric case are obtained.

At each lattice site and for each time $t > 0$ we introduce two random variables, $\zeta_i(t)$ that takes on values 0 or 1 with equal probability, and $\theta_i(t)$ which is 0 with probability $(1-p)$ and 1 with probability p. The stochastic dynamics is defined by

$$\sigma_i(t+1) = [1-\theta_i(t+1)]\left\{\zeta_i(t+1)\sigma_{i-1}(t) + [1-\zeta_i(t+1)]\,\sigma_{i+1}(t)\right\} + \theta_i(t+1) \tag{2.1}$$

Given that all the random variables ζ and θ are statistically independent, one can easily verify that the rule (2.1) correctly incorporates the dynamics as described in the preceding paragraphs.

In calculating the averages, we can use the properties $\sigma^2 = 1$, $\zeta^2 = \zeta$, $\theta^2 = \theta$ at fixed time and lattice coordinate. Furthermore, $\overline{\zeta_i(t)} = \frac{1}{2}$, $\overline{\theta_i(t)} = p$, where the overbars denote statistical averages.

However, we still have to specify the initial values at $t = 0$. Either the values $\sigma_i(0)$ can be given deterministically or quantities involving the later-time values $\sigma_i(t > 0)$ can be averaged over the distribution of the initial conditions. Here we prefer the latter option; we assume that the initial values are random and uncorrelated, with $\overline{\sigma_i(0)} = \mu$. Thus, $-1 \leq \mu \leq 1$ is the initial magnetization: each $\sigma_i(0)$ takes on values $+1$ and -1 with respective probabilities $(1+\mu)/2$ and $(1-\mu)/2$.

Let us now define the average quantities that will be considered in this study. Firstly, due to translational invariance of the initial conditions and of the dynamical rule (2.1) after averaging over the random variables, the magnetization, $m(t) = \overline{\sigma_i(t)}$, depends only on time and satisfies the recursion $(t \geq 0)$

$$m(t+1) = (1-p)m(t) + p \tag{2.2}$$

with

$$m(0) = \mu \tag{2.3}$$

Similarly, the two-point correlation function,

$$G_n(t) = \overline{\sigma_i(t)\sigma_{i+n}(t)} \tag{2.4}$$

depends only on the distance $n = 0, 2, 4, \ldots$ between the spins. The recursion relation for the two-point function for $n > 0$ and $t > 0$ is easily derived from (2.1):

$$G_n(t+1) = \frac{1}{4}(1-p)^2 \left[G_{n-2}(t) + 2G_n(t) + G_{n+2}(t)\right] + 2p(1-p)m(t) + p^2 \tag{2.5}$$

where the initial and boundary conditions are

$$G_{n=0}(t \geq 0) = 1 \quad \text{and} \quad G_{n>0}(t=0) = \mu^2 \tag{2.6}$$

The decoupling of the equations for the correlation functions should be obvious at this stage. Due to the linearity of (2.1) in σ, the k-point averages at $(t+1)$ are determined by the k, $k-1$, $\ldots$-point averages at time t. This property is further amplified for the connected correlation function defined by

$$C_n(t) = G_n(t) - m^2(t) \tag{2.7}$$

Indeed, the appropriate recursion is

$$C_n(t+1) = \frac{1}{4}(1-p)^2 \left[C_{n-2}(t) + 2C_n(t) + C_{n+2}(t)\right] \tag{2.8}$$

so that the $m(t)$-dependence enters only via the boundary conditions. The equivalents of relations (2.6) are

$$C_{n=0}(t \geq 0) = 1 - m^2(t) \quad \text{and} \quad C_{n>0}(t=0) = 0 \tag{2.9}$$

where we used (2.3).

The probability to find an interface at i, in the interstice between the two spins $\sigma_{i\pm1}$, is given by

$$\rho(t) = \frac{1}{2}\left[1 - G_2(t)\right] = \frac{1}{2}\left[1 - m^2(t) - C_2(t)\right] = \frac{1}{2}\left[C_0(t) - C_2(t)\right] \tag{2.10}$$

Similarly to $m(t)$, this quantity is translationally invariant. Both m and ρ can be also considered as the order-parameter and interface-number densities if we allow for the fact that they are defined per site of the lattice of twice the spacing of the original $1D$ linear system of sites labeled by i. In fact, all our calculations will be with dimensionless variables such as distance n and time t. One can of course introduce dimensional length and time scales which has been a common practice especially in the continuum limit. However, we found that no new useful physical insight in gained, while the equations become more complicated. Thus, we use the dimensionless variables throughout.

Before we outline the extension to $D > 1$, which will be detailed elsewhere, let us emphasize three appealing features of the $1D$ model: the property that only two parent spins "vote" at each time step, the linearity of the evolution rule (2.1), and the fact that for $p = 0$ there are already many results available, in particular, the relation to the interface motion and the interpretation of cluster coarsening due to interface annihilation.

The simplicity of two-spin voting and the linearity of the dynamical rule (implying, essentially, solvability) can be extended to $D > 1$ by using the idea of updating along different axes in each time step.[2] Consider spins $\sigma_{i_1 i_2 \ldots i_D}(t)$. For time steps $t = 0 \to 1$, $D \to D+1$, $2D \to 2D+1$, ..., the rule (2.1) is used along axis 1, i.e., with i_1 varied as in (2.1) while $i_2, \ldots, i_D$ kept the same on both sides of the relation. Similarly, for time steps $1 \to 2$, $D+1 \to D+2$, ..., the update relation involves the index i_2 along axis 2, and so on. In D time steps, the cycle of the axis indices is complete.

Regarding the availability of exact results for $p = 0$ and the interpretation of the dynamics of the broken bonds connecting $\pm$ spin pairs, the $D > 1$ results, see Ref. 10 and literature cited therein, are understandably less numerous than those available for $D = 1$. In fact, it has been argued[1,15–17] that symmetric voter-model type dynamics cannot lead to cluster coarsening in $D = 3$ and higher. In $D = 2$ the clusters do grow[1,15–17] but the process can no longer be described by a simple cluster-size scaling.[1,2] Quite generally, many open questions remain for $D > 1$.

3. GENERATING FUNCTION FORMULATION

The recursion relation (2.2) for the magnetization is trivial to solve,

$$m(t) = 1 - (1-\mu)(1-p)^t \tag{3.1}$$

However, the solution for the correlation function can be obtained in a simple form only in terms of the generating functions

$$B_n(v) = \sum_{t=0}^{\infty} v^t C_n(t) \tag{3.2}$$

Indeed, relations (2.8) yield, for $n = 2, 4, \ldots$,

$$B_n = \frac{v}{4}(1-p)^2 \left(B_{n-2} + 2B_n + B_{n+2}\right) \tag{3.3}$$

while for $n = 0$ we get

$$B_0(v) = \frac{2(1-\mu)}{1-(1-p)v} - \frac{(1-\mu)^2}{1-(1-p)^2 v} \tag{3.4}$$

where we used the conditions (2.9) in deriving both (3.3) and (3.4).

The second-order difference equation (3.3) has two linearly independent solutions of the form

$$B_n(v) \propto b^{n/2} \tag{3.5}$$

where $b(v)$ is a root of the quadratic characteristic equation. However, one can check that only one of the two roots yields the solution which converges exponentially to zero as $n \to \infty$ (for fixed v is the vicinity of 0). The other root yields exponentially divergent terms. After some algebra we arrive at the expression

$$B_n(v) = B_0(v) \left\{ \frac{\left[1 - \sqrt{1-(1-p)^2 v}\right]^2}{(1-p)^2 v} \right\}^{n/2} \tag{3.6}$$

where $n = 2, 4, \ldots$.

The result (3.6), when expanded in powers of v, yields $C_n(t)$ as the t^{th} Taylor series coefficient. However, the expressions thus obtained involve double sums and are rather unilluminating. The continuum limit results derived in Sections 5 and 6 provide a more useful source of physical insight on the nature of the dynamics.

Our main interest presently will be in the expression of the generating function for the interface density $\rho(t)$, see (2.10). This quantity is the t^{th} Taylor coefficient of the function

$$\frac{1}{2}\left[B_0(v) - B_2(v)\right] \tag{3.7}$$

Explicit calculation yields the result

$$\rho(t) = 2(1-\mu)(1-p)^t \left[1 - S_t(1-p)\right] - (1-\mu)^2 (1-p)^{2t} \left[1 - S_t(1)\right] \tag{3.8}$$

where we defined the finite sum

$$S_t(\alpha) = \sum_{k=0}^{t} \frac{(2k)!\, \alpha^k}{k!\,(k+1)!\, 2^{2k+1}} \tag{3.9}$$

Note that

$$S_\infty(\alpha) = \left(1 - \sqrt{1-\alpha}\right)/\alpha \tag{3.10}$$

where the $t = \infty$ Taylor series converges for all α in $[0, 1]$.

4. DIFFUSION AS OPPOSED TO SPIN-FLIP

It is instructive to consider two models which represent the extremes of diffusion only or no diffusion at all, as far as interfacial dynamics is concerned. If we set $p = 0$ in our model, the only processes are those of interface hopping and pairwise annihilation. Thus, the model falls in the class of the diffusion-limited chemical reactions A+A→inert, which, as well as related models, were studied extensively[5−14] in $1D$. The density of interfaces reduces to

$$\frac{\rho(t)}{\rho(0)} = \sum_{k=t+1}^{\infty} \frac{(2k)!}{k!\,(k+1)!\, 2^{2k}} \qquad (p = 0) \tag{4.1}$$

where generally,

$$\rho(0) = (1-\mu^2)/2 \tag{4.2}$$

The density of interfaces decreases monotonically and smoothly for discrete time steps $t = 0 \to 1 \to 2 \to \ldots$. For large times, we have

$$\rho(t) = 2\rho(0)/\sqrt{\pi t} \qquad (p = 0) \tag{4.3}$$

where the $\sim t^{-1/2}$ law is consistent with the previous exact calculations for these reactions.

The other extreme would be to have no diffusion at all. For this, however, we have to modify our model. Thus, let us consider a model of ± 1 spins with the only dynamical process consisting of spin-flips $-1 \to +1$ with probability p. Since the spins are uncorrelated, the model is trivial to solve. Indeed, the dynamical equation (2.1) is replaced by

$$\sigma_i(t+1) = [1 - \theta_i(t+1)]\, \sigma_i(t) + \theta_i(t+1) \tag{4.4}$$

where we now assume that the spins are located on the even sublattice at all times.

The magnetization obeys the same equation (2.2), which simply reflects the fact that in our more complicated model diffusion of interfaces conserves the order parameter. However, in the new, uncorrelated-spin model (USM), all the k-point correlations factorize trivially, and as a result the connected correlations vanish identically (for distinct k coordinates). Specifically, we get

$$\rho^{\rm USM}(t) = (1 - m^2)/2 = (1 - \mu)(1 - p)^t - \frac{1}{2}(1 - \mu)^2(1 - p)^{2t} \tag{4.5}$$

$$C_n^{\rm USM}(t) = \delta_{n,0}\, [1 - m^2(t)] \tag{4.6}$$

The time-dependent length scale of interest in cluster growth is the average size of the dominant, +, clusters. More generally, one may consider the cluster size distribution, which was not obtained analytically. (For some asymptotic results in the diffusion-only model see Ref. 6.) One measure of the cluster size is $\rho^{-1}(t)$. For the diffusion-only model this cluster size measure grows according to $\sim \sqrt{t}$ for large times. For the USM, it grows as $\sim (1-p)^{-t}$ (assuming $0 < p < 1$).

Note, however, that this quantity is related to the <u>short-distance</u> properties of the two-point correlations, see (2.10). The moment or decay-tail definitions of the "correlation" length scales (defined in Section 6) are typically used to probe the fixed-time <u>large-distance</u> behavior of the two-point correlations in strongly fluctuating systems. The various length scales are not necessarily related. For the diffusive model (symmetric, phase coexistence case, $p = 0$), it turns out that all the length scales behave according to $n \sim \sqrt{t}$ (Section 6). The USM example is instructive as the opposite extreme: the two-point correlations are zero-range; see (4.6). However, the cluster size measure ρ^{-1} is well defined and diverges as $t \to \infty$. The length scale properties will be further explored in Section 6.

5. CONTINUUM LIMIT

The continuum limit has been the standard framework for writing phenomenological equations in cases which are not exactly solvable, or where the precise microscopic dynamics is not known or specified. If fact, the continuum limiting description provides a useful guide for the identification of the "universality classes" or at least general classes of models with similar properties. A simple-minded continuum limiting procedure would amount to the assertion that for $t \gg 1$ and $n \gg 1$ the discrete variation can be replaced by smooth functional dependence on t and n. Formally, one then uses the expression

$$f(t+\Delta t, n+\Delta n) = \left\{ \exp\left[\lambda \Delta t \frac{\partial}{\partial t}\right] \exp\left[\Lambda \Delta n \frac{\partial}{\partial n}\right] f(t,n) \right\}_{\lambda=1,\, \Lambda=1} \tag{5.1}$$

to expand in the derivatives, which are presumably small. The order of the expansion is conveniently monitored by collecting powers of λ and Λ before setting these variables to 1 in the final expressions.

If we apply this procedure to the equation (2.2) for $m(t)$, and keep the leading t-derivative, we obtain the equation

$$\frac{dm}{dt} = -pm + p \tag{5.2}$$

with the solution

$$m(t) = 1 - (1-\mu)e^{-pt} \tag{5.3}$$

where we used (2.3). However, this result differs from the exact expression (3.1), which is, in fact, perfectly well defined for all real $t \geq 0$. The source of the difficulty is clearly in that the t-derivatives are small only for $p \ll 1$. This is an illustration of the well-known property that the continuum approximation can be used only in a limited part of parameter space.

A better controlled procedure is to use properly rescaled variables so that the parameters λ and Λ in the equivalent of (5.1) are actually small. For our problem, we set

$$\tau = pt \quad \text{and} \quad \lambda = p \tag{5.4}$$

$$x = \sqrt{p}\, n \quad \text{and} \quad \Lambda = \sqrt{p} \tag{5.5}$$

where the t-rescaling is suggested by our consideration of $m(t)$, while the n-rescaling is implied by the diffusive combination n/t^2 which is expected to survive the $p \to 0$ limit.

In terms of the new variables, the relation (2.8) for the two-point function has the leading terms in order p. The "continuum limit" two-point function $C(x,\tau)$ satisfies the relation obtained by collecting these terms,

$$\frac{\partial C}{\partial \tau} = -2C + \frac{\partial^2 C}{\partial x^2} \qquad (x > 0) \tag{5.6}$$

where the initial and boundary conditions (2.9) are replaced by

$$C(x=0, \tau \geq 0) = 1 - m^2(\tau) \quad \text{and} \quad C(x>0, \tau=0) = 0 \tag{5.7}$$

with

$$m(\tau) = 1 - (1-\mu)e^{-\tau} \tag{5.8}$$

The interface density in the continuum limit is approximated as follows,

$$\frac{\rho}{\sqrt{p}} \simeq \left[-\frac{\partial C(x,\tau)}{\partial x} \right]_{x=0} \tag{5.9}$$

The reader should keep in mind that the continuum limit is an approximation valid asymptotically for $0 \leq p \ll 1$, $t \gg 1$, $n \gg 1$. The results must be properly interpreted. For instance, if taken literally, relation (5.9) would imply that the interface density is infinite at $\tau = 0$ because the initial conditions for $C(x,\tau)$ are step-like. In fact, the divergence is in the regime where the continuum limiting approximation breaks down; see the next section. The rescaling (5.4)-(5.5) also obscures the $p = 0$ case. Indeed, the results must be properly expressed in terms of the variable $x/\tau^2 = n/t^2$ before taking the limit $p \to 0$. If fact, $p = 0$ is reminiscent of the "critical-point" limit in which there are no small parameters to rescale n and t. Instead, only their "scaling combination" enters in the continuum limit.

The solution of the equation (5.6) with conditions (5.7) is obtained by the Laplace Transform method. We omit the mathematical details and only quote the final expression,

$$\int_0^\infty e^{-\omega\tau} C(x,\tau)d\tau = \left[\frac{2(1-\mu)}{\omega+1} - \frac{(1-\mu)^2}{\omega+2} \right] e^{-\sqrt{\omega+2}\,x} \tag{5.10}$$

which inverse-transforms to

$$\frac{C(x,\tau)}{1-\mu} = e^{-\tau} \left[e^{x}\mathrm{erfc}\left(\frac{x}{2\sqrt{\tau}} + \sqrt{\tau} \right) + e^{-x}\mathrm{erfc}\left(\frac{x}{2\sqrt{\tau}} - \sqrt{\tau} \right) - (1-\mu)e^{-\tau}\mathrm{erfc}\left(\frac{x}{2\sqrt{\tau}} \right) \right] \tag{5.11}$$

where

$$\mathrm{erfc}(\alpha) = \frac{2}{\sqrt{\pi}} \int_\alpha^\infty e^{-\beta^2} d\beta \tag{5.12}$$

is one of the standard error functions, the properties of which are well known. Thus, the expression (5.11) can be used to analyze various properties of connected two-point correlations. Some such results will be presented in the next section.

6. LENGTH SCALES AND BREAKDOWN OF SCALING

Let us consider the large-x behavior of $C(x,\tau)$ for fixed $\tau > 0$ (and $\mu \neq 1$). All three terms in (5.11) then follow the asymptotic large-argument behavior of the error function. The result turns out to be

$$C(x \to \infty, \tau) \propto e^{-2\tau} \left[\frac{\sqrt{\tau}}{x} \exp\left(-\frac{x^2}{4\tau} \right) \right] \tag{6.1}$$

where we omitted the proportionality constant. The decay-tail length scale, n_{tail}, is thus determined by the dependence on the diffusional combination $x^2/\tau = n^2/t$,

$$n_{\text{tail}} \sim \sqrt{t} \tag{6.2}$$

Consider next the moment-definition length scales. We define the k^{th} moment,

$$M_k(\tau) = \int_0^\infty x^k C(x,\tau)dx \tag{6.3}$$

and the associated time-dependent length, $n_k(t)$,

$$\sqrt{p}\, n_k = \left(M_k/M_0\right)^{1/k} \tag{6.4}$$

In the evaluation of M_k, the contribution due to the first term in (5.11) can be always used in its large-argument form, while the third term is originally a function of the diffusional combination (times $e^{-2\tau}$). The second term, however, can be written in such a diffusional-scaled form only for

$$x \gg a_1\sqrt{\tau} + a_2\tau \tag{6.5}$$

where from now on the coefficient notation a_j will be defined to stand for "a slow varying function of τ, of order 1, possibly k-dependent (when implied by context)." The diffusional contribution to the moments is

$$M_k^{(\text{diff})} = a_3\tau^{(k+1)/2}\, e^{-2\tau} \tag{6.6}$$

In the range of smaller x, not satisfying (6.5), the error function in the second term in (5.11) becomes of order 1. In fact, the fixed-x, large-time behavior

$$C(x,\tau\to\infty) \propto e^{-\tau-x} \tag{6.7}$$

is explicit in the Laplace-transformed form (5.10) due to the rightmost pole singularity at $\omega = -1$. The added contribution due to this exponential behavior is of the form

$$M_k^{(\text{exp})} = a_4 e^{-\tau} \int_0^{a_1\sqrt{\tau}+a_2\tau} x^k e^{-x} dx \tag{6.8}$$

For small τ, the intergation will yield the same power of τ as in (6.6). Thus, the moment length scales behave according to

$$n_k \sim \left(\tau^{(k+1)/2}/\tau^{1/2}\right)^{1/k} p^{-1/2} = \sqrt{t} \qquad (t \ll 1/p) \tag{6.9}$$

However, as τ increases, the integral in (6.8) saturates at a value of order 1. Since the remaining time dependence, $e^{-\tau}$, dominates that of the diffusive contribution (6.6), the length scales saturate at

$$n_k \sim 1/\sqrt{p} \qquad (t \gg 1/p) \tag{6.10}$$

The crossover between the limiting behaviors occurs at $t \sim 1/p$ and is difficult to evaluate in closed form.

We next turn to the density of interfaces and the associated length scale. A direct calculation of the right-hand side of (5.9) yields

$$\frac{\rho}{\sqrt{p}} \simeq \frac{1-\mu^2}{\sqrt{\pi\tau}} e^{-2\tau} + 2(1-\mu)e^{-\tau}\mathrm{erf}(\sqrt{\tau}) \tag{6.11}$$

where we kept the approximation sign to emphasize that this result applies only for $t \gg 1$ (as well as $p \ll 1$). Note that $\mathrm{erf}(\alpha) = 1 - \mathrm{erfc}(\alpha)$. The limit $p = 0$ is thus correctly reproduced; see (4.3). For small τ, the first term in (6.11) dominates, and the associated length scale behaves according to

$$n_\rho \propto \rho^{-1} \sim \sqrt{t} \qquad (t \ll 1/p) \tag{6.12}$$

However, for large τ the second term takes over. Noting that the function $\mathrm{erf}(\alpha)$ approaches 1 for large α, we conclude that

$$n_\rho \sim e^{pt}/\sqrt{p} \qquad (t \gg 1/p) \tag{6.13}$$

In the theories of structure-factor scaling,[4] where the structure factor is defined as the spatial Fourier transform of $C(n,t)$, assuming continuous coordinate n and time t, the momentum, q, dependence is scaled in the form $\hat{n}(t)q$. In the direct-space notation this amounts to assuming that the coordinate dependence enters via $n/\hat{n}(t)$. It is tempting to associate $\hat{n}(t)$ with a typical cluster size measure. In practice, $\hat{n}$ is determined as the inverse of some momentum scale found at low or fixed q-values, corresponding to large or intermediate coordinate values n.

Our results support this picture <u>at coexistence</u>, i.e., at $p = 0$. Indeed, due to the critical-point-like scaling expressed by the diffusive scaling combination n^2/t, all length scales defined at short or large distances are essentially the same. The identification $\hat{n} \sim \sqrt{t}$ is unambiguous. However, explicit expressions obtained for $p > 0$ indicate two difficulties with the structure-factor scaling when the growth of the stable phase occurs off coexistence. Firstly, the identification of a unique length scale is no longer possible for large times for which the cluster size distribution deviates significantly from the symmetric case. All three length scales estimated behave differently for large t. Secondly, the two-point correlation function no longer has simple scaling properties. In fact, a more general conclusion, alluded to in Section 4, is that in such cases the length scale $n_\rho(t)$ is the appropriate one to use as a typical + cluster size. However, it is characteristic only of the <u>short-distance</u> coordinate dependence of the two-point function.

In summary, we presented a solvable $1D$ model of cluster growth. Our results indicate that the ideas of structure-factor scaling apply only to cluster coarsening at coexistence. Off coexistence, a typical stable-phase cluster size measure reflects only the short-distance properties of the two-point correlations; the full correlation function no longer obeys scaling.

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