UWThPh-1992-30

An Ergodic Abelian Skeleton for Quantum Systems

Heide Narnhofer
Institut für Theoretische Physik
Universität Wien

Abstract

Under the assumption that there exists an optimal stationary coupling of a dynamical quantum system with a dynamical classical system we prove that the quantum system contains an ergodic classical system.

1 Introduction

Dynamical entropy is a powerful concept in classical ergodic theory [1]. It describes in a quantitative way how information increases by repeated measurements in the course of time and how quickly correlations disappear and measurements resp. operators become independent. It was natural to translate this concept also to the noncommutative case. The first attempt was done in [2] for type II_1 algebras. It was based on the idea that the conditional expectation of a translated operator with respect to a fixed subalgebra should become trivial in the course of time. This concept was generalized in [3] replacing the concept of conditional expectations which, in general, do not exist in the type III case by comparison with finite abelian algebras. In [4] it was shown that this comparison as proposed in [3] is in fact much stronger than it seemed: we do not only compare finite-dimensional noncommutative algebras with finite-dimensional abelian algebras, but these algebras are imbedded in a large abelian algebra that carries itself an ergodic dynamics.

Abelian algebras with ergodic dynamics are well studied and especially we know the behaviour of multicorrelations. It is evident that we wonder how the behaviour of these multicorrelations is inherited by the noncommutative algebra for which the abelian one is a good model.

The first result on these correlations is stated in [4]: the unitary operator implementing the automorphism must have an absolutely continuous spectrum. So at least for some operators

$$\lim_{t\to\infty} \omega(A\tau_t B) = \omega(A)\omega(B). \tag{1.1}$$

In the abelian case it follows immediately that also

$$\lim_{t\to\infty} \omega(A\tau_t BC\tau_t D) = \omega(AC)\omega(BD). \tag{1.2}$$

In the quantal situation weak asymptotic abelianness (equivalent to (1.1)) does not imply strong asymptotic abelianness (equivalent to (1.2) for faithful states) [5]. Nevertheless there are hints that it is strong asymptotic abelianness that plays the essential role to give positive dynamical entropy. In [6] the stronger concept of an entropic K-system was defined: the idea in this definition was to demand that finite subalgebras should become completely independent in the course of time where independence was expressed by the additivity of the entropy H. One should remember that there are many possibilities to define independence in the quantal situation [7]. In [8] it was shown that in the tracial situation the independence expressed by additivity of H can only hold if the automorphism is strongly asymptotically abelian.

Another support for the suspicion that strong asymptotic abelianness is necessary can be found in [9]. Here it is shown that the shift on the II_1 factor defined by the free group gives zero dynamical entropy. This shift let the algebra increase enormously, but it is very noncommutative and this noncommutativity hinders the dynamical entropy to develop.

In this paper we will use the definition of [4] and will assume that there exists an optimal abelian model. We will show that the existence of such a model implies the existence of an abelian subalgebra in $\mathcal{A}$ invariant under the automorphism. Of course, the assumption that such an abelian model exists can be very restrictive. On the other hand, examples without an abelian subalgebra invariant at least under a subgroup and a positive lower bound on the dynamical entropy are missing. So we consider this result again as support that positive dynamical entropy is very restrictive for the automorphism and looks like a footprint of something abelian in the algebra.

2 The Abelian Model

We present here the definition of the dynamical entropy as given in [4].

Definition (2.1) [2,3]. Let λ be a state on an algebra $\mathcal{M}$. Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathcal{M}$. Then

$$\begin{aligned} H_\lambda(\mathcal{B}|\mathcal{A}) &= \sup_{\lambda=\sum\lambda_i} \sum\{S(\lambda|\lambda_i)_\mathcal{B} - S(\lambda|\lambda_i)_\mathcal{A}\} \\ &= \sup \sum \lambda_i(1)\{S(\lambda|\widehat{\lambda}_i)_\mathcal{B} - S(\lambda|\widehat{\lambda}_i)_\mathcal{A}\} \end{aligned}$$

where $\lambda = \sum\lambda_i$ is a decomposition of λ into positive linear functionals λ_i over $\mathcal{M}$ and $\widehat{\lambda}_i(\cdot) = \lambda_i(\cdot)/\lambda_i(1)$ are the corresponding normalized states.

Lemma (2.2) Let $\mathcal{M} = \mathcal{B}\otimes\mathcal{A}$ and $\mathcal{B}$ abelian and finite-dimensional. Define $\mu = \lambda_{|\mathcal{B}}$ and $\omega = \lambda_{|\mathcal{A}}$. Then

$$H_\lambda(\mathcal{B}|\mathcal{A}) = H_\mu(\mathcal{B}) - S(\mu\otimes\omega|\lambda)_{\mathcal{A}\otimes\mathcal{B}}.$$

Proof: Let $\{p_i\}$ be the minimal projections in $\mathcal{B}$. First we express $S(\mu\otimes\omega|\lambda)$ with the help of the $\{p_i\}$. Next we show that these $\{p_i\}$ give the optimal decomposition of $H_\lambda(\mathcal{B}|\mathcal{A})$.

Consider $\lambda_i(p_j\otimes a) = \lambda(p_ip_j\otimes a)$. Then

$$\mu = \sum \lambda_i|_\mathcal{B} = \sum\lambda(p_i)\mu_i \qquad \text{with } \mu_i(p_j) = \delta_{ij},$$

$$\omega = \sum \lambda_i|_\mathcal{A} =: \sum\lambda(p_i)\omega_i \tag{2.3}$$

so that

$$\lambda = \sum\lambda(p_i)\mu_i\otimes\omega_i.$$

Therefore

$$H_\mu(\mathcal{B}) = -\sum\lambda(p_i)\ln\lambda(p_i) = \sum\lambda(p_i)S(\mu|\mu_i)_\mathcal{B}$$

$$S(\mu\otimes\omega|\lambda)_{\mathcal{A}\otimes\mathcal{B}} = \sum\lambda(p_i)S(\omega|\omega_i)_\mathcal{B}.$$

Next we show that the decomposition $\{\lambda_i\}$ optimizes (2.1). We start with some decomposition $\lambda = \sum_\alpha\lambda_\alpha$ and refine it to a decomposition $\lambda_{\alpha k}(\cdot) = \lambda_\alpha(p_k\cdot)$. This refinement improves (2.1) according to

$$\begin{aligned} &\sum_{\alpha,i}[S(\lambda|\lambda_{\alpha i})_\mathcal{B} - S(\lambda|\lambda_{\alpha i})_\mathcal{A}] - \sum_\alpha[S(\lambda|\lambda_\alpha)_\mathcal{B} - S(\lambda|\lambda_\alpha)_\mathcal{A}] = \\ &= \sum_\alpha\sum_i \lambda_{\alpha i}(1)[S(\widehat{\lambda}_\alpha|\cdot\widehat{\lambda}_{\alpha i})_\mathcal{B} - S(\widehat{\lambda}_\alpha|\widehat{\lambda}_{\alpha i})_\mathcal{A}] \geq 0 \end{aligned}$$

because

$$\widehat{\lambda}_{\alpha i} \leq \frac{\lambda_{\alpha i}(1)}{\lambda_\alpha(1)}\widehat{\lambda}_\alpha, \qquad S(\widehat{\lambda}_\alpha|\widehat{\lambda}_{\alpha i})_\mathcal{B} = -\ln\frac{\lambda_{\alpha i}(1)}{\lambda_\alpha(1)},$$

and dominates $S(\widehat{\lambda}_\alpha|\widehat{\lambda}_{\alpha i})_\mathcal{A}$.

Therefore we can restrict ourselves to decompositions finer than $\{\lambda_i\}$ (2.3), i.e., again $\lambda = \sum_{i,\alpha} \lambda_{i\alpha}$. Then

$$\sum_\alpha \lambda_{i\alpha}(1) S(\lambda|\widehat{\lambda}_{i\alpha})_{\mathcal{B}} = S(\lambda|\lambda_i)_{\mathcal{B}}$$

because for $\mathcal{B}$ the refinement is trivial, whereas

$$\sum_\alpha \lambda_{i\alpha}(1) S(\lambda|\widehat{\lambda}_{i\alpha})_{\mathcal{A}} \geq S(\lambda|\lambda_i)_{\mathcal{A}}$$

due to the convexity of S. Therefore a refinement of $\{\lambda_i\}$ only decreases the right hand side of (2.1) and the decomposition $\{\lambda_i\}$ is optimal and gives (2.2).

We turn to the final definition and follow [4]:

Definition (2.4) Let $(\mathcal{M}, \omega, \alpha)$ be a dynamical system, i.e., $\mathcal{M}$ a von Neumann algebra, α an automorphism and $\omega = \omega \circ \alpha$ an invariant state. A stationary coupling with a classical dynamical system $(\mathcal{B}, \sigma, \mu)$ is a state λ over the C^* algebra $\mathcal{M} \otimes \mathcal{B}$ such that $\lambda \circ \alpha \otimes \sigma = \lambda$ and $\lambda_{|\mathcal{M}} = \omega$ and $\lambda_{|\mathcal{B}} = \mu$. We call $(\mathcal{B} \otimes \mathcal{M}, \lambda, \sigma \otimes \alpha)$ an abelian model.

Definition (2.5) [4]. Let λ be a stationary coupling with $(\mathcal{B}, \sigma)$, let P be a finite partition of $\mathcal{B}$ and P_- its past. Then define

$$h(P, \lambda) = H_\mu(P|P_-) - H_\lambda(P|\mathcal{M} \otimes P_-).$$

Then the dynamical entropy $h_\omega(\alpha)$ of $(\mathcal{M}, \omega, \alpha)$ is given by

$$h_\omega(\alpha) = \sup h(P, \lambda)$$

where the supremum is taken over all stationary couplings and all partitions P.

Remark: In [4,10] it is shown that in fact $(\mathcal{B}, \sigma)$ can be taken as Bernoulli shift which can be realized on the state space of $\mathcal{M}$.

In [4] it is proven:

Theorem (2.6) Define

$$\widehat{h}(Q, \lambda) = H_\mu(Q|Q_-) - H_\lambda(Q|\mathcal{M}),$$

again with Q a subalgebra of an abelian model with a stationary coupling. Then

$$\sup \widehat{h}(Q, \lambda) = \sup h(P, \lambda).$$

Remark: Since $H(\mathcal{B}|\mathcal{A})$ is decreasing in the second argument

$$\widehat{h}(P, \lambda) \leq h(P, \lambda). \tag{2.7}$$

Therefore it becomes harder to attain the supremum of $\widehat{h}$. Especially it is not possible to take the supremum only over Bernoulli shifts – or at least this is not proven. A good model P for $h(P, \lambda)$ can be transformed into a good model Q for $\widehat{h}(Q, \lambda)$, but Q of higher dimension, long reaching correlation, and to a large extent with trivial coupling in λ, and still $\widehat{h}$ remains somewhat smaller than h. Therefore it is a completely open problem whether the supremum in $\widehat{h}$ can really be attained. Nevertheless, in the following we will assume that this optimal model exists and and look what are the consequences.

3 The Abelian Subalgebra

Theorem (3.1) Let the supremum in (2.6) be attained, i.e.,

$$\widehat{h}(Q,\lambda) = \sup_{P,\nu} \widehat{h}(P,\nu).$$

Then there exists an abelian algebra $\mathcal{B} \subset \mathcal{M}$ with $\alpha\mathcal{B} = \mathcal{B}$ and $h_{\omega_{|\mathcal{B}}}(\alpha) > 0$ if $\widehat{h}(Q,\lambda) > 0$.

Proof: We split the proof into several steps. First we observe that $\widehat{h}(Q,\lambda) = h(Q,\lambda)$ as consequence of (2.6) and (2.7). Therefore

$$H_\lambda(Q|\mathcal{M} \otimes \bigvee_{n=1}^{\infty} \sigma^{-n} Q) = H_\lambda(Q|\mathcal{M}). \tag{3.2}$$

Lemma (3.3) Let γ be a completely positive map from an abelian algebra R into Q with $\gamma\bar{\sigma} = \sigma\gamma$, and $\lambda \circ \gamma = \bar{\lambda}$, then

$$H_{\bar{\lambda}}(R|\mathcal{M} \otimes \bigvee \bar{\sigma}^{-n} R) = H_{\bar{\lambda}}(R|\mathcal{M}).$$

Proof: Because of the monotonicity of the relative entropy under completely positive maps

$$H(Q|\mathcal{M} \otimes \bigvee \sigma^{-n} Q) \leq H(Q|\mathcal{M} \otimes \bigvee \bar{\sigma}^{-n} R) \leq H(Q|\mathcal{M}) = H(Q|\mathcal{M} \otimes \bigvee \sigma^{-n} Q).$$

Since Q and R are abelian we have [4]

$$H(R|\mathcal{M}) + H(\sigma^{-n} Q|\mathcal{M} \otimes R) = H(R \otimes \sigma^{-n} Q|\mathcal{M}) = H(\sigma^{-n} Q|\mathcal{M}) + H(R|\mathcal{M} \otimes \sigma^{-n} Q)$$

where λ is in an obvious manner defined on $R \otimes \mathcal{M} \otimes \sigma^{-n} Q$. The above equality together with (3.2) gives

$$H(R|\mathcal{M}) \geq H(R|\mathcal{M} \otimes \bar{\sigma}^{-n} R) \geq H(R|\mathcal{M} \otimes \sigma^{-n} Q) = H(R|\mathcal{M}).$$

In the following we will choose R resp. γ in such a way that for the minimal projections p_i in R we have

$$\lambda(p_i) \neq 0, \qquad \widehat{\lambda}_i(a) = \frac{\lambda(p_i a)}{\lambda(p_i)} \geq c_i \lambda(a), \qquad c_i > 0. \tag{3.4}$$

The so defined $\widehat{\lambda}_i$ give the optimal decomposition for $H(R|\mathcal{M})$ respectively for $H(R|\mathcal{M} \otimes \bar{\sigma}^{-n} \mathcal{M})$. This implies

$$S(\lambda|\widehat{\lambda}_i)_{\mathcal{M} \otimes \bar{\sigma}^{-n} R} = S(\lambda|\widehat{\lambda}_i)_{\mathcal{M}}. \tag{3.5}$$

Lemma (3.6) Let $\mathcal{M}$ and $\mathcal{B}$ be finite dimensional and

$$S(\lambda|\omega)_{\mathcal{M} \otimes \mathcal{B}} = S(\lambda|\omega)_{\mathcal{M}} < \infty$$

with $c_1 \lambda < \omega < c_2 \lambda$. Define in the GNS construction corresponding to λ, ω with Λ and Ω in the same natural cone

$$Q_\lambda \mathcal{H} = \overline{\pi(\mathcal{M})|\Lambda\rangle}, \qquad Q_\omega \mathcal{H} = \overline{\pi(\mathcal{M})|\Omega\rangle},$$

then $Q_\lambda = Q_\omega$. Let $D^t_{\lambda\omega} \in \mathcal{M} \otimes \mathcal{B}$ be the intertwiner of the modular operators Δ_λ and Δ_ω, then

$$[Q_\lambda, D^t_{\lambda\omega}] = 0.$$

Proof: We write

$$\begin{aligned}\lambda(a) &= \langle\Lambda|\pi(a)|\Lambda\rangle \\ \omega(a) &= \langle\Omega|\pi(a)|\Omega\rangle.\end{aligned}$$

Then we define the relative modular operator $\Delta_{\lambda\omega}$ by [11,12]

$$\langle\Omega|\pi(a)\Delta_{\lambda\omega}\pi(a^\dagger)|\Omega\rangle = \langle\Lambda|\pi(a^\dagger)\pi(a)|\Lambda\rangle \qquad \forall\, a \in \mathcal{M}\otimes\mathcal{B}. \tag{3.7}$$

Then

$$S(\lambda|\omega)_{\mathcal{M}\otimes\mathcal{B}} = -\langle\Omega|\ln\Delta_{\lambda\omega}|\Omega\rangle$$

and

$$S(\lambda|\omega)_{\mathcal{M}} = -\langle\Omega|\ln(Q_\omega\Delta_{\lambda\omega}Q_\omega)|\Omega\rangle. \tag{3.8}$$

Due to our assumptions on λ and ω all these expressions are well defined. Now [12]

$$\begin{aligned}0 &= \langle\Omega|\ln(Q_\omega\Delta_{\lambda\omega}Q_\omega) - \ln\Delta_{\lambda\omega}|\Omega\rangle \\ &= \int_0^\infty dt\langle\Omega|\frac{1}{Q_\omega\Delta_{\lambda\omega}Q_\omega + t} - \frac{1}{\Delta_{\lambda\omega}+t}|\Omega\rangle \\ &= \int_0^\infty dt\langle\Omega|\frac{1}{\Delta_{\lambda\omega}+t}(1-Q_\omega)\frac{1}{(1-Q_\omega)\frac{1}{\Delta_{\lambda\omega}+t}(1-Q_\omega)}(1-Q_\omega)\frac{1}{\Delta_{\lambda\omega}+t}|\Omega\rangle\end{aligned}$$

so that for all $t \in \mathbf{R}^+$

$$(1-Q_\omega)\frac{1}{\Delta_{\lambda\omega}+t}|\Omega\rangle = 0.$$

By analytic continuation this holds for all t in the resolvent set of $\Delta_{\lambda\omega}$ and thus also for functions of $\Delta_{\lambda\omega}$, especially for

$$Q_\omega(\Delta_{\lambda\omega})^\gamma|\Omega\rangle = (\Delta_{\lambda\omega})^\gamma|\Omega\rangle \tag{3.9}$$

as long as $(\Delta_{\lambda\omega})^\gamma|\Omega\rangle$ is well defined. Especially the powers of the relative modular operator and the intertwiner that belongs to the algebra are related by

$$(\Delta_{\lambda\omega})^{it}|\Omega\rangle = D^t_{\lambda\omega}|\Omega\rangle.$$

(Notice $\Delta_{\lambda\omega} \notin \mathcal{M}\otimes\mathcal{B}$.) With $\gamma = 1/2$ we have

$$\Delta^{1/2}_{\lambda\omega}|\Omega\rangle = |\Lambda\rangle.$$

Therefore

$$Q_\omega|\Lambda\rangle = Q_\omega\Delta^{1/2}_{\lambda\omega}|\Omega\rangle = \Delta^{1/2}_{\lambda\omega}|\Omega\rangle = |\Lambda\rangle.$$

Now we can define Q_λ also as the minimal projection in $\pi(\mathcal{M})'$ that satisfies $Q_\lambda|\Lambda\rangle = |\Lambda\rangle$. Therefore $Q_\omega \geq Q_\lambda$. On the other hand, $Q_\omega|\Lambda\rangle = |\Lambda\rangle$ implies $|\Lambda\rangle = \pi(m)|\Omega\rangle$ for some $m \in \mathcal{M}$. $\pi(m)$ is invertible because $|\Lambda\rangle$ is separating for $\mathcal{M}$. Therefore

$$Q_\lambda\pi(m)^{-1}|\Lambda\rangle = Q_\lambda|\Omega\rangle = |\Omega\rangle$$

and also $Q_\lambda \geq Q_\omega$ and thus $Q_\lambda = Q_\omega$. Next

$$(\Delta_{\lambda\omega})^\gamma|\Omega\rangle = D^{i\gamma}_{\lambda\omega}|\Omega\rangle$$

and

$$Q_\omega(\Delta_{\lambda\omega})^\gamma|\Omega\rangle = D^{i\gamma}_{\lambda\omega}|\Omega\rangle$$

implies

$$D^{i\gamma}_{\lambda\omega} \in \mathcal{M}\otimes\mathbf{1} \qquad \text{and} \qquad [Q_\omega, D^{i\gamma}_{\lambda\omega}] = 0.$$

To generalize the result to arbitrary $\mathcal{M}$ we notice:

$$S(\lambda|\omega)_\mathcal{M} = \sup S(\lambda|\omega)_{\mathcal{M}_n} = \lim_{n\to\infty} S(\lambda|\omega)_{\mathcal{M}_n}$$

for an increasing sequence $\mathcal{M}_1$ of finite dimensional algebras.

Further $S(\lambda|\omega)_{\mathcal{M}_n\otimes\mathcal{B}} \geq S(\lambda|\omega)_{\mathcal{M}_n}$, therefore (notice that we can consider $\mathcal{B}$ to be finite dimensional)

$$S(\lambda|\omega)_{\mathcal{M}\otimes\mathcal{B}} - S(\lambda|\omega)_\mathcal{M} = \lim_{n\to\infty} S(\lambda|\omega)_{\mathcal{M}_n\otimes\mathcal{B}} - S(\lambda|\omega)_{\mathcal{M}_n}.$$

Since $\left((1-Q)\dfrac{1}{\Delta_{\lambda\omega}+t}(1-Q)\right)^{-1} > (1-Q)t$, we have $\lim_{n\to\infty}(1-Q^n_\omega)\dfrac{1}{\Delta_{\lambda\omega,n}+t} = 0\ \forall\, t>0$. Thus (3.9) can be replaced by

$$\lim_{n\to\infty}\left[Q^n_\omega\Delta^\gamma_{\lambda\omega,n}|\Omega\rangle - \Delta^\gamma_{\lambda\omega,n}|\Omega\rangle\right] = 0.$$

Due to our assumption on λ and ω in addition

$$\lim_{n\to\infty}\Delta^\gamma_{\lambda\omega,n}|\Omega\rangle = \Delta^\gamma_{\lambda\omega}|\Omega\rangle,$$

$$\lim_{n\to\infty} Q^n_\omega\Delta^\gamma_{\lambda\omega}|\Omega\rangle = Q_\omega\bar{\Delta}^\gamma_{\lambda\omega}|\Omega\rangle,$$

where Δ resp. $\bar{\Delta}$ are the relative modular operators corresponding to $\mathcal{M}\otimes\mathcal{B}$ resp. $\mathcal{M}$.

Definition (3.10) Consider the imbedding ι from $\mathcal{M}$ into $\mathcal{M}\otimes\mathcal{B}$. Let $\omega = \lambda\circ\iota$ be the state over $\mathcal{M}$ with modular automorphisms Θ_ω, and Θ_λ the modular automorphism corresponding to λ. Then

$$\lambda(\Theta^{i/2}_\lambda(a)\iota(m)) = \omega(\gamma(a)\Theta^{-i/2}_\omega m) \quad a\in\mathcal{M}\otimes\mathcal{B},\ m\in\mathcal{M}.$$

Observation (3.11) For finite dimensional $\mathcal{M}$ and $\mathcal{B}$ the map $\gamma:\mathcal{M}\otimes\mathcal{B}\to\mathcal{M}$ can be expressed in the GNS construction with the help of the operator Q_ω in the following way:

$$\langle\Omega|a\Delta^{1/2}_\omega m|\Omega\rangle = \langle\Omega|\gamma(a)(Q_\omega\Delta_\omega Q_\omega)^{1/2}m|\Omega\rangle$$

and $\gamma(a)$ is defined as the unique operator satisfying

$$Q_\omega\Delta^{1/2}_\omega a|\Omega\rangle = (Q_\omega\Delta_\omega Q_\omega)^{1/2}\gamma(a)|\Omega\rangle.$$

We apply these results to (3.5). Now Ω_i corresponds to $\lambda_{i|\mathcal{M}\otimes\bar\sigma^{-n}R}$ and Ω to $\lambda_{|\mathcal{M}\otimes\bar\sigma^{-n}R}$, $\Delta_i \equiv \Delta_{\lambda_i,\lambda}$. Then we recall that $\bar\sigma^{-n}p_j$ belongs to the center of $\dot\pi(\mathcal{M}\otimes\bar\sigma^{-n}R)''$, so is invariant under the modular automorphism. From (3.9) we get with $m\in\mathcal{M}_k$

$$\lim_{k\to\infty}\langle\Omega|m\bar\sigma^{-n}p_jQ_k\Delta^{-1}_{i,k}|\Omega\rangle = \lim_{k\to\infty}\langle\Omega|\bar\sigma^{-n}p_jQ_k\cdot m\Delta^{-1}_{i,k}|\Omega\rangle = \langle\Omega|\Theta^{1/2}\gamma(\bar\sigma^{-n}p_j)m\Delta^{-1}_i|\Omega\rangle =$$

$$= \langle \Omega_i | \Theta^{1/2} \gamma(\bar\sigma^{-n} p_j) m | \Omega_i \rangle = \lim_{k\to\infty} \langle \Omega | m \bar\sigma^{-n} p_j \Delta_{i,k}^{-1} | \Omega \rangle = \langle \Omega_i | \bar\sigma^{-n} p_j m | \Omega_i \rangle$$

where we let act Q_k once to the left side and then to the right side. Here Θ is the modular automorphism on $\mathcal{M}$ corresponding to ω. Now we use the explicit form of λ_i and continue to

$$\begin{aligned} \langle \Omega | p_i \Theta^{1/2} \gamma(\bar\sigma^{-n} p_j) m | \Omega \rangle &= \langle \Omega | p_i \bar\sigma^{-n} p_j m | \Omega \rangle = \\ \langle \Omega | \Theta^{1/2} (\gamma(p_i) \gamma(\bar\sigma^{-n} p_j) m | \Omega \rangle &= \langle \Omega | \Theta^{1/2} \circ \gamma(p_i \bar\sigma^{-n} p_j) m | \Omega \rangle. \end{aligned}$$

Notice that we can extend γ as map from $\mathcal{M} \otimes \bar\sigma^{-n} R$ into $\mathcal{M}$ to a map from $R \otimes \mathcal{M} \otimes \bar\sigma^{-n} R$ into $\mathcal{M}$. The above equality holds for all $m \in \mathcal{M}$. Therefore

$$\gamma(p_i)\gamma(\bar\sigma^{-n} p_j) = \gamma(p_i \bar\sigma^{-n} p_j). \tag{3.12a}$$

In the same way we can consider

$$\lim_{k\to\infty} \langle \Omega | \Delta_{i,k}^{-1} Q_k \bar\sigma^{-n} p_j m | \Omega \rangle = \lim_{k\to\infty} \langle \Omega | \Delta_{i,k}^{-1} \bar\sigma^{-n} p_j m | \Omega \rangle$$

and get

$$\gamma(\bar\sigma^{-n} p_j)\gamma(p_i) = \gamma(\bar\sigma^{-n} p_j p_i) = \gamma(p_i \bar\sigma^{-n} p_j) = \gamma(p_i)\gamma(\bar\sigma^{-n} p_j) \tag{3.12b}$$

$\forall\, i, j$, and $n \in \mathbf{Z} \setminus 0$. Since we started with a stationary coupling

$$\gamma(\bar\sigma p) = \alpha \gamma(p). \tag{3.13}$$

Conclusion (3.14) The algebra $\mathcal{B}_i = \{\alpha^n \gamma(p_i), i \text{ fixed}, n \in \mathbf{Z}\}''$ satisfies the properties of Theorem (3.1).

We want to find some further information on $\mathcal{B}$. We had to make assumptions on R (3.4). But now we can take functions f_i of $\gamma(p_i)$ and still keep

$$[f_i, \alpha^n f_j] = 0.$$

These f_i approximate minimal projections in $\gamma(R)'' = \mathcal{B}_0$. If we now consider decompositions of $\omega = \sum \omega_i$ with $\omega_i(a) = \omega(\Theta_\omega^{i/2} b_i a)$ and $b_i \in \mathcal{B}_0$, $\sum b_i = 1$, then the optimal decomposition will be attained if the b_i are minimal projections in $\mathcal{B}_0$ (again by the convexity of the relative entropy). Whether we can choose these minimal projections commuting with one another is an open problem. At least a counterexample has not been found.

It remains to check when the converse can be true: Start with an abelian algebra $\mathcal{B} \subset \mathcal{M}$ with $\alpha\mathcal{B} = \mathcal{B}$. Then take some finite subalgebra $\mathcal{B}_0 \subset \mathcal{M}$ and consider $J\mathcal{B}_0 J \otimes \mathcal{M}$ as abelian model.

We start with the state ω over $\mathcal{M}$ with corresponding GNS vector Ω and modular automorphism Θ implemented by e^{iHt}. We assume $H(\mathcal{B}_0 | \mathcal{M} \otimes \alpha^{-n} \mathcal{B}_0) = H(\mathcal{B}_0 | \mathcal{M})$. According to (3.3), (3.4), we chosse $b_i = \exp V_i \in \mathcal{B}_0$. Also for this abelian model

$$H(\widetilde{\mathcal{B}}_0 | \mathcal{M} \otimes \alpha \widetilde{\mathcal{B}}_0) = H(\widetilde{\mathcal{B}}_0 | \mathcal{M}). \tag{3.15}$$

The GNS representation for $\mathcal{M} \otimes \alpha \widetilde{\mathcal{B}}_0$ of ω resp. of $\omega(\Theta b_1 \cdot)$ reads

$$\left| \begin{matrix} J\alpha b_1 J\Omega \\ \vdots \\ J\alpha b_n J\Omega \end{matrix} \right\rangle \qquad\qquad \left| \begin{matrix} J b_1 \alpha b_1 J\Omega \\ \vdots \\ J b_1 \alpha b_n J\Omega \end{matrix} \right\rangle. \tag{3.16}$$

The projector Q resp. Q_1 map onto the Hilbert spaces

$$\left| \begin{array}{c} mJ\alpha b_1 J\Omega \\ \vdots \\ mJ\alpha b_n J\Omega \end{array} \right\rangle \qquad\qquad \left| \begin{array}{c} m\Theta b_1 J\alpha b_1 J\Omega \\ \vdots \\ m\Theta b_1 J\alpha b_n J\Omega \end{array} \right\rangle$$

and evidently these spaces coincide, so $Q_1 = Q$. (3.15) only holds if $\Delta_1^\gamma \in \mathcal{M} \otimes \mathbf{1}$ for all γ. Explicitly this means

$$\exp[-(H+\sigma V_i)\gamma] \left(\exp[\frac{H+\sigma V_i}{2}] \exp[-\frac{H}{2}] \exp V_1 \exp[\frac{H}{2}] \exp[-\frac{H+\sigma V_i}{2}] \right)^{2\gamma} \exp[(H+\sigma V_i)\gamma]$$

has to be independent of i for all γ.

We can assume V_i to be small and expand in V_i. To leading order we obtain

$$1 + \gamma V_1 + \gamma^2 [H, V_1] + \gamma^2 [H, [\alpha \widetilde{V}_i, V_1]] + \ldots \tag{3.17}$$

with

$$\widetilde{V}_i = \int_0^{1/2} d\gamma \; e^{\gamma H} V_i \; e^{-\gamma H}.$$

If $\Theta^{it} V_j = V_j$ this is satisfied, so especially if ω is the tracial state. But it can also hold in more general situations, but is evidently a further condition on $\mathcal{B}_0$.

4 Examples of Abelian Subalgebras

The examples where the dynamical entropy could be calculated, i.e., the shift on the lattice algebra [3,13] and quasifree time evolutions [14,15,16] were always based on the fact that one could find abelian subalgebras and these algebras were pointwise nearly invariant under the modular automorphisms. These abelian models were in general not optimal but approximated the dynamical entropy.

We consider as typical example the quasifree automorphism α on the C^*-algebra over $L^2([0,1])$ which by Fourier transform is identical with the CAR algebra over $\ell^2(Z)$. We define

$$\alpha a(f(p)) = a(e^{i\pi p} f(p)), \qquad p \in [0,1]. \tag{4.1}$$

Evidently α^2 is the shift on the lattice. We consider the tracial state where

$$h_\omega(\alpha) = \frac{1}{2} h_\omega(\alpha^2) = \frac{1}{2} \ln 2. \tag{4.2}$$

An abelian model is constructed by refering to α^2. We start with the abelian subalgebra $\mathcal{B} = \bigcup_{x \in Z} \{a_x^\dagger a_x\}''$ that is invariant under α^2. As abelian model we take $\mathcal{B} \otimes \mathcal{B}$ which consists of projections P_I, Q_J, corresponding to projections p_I resp. p_J in $\mathcal{B}$. Then we define the state λ on $\mathcal{M} \otimes (\mathcal{B} \otimes \mathcal{B})$ by

$$\lambda((P_I \otimes Q_J)a) = \frac{1}{2} \omega(p_I a) \omega(p_J) + \frac{1}{2} \omega(p_J \alpha^{-1} a) \omega(p_I). \tag{4.3}$$

On $\mathcal{B} \otimes \mathcal{B}$ we define the automorphism σ by

$$\sigma P_I = Q_I, \qquad \sigma Q_I = P_{\alpha^2 I}.$$

This makes λ to a stationary coupling

$$\lambda(\sigma \otimes \alpha((P_I \otimes Q_J)a) = \frac{1}{2}\omega(p_I a)\omega(p_J) + \frac{1}{2}\omega(\alpha^2 p_J \alpha a)\omega(p_I) = \lambda((P_I \otimes Q_J)a).$$

To calculate $h(\alpha)$ we use the subalgebra $Q = \{a_x^\dagger a_x\} \otimes \{a_x^\dagger a_x\}$ which gives ($Q_- = \bigvee_{k=1}^\infty \sigma^{-k} Q$)

$$H(Q|Q_-) = \ln 2,$$

$$H(Q|\mathcal{M} \otimes Q_-) = \lim \frac{1}{n} H\left(\bigvee_{k=1}^{n} \sigma^k Q|\mathcal{M}\right) = \frac{1}{2}\ln 2,$$

$$H(Q|\mathcal{M}) = \frac{3}{4}\ln 2. \tag{4.3}$$

By enlarging Q (compare the construction in [4]) we can only approximate but cannot reach $\widehat{h}$, in correspondence to the fact that $[\alpha a_x^\dagger a_x, a_x^\dagger a_x] \neq 0$. Whether an abelian subalgebra $\mathcal{C}$ with $\alpha\mathcal{C} = \mathcal{C}$ can be found remains an open question. What remains valid is the fact that α is asymptotic abelian for the even subalgebra. This holds in all known examples. Whether it is a general fact demands a better understanding of possible stationary couplings.

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