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Covariant QED without Indefinite Metric

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Abstract

We construct for the linearized Higgs model a representation of the field operators in a Hilbert space $\mathcal{H}$ with the following features:

1. $\mathcal{H}$ has a positive definite metric but is nonseparable.
2. The vacuum is gauge invariant.
3. The gauge variant operators exist only in their exponentiated form as unitaries.
4. There is a subspace of $\mathcal{H}$ where $A_{\mu}{}^{,\mu}$ is represented by 0.

1 Introduction

Quantum electrodynamics is the experimentally most precisely tested quantum theory and yet in its mathematical formulation there remain some disturbing features. Even if one disregards the infinite renormalization constants and the still open question of the convergence of the perturbation series we are left with some problems typical of gauge theories.

(i) The canonical formulation of the theory uses the gauge dependent potentials A which transform under gauge transformations as $A \to A + d\Lambda$. The vacuum should be an invariant state and not prejudize a particular gauge. Yet a gauge invariant state $|0\rangle$ seems impossible since for a c-number Λ

$$\langle 0|A|0\rangle \to \langle 0|A|0\rangle + d\Lambda. \tag{1.1}$$

(ii) The theory contains the generator of gauge transformations $\chi = \delta A$ which must satisfy with A commutation rules like x and p since χ shifts A

$$[A, \chi] = i. \tag{1.2}$$

The equivalence of the field equations with Maxwell's equations requires $\chi = 0$ but as operator equation this contradicts (1.2). So one poses this constraint as a supplementary condition on the good vectors $|g\rangle$ in Hilbert space

$$\chi|g\rangle = 0. \tag{1.3}$$

But there is still trouble ahead since v. Neumann's uniqueness theorem tells us that for the Weyl form of (1.2) there is essentially only the usual representation where the operators have a purely continuous spectrum. Thus vectors of the type $|g\rangle$ cannot exist because they would be eigenvectors of χ. The situation is similar to the "improper" eigenfunctions of the operator x which are something like δ-functions. They are not normalizable, $\int \delta^2 = \infty$, and so one introduces an indefinite metric η in Hilbert space such that $\langle g|\eta|g\rangle < \infty$ [1,2,3]. This creates more problems in the physical interpretation ("negative probabilities") and in formulating things in a mathematically well defined way (even unitaries $U^*\eta U = \eta$ are no longer bounded operators and the road is paved with domain questions).

Even though people live happily with this construct we want to point out that there is no need to abandon the standard C^*-algebraic formulation of quantum mechanics which excludes negative probabilities. In fact, there are general theorems which tell us that the difficulties mentioned before cannot really exist but must be artefacts of an inappropriate mathematics. The point is that the gauge transformations generated by χ are an amenable group and there is a general theorem if an automorphism group of a C^*-algebra is amenable then there are invariant states ω [4,5]. Furthermore, since the gauge transformations are inner automorphism the vector corresponding to the invariant state must be an eigenvector of the corresponding unitaries. Thus there must be representations where χ has a point spectrum and the goal of this paper is to study them. In doing so we follow a program pioneered by A. Hurst et al. who have for a long time advocated this point of view [6,7,8,9]. The novel feature which turns up is that gauge dependent quantities S do not exist as operators in Hilbert space but only the associated unitaries $e^{i\alpha S}$. A similar construction of a state can be found in [10]. Since for the gauge invariant quantities the representation is the standard one we consider this price not too high for getting rid of negative probabilities.

2 The Weyl Algebra

First we want to recall that von Neumann's uniqueness theorem [11] can be circumvented and it is possible to find proper eigenvalues to, let's say x, if one is ready to pay a price.

The Weyl algebra $\mathcal{W}$ is generated by the unitaries $W(r,s) = e^{i(pr+xs)}$, $(r,s) \in \mathbf{R}^2$ which satisfy the relations

$$W(r,s)W(r',s') = e^{i\sigma((r,s),(r',s'))}W(r+r',s+s') \tag{2.1}$$

with the symplectic form

$$\sigma((r,s),(r',s')) = \frac{1}{2}(s'r - r's). \tag{2.2}$$

We have

$$W(r,s)^* = W(-r,-s) = W^{-1}(r,s) \tag{2.3}$$

and the C^*-algebra $\mathcal{W}$ consists of the elements $\sum_i c_i W(r_i,s_i)$, $c_i \in \mathbf{C}$, $\sum_i |c_i| < \infty$. Their norm satisfies

$$\left\|\sum_i c_i W(r_i,s_i)\right\| \leq \sum_i |c_i|, \qquad \left\|\sum_{i=1}^{2} c_i W(r_i,s_i)\right\| = \sum_{i=1}^{2} |c_i|. \tag{2.4}$$

Note that the map $(r,s) \to W(r,s)$ is not continuous from $\mathbf{R}^2$ into $\mathcal{W}$ (in the norm topology). In $\mathcal{W}$ the inner automorphism group of $\mathcal{W}$ isomorphic to $\mathbf{R}^2$ contains the shifts which are

$$S_{(\alpha,\beta)}W(r,s) = W(\alpha,\beta)W(r,s)W^{-1}(\alpha,\beta) = e^{i\alpha s - i\beta r}W(r,s). \tag{2.5}$$

Since $\mathbf{R}^n$ is amenable general theorems tell us that there are S-invariant states. In fact, the only state invariant under S is given by

$$\omega_{00}(W(r,s)) = \delta_{r,0}\delta_{s,0} \tag{2.6}$$

where $\delta_{r,s} = 1$ for $r = s$, 0 otherwise. The state

$$\omega_0(W(r,s)) = \delta_{r,0} \tag{2.7}$$

is invariant only under the subgroup $S_{(0,\mathbf{R})}$. The tracial state ω_{00} is faithful:

$$\omega_{00}\big(\sum_i c_i^* w^*(r_i,s_i) \sum_j c_j W(r_j,s_j)\big) = \sum_i |c_i|^2 > 0$$

unless all c_i are zero. ω_0 is not since

$$\omega_0((1 - W^*(0,s))(1 - W(0,s))) = 0 \quad \forall\, s \in \mathbf{R}. \tag{2.8}$$

However, $\mathcal{W}$ is simple so that each state gives a faithful GNS-representation. Denoting the cyclic vector in the representation π_{00} (resp. π_0) given by ω_{00} (resp. ω_0) by $|0,0\rangle_0$ (resp. $|0\rangle$) and defining $|r,s\rangle_0 = \pi_{00}W(r,s)|0,0\rangle_0$ we see

$${}_0\langle r',s'|r,s\rangle_0 = \delta_{r,r'}\delta_{s,s'}.$$

Since $(r,s) \in \mathbf{R}^2$ we have an uncountable set of orthogonal vectors and thus a representation in a nonseparable Hilbert space. Its elements are of the form $\sum_{i,j} c_{ij}|r_i,s_j\rangle_0$ with the norm $\sum_{i,j}|c_{ij}|^2 < \infty$ and π_{00} is given by

$$\pi_{00}(W(r,s))|r',s'\rangle_0 = e^{i\sigma((r,s),(r',s'))}|r+r',s+s'\rangle_0. \tag{2.9}$$

Note that $(r,s) \to \pi_{00}(W(r,s))$ is not strongly continuous and thus one cannot define the infinitesimal generators p and x, not even as unbounded operators. This phenomenon can only occur in the non-separable Hilbert space. Since $(r,s) \to \pi_{00}(W(r,s))$ is weakly measurable in a separable Hilbert space this would be enough to guarantee the existence of p and x.

If we again define $|r,s\rangle = \pi_0(W(r,s))|0\rangle$ in the representation π_0, the analog of (2.9) is still valid but the scalar product is changed to

$$\langle r', s'|r, s\rangle = \delta_{r,r'} e^{ir(s'-s)/2}.$$

Thus $|0,s\rangle = |0\rangle$ and $|r,s\rangle = e^{-irs/2}|r\rangle$ with $\langle r'|r\rangle = \delta_{r,r'}$ and the Hilbert space is still non-separable. In contrast to ω_{00} now ω_0 corresponds to an irreducible representation π_0: Assume $A\pi_0(W(r,s)) = \pi_0(W(r,s))A$ for all $r,s \in \mathbf{R}^2$. Then

$$\begin{aligned} &A\pi_0(W(r,s))|\bar{r},0\rangle = \pi_0(W(r,s))A|\bar{r},0\rangle = \\ &= A\pi_0(W(r,s)\cdot W(\bar{r},\bar{s}))e^{i\bar{s}\bar{r}/2}|0,0\rangle = \pi_0(W(r,s)A\pi_0(W(\bar{r},\widehat{\bar{s}}))e^{-i\widehat{\bar{s}}\bar{r}/2}|0,0\rangle \\ &= \pi_0(W(r+\bar{r},s+\bar{s}))e^{i(\bar{s}\bar{r}+r\bar{s}-s\bar{r})/2}A|0,0\rangle = \pi_0(W(r+\bar{r},s+\widehat{\bar{s}}))e^{i(\widehat{\bar{s}}\bar{r}+r\widehat{\bar{s}}-s\bar{r})/2}A|0,0\rangle. \end{aligned}$$

This implies

$$\pi_0(W(0,\bar{s}-\widehat{\bar{s}}))A|0,0\rangle = A|0,0\rangle.$$

But the only eigenstate to $W(0,s)$ with eigenvalue 1 is $|0,0\rangle$ which is separating for the commutant. Therefore $A = \lambda 1$ and the representation is irreducible. Though the representation π_0 of $W(r,s)$ is not strongly continuous in both variables it is in the second. This means that in this representation the generator p does not exist but x does and is a selfadjoint operator. This corresponds to the fact that the cyclic vector $|0\rangle$ is the limit $b \to 0$ of a wave function $\sim \frac{1}{\sqrt{b}} e^{-(x/b)^2}$ in the standard Schrödinger representation. By compactness there must be weak accumulation points and in Hilbert space this vector converges weakly to zero. However, the weak* limit of the corresponding state is ω_0. Note that now x does not have a continuous spectrum but $\mathbf{R}$ as a point spectrum, $|r\rangle$ being the eigenvectors. There is no contradiction to the Heisenberg commutation relations since p does not exist as operator.

3 The Two-Dimensional Oscillator

As next warming up exercise we ask for which algebraic structures the motion

$$\vec{a}(t) = \vec{a}(0)\cos t + \dot{\vec{a}}(0)\sin t, \qquad \vec{a} \in \mathbf{R}^2, \tag{3.1}$$

is a one parameter group of automorphisms: Classically the equations

$$a_1 = -\dot{a}_0, \qquad a_0 = \dot{a}_1 \tag{3.2}$$

define a time invariant submanifold of phase space but with the usual quantization one cannot put $a_0 - p_1 = a_1 + p_0 = 0$, this would contradict the commutation relations $[a_1 + p_0, a_0 - p_1] = -2i$. We shall now show how these constraints can be realized by some representation π_0.

First we note that (3.1) can be derived from the Hamiltonian

$$2H = p_1^2 + a_1^2 - p_0^2 - a_0^2 \tag{3.3}$$

for which $\dot{a}_1 = p_1$, $\dot{a}_0 = -p_0$ and this change in sign removes the contradiction with the commutation relations (which exist equally in their better defined Weyl form). The commutation relations $[a_j, p_n] = i\delta_{jn}$ suggest that the variables

$$\chi = a_1 - p_0, \qquad S = (a_1 + p_0)/2, \qquad \Pi_\chi = (a_0 + p_1)/2, \qquad \Pi_S = -a_0 + p_1 \tag{3.4}$$

are conjugate

$$[\chi, S] = [\Pi_\chi, \Pi_S] = [\Pi_\chi, S] = [\Pi_S, \chi] = 0, \qquad [S, \Pi_S] = [\chi, \Pi_\chi] = i. \tag{3.5}$$

In terms of these variables we have

$$H = S\chi + \Pi_\chi \Pi_S \tag{3.6}$$

and the time evolution is

$$\dot{S} = \Pi_\chi, \qquad \dot{\chi} = \Pi_S, \qquad \dot{\Pi}_\chi = -S, \qquad \dot{\Pi}_S = -\chi. \tag{3.7}$$

With the commutation relations (3.5) we see that $S(t)$ and $\chi(t)$ generate time invariant abelian subalgebras but the two do not commute with each other. To give a meaning to the submanifold (3.2) on which $\chi(t)$ is zero we replace the ill-defined relations (3.5) by the ones for the Weyl operators

$$W(\alpha,\beta,\gamma,\delta)=\exp[i(\alpha S+\beta\Pi_S+\gamma\chi+\delta\Pi_\chi)]. \tag{3.8}$$

In the Hilbert space corresponding to the state

$$\omega_0(W(\alpha,\beta,\gamma,\delta))=\delta_{\alpha 0}\delta_{\delta 0}$$

we have as in Sect. 1

$$W(0,\beta,\gamma,0)|0\rangle=|0\rangle \tag{3.9}$$

or χ and $\Pi_S=\dot{\chi}$ exist as operators and

$$\chi|0\rangle=\dot{\chi}|0\rangle=0. \tag{3.10}$$

The state ω_0 is invariant under the automorphism group (3.1) and it is thus implemented by a one parameter group of unitaries. Formally in the notation implied by (3.1)

$$\vec{a}(t)=U_t^{-1}\vec{a}(0)U_t. \tag{3.11}$$

But since S and $\dot{S}$ do not exist in this representation one has to write it

$$W(\alpha,\beta,\gamma,\delta)(t)=U_t^{-1}W(\alpha,\beta,\gamma,\delta)U_t \tag{3.12}$$

with

$$W(\alpha,\beta,\gamma,\delta)(t)=W(\alpha\cos t-\delta\sin t,\beta\cos t+\gamma\sin t,\gamma\cos t-\beta\sin t,\delta\cos t+\alpha\sin t). \tag{3.13}$$

Since the W's are not strongly continuous in α and δ the U's cannot be in t either. Thus Stone's theorem does not apply and U_t cannot be written e^{-iHt}. The formal expression (3.6) is not an operator in this representation only $H|0\rangle=0$ makes sense.

Remarks (3.14): In QED [12] instead of $\chi(t)|0\rangle=0$ the weaker condition $\chi^-(t)|0\rangle=0$ is postulated. χ^- is the negative frequency part of $\chi(t)=(e^{-i\omega t}/2)(a_1+i\dot{a}_1-ia_0+\dot{a}_0)+h.c.\equiv\chi^-+\chi^+$, so in fact χ^- is the sum of the usual annihilation operators. Therefore with the normal Hamiltonian $2H=p_1^2+a_1^2+p_0^2+a_0^2$ there is such a vector in the usual representation

$p_k = \frac{1}{i}\frac{\partial}{\partial a_k}$, namely corresponding to the wave function $|0\rangle \sim e^{-(a_0^2+a_1^2)/2}$. With this Hamiltonian $[\chi(t), S(t')] = 0$ such that all vectors $|f\rangle \equiv \exp[i\int dt f(t)S(t)]|0\rangle$ are annihilated by χ^-. Thus in this representation there is no vector with $\chi(t)|0\rangle = 0$ but an infinite dimensional subspace spanned by the $|f\rangle$'s with $\langle f'|\chi(t)|f\rangle = 0$.

4 Quantumelectrodynamics

The next model we will study is the electromagnetic field coupled to a scalar field ϕ [13,14]. First we consider the classical equation. As in the Higgs model we assume the absolute value of ϕ to be fixed by some mechanism so that we set $\phi(x) = e^{iS(x)}$. In this case the Lagrangean of the scalar field becomes ($g_{00} = +1$)

$$\mathcal{L}_s = \frac{1}{2}(\phi_{,\mu} + ieA_\mu\phi)(\phi^*_{,\mu} - ieA_\mu\phi^*) = \frac{1}{2}(S_{,\mu} + eA_\mu)(S^{,\mu} + eA^\mu) \tag{4.1}$$

which leads to a current

$$J_\mu = -e(S_{,\mu} + eA_\mu). \tag{4.2}$$

Thus Maxwell's equations

$$F_{\mu\nu}^{;\nu} = J_\mu \tag{4.3}$$

are complemented by

$$F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu} = e^{-2}(J_{\nu,\mu} - J_{\mu,\nu}), \qquad J_\mu{}^{,\mu} = 0. \tag{4.4}$$

¿From this we infer from (4.3) and (4.4)

$$e^2 F_{\mu\nu}{}^{,\nu} = -J_{\mu,\nu}{}^{\nu} = e^2 J_\mu \tag{4.5}$$

and thus again from (4.4)

$$F_{\mu\nu,\sigma}{}^{\sigma} = -e^2 F_{\mu\nu} \tag{4.6}$$

For a canonical formalism one uses frequently for the electromagnetic Lagrangean

$$\mathcal{L}_e = -\frac{1}{2}A_{\mu,\nu}A^{\mu,\nu}. \tag{4.7}$$

The Euler equations of the total Lagrangean $\mathcal{L} = \mathcal{L}_e + \mathcal{L}_s$ are

$$A_{\mu,\nu}{}^{\nu} = J_\mu = -\frac{\partial \mathcal{L}}{\partial A_\mu}, \qquad J_\mu{}^{,\mu} = 0. \tag{4.8}$$

They are equivalent to (4.3) iff

$$\chi := A_\mu{}^{,\mu} = const. \tag{4.9}$$

Keeping χ as dynamical variable we derive from the Euler equation (4.8) $J_\mu{}^{,\mu} = 0$ in addition to (4.6) the relations

$$\chi_{,\sigma}{}^{\sigma} = 0, \qquad S_{,\sigma}{}^{\sigma} = -e\chi. \tag{4.9}$$

Remarks:

1. Whereas the original system of equations is invariant under the gauge transformations $A_\mu \to A_\mu + \Lambda_{,\mu}$, $S \to S - e\Lambda$ (4.8) allows only for gauge transformations with $\Lambda_{,\mu}{}^{\mu} = 0$. χ is then gauge invariant, S is not.

2. The equations (4.9) were studied in [15] as the prototype of a dipole ghost. There it was shown that for its quantization there is no need for a Hilbert space with indefinite metric. There we had no constraint $\chi = 0$ and we could use a standard representation in terms of creation and annihilation operators. This corresponds in the toy model in Section 3 where both x and p exist as operators. A discussion in the framework of indefinite metric can be found in [16,17].

3. It has to be remarked that the particles of the F-field acquire a mass $|e|$ purely by the equations of motion (4.5). There is no breaking of the gauge group $A_\mu \to A_\mu + \Lambda_{,\mu}$, $S \to S - e\Lambda$ and in (4.20) we shall specify a gauge invariant vacuum.

For the quantization we shall restrict ourselves for simplicity to one space dimension since the transversal degrees of freedom do not add anything to our argument. In two dimensions the dual of $F_{\mu\nu}$ is a scalar which we denote by

$$F = F_{10} = \dot{A}_1 - A_0'. \tag{4.10}$$

Since the equations for F and for χ and S decouple we shall introduce these variables instead of A_0, A and S. From the Lagrangean (4.8) we have the canonical variables

$$\Pi_0 = -\dot{A}_0, \qquad \Pi_1 = \dot{A}_1 \quad \text{and} \quad \Pi_S = \dot{S} + eA_0 \tag{4.11}$$

and using the field equations we find that formally

$$S, \qquad \chi = -\Pi_0 - A_1', \qquad F = \Pi_1 - A_0' \tag{4.12}$$

obey with

$$\dot{S} = \Pi_S - eA_0, \qquad \dot{\chi} = A_0'' - e\Pi_S - \Pi_1', \qquad \dot{F} = A_1'' - eS' - e^2 A_1 + \Pi_0'$$

at equal times the commutation relations

$$i\delta(x-x') = [S(x), \dot{S}(x')] = e^{-1}[\dot{S}(x), \chi(x')] = e^{-1}[\dot{\chi}(x), S(x')] = e^{-2}[F(x), \dot{F}(x')], \tag{4.13}$$

all the other commutators being zero. Because of the field equations

$$F_{,\nu}{}^{\nu} = -e^2 F, \qquad \chi_{,\nu}{}^{\nu} = 0, \qquad S_{,\nu}{}^{\nu} = -e\chi \tag{4.14}$$

we can express the fields arbitrary times by their initial values with the usual invariant functions Δ, D, $\mathcal{D}$ which satisfy

$$\begin{gathered} \Delta_{,\nu}{}^{\nu} + e^2\Delta = 0, \qquad D_{,\nu}{}^{\nu} = 0, \qquad \mathcal{D}_{,\nu}{}^{\nu} = D, \\ \Delta(-x) = -\Delta(x), \qquad D(-x) = -D(x), \qquad \mathcal{D}(-x) = -\mathcal{D}(x), \qquad x \in \mathbf{R}^2, \\ \delta(x) = \dot{\Delta}(0,x) = \dot{D}(0,x). \end{gathered} \tag{4.15}$$

and therefore

$$D(t,x) = \frac{1}{2}[\Theta(t-|x|) - \Theta(|x|-t)], \qquad \mathcal{D}(t,x) = \frac{1}{4}(t^2 - x^2)D(t,x).$$

Thus for arbitrary times (4.13) generalizes to

$$\begin{aligned} [F(x), F(x')] &= i\Delta(x-x')e^2 \\ [\chi(x), \chi(x')] &= 0 \\ [\chi(x), S(x')] &= ieD(x-x') \\ [S(x), S(x')] &= iD(x-x') + e^2 i\mathcal{D}(x-x'). \end{aligned} \tag{4.16}$$

To give these relations a meaning for elements of a C^*-algebra we define the Weyl algebra W by

$$W(f,c,s) = \exp[i\int d^2x(f(x)F(x) + c(x)\chi(x) + s(x)S(x))], \qquad f,c,s \in C_0^\infty(\mathbf{R}^2),$$

$$\|W(f,c,s) - W(f',c',s')\| = \begin{array}{ll} 0 & \text{if } f-f' = (\Box + e^2)g_1, c-c' = \Box g_2, s-s' = -e\Box(c-c'),\ g_i \in \mathcal{S} \\ 2 & \text{otherwise.} \end{array} \tag{4.17}$$

($\Box = \partial^2/\partial x^\mu \partial x_\mu$.) Their algebraic properties are determined by

$$\begin{aligned} W(f,c,s)W(\widehat{f},\widehat{c},\widehat{s}) &= W(f+\widehat{f}, c+\widehat{c}, s+\widehat{s})\cdot \\ &\cdot \exp\{i\textstyle\int d^2x[f(x)e^2\Delta(x-x')\widehat{f}(x') + e(c(x)D(\widehat{x}-x')\widehat{s}(x') \\ &- s(x)D(x-x')\widehat{c}(x')) + s(x)(D(x-x') + e^2\mathcal{D}(x-x'))\widehat{s}(x)]\}. \end{aligned} \tag{4.18}$$

In generalization of our previous state (2.7) the vacuum state is given by

$$\omega_0(W(f,c,s)) = \begin{array}{ll} e^{-e^2 f\Delta^+ f} & s \equiv 0 \\ 0 & s \not\equiv 0. \end{array} \tag{4.19}$$

Remarks (4.20)

1. The usual [3] argument for the necessity of an indefinite metric runs as follows. From (4.16) one infers
$$\langle 0|S(x)S(x')|0\rangle = D^+(x-x') + e^2\mathcal{D}^+(x-x')$$
where
$$\mathcal{D}^+(x) = \int \frac{dp}{2\pi}\Theta(p_0)e^{ipx}\delta'(p^2).$$
Since δ' does not have a definite sign this formula is not possible in Hilbert space. In our formulation S is not an operator but only $\exp[i\int dx S(x)s(x)]$ exists and its expectation value is not $\exp[-\frac{1}{2}\int s(x)\mathcal{D}^+(x-x')s(x')]$ but zero if $s \neq 0$.

2. Naively one might think that the Higgs model contradicts Goldstone's theorem [18–20] because one has a broken symmetry and the physical particles have mass > 0. There is no formal contradiction in our approach since our state is gauge invariant.

3. Note that for $e = 0$ the transformation from A_i, Π_i to (F, χ), $(\dot{F}, \dot{\chi})$ becomes singular $(F' \equiv \dot{\chi})$ and the latter algebra abelian. The A's are expressible as $A_\mu = e^{-1} S_{,\mu} + \varepsilon_{\mu\nu} F^{,\nu}$ but for $e \to 0$ this does not tend to a limit.

We define a Poincaré transformation

$$\sigma_P W(f, c, s) = W(f_P, c_P, s_P) \tag{4.21}$$

with

$$\begin{aligned} f_P(x) &= f(\Lambda^{-1} x - \lambda) \\ c_P(x) &= c(\Lambda^{-1} x - \lambda) \\ s_P(x) &= s(\Lambda^{-1} x - \lambda). \end{aligned} \tag{4.22}$$

Our state is σ_P invariant and therefore σ_P is unitarily implementable. However it is not strongly continuous because

$$\|(W(0,0,t) - W(0,0,s))|\Omega\rangle\| = 2 \quad \text{for } t \neq 0,\ s \equiv 0. \tag{4.23}$$

Therefore there is no generator for the Poincaré group. This corresponds to the fact that the Hamiltonian should contain $\int (\nabla S)^2$ which in our representation is not defined. However, if we restrict our interest on $W(f, c, 0)$ in the corresponding subspace $W(f, c, 0)|\Omega\rangle$ the Poincaré transformation acts strongly continuously, such that the restricted generators are given by $H = \frac{1}{2} \int ((\nabla F)^2 + e^2 F^2) dx$. Furthermore, ω_0 is gauge invariant and this group is strongly continuous in the ω_0-representation. It is implemented by $W(0, c, 0)$.

ω_0 corresponds to an irreducible representation for $\mathcal{M} = \{W(f, c, s)\}$, though in a non-separable Hilbert space. It corresponds to an irreducible though not faithful representation of $\mathcal{M}_{\text{phys}} = \{W(f, c, 0)\}$ in the same way as ω did for $\{e^{isx}\}$. And again it can be obtained as the limit of pure regular states: We follow the calculations in [15]. There we considered the dipole ghost, though without the operator F, that anyhow commutes with χ and S and therefore lives in a separate factor of the tensor product. If we define $B = S + \chi/2$ (where for simplicity we now put $e = 1$), then $[\chi, \dot{\chi}] = [B, \dot{B}] = 0$, whereas $[\chi, \dot{B}] = [B, \dot{\chi}] = i\delta(x - x')$. We can use the result of [15] to write

$$\widetilde{\chi}(k) \quad = \quad \{\alpha_1(k) + \alpha_1^\dagger(k) - i(\alpha_2(k) - \alpha_2^\dagger(k))\}/2\sqrt{k}$$

$$
\begin{aligned}
\dot{\chi}(k) &= -\{\alpha_1(k) + \alpha_1^\dagger(k) + i(\alpha_2(k) - \alpha_2^\dagger(k))\}\sqrt{k}/2 \\
\widetilde{B}(k) &= \{\alpha_2(k) + \alpha_2^\dagger(k) - i(\alpha_1(k) - \alpha_1^\dagger(k))\}/2\sqrt{k} \\
\dot{\widetilde{B}}(k) &= -\{\alpha_2(k) + \alpha_2^\dagger(k) + i(\alpha_1(k) - \alpha_1^\dagger(k))\}\sqrt{k}/2
\end{aligned}
$$

with $\alpha_1(k)$, $\alpha_2(k)$ bosonic annihilation operators.

We consider quasifree states with

$$\langle \alpha_1(k)\alpha_1^\dagger(k')\rangle = \langle \alpha_2(k)\alpha_2^\dagger(k')\rangle = \rho(k)\delta(k-k')$$

$$\langle \alpha_1(k)\alpha_1(k')\rangle = -\langle \alpha_2(k)\alpha_2(k')\rangle = \sigma(k)\delta(k-k')$$

$$\langle \alpha_1^{(\dagger)}(k)\alpha_2^{(\dagger)}(k')\rangle = 0.$$

Calculating

$$\langle \chi^2\rangle = (4\rho - 2 - 2\sigma - 2\bar{\sigma})/4k, \qquad \langle \dot{\chi}^2\rangle = (4\rho - 2 - 2\sigma - 2\bar{\sigma})k/4$$

and

$$\langle B^2\rangle = (4\rho - 2 + 2\sigma + 2\bar{\sigma})/4k,$$

we can choose $\sigma = \sqrt{\rho(\rho-1)}$ such that ω is pure and take the limit $\rho \to \infty$. Then $\lim\langle\chi^2\rangle_\rho = 0$ whereas $\lim\langle B\rangle_\rho$ diverges, in fact, we should calculate $\langle e^{iB\gamma}\rangle$ which oscillates faster and faster.

In [15] it was mentioned that there does not exist a pure state that is space- and also time-invariant. But silently is was assumed that in the representation the Weyl operators should be strongly continuous (regular). A typical term for the time dependence is

$$\langle \alpha_1^\dagger(k,t)\alpha_1(k',t)\rangle = \delta(k-k')[\rho_1\cos^2 kt + \rho_2 \sin^2 kt - 1 + (2\rho_1 - 1 - 2\sigma_1)\frac{t^2}{k^2}\cos^2 kt +$$

$$+ (2\rho_2 - 1 - 2\sigma_2)\left(\frac{t}{k} + \frac{1}{k^2}\right)^2 \sin^2 kt].$$

If $\rho_1 = \rho_2 \to \sigma_1 + 1/2 = \sigma_2 + 1/2$ this becomes time independent and ω invariant. This should be compared with the result of [17]: In the representation with indefinite metric the time evolution is implementable but in terms of creation and annihilation operators with pseudocanonical commutation relations.

We should also compare this result with the model discussed in [21]: Again there is a fourth order differential equation. This makes the Hamiltonian unbounded from below. But the singularity is not drastic enough to forbid time independent regular representations. Only the positivity condition is violated. Again this can be cured by escaping to indefinite metric, and again this indefinite metric corresponds to an irregular representation that is regular only in the physical subspace on the physical observables.

5 External Perturbations

One might be afraid that representations like π_0 where the Weyl operators are not strongly continuous in all variables will lead to difficulties in perturbation theory when one wants to expand in powers of the parameter of a perturbation. We shall investigate how serious these troubles might become in the simplest typical cases. For this purpose we shall couple the oscillators to an external (= c-number) force and the Higgs model to an external current. In both cases we shall see that for good couplings the perturbation expansion exists already for the time evolution of the algebra independent of a particular state and is not related to the vacuum satisfying a constraint.

In the case of oscillators we generalize the Hamiltonian (3.6) to

$$H = S\chi + \Pi_\chi \Pi_S - j\chi \tag{5.1}$$

where $j(t)$ is some c-number $L^1(\mathbf{R})$ function. This changes the canonical equations to

$$\dot{S} = \Pi_\chi, \qquad \dot{\Pi}_\chi = j - S, \qquad \dot{\chi} = \Pi_S, \qquad \dot{\Pi}_S = -\chi. \tag{5.2}$$

They are solved by

$$\begin{aligned} S(t) &\equiv S\cos t + \Pi_\chi \sin t + \textstyle\int_0^t dt' j(t') \sin(t-t') \\ \chi(t) &= \chi \cos t + \Pi_S \sin t \\ \Pi_S(t) &= -\chi \sin t + \Pi_S \cos t \\ \Pi_\chi(t) &= -S \sin t + \Pi_\chi \cos t + \textstyle\int_0^t dt' j(t') \cos(t-t'). \end{aligned} \tag{5.3}$$

In quantum theory these classical solutions induce an automorphism $\Theta_j(t)$ of the Weyl algebra ($\Theta_j(t) e^{i\alpha S} := e^{i\alpha S(t)}$, etc.) and keeping the notation (3.13), we see $W(\alpha, \beta, \gamma, \delta)(t) =$

$\Theta_0(t)W(\alpha,\beta,\gamma,\delta)$ and

$$\Theta_j(t)(W(\alpha,\beta,\gamma,\delta)) = W(\alpha,\beta,\gamma,\delta)(t)\exp[i\int_0^t dt' j(t')(\alpha\sin(t-t') + \delta\cos(t-t'))]. \tag{5.4}$$

Since even $\Theta_0(t)$ was not norm continuous in t we see that $\Theta_j(t)$ cannot be either. On the other hand, the j-dependence is contained in a finite phase factor such that the map $L^1(\mathbf{R})\times\mathcal{W}\to\mathcal{W}$: $j, W \to \Theta_j(t)W$ is norm continuous. The j part is even continuous in t such that $\Theta_0(-t)\Theta_j(t)$ as map $\mathcal{W}\to\mathcal{W}$ is norm continuous in t. Furthermore since

$$\Theta_0(-t)\Theta_j(t)W(\alpha,\beta,\gamma,\delta) = W(\alpha,\beta,\gamma,\delta)e^{i(\alpha a(t)+\delta d(t))} \tag{5.5}$$

with the c-numbers

$$a(t) = \int_0^t dt' j(t')\sin(t-t'), \qquad d(t) = \int_0^t dt' j(t')\cos(t-t'),$$

$\Theta_0(-t)\Theta_j(t)$ is just a shift. This shows that it is an inner automorphism and thus in any representation unitarily implemented:

$$\Theta_0(-t)\Theta_j(t)W(\alpha,\beta,\gamma,\delta) = W(0,-a/2,d/2,0)W(\alpha,\beta,\gamma,\delta)W(0,a/2,-d/2,0).$$

Therefore for these perturbations a power series expansion in j exists even on the level of the dynamical C^*-system $(\mathcal{W},\Theta)$ without reference to a particular state. The state $\omega_0(W(\alpha,\beta,\gamma,\delta)) = \delta_{\alpha 0}\delta_{\delta 0}$ is invariant under $\Theta_j(t)$ and the corresponding vector $|0\rangle$ (3.9) has eigenvalue 1 for the generators $W(0,\beta,\gamma,0)$ for $\Theta_0(-t)\Theta_j(t)$. This means that the constraint is still that the $\Theta_j(t)$ invariant subalgebra generated by $\chi(t)$ annihilates $|0\rangle$. We could also consider a representation generated by

$$\bar{\omega}_0(W(\alpha,\beta,\gamma,\delta)) = \delta_{\beta 0}\delta_{\gamma 0}. \tag{5.6}$$

This state is not invariant under $\Theta_j(t)$ but

$$\bar{\omega}_0(\Theta_j(t)W(\alpha,\beta,\gamma,\delta)) = e^{i(\alpha a(t)+\delta d(t))}\bar{\omega}_0(W(\alpha,\beta,\gamma,\delta)). \tag{5.7}$$

For this representation the cyclic vector is no longer an eigenvector for the generator $W(0,\beta,\gamma,0)$ of $\Theta_0(-t)\Theta_j(t)$. This could be anticipated from the inhomogeneous time dependence of S (5.3) and thus for $j\neq 0$ the cyclic vector $|\bar{0}\rangle$ for (5.6) cannot satisfy $S(t)|\bar{0}\rangle = 0\;\forall t$. Since we assumed

$j(t) \in L^1(\mathbf{R})$ we note that $\lim_{t\to\pm\infty} \Theta_0(-t)\Theta_j(t)$ exists as norm limit when applied to $W \in \mathcal{W}$ and thus the Møller operators and the S matrix exist in any representation. Turning to the Higgs model we couple to an external current by adding to (4.1) and (4.7)

$$\mathcal{L}_i = -A_\mu j_\mu. \tag{5.8}$$

To keep gauge invariance we impose on j the conservation equation $j^\mu{}_{,\mu} = 0$ in which case the Euler equations from $\mathcal{L} = \mathcal{L}_e + \mathcal{L}_s + \mathcal{L}_i$, namely,

$$A_{\mu,\nu}{}^{\nu} = -e(S_{,\mu} + eA_\mu) + j_\mu, \qquad (S_{,\mu} + eA_\mu)^{,\mu} = 0 \tag{5.9}$$

imply again for $\chi = A_\mu{}^{,\mu}$, $\Box S \equiv S_{,\mu}{}^{\mu}$ etc.

$$\Box\chi = 0, \qquad \Box S = -e\chi. \tag{5.10}$$

(5.9) are equivalent to Maxwell's equations

$$F_{\mu\nu}{}^{,\nu} = J_\mu + j_\mu \tag{5.11}$$

whenever the constraint $\chi = const.$ holds. In two dimensions with the notation (4.10), in addition to (5.10) we find

$$(\Box + e^2)F = j, \qquad j = j_{1,0} - j_{0,1}. \tag{5.12}$$

We have seen the $\mathcal{W}$ tensors into $W(f,0,0)$ and $W(0,c,s)$, though the field algebra is no longer characterized by (4.17) but by initial data like (3.8) which anyhow is a one to one correspondence. Only $W(f,0,0)$ is affected by j whereas the rest is not since (5.10) is unchanged.

Since again the scattering automorphism corresponds to a phase for the Weyl operators, the automorphism is inner and the representations equivalent for the full field algebra. For the gauge invariant observable algebra this is not necessarily so (compare [7]).

If $e \neq 0$, then ω_0 (4.19) on $W(f,0,0)$ is irreducible and every shift is inner. Therefore the representations are equivalent. For $e = 0$ the relations between F and χ, given by the evolution equation, involve j and the representations of $W(f,c,0)$ for different j become inequivalent.

6 Conclusions

In the usual canonical formulation of QED the field equations are equivalent to Maxwell's equation provided $\chi = A_\mu{}^{,\mu}$ vanishes. In quantum theory one faces the difficulty that not every ideal is a twosided ideal hence there is no representation Π with $\Pi_{|\chi} = 0$. There can be just a subspace $\{|g\rangle\}$ in Hilbert space (the "physical space") such that $\Pi(\chi)|g\rangle = 0$. Even the existence of such vectors $|g\rangle$ is problematic and usually one tries to reach mathematical consistency by introducing an indefinite scalar product in Hilbert space to obtain $\langle g|g\rangle < \infty$. In such a representation one loses positivity and norm and with them important tools of analysis. Therefore we propose a conservative procedure where the C^*-algebra of the fields $\mathcal{A}$ obeys the canonical commutation rules in their Weyl form and is represented in a Hilbert space $\mathcal{H}$ by an isomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. It turns out that the existence of a physical subspace $\{|g\rangle : \chi|g\rangle = 0\}$ requires $\mathcal{H}$ to be non-separable and the Weyl operators are not strongly continuous in some arguments. As a consequence the gauge variant fields do not exist as selfadjoint operators but only their associated unitaries. However, there is no real difficulty since observables as gauge invariant quantities exist and their representation in the physical space [22] is the usual one. To this extent the results of this paper only assure that (disregarding divergences in perturbation theory) the C^*-algebraic formulation is adequate for QED. There are of course other methods of treating the problem and even in our conservative procedure there is a one-parameter family of Lagrangeans which lead to different field equations which are equivalent mod $\chi = 0$ and are equally good. However they lead to different commutation relations for the A and it would be unpleasant if the procedure described here were only to work for some of them. As will be examined in the appendix this is not so but as long as one of the canonical conjugate variables does not vanish identically, our method works equally well.

Appendix A

Here we shall verify that the commutation relations from a particular class of Lagrangeans

$$\mathcal{L} = \frac{1}{2}\{-A_{\mu,\nu}A^{\mu,\nu} + \alpha A_{\mu,\nu}A^{\nu,\mu} + (S_{,\mu} + eA_\mu)^2\} \tag{A.1}$$

with the field equations

$$A_{\mu,\nu}{}^{\nu} - \alpha A^{\nu}{}_{,\mu\nu} = -eS_{,\mu} - e^2 A_\mu \tag{A.2}$$

leads $\forall\, \alpha \neq 1$ to commutative relations which are consistent with the constraint $A^{\mu}{}_{,\mu}|0\rangle = \chi|0\rangle = 0\ \forall\,(x,t)$. (For simplicity we shall use the Heisenberg version.) If we start with 2 dimensions the 3 conjugate fields are

$$\Pi_0 = -(1-\alpha)\dot{A}_0, \qquad \Pi_1 = \dot{A}_1 - \alpha A_0', \qquad \Pi_S = \dot{S} + eA_0.$$

This gives for the constraint variable $\chi = \dot{A}_0 - A_1'$

$$\begin{aligned}
[\chi(x,t),\dot{\chi}(x',t)] &= \left[-\frac{1}{1-\alpha}\Pi_0 - A_1', \frac{1-\alpha}{1-\alpha}A_0'' - \frac{1}{1-\alpha}\Pi_1' - \frac{e}{1-\alpha}\Pi_S\right] \\
&= -\frac{1}{1-\alpha}[\Pi_0(x,t), A_0''(x',t)] + \frac{1}{1-\alpha}[A_1'(x,t), \Pi_1'(x',t)] \\
&= i\delta''(x-x')\left(\frac{1}{1-\alpha} - \frac{1}{1-\alpha}\right) = 0
\end{aligned}$$

$$\begin{aligned}
[F(x,t),\dot{F}(x',t)] &= [\Pi_1 - (1-\alpha)A_0', (1-\alpha)A_1'' + \Pi_0' - eS' - e^2 A_1] \\
&= [\Pi_1, (1-\alpha)A_1'' - e^2 A_1] - (1-\alpha)[A_0', \Pi_0'] \\
&= ie^2\delta(x-x')
\end{aligned}$$

$$[\chi(x,t), F(x',t)] = -\left[\frac{1}{1-\alpha}\Pi_0 + A_1', \Pi_1 - (1-\alpha)A_0'\right] = 0$$

$$\begin{aligned}
[\dot{F}(x,t),\dot{\chi}(x',t)] &= \left[(1-\alpha)A_1'' + \Pi_0' - eS' - e^2 A_1, A_0'' - \frac{1}{1-\alpha}\Pi_1' - \frac{e}{1-\alpha}\Pi_S\right] \\
&= \frac{1}{1-\alpha}[(1-\alpha)A_1'' - e^2 A_1, \Pi_1'] + [\Pi_0', A_0''] + \frac{e^2}{1-\alpha}[S', \Pi_S] = 0.
\end{aligned}$$

The two remaining commutators $[\dot{F},\chi]$ and $[F,\dot{\chi}]$ vanish obviously since no conjugate quantities occur. Thus $\forall\,\alpha \neq 1$, χ with $\dot{\chi}$ generate an abelian subalgebra and F and $\dot{F}$ generate a massive bosonic field which becomes abelian for $e^2 \to 0$. For $e = 0$, in addition, the fields χ, $\dot{\chi}$ and F and $\dot{F}$ become linearly dependent. So the constraint $\chi \equiv 0$ implies $F \equiv 0$. By a somewhat more involved calculation one verifies that also $3+1$ dimensions $\chi = A_k{}^{,k}$ and $\dot{\chi}$ commutes for all α such that $\chi(t)|0\rangle = 0$ can be imposed consistently.

Appendix B

We want to discuss on the example of the Weyl algebra how a state that satisfies the constraint can be constructed.

We consider the automorphism group

$$\alpha_t W(r,s) = e^{irt} W(r,s).$$

Starting with some state $\omega(W(r,s))$ we can take some invariant mean (which exists) that will have the form

$$\bar{\omega}(W(r,s)) = \delta_{r0} f(s).$$

For this state we can construct its GNS representation and find in its commutant operators $M(0,s)$ with

$$\langle\Omega|\pi(W^*(r,t))M(0,s)\pi(W(\bar{r},\bar{t}))|\Omega\rangle = \delta_{r\bar{r}} f(s+\bar{t}-t)e^{ir(t-\bar{t})/2}$$

or

$$M(0,s)\pi(W(\bar{r},\bar{t}))|\Omega\rangle = \pi(W(\bar{r},\bar{t})\cdot W(0,s))|\Omega\rangle$$

as well as operators $M(p,0)$ with

$$M(p,0)\pi(W(\bar{r},\bar{t}))|\Omega\rangle = \pi(W(\bar{r},\bar{t})\cdot W(p,0))|\Omega\rangle.$$

$\{M(0,s)\}''$ is invariant under α_t and maximal abelian in the commutant of $\pi(\mathcal{W})$. Therefore it implies a unique decomposition of $\bar{\omega}$ into extremal α_t invariant states. Since $\{M(0,s)\}''$ is maximal abelian in $\{M(r,s)\}''$ they are at the same time irreducible for $\{W(r,s)\}''$. Furthermore they correspond to vectors with pure point spectrum of $W(0,s)$.

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