Conditional symmetries and conditional constants of motion for dynamical systems

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Summary. We apply conditional Lie-point symmetries to the study of dynamical systems, discussing in particular their relation to conditional constants of motion; we also briefly discuss the relation between conditional symmetries, conditional constants of motion, and invariant manifolds.

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Introduction

Symmetry methods are among the most effective tools in studying nonlinear differential equations, both ODEs and PDEs [1-4]. Here we are specially interested in ODEs, which we can write in full generality as a first order system; for ease of notation we will assume the system is autonomous. (This is no restriction since if we have a nonautonomous system, we can always add one variable x_0 and one equation $\dot{x}_0 = 1$, changing it into an equivalent autonomous system).

We will write $\xi = (x_1, \ldots, x_N) \in M \subseteq \mathbf{R}^N$ and write the ODE Δ as

$$\Delta \ : \qquad \dot{\xi} = f(\xi) \qquad\qquad f : M \to TM \tag{1}$$

If we know the algebra $\mathcal{G}_\Delta$ of exact symmetries of (1), we can choose symmetry adapted variables $w = w(\xi)$ on M, such that

$$\dot{w}^i = 0 \qquad i = 1, ..., m \ , \ m \leq N \tag{2}$$

where $m \leq N$ depends on $\mathcal{G}_\Delta$ and, more precisely, on the dimension of the maximal solvable subalgebra of $\mathcal{G}_\Delta$ (see [1,2] for exact conditions and results); symmetry reduction of ODEs is also described and discussed in [5,6], and e.g. in [7-9] in the context of Hamiltonian mechanics.

In the application of symmetry methods to PDEs, a prominent role is played also by "non classical" [10], or "conditional" [11] symmetries. These are also related to compatibility conditions of the original PDE and a differential constraint, or "side condition" [12,13] imposed on it, see [14,15]; a relation also exists to the so called "direct reduction" method [16], see [11, 14,15,17]; see also [18,19] for related discussion.

Quite surprisingly, the application of this method to ODEs and dynamical systems (DS) - by which we mean ODEs in the form (1) - seems not to have been considered in the Lie-theoretic literature; our goal here is to study this question.

It should be mentioned that the application of symmetry methods to equations of the simple form (1) can run into unexpected troubles - not met for PDEs - precisely because of the semplicity of the equations [20], which introduces degeneracies. A typical problem is constituted by the fact that $\mathcal{G}_\Delta$ turns out to be an infinite-dimensional Lie algebra [4]; this fact is better understood by remarking that, if I_Δ is the algebra of conserved quantities for the equation Δ, then $\mathcal{G}_\Delta$ is a module over I_Δ [20-23], as discussed below (see section 2); in the presence of nontrivial conserved quantities, $\mathcal{G}_\Delta$ will be infinite dimensional as a Lie algebra; this does not forbid that it is finite dimensional as a module over I_Δ.

Considering conditional symmetries (CS) will naturally lead us to consider "conditional constants of motion" (CCM); it will moreover turn out that certain subsets of CS for Δ have a structure of module over an associated subset of CCM.

Pretty much as in the PDE case, use of CS will permit us to determine a set of special solutions to the nonlinear DS under study. Actually, CS and CCM will be related to *invariant manifolds* of the DS, and the methods developed in the PDE frame can help to identify these. We will discuss this issue, and in particular the case in which the DS admits a fixed point $\bar{\xi}$, so that it is natural to consider the stable, unstable and center manifolds [24-27] for this fixed point; similar considerations would apply if considering a periodic solution rather than a stationary one, or even more general invariant sets. We will then be specially interested in the case the DS (1) depends on external control parameter and the fixed point $\bar{\xi}$ undergoes a bifurcation; this will also permit to make contact with recent works of ours [21,28-30].

We should point out that our results will be in the direction of a pioneering work by Wulfman [31]. Also, after the completion of this work we came across a paper by Sarlet, Leach and Cantrijn [32], who consider CCM in the context of hamiltonian systems; they call them "configurational invariants", and discuss their relation to a weak form of complete integrability (which in our present vocabulary would be "conditional complete integrability"). Although the paper [32] does not mention conditional symmetries - among other good reasons, because they had not been introduced yet - the reader will have no difficulties in placing it into the present framework; this will also exempt us from dealing specifically with hamiltonian dynamical systems.

Formulas are numbered consecutively in each section; obviously $(i.j)$ will refer to formula j of section i, and we will omit the section index when referring to formulas in the same section.

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1. Lie-point symmetries for dynamical systems

Let M be a smooth manifold, which we think as embedded in $\mathbf{R}^N$; a *Dynamical System* (DS) on M is a system of autonomous first order ODEs

$$\dot{x} = f(x) \tag{1}$$

$$x = (x^1, \ldots, x^N) \in M \quad , \qquad f : M \to TM \tag{2}$$

where f is a smooth function. If f is smooth of class $\mathcal{C}^k$ (analytic), we talk of a $\mathcal{C}^k$ (analytic) DS; unless otherways specified, in the following we consider smooth, i.e. $\mathcal{C}^\infty$, DS.

The DS (1) corresponds to the flow on M under the action of the *vector field* (VF)

$$\hat{f} = f(x)\partial_x \equiv f^i(x)\frac{\partial}{\partial x^i} \tag{3}$$

The set of tangent (smooth) VF on M equipped with the Lie commutator $[.,.]$ form a Lie algebra, which we denote by $\mathcal{M}$. Clearly, to any VF $\hat{f} \in \mathcal{M}$ we can associate a smooth function $f : \mathbf{R}^N \to \mathbf{R}^N$, $f : x \in M \to T_x M$ (actually, a section of the tangent bundle of M, TM), which gives its component expansion in the x coordinates, as in (3). The space of such smooth functions $M \to TM$ will be denoted by $\mathcal{V}$; this is naturally equipped with a Lie-Poisson bracket $\{.,.\}$ induced by $[.,.]$, which is given by

$$\{f, g\} = (f \cdot \nabla) g - (g \cdot \nabla) f \tag{4}$$

A VF $\hat{s} \in \mathcal{M}$ is a *Lie-point* (LP) symmetry[1] of the DS (1) if under its flow solutions of (1) are transformed into (generally, different) solutions of (1). In the present setting, this is equivalent to the condition $[\hat{f}, \hat{s}] = 0$ or, in terms of (4),

$$\{f, s\} = 0 \tag{5}$$

Remark 1. Notice that, although in the present setting both f and s do not involve neither depend on time, so that they give *Lie-point time-independent* (LPTI) vector fields, the case of nonautonomous VF is readily recovered by adding a new coordinate x_0, whose evolution is given by $\dot{x}_0 = 1$. If s does depend on time but f does not, and we keep the time variable distinct from the space ones, eq.(5) would be replaced by

$$s_t + \{f, s\} = 0 \tag{5'}$$

More in general [1-4], LP symmetries would be VF on $\mathbf{R} \times M$, of the form

$$\eta = \varphi^i(x,t) \frac{\partial}{\partial x^i} + \tau(x,t) \frac{\partial}{\partial t} \tag{6}$$

Under this, i.e. under $e^{\varepsilon \eta}$ with $\varepsilon \to 0^+$,

$$t \to t + \delta t \ , \quad x^i \to x^i + \delta x^i \ , \quad \dot{x}^i \to \dot{x}^i + \delta \dot{x}^i \tag{7}$$

$$\begin{aligned} \delta t = \varepsilon \tau(x,t) \quad &, \quad \delta x^i = \varepsilon \varphi^i(x,t) \\ \delta \dot{x}^i = \varepsilon [D_t(\varphi^i - \tau \dot{x}^i) + \tau \ddot{x}^i] &= \varepsilon [\partial_t \varphi^i + \dot{x}^j \partial_j \varphi^i] \end{aligned} \tag{8}$$

where $\partial_i \equiv \partial / \partial x^i$, and we have used the prolongation formula [1-4].

Applying this to (1), we get on solutions (i.e. for $\dot{x}^i = f^i(x)$)

$$\partial_t \varphi^i + f^j \partial_j \varphi^i - f^i \partial_t \tau - f^i f^j \partial_j \tau = \varphi^j \partial_j f^i$$

[1] More precisely, a VF would be the generator of a LP symmetry; we will use this innocent abuse of language, which should not cause any confusion, for ease of discussion.

which, if η does not involve a reparametrization of time (if not possibly a trivial shift), so that $\tau = const.$, reads

$$\partial_t \varphi^i + \{f, \varphi\} = 0$$

and we recover (5'), or (5) if φ does not depend on time.

Remark 2. Notice that if τ is not identically zero, we can divide (6) by τ, reaching the form $\partial_t + \varphi\partial_x$; anyway, this could introduce singularities, corresponding to zeroes of τ, and take us out of the class of smooth VF.

Remark 3. It is particularly convenient, in the study of autonomous DS, to consider VF for which $\tau = 0$, $\varphi = \varphi(u)$; some reasons for this will be recalled below, for other see e.g. [20-23].

It is easy to see, e.g. by Jacobi identity, that the VF satisfying (5) - or even (5') - form a Lie algebra under $[.,.]$ or, which is the same, under $\{.,.\}$. This algebra will be denoted by $\mathcal{G}_f$.

2. Module structure of the symmetry algebra

Let us now consider the case (1.1) admits a *constant of motion* (CM); i.e., there is a function $P(x) : M \to \mathbf{R}$ such that

$$\frac{d}{dt}P(x) \equiv (\dot{x} \cdot \nabla)P = (f(x) \cdot \nabla)P(x) = 0 \tag{1}$$

In this case, it is easy to see that if $\{f, s\} = 0$, then also $\{f, g(P(x))s\} = 0$ for smooth g. Indeed,

$$\{f, g(P(x))s\} = g'(P(x))s(x)(f(x) \cdot \nabla)P(x) + g(P(x))\{f, s\} = g(P(x))\ \{f, s\} \tag{2}$$

The constants of motion form an algebra under addition (i.e. if $P_1(x)$, $P_2(x)$ are CM, then also $P_+(x) = P_1(x) + P_2(x)$ is a CM); this algebra will be denoted by I_f.

Suppose now that $s_1, s_2 \in \mathcal{G}_f$, $P_1, P_2 \in I_f$; the above (2) and bilinearity of $\{.,.\}$ show that then

$$P_1(x)s_1(x) + P_2(x)s_2(x) \in \mathcal{G}_f \qquad \forall P_1, P_2 \in I_f\ ,\ \forall s_1, s_2 \in \mathcal{G}_f \tag{3}$$

This shows that $\mathcal{G}_f$ is an infinite dimensional algebra for nontrivial[2] I_f; More in general it proves that

[2] I_f does always contains the functions which are constant on the whole M, which are trivially also CM.

Lemma 1. *$\mathcal{G}_f$ is a module over the algebra I_f.*

Notice that although $\mathcal{G}_f$ is infinite dimensional as a Lie algebra, it is in general, for $f \not\equiv 0$, finite dimensional as a module. For definition and properties of modules, see any text on algebra, or e.g. [33].

Remark 4. If we see (1) as an equation determining the $P(x)$ which are constants of motions for f, this is a first order linear PDE and its solutions are therefore detrmined by solving the associate characteristic system [34,35]

$$\frac{dx^1}{f^1(x)} = \ldots = \frac{dx^N}{f^N(x)}$$

3. Symmetry reduction for ODEs and symmetry adapted coordinates

Let us briefly recall how the knowledge of $\mathcal{G}_f$, or part thereof, can be used to simplify the DS (1.1). If we know a single VF $\hat{s} = s(x)\partial_x \in \mathcal{G}_f$, we can change coordinates and pass to coordinates $w(x)$ such that, say,

$$\hat{s} = \frac{\partial}{\partial w^N} \tag{1}$$

i.e. such that $\hat{s}$ is along one of the coordinates, which are therefore called *symmetry adapted* [2].

The commutation relation $[\hat{f}, \hat{s}] = 0$ being satisfied indipendently of the coordinate choice, in the w coordinates $\hat{f}$ will be independent of w^N. This means that the system (1.1) will be transformed, in the coordinates w, into

$$\dot{w} = g(w) \quad ; \quad \partial g(w)/\partial w^N = 0 \tag{2}$$

so that we are reduced to a system of $(N-1)$ equations for the first $(N-1)$ variables, plus an equation giving the evolution of the variable w^N in terms of the others, i.e. of solutions of the $(N-1)$ dimensional system.

Example 1. Let us consider the system

$$\begin{aligned} \dot{x} =& \alpha(r^2)x - \beta(r^2)y \\ \dot{y} =& \beta(r^2)x + \alpha(r^2)y \end{aligned} \tag{3}$$

where $r^2 = x^2 + y^2$, and α, β are smooth functions. This is symmetric under rotations, generated by the VF $\hat{s} = -y\partial_x + x\partial_y$. Passing to polar coordinates (r, ϑ), we have $\hat{s} = \partial_\vartheta$, and the time evolution is given by

$$\begin{aligned} \dot{r} =& \alpha(r^2) \\ \dot{\vartheta} =& \beta(r^2) \end{aligned} \tag{4}$$

Remark 5. As shown concretely by this example, the passage to symmetry adapted coordinates can require a singular change of coordinates.

In the case we know a subalgebra $\mathcal{G}_0 \subseteq \mathcal{G}_f$, the procedure can be iterated d times, where d is the dimension of the maximal solvable subalgebra in $\mathcal{G}_0$ [1,2]; it should be stressed that the VF in $\mathcal{G}_0$ to be successively rectified should be chosen in the appropriate order, see [1]. In this way (1.1) is reduced to an $(N-d)$ dimensional system plus d "quadrature" equations.

If we know a constant of motion $P(x)$, we can also take advantage of this by passing to coordinates $z(x)$, chosen in such a way that $z^N = P(x)$; in these new coordinates, (1.1) will read

$$\begin{aligned} \dot{z}^i =& h^i(z) \qquad i = 1, ..., N-1 \\ \dot{z}^N =& 0 \end{aligned} \tag{5}$$

Notice that the h^i will depend on all the z coordinates, including z^N.

Example 2. If $\alpha = 0$ in (3), r is a CM; indeed with $\alpha = 0$ the polar coordinate equations (4) are just

$$\dot{r} = 0 \quad ; \quad \dot{\vartheta} = \beta(r^2) \tag{6}$$

More in general, if we know p functionally independent CM, $P_1(x), ..., P_p(x)$, we can pass to coordinates $z_1, ..., z_{N-p}, \vartheta_1, ..., \vartheta_p$, with $\vartheta_i \equiv P_i(x)$; in these coordinates,

$$\begin{aligned} \dot{z}^i =& h^i(z, \vartheta) \\ \dot{\vartheta}^j =& 0 \end{aligned} \tag{7}$$

Remark 6. The CM can be seen as invariant functions for the one-parameter group $e^{\lambda \hat{f}}$. Then a celebrated theorem of Hilbert (for polynomial functions; extended by Schwarz [36,37] to the smooth case) ensures that [38-41] there exists a basis $\Theta_1, ..., \Theta_d$ of invariant functions (respectively, polynomials), such that any $P(x) \in I_f$ (polynomial $P(x) \in I_f$) can be written as a function (as a polynomial) of the Θ's; i.e. as $P(x) = \widetilde{P}(\Theta(x))$.

Notice that the ϑ's enter in (7) only as parameters; the problem is therefore reduced to an $(N-p)$ dimensional one with p parameters. Fixing the values of the ϑ's identifies a submanifold $M_\vartheta \subset M$, by definition invariant under the flow of (1); one is therefore legitimate to consider the reduction of (1.1) to this

$$\dot{x} = f_\vartheta(x) \qquad x \in M_\vartheta \ , \ f_\vartheta \equiv f|_\vartheta : M_\vartheta \to TM_\vartheta \tag{8}$$

Remark 7. Notice that not necessarily $\dim(M_\vartheta) = [\dim(M) - p]$: e.g., in the example 2 above, for $r = 0$ we have a zero dimensional $M_\vartheta = \{0\}$.

In the following, we will assume that the reduction corresponding to global CM and/or symmetries has already been performed.

Remark 8. It is clear from (7) that $\hat{s} \in \mathcal{G}_f$ will be, in the (z, ϑ) coordinates, of the form

$$\hat{s} = r(z, \vartheta)\partial_z + t(\vartheta)\partial_\vartheta \tag{9}$$

as can be easily verified by direct computation.

Remark 9. It should be stressed that $(f \cdot \nabla)P = 0$ does *not* imply $(s \cdot \nabla)P = 0$ for $\{f, s\} = 0$, as the following example shows.

Example 3. Let us consider again (3), with $\alpha = 0$, $\beta(r^2) = 1$. In this case r is a CM, but the system is symmetric under $\hat{s} = x\partial_x + y\partial_y$, which generates scaling transformations and does *not* leave r invariant.

4. Topology of trajectories and LP symmetries

It should be briefly recalled that LP transformations, being smooth and locally invertible[3], cannot transform a subset $A \subset M$ into a topologically different one.

This applies in particular to the trajectory of a point $x \in M$ under the flow of (1), so that trajectories of topologically "special" types, e.g. fixed points, periodic orbits, or orbits filling densely a (topological) k-dimensional torus, cannot be transformed to orbits of different types. Isolated fixed points, periodic orbits, k-dimensional invariant tori, are therefore invariant under LP transformations [20,23].

Remark 10. Notice that if we consider full solutions instead than trajectories, i.e. the graph $(t, x(t))$ instead than the set $\{x(t)\ ,\ t \in \mathbf{R}_+\}$, we do not have topological difference between graphs of solutions corresponding to topologically different trajectories.

5. Smooth structures and Lie-point symmetries

It should be understood that preservation of topology of trajectories is not the only limit on the type of solutions that can be connected by a LP transformation. Roughly speaking, preservation of topology just follows from the fact that LP transformations are one to one (invertible) and continuous; the fact that they are not only $\mathcal{C}^0$ but $\mathcal{C}^\infty$, or analytic, poses further constraints, as we are going to discuss in this section [20,23,42].

[3] The invertibility is global if we consider groups of LP transformations.

A first obvious remark is that a polynomial (in general, a $\mathcal{C}^\infty$) LP transformation cannot transform a $\mathcal{C}^\infty$ solutions into a $\mathcal{C}^k$ one, $k<\infty$ (this will play a role in the discussion of center manifolds).

Another useful remark is that a polynomial LP transformation cannot connect two trajectories that diverge exponentially. More in general, let $x(t)$ be the solution to (1.1) with initial datum $x(0)=x_0$; this identifies an invariant curve under $\hat{f}$ in M, the trajectory γ_0 of $x(t)$. Suppose now that the linearization of $\hat{f}$ at γ_0 has positive eigenvalues in some direction $\xi \in T_{x_0}M$ (transversal to γ_1); let $\hat{s}$ be a VF such that $s(x_0)=\xi$, and let $x_1 = e^{\varepsilon \hat{s}} x_0$ (with ε small). Let us consider an interval of γ_0 of lenght $\ell > 0$, and a tubular neighbourhood u_δ of γ_0 of radius δ. Then the trajectory γ_1 issued from x_1 will get out of u_δ for ℓ long enough (the trajectories γ_0, γ_1 diverge with positive Lyapounov exponent [43]). By the uniform continuity of LP transformations, γ_1 and γ_0 cannot be connected by a LP transformation.

There is a point in the above reasoning that should be emphasized: the separation is exponential in the curvilinear coordinate along γ_0 (by taking δ small enough, we can always choose a system of coordinates in u_δ such that γ_0 corresponds to, say, $x_2 = x_3 = ... = x_N = 0$, so the exponential in x_1 is well defined), but not necessarily exponential in time.

For given δ_0 and ℓ, by taking $\varepsilon < \delta_0$ small enough, we can guarantee that $|e^{\varepsilon \hat{s}} x - x| < \delta_0 \ \forall x \in \gamma_0(\ell)$; but for any δ_0 , ℓ we can find a point $x_1 = e^{\varepsilon \hat{s}} x_0$, $\varepsilon < \delta_0$, such that $|\gamma_1(\lambda) - \gamma_0(\lambda)| > \delta_0$ for $\lambda_0 < \lambda < \ell$.

Notice that exponential separation of solutions in time does not forbid that their two trajectories are connected by a LP transformation, as the example 4 below shows; also, the trajectories can separate exponentially in the curvilinear coordinate even if the solutions do not separate exponentially in time, see example 5.

Example 4. Let us consider the simple system

$$\begin{aligned}\dot{x} =& x\\ \dot{y} =& y\end{aligned}$$

which has solutions $(x(t), y(t)) = (x_0 e^t, y_0 e^t)$, so that the trajectories are just straight lines $y = cx$, percurred with exponentially increasing speed. Let us consider the motions with initial data

$$p_0 = (a, 0) \quad ; \quad p_1 = (a, \varepsilon)$$

Then, $\hat{s} = \mathrm{arctg}(\varepsilon/a)(x\partial_y - y\partial_x)$ transforms the trajectory $y=0$ issuing from p_0 into the trajectory $y = (\varepsilon/a)x$ issuing from p_1.

Example 5. Let us consider the system

$$\begin{aligned}\dot{x} =& f(x)\\ \dot{y} =& f(x)\cdot y\end{aligned}$$

for which $dy/dx = y$, i.e. the trajectories are given by

$$y = ce^x$$

with c a constant. The trajectories $y = 0$ issuing from $p_0 = (a, 0)$ and $y = \varepsilon e^x$ issuing from $(a, \varepsilon e^a)$ diverge exponentially in t. If $f(x)$ is such that $e^{x(t)}$ is not exponential in t, e.g. $f(x) = \log(x)$, the solutions do not diverge exponentially in time.

6. Conditional symmetries. General setting

Let us briefly recall the setting for the determination (and use) of conditional symmetries, following [11] (see [12-19] for further details and related topics); the specifities of the DS case will be discussed in the next section.

Let us consider a differential equation Δ and its symmetry algebra $\mathcal{G}_\Delta$; we will denote by y the independent variables and by u the dependent ones. Denoting a VF η as

$$\eta = \varphi^\alpha(y, u) \frac{\partial}{\partial u^\alpha} + \xi^i(y, u) \frac{\partial}{\partial y^i} \tag{1}$$

if Δ is of order n, the VF $\eta \in \mathcal{G}_\Delta$ are those for which

$$\begin{cases} \eta^{(n)} \cdot \Delta = 0 \\ \Delta = 0 \end{cases} \tag{2}$$

where $\eta^{(n)}$ is the n-th prolongation of η [1-4].

As already recalled, knowledge of $\mathcal{G}_\Delta$ allows for a reduction of the equation Δ; in particular, one can look for solutions $u = f(y)$ that are invariant under a subgroup $\mathcal{G}_0 \subseteq \mathcal{G}_\Delta$. With this invariance ansatz, eq.(2) reduces to simpler ones: indeed, now we can express $u(y)$ in terms of the (differential) invariants of $\mathcal{G}_0$.

It should be noted that, given a VF $\eta_0 = \varphi_0 \partial_u + \xi_0 \partial_y$, the condition of invariance of $u = f(y)$ under this reads

$$\Delta_0 \equiv \varphi_0^\alpha + \xi_0^i \frac{\partial u^\alpha}{\partial y^i} = 0 \tag{3}$$

so that if we know a priori that $\eta_0 \in \mathcal{G}_\Delta$, and therefore that (2) is satisfied, the solutions invariant under η_0 can be seen as solutions of the system made of (2) and (3), and in particular of

$$\begin{cases} \Delta = 0 \\ \Delta_0 = 0 \end{cases} \tag{4}$$

Notice that it can happen that a solution $u = f(y)$ of Δ is also invariant under VF which are *not* in $\mathcal{G}_\Delta$.

Example 6. Deferring to the next section the consideration of examples in the frame of DS, consider the PDE

$$(\partial_1^2 + \partial_2^2) u = -2\alpha u + \alpha (y_1^2 + y_2^2) u + \beta y_2^2 u + (\beta/\alpha) y_2 \partial_2 u \tag{5}$$

where α, β are real constants, $u \in \mathbf{R}$, $y \in \mathbf{R}^2$, and $\partial_k = \partial / \partial y^k$. This is easily seen to be not invariant for rotations in the (y_1, y_2) plane, i.e. under the VF $\eta_0 = (y_2 \partial_1 - y_1 \partial_2)$. On the other side, the function

$$f(y) = \exp \left[-\frac{1}{2} (y_1^2 + y_2^2) \right] \tag{6}$$

is obviously rotationally invariant, and gives a solution to (5).

This suggests that symmetry reduction could be possible, and useful, also considering symmetries (of solutions) which are *not* in $\mathcal{G}_\Delta$.

Solving the original equation $\Delta = 0$ with the ansatz of invariance under η_0 amounts to solution of (4); notice that there we do not care about the first of (2). If we give an arbitrary η_0, anyway, the system (4) will in general not have solutions, so that this method is useful only if we are also able to determine the Δ_0 compatible with the original Δ, i.e. the η_0 that can be symmetries of solutions of Δ. A method for determining these, and the corresponding invariant solutions, does indeed exist, and we do now briefly illustrate it.

Before doing this, it is maybe worth remarking explicitely that although we always discuss invariance under a single VF for ease of notation, the whole discussion is immediately extended to an algebra of VF by considering invariance under all its generators at the same time.

Let us now consider η_0 to be a not specified VF: i.e., the functions φ_0, ξ_0 are not known; the solutions of Δ which are also invariant under η_0 will again be given by solutions to (4), but this should now be seen not as an equation for $u = f(y)$, but as an equations for the set of unknown functions $\varphi_0(y, u)$, $\xi_0(y, u)$ and $f(y)$.

We can apply to (4) the known methods for solving a system of PDEs (notice that even if Δ was an ODE, Δ_0 will be a PDE), among which symmetry methods.

Let us indeed determine the symmetry of (4); the determining equations for this amount to

$$\begin{cases} \eta^{(n)} \cdot \Delta = 0 \\ \eta^{(1)} \cdot \Delta_0 = 0 \\ \Delta = 0 = \Delta_0 \end{cases} \tag{7}$$

Notice that if we choose to consider $\eta_0 = \eta$, which is surely legitimate as η_0 is completely generic, the second of these is automatically satisfied, since $\Delta_0 = 0$. This means that (7) is equivalent to (2) and (3), i.e. it poses no restriction on η_0.

Therefore, we can apply to (4) the usual symmetry reduction methods, and obtain solutions in this way. Such solutions of (4) give us a VF η_0 and a function $u = f(y)$, invariant under η_0, which is also a solution to the original equation $\Delta = 0$. The VF η_0 is also a symmetry of (4), and is therefore obtained by the corresponding determining equations (7).

Notice that we can solve (7) by the well known algorithms for determining equations [1-4], also by means of computer algebra if desired [44,45]; once this is done, we can choose a specific η_0 solution of (7) and pass to consider (4), i.e. determine the invariant solution $u = f(y)$ to the original equation.

The VFs η_0 solution of (7) are symmetries of (4) but not of Δ alone, except of course for the trivial case $\eta_0 \in \mathcal{G}_\Delta$. They are therefore called *conditional symmetries* (CS) of Δ, as they are symmetries of Δ once this is subject to the additional condition (3).

Notice that this additional (or side) condition, i.e. Δ_0, depends on η_0 itself. Therefore, the conditional symmetries of a given equation Δ do not in general form an algebra, as they are ordinary symmetries of different systems.

Notice also that the space of CS of Δ does naturally carry an action of $\mathcal{G}_\Delta$, and the CS can be divided into conjugacy classes under this action, leading to a classification.

It should be stressed that, although the introduction of conditional symmetries was motivated by the search for invariant solutions, we can very well have solutions to (7) under which *no* solutions of Δ - neither therefore of (4) - are invariant. Although these VF do not lead to symmetry reduction of Δ as above, they are nevertheless CS; indeed, they exchange among themselves a set of solutions to Δ, those which are also solutions of the corresponding Δ_0 (see the discussion and definitions 1,1' and 4 below).

As already recalled, in this section we have followed [11], to which we also refer for further details and examples; see also [12-19] for related approaches and questions.

7. Conditional symmetries and conditional constants of motion for dynamical systems

Let us now consider how the general discussion on CS given in the previous section does apply to dynamical systems. In this case, the Δ of previous section will be a system of first order, autonomous ODEs as in (1.1), i.e.

$$\Delta \equiv \dot{x} - f(x) = 0 \tag{1}$$

where as usual $x \in M \subseteq \mathbf{R}^N$, $f : M \to TM$. Correspondingly, we write generic LP VF on $M \times \mathbf{R}$ as

$$\eta = \sigma(x,t)\partial_x + \tau(x,t)\partial_t \tag{2}$$

where $\sigma \partial_x \equiv \sum_i \sigma^i \partial / \partial x^i$.

The equation Δ_0, see (6.3), is in this case

$$\sigma(x,t) + \tau(x,t) \dot{x} = 0 \tag{3}$$

which using (1) reads simply

$$\sigma(x,t) + \tau(x,t) f(x) = 0 \tag{4}$$

Remark 11 As already discussed, we can consider $M' = \mathbf{R} \times M \subseteq \mathbf{R}^{N+1}$ instead than M, with $\dot{x}_0 = 1$, in order to have a time-independent formalism. With this, (1) remains unchanged but $x \in M'$ and $f_0(x) = 1$, while $\eta = \sigma^0 \partial / \partial x^0 + \sum_i \sigma^i \partial / \partial x^i$ and (4) reads

$$\sigma^i(x) = - \sigma^0(x) f^i(x) \tag{5}$$

We will anyway not use this setting.

We want to concentrate our attention on autonomous, time-independent VFs, and consider CS of this form; this means that the considered VF will be of the form

$$\eta = \hat{s} = s(x) \partial_x \equiv s^i(x) \frac{\partial}{\partial x^i} \tag{6}$$

rather than (2). (The above remark 11 would then permit to reinterprete our results in a more general setting, which we will not do explicitely.)

Now $\eta^{(1)} \cdot \Delta = 0$ will give, before restriction to solution of (1),

$$\dot{x}^j \partial_j s^i - s^j \partial_j f^i = 0 \tag{7}$$

while a solution $x = \xi(t)$ is invariant under (6) if

$$\Delta_0 \equiv s(\xi(t)) = 0 \tag{8}$$

Notice that if we do not ask invariance under (6) of the full solution $x = \xi(t)$, but only of its trajectory $\gamma = \{\xi(t) | f(\xi(t))| \cdot |s(\xi(t))| = \pm \left(f(\xi(t)), s(\xi(t)) \right) (9) where (.,.) denotes scalar product in \mathrm{T}_x M;$ in geometrical terms, this is just the requirement that

$$\hat{s} : \gamma \to T\gamma \tag{10}$$

We could therefore repeat the discussion of sect.6, which would just require to consider (1),(7) and (8) (or (10) if we are considering invariance of trajectories) instead than (6.1) and (6.3). We will not bother the reader with such a repetition, and just state some definitions which are natural in view of it.

Definition 1. *The VF (6) is a (LPTI) conditional symmetry of the DS (1) if and only if there are solutions $x = \xi(t)$ to this, such that (8) is satisfied for all t.*

Similarly, we say that $\hat{s}$ is a *configuration symmetry* of a solution $x = \xi(t)$ (respectively, of the DS Δ) if the trajectory $\gamma = \gamma_\xi$ is invariant under $\hat{s}$, i.e. if (10) is verified for $\gamma = \gamma_\xi$ (respectively, if all trajectories are invariant under $\hat{s}$, i.e. if (10) is verified for any solution trajectory γ). With this, we have the

Definition 2. *The VF (6) is a (LPTI) conditional configuration symmetry of the DS (1) if and only if there are solutions $x = \xi(t)$ to this, such that (10) is satisfied for $\gamma = \gamma_\xi$ the corresponding trajectory.*

The discussion of the previous section shows that (LPTI) conditional symmetries are also obtained as ordinary (LPTI) symmetries of the system

$$\begin{cases} \Delta = 0 \\ \Delta_0 = 0 \end{cases} \tag{11}$$

where Δ, Δ_0 are given by (1),(8); the same holds for conditional configuration symmetries, provided Δ_0 is now given instead by (9). This would suggest an alternative definition of CS, i.e. the

Definition 1'. *The VF (6) is a (LPTI) conditional symmetry of the DS (1) if and only if it is an ordinary (LPTI) symmetry of the system (11).*

It should be stressed that Definition 1 and Definition 1' are *not equivalent*, as example 7 below does clearly show.

In the same spirit, it is entirely natural to consider, beside conditional symmetries, also *conditional constants of motion* (CCM); as already recalled, these were introduced ante litteram by Sarlet, Leach and Cantrijn [32]; they are also implicitely considered in [11] (see (2.9) of that paper), provided one specializes the discussion of Levi and Winternitz to the case of DS instead than considering the general PDE case.

Indeed, in the same way as a conditional symmetry is a symmetry of (11), i.e. of Δ subject to the additional condition $\Delta = 0$, we have that

Definition 3. *A function $P(x)$, $P : M \to \mathbf{R}$, is a conditional constant of motion of Δ if and only if it is a constant of motion for the system (11).*

Notice that, as for ordinary constants of motion, every DS admits CCM, given by constant functions.

It should be mentioned that symmetries of Δ can be also CS for it, and in the same way CM are also CCM; such CS and CCM are not interesting in this context, since they are not really "conditional", and are dealt with - and utilized - by standard methods; they will be seen here as trivial. In the following, it will be implicit that we consider nontrivial CS and CCM.

Also, it should be stressed that since by definition (11) admits solutions only for Δ_0 corresponding to a CS, definition 3 is meaningful only for this case; i.e., CCM are associated with CS, as we are going to discuss in greater detail in the following.

Before doing this, it is worth considering again, from the present point of view, simple examples given in previous sections.

Example 7. Let us consider the simple system of examples 1-3, namely (say in $\mathbf{R}^2\backslash\{0\}$ to eliminate the trivial solution)

$$\begin{aligned} \dot{x} =& \alpha(r^2)x - \beta(r^2)y \\ \dot{y} =& \beta(r^2)x + \alpha(r^2)y \end{aligned} J \tag{12}$$

which in polar coordinates reads simply

$$\begin{aligned} \dot{r} =& \alpha(r^2) \\ \dot{\vartheta} =& \beta(r^2) \end{aligned} \tag{13}$$

If $\alpha(r^2)$ has some nontrivial zero, say $r = r_0$, then the solutions on the invariant circle r_0, i.e. the solution of the system (11) where Δ is given by (13) and Δ_0 by $r = r_0$, are exchanged among themselves by rotations, i.e. by action of the VF $\hat{s} = x\partial_y - y\partial_x = \partial_\vartheta$. It should be stressed that while this shows that $r^2 = x^2 + y^2$ is a CCM in the sense of definition 3, and that ∂_ϑ is a conditional configuration symmetry in the sense of definition 2, some extra care should be taken before concluding that ∂_ϑ is also a CS. Indeed, ∂_ϑ will *not* be a symmetry of (11) in the sense of definition 1 with this choice of Δ and Δ_0, neither any solution of (12) (other than the trivial one $x = y = 0$ that we have excluded before) can be invariant as a full solution[4]. One gets easily convinced that

$$\eta = \beta(r_0^2)\partial_t + [x\partial_y - y\partial_x] = \beta(r_0^2)\partial_t + \partial_\vartheta \tag{14}$$

is a conditional symmetry for (12) (if, as before, $\alpha(r_0^2) = 0$).

The above example shows that requiring the CS of a DS to be time-independent can be a too restrictive condition, especially in view of definition 1, so that one should consider CS of general form. It is maybe also worth giving an example of a case in which there are CS which are time-independent.

Example 8. Let us consider the system ($r^2 = x^2 + y^2$)

$$\begin{aligned} \dot{x} =& \alpha(r^2)x - \beta(r^2)y \\ \dot{y} =& \beta(r^2)x + \alpha(r^2)y \\ \dot{z} =& f(z) + g(x,y)r^2 \end{aligned} \tag{15}$$

[4] It should be stressed that ∂_ϑ is, anyway, a symmetry of (11); i.e. it is a CS in the sense of definition 1'.

It is immediate to check that $\partial_\vartheta = x\partial_y - y\partial_x$ is now a CS, corresponding indeed to solutions on the z axis, i.e. with $r = 0$.

Notice that unless $g(x,y) = g(r)$, ∂_ϑ is not an ordinary symmetry for (15); notice also that it is indeed an ordinary symmetry for (11), and therefore a CS in the sense of definition 1', as it clearly exchanges solutions into solutions. It is the fact that it does not leave any nontrivial solution invariant that makes that it does not qualify as a CS in the sense of definition 1 as well.

The above discussion clearly explains why ordinary symmetries *can* be also CS, but are not necessarily such in the sense of definition 1 (while they are necessarily such in the sense of definition 1'). In particular, it shows that necessarily if a VF η is a symmetry but not a CS for Δ, then there is no solution of Δ which is invariant under η.

Notice that if we do not focus on invariant solution, the definition 1 is not so natural; it would indeed be also useful to consider as CS those VF which, without necessarily leaving invariant the set $\mathcal{S}$ of all solutions to Δ, leave invariant some set $\mathcal{S}_0 \subseteq \mathcal{S}$; if $\mathcal{S}_0$ amounts to a single solution, we recover definition 1. At the risk of adding some confusion to the already intricated nomenclature for conditional or similar symmetries, we will call these *weak conditional symmetries* (WCS); i.e., we have the

Definition 4. *A VF η is a weak conditional symmetry for the DS Δ if and only if it leaves invariant (not necessarily pointwise) a set $\mathcal{S}_0$ of solutions to Δ.*

Notice that with this, a CS is also necessarily a WCS; notice also that CS in the sense of definition 1' are necessarily WCS.

Also, in this way conditional configuration symmetries appear as a special class of WCS, the invariant set $\mathcal{S}_0$ being that of solutions with different initial data $x(0)$ along the same trajectory, i.e. the set of solutions having the same trajectory.

In conformity with the general spirit of the present paper, we will be specially interested in time-independent WCS; notice that these will also be "weak conditional configuration symmetries", in the sense that they will leave invariant (not necessarily pointwise) a subset $\mathcal{T}_0$ of the set $\mathcal{T}$ of trajectories of solutions to Δ.

8. Conditional symmetries, conditional constants of motion, and invariant manifolds

It is clear from the discussion of sect.7, as the reader may have already remarked, that a close connection does also exists between CS and CCM of a DS on one side, and its *invariant manifolds* on the other; we are now going to discuss this.

Let us first consider LPTI CS; in this case the additional condition Δ_0 in (7.11) is given by (7.8). In this way, any VF $\hat{s}$ identifies a manifold $S_0 \subseteq M$, and solutions to (7.11) are nothing else than the solutions to Δ which lie entirely in S_0. These are associated to the set $\Xi_0 \subseteq S_0$ of points x for which $\dot{x}$ lies in TS_0, i.e.

$$\Xi_s = \{x \in S_0 \ / \ \dot{x} \equiv f(x) \in T_x S_0\} \tag{1}$$

It is clear that Ξ_s is an invariant manifold for the DS (7.1), i.e. (by definition)

$$f : \Xi_s \to T\Xi_s \tag{2}$$

Notice that a different CS $\hat{s}'$ with $S_0' = \Xi_s$, would yield the same invariant solutions as $\hat{s}$; this fact, ease of notation, and the autonomous interest of this case, suggest to consider in particular the case $\Xi_s = S_0$, i.e. that S_0 is itself an invariant manifold for the dynamical system,

$$f : S_0 \to TS_0 \tag{3}$$

In both cases, (2) and (3), the determination of CS greatly simplifies or solves the problem of determining invariant manifolds for the DS under study. It should be stressed that this simple remark does at once put at our disposition the powerful - and, what's more, *completely algorithmical* - methods developed for the study of symmetries and CS of differential equations, for studying invariant manifolds of dynamical systems.

Let us now concentrate on the case (3): the invariant manifold being determined by (7.8), it follows at once that $s^i(x)$ are CCM for the DS $\dot{x} = f(x)$ under study. More in general, any function $P(x)$ such that S_0 (or Ξ_s in case (2)) is a level set of P will be a CCM for this DS. Similarly, in case (2) any function $P(x)$ such that Ξ_s is a level set of P will be a CCM for the DS. In this case we can first restrict $P(x)$ to S_0, and consider more simply level sets of this restriction.

Remark 12. Clearly, if $P(x)$ is a CCM, also $P_c(x) = P(x) + c$ is a CCM, for c a constant. Denoting by I_0 the algebra of constant functions on M, this and previous remarks on trivial CS and CCM suggest that - in order to avoid unneeded redundancies - one should actually consider constants of motion, and CCM, modulo I_0.

Notice now that S_0 is a level set of $P(x)$ if and only if

$$(v(x) \cdot \nabla) P(x) = 0 \tag{4}$$

for all $\hat{v} : S_0 \to TS_0$; this last condition just means that

$$(v(x) \cdot \nabla) s(x) = 0 \quad \text{ on } s(x) = 0 \tag{5}$$

We have therefore a simple method to determine the $P(x)$ associated to a given CS $\hat{s}$. Indeed, once $s(x)$ is known, (5) is a linear equation for v,

$$A(x) v(x) = 0 \tag{6}$$

where

$$A_{ij}(x) = \left[\frac{\partial s^i(x)}{\partial x^j} \right]_{S_0} \tag{7}$$

Once the general solution $v(x)$ of this is known, we can pass to determine $P(x)$ from (4), i.e. from the characteristic equation

$$\frac{\partial x^1}{v^1(x)} = ... = \frac{dx^N}{v^N(x)} \tag{8}$$

A close connection also exists between conditional configuration symmetries and CCM. Indeed, if $\hat{s}$ is a conditional configuration symmetry, the trajectories invariant under it provide invariant manifolds; such trajectories correspond, once $s(x)$ is fixed, to the manifold χ_s of points satisfying (7.9), i.e.

$$\chi_s = \{ x \in M \ / \ |f(x)| \cdot |s(x)| = (f(x), s(x)) \} \tag{9}$$

Notice that necessarily $S_0 \subseteq \chi_s$, which corresponds to the fact that any CS is a fortiori a conditional configuration symmetry. We can then proceed as in (4)-(8), with the role of S_0 played by χ_s.

It is also interesting to consider weak CS, and correspondingly weak configuration symmetries. In this case again we have a connection with CCM.

Indeed, for $\hat{s}$ a WCS, let us consider the invariant set $\mathcal{S}_0$, see definition 4; the trajectories of these form a manifold (recall f and s are smooth functions), which we call again S_0 and which is by construction invariant under the flow of $\hat{f}$ (and also of $\hat{s}$, obviously); we can then apply again the discussion (4)-(8).

Notice that in this way a WCS is also characterized as a VF $\hat{s}$ such that it exists a manifold $S_0 \subseteq M$ for which

$$\begin{gathered} \{f, s\}|_{S_0} = 0 \\ \hat{f} : S_0 \to TS_0 \ ; \quad \hat{s} : S_0 \to TS_0 \end{gathered} \tag{10}$$

Finally, let us consider a weak configuration symmetry, as defined in sect.7; again the invariant set of trajectories $\mathcal{T}_0$ defines a manifold in M (the union of points $x \in \gamma$ for $\gamma \in \mathcal{T}_0$) which is invariant under both $\hat{f}$ and $\hat{s}$, and the same considerations apply.

9. Module structure of conditional symmetries

The connection between CS and CCM discussed in the previous section also makes that there is a module structure of CS over the corresponding CCM, pretty much analoguous to the model structure of $\mathcal{G}_\Delta$ over I_Δ discussed in sect.2.

Indeed, let us reverse the point of view of the previous section, and consider manifolds $S_0 \subset M$ identified by

$$s(x) = 0 \tag{1}$$

Suppose to have chosen S_0 invariant under $\hat{f}$, so that any VF $\eta : S_0 \to TS_0$ is a WCS for Δ; moreover $\hat{s}$ is then a CS for Δ. More in general, we could have several VF which are CS for Δ and which leave invariant one (or more, or all) solution lying in S_0.

It should be stressed that, once S_0 has been fixed, the associated WCS form a Lie algebra, which we will denote by $\mathcal{G}_S^W$; this follows from the definition of WCS or equivalently from (8.10).

As for CS, those which leave invariant *some* solutions lying in S_0 (not necessarily the same solutions for different CS) do *not* form an algebra; on the other side, if we fix a given set of solutions lying in S_0, possibly all of them, the CS leaving these invariant *do* form an algebra. To be specific, let us consider the CS leaving invariant all the solutions in S_0, and let us call their algebra $\mathcal{G}_S^C$; clearly,

$$\hat{s} \in \mathcal{G}_S^C \subseteq \mathcal{G}_S^W \tag{2}$$

Let us also consider the CCM associated to S_0, i.e. the funstions $P : M \to \mathbf{R}$ for which S_0 is a level set (their relation with S_0 is discussed in sect.8); these do clearly form an algebra, which we denote as I_S. It is then immediate to remark, as in sect.2, that $\mathcal{G}_S^C$ and $\mathcal{G}_S^W$ are not only algebras, but do also have the structure of modules over I_S.

Similar remarks, and corresponding results, hold when considering conditional, and weak conditional, configuration symmetries.

10. Hyperbolic fixed points, stable and unstable manifolds

In the last two sections we have considered the relation between CS (and CCM) and invariant manifolds, under several points of view which were natural for the investigation of symmetry properties. We want now to reverse our point of view, i.e. consider this relation, and symmetry properties, from the point of view of dynamical system theory. This discussion, developed in this and the next two sections, is strongly related to [29], to which we refer for further details in the context of ordinary symmetries.

It should be mentioned that a number of results are already known in this direction, mainly concerned with ordinary Lie-point symmetries rather than conditional ones [20,21,28,29]; we will briefly recall them, referring the interested reader to original papers for details and proofs.

First of all, as already remarked in sect.4, if the smooth DS

$$\dot{x} = f(x) \tag{1}$$

admits an isolated fixed point x_0,

$$f(x_0) = 0 \tag{2}$$

then any LPTI (ordinary) symmetry $\hat{s} = s(x)\partial_x$ must satisfy

$$s(x_0) = 0 \tag{3}$$

Notice that (3) is *not* a necessary condition for $\hat{s}$ to be a CS, but is sufficient for this ($x(t) = x_0$ being the invariant solution).

Let us now suppose x_0 to be hyperbolic [24-27], i.e. let us consider the linear operator

$$A \equiv (Df)(x_0) \tag{4}$$

and suppose that all the eigenvalues of A have nonzero real part. In this case, to x_0 are associated local unique *stable* and *unstable manifolds* [24-27], W_s and W_u, which are in particular invariant manifolds under the flow of $\hat{f}$ (see [24-27] for the role they play in DS theory).

It is remarkable that the situation described above for the fixed point case is, mutatis mutandis, met again here: namely, it can be proved [21,28,29] that a necessary (but clearly not sufficient) condition for $\hat{s} = s(x)\partial_x$ to be an ordinary LP symmetry of (1) is that

$$\hat{s} : W_s \to TW_s \quad ; \quad \hat{s} : W_u \to TW_u \tag{5}$$

This implies that the restrictions of $\hat{s}, \hat{f}$ to W_s and W_u are well defined, and indeed it is immediate to see that a condition (again, necessary but not sufficient) for $\hat{s}$ to be an ordinary symmetry is that

$$[\hat{s}, \hat{f}]|_{W_s} = 0 \quad ; \quad [\hat{s}, \hat{f}]|_{W_u} = 0 \tag{6}$$

Conversely, each of these is clearly a sufficient - but not necessary - condition for $\hat{s}$ to be a CS[5] of (1).

Also, the invariant manifolds W_s, W_u will be identified by equations

$$w_s(x) = 0 \quad ; \quad w_u(x) = 0 \tag{7}$$

[5] At first sight, one could think this just guarantees to have a WCS, but (6) also guarantees (3).

Notice that these are not unique, although W_s and W_u are: any function with the same zero level set would do; clearly, functions vanishing on W_s (on W_u) form an algebra under sum, which we call I_W^s (I_W^u).

Clearly, the $w_s(x), w_u(x)$ are automatically CCM for (1); as discussed above, the functions satisfying the first (the second) of (6) will form a module over the algebra I_W^s (I_W^u).

The above considerations extend, a fortiori, to conditional configuration symmetries.

It should also be mentioned that the property of being invariant under any LPTI (ordinary) symmetry, is not peculiar to stable and unstable manifolds, but does actually extend to any transversally hyperbolic manifold (more in general, set) [29].

11. Nonhyperbolic fixed points, center manifolds

The case in which the fixed point x_0 of the DS (10.1) is not hyperbolic, i.e. the linear operator A given by (10.4) has some eigenvalue with zero real part, is slightly more complicated. In this case (see [29] for precise conditions and statement of results), together with stable and unstable manifolds, one has to consider center manifolds W_c [24-27].

It should be stressed that the center manifold is *not* uniquely defined, and that even if the DS is $\mathcal{C}^\infty$ the center manifold could happen to be nonsmooth, or smooth only of class $\mathcal{C}^k$ [6]. Anyway, each of these center manifolds is still an invariant manifold under the flow of $\hat{f}$; nevertheless, the result of the hyperbolic case do not extend immediately to this setting.

The obstacle to such an extension is intimately related to the non-unicity of the center manifold, i.e. to terms beyond all orders in the perturbative expansion of the center manifold (these are in turn related to resurgent functions [6,46]); a discussion of these issues is given in [29], and it would be well out of place to reproduce it here.

We will therefore consider only the *Poincaré- Dulac center manifold* W_c^0, which we define to be the (analytic) center manifold constructed perturbatively or, equivalently, the (infinite order) jet of center manifolds [6,35]. This is uniquely defined, as all the center manifolds differ indeed for non-perturbative terms and they all share the same jet.

Example 9. Let us consider the simple DS

$$\begin{aligned} \dot{x} &= -\, x^3 \\ \dot{y} &= -\, y \end{aligned} \tag{1}$$

It is immediate to see that all the curves

$$y = \alpha e^{-1/x^2} \tag{2}$$

are center manifolds, for $\alpha \in \mathbf{R}$. the perturbative (Taylor) expansion of all these is anyway the same, and just gives $y = 0$. Notice that although all the center manifolds are $\mathcal{C}^\infty$, the one with $\alpha = 0$ is the only one to be analytic.

If we consider the Poincaré- Dulac center manifold, the same results given in the previous section apply [28,29], in particular for what concerns (10.5), (10.6), (10.7), up to a simple rephrasing amounting to write W_c instead than W_s or W_u.

In general, i.e. for generic center manifolds W_c, even $\hat{s} : W_c \to TW_c$ is not true, and we can only be sure that a LPTI symmetry transforms a solution lying on a center manifold W_c^a into a solution lying on a center manifold W_c^b (possibly, but in general not, the same), see [28,29].

12. Bifurcation of fixed points

In the case the DS (10.1) depends on a real control parameter λ, so that we actually deal with

$$\dot{x} = f(\lambda; x) \tag{1}$$

and x_0 is a fixed point for all values of λ,

$$f(\lambda; x_0) = 0 \qquad \forall \lambda \tag{2}$$

which is undergoing a simple bifurcation [25,26,47,48] at $\lambda = \lambda_0$, the setting and results of the previous two sections combine nicely.

Indeed, let us suppose x_0 is stable for $\lambda < \lambda_0$ and only one real eigenvalue (or a pair of complex conjugate ones) crosses the imaginary axis with positive speed at $\lambda = \lambda_0$ (we suppose here the reader to have some familiarity with bifurcation theory; see e.g. [25,26,47,48] for introduction and discussion). Then, beside stable manifolds, for fixed λ the system (1) has a center manifold W_c for $\lambda = \lambda_0$, and an unstable manifold $W_u(\lambda)$ for $\lambda > \lambda_0$. The theorem on persistence of transversally hyperbolic manifolds [25] ensures that the center manifold can be uniquely defined by requiring it to be the limit of $W_u(\lambda)$ for $\lambda \to 0^+$.

In this case again we do not meet the problems related to non-uniqueness of the center manifold; moreover, since $W_u(\lambda)$ corresponds to its perturbative expansion, this limit is precisely W_c^0, the Poincaré- Dulac center manifold. Once again, we refer to [29] for details.

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