On an example of genuine quantum chaos

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Abstract: The first example of a quantum system with the genuine quantum chaos is presented.

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Much progress has been achieved in understanding and calculating various properties of quantal and semiquantal models with chaotic behaviour during the last two decades. A prominent part of these investigations deals with the study of semiquantal models. Here by a semiquantal model we understand a classical dynamical system which can be derived from a purely quantum–mechanical system by limiting procedure. It still remains , however, to determine whether or not the chaotic behaviour of the semiquantal model reflects "true" chaos in the quantum system.

The first question we meet concerns the definition of "quantum chaos". Several definitions exist and their interconnections have not been fully elucidated yet.

In the present paper, by genuine quantum chaos we understand the positivity of the quantum characteristic exponents which were recently introduced and discussed by us (cf [1], [2], [3]). Although it is easy to give a (mathematical) model with positive quantum characteristic exponent (see [2]), there still is no model of a purely quantum–mechanical system with genuine quantum chaos. In order to improve this situation we reexamine the Milburn model (see [4], [5], [6]). We will show that this model can be considered as an example of a quantum system with genuine quantum chaos. For this reason, one can say that our approach to the description of chaotic behaviour of quantum dynamical systems seems to be fully justified.

Let us turn to the description of the Milburn model. It is the quantum optical counterpart of a parametrically kicked nonlinear oscillator. Its hamiltonian H is of the form (cf. [4], [6]):

$$H = H_{NL} + H_{PA} \tag{1}$$

where

$$H_{NL} = \frac{\chi}{2}(a^*)^2 a^2 \tag{2}$$

and

$$H_{PA} = i\hbar\frac{\kappa}{2}[(a^*)^2 - a^2] \cdot \sum_{n=-\infty}^{+\infty} \delta(t - n\tau) \tag{3}$$

In above formulae, χ is a constant proportional to the third order nonlinear susceptibility of the medium, a stands for the boson annihilation operator, κ is the coupling constant (the product of the pump field amplitude and the second–order nonlinearity in the parametric gain medium). Finally, τ is the period of free evolution (i.e. the evolution described by H_{NL}) between each pump pulse. Let us remark that these pump pulses mimick kicks of the harmonic oscillator. The Heisenberg equation of motion for the nonlinear hamiltonian H_{NL} gives the following evolution for a:

$$a(t) = e^{-i\chi t a^* a} a(0). \tag{4}$$

Then, the time evolution of the system can be described (see [4], [6]) by the equation:

$$a(t_n^+) = a(t_n^-)\cosh r + a^*(t_n^-)\sinh r \tag{5}$$

where t_n^+ (t_n^-) is the time just after (before) the passage of the n-th pulse, r is the effective constant for the kick and the time dependence of a on t is given by (4).

Now, we want to calculate quantum characteristic exponents λ^q for the evolution $\mathcal{U}$ given by (4) and (5), i.e. $\mathcal{U}a(t_n^+) = a(t_{n+1}^+)$ and a quantum characteristic exponent is defined by

$$\lambda^q(\mathcal{U}; x, y) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log \|(D_x \mathcal{U}^n)(y)\|$$

where $(D_x\mathcal{U}^n)(y)$ denotes derivatives of $\mathcal{U}$ composed with itself n times at x in direction y.

To do so, let us rewrite (4) and (5) in terms of selfadjoint operators Φ and Π, where Φ and Π are defined by:

$$a = \Phi + i\Pi \tag{6}$$

Then, (4), (5) and their conjugate equations can be written as:

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{\frac{-i\mu}{2}} \begin{pmatrix} e^r \cos\mu B_0 & e^r \sin\mu B_0 \\ -e^{-r}\sin\mu B_0 & e^{-r}\cos\mu B_0 \end{pmatrix} \begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \tag{7}$$

where $\mu = \chi\tau$ and $B_0 = a^*a = (\Phi^2 + \Pi^2 - \frac{1}{2})$.

Now, let us change the time evolution (7) slightly. Namely, we will impose a control over the possible number of photons in the propagator of free evolution. We ought to do this modification in order to have a real influence of kicks on the mode. In order to carry out the considered change let us denote by p_n the projection on the n–photon state. Clearly

$$[H_{NL}, P_N] = 0 = [a^*a, P_N]$$

where $P_N = \sum_{n=1}^{N} p_n$. Thus we replace (7) by

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{\frac{-i\mu}{2}} \begin{pmatrix} e^r \cos\mu B & e^r \sin\mu B \\ -e^{-r}\sin\mu B & e^{-r}\cos\mu B \end{pmatrix} \begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \tag{8}$$

with $B = (B_0 \cdot P_N) \oplus P_N^{\perp}$.

Next, let us consider the case:

$$A^2 \equiv \cos^2\mu B \cosh^2 r - 1 > \epsilon 1 \tag{9}$$

with arbitrary small $\epsilon > 0$. The opposite case $A^2 < -\epsilon 1$ can be treated in a similar way. Let us observe that the following condition for the parameter μ:

$$\mu \in X = \{x \in R; x > 0, x \cdot n \neq (2k-1) \cdot \frac{\pi}{2} \text{ for any k} \in \mathcal{N} \text{ and n} \in \mathrm{N}_0\} \tag{10}$$

where $N_0 = \{1, \ldots N\}$ implies the nontriviality of the operator A ($A \neq -1$). Then, for large enough r, the equality (9) is satisfied. Moreover, let us assume that

$$\mu \in X \cap Y \equiv Z = \{x \in R; x > 0, x \cdot n \neq \frac{k}{2} \cdot \pi \text{ for any } \mathrm{k} \in \mathcal{N} \text{ and } \mathrm{n} \in \mathrm{N}_0\} \tag{11}$$

where

$$Y = \{x \in R; x > 0, x \cdot n \neq k \cdot \pi \text{ for any } \mathrm{k} \in \mathcal{N} \text{ and } \mathrm{n} \in \mathrm{N}_0\} \tag{12}$$

The condition $\mu \in Y$ implies that the function $\sin^{-1}\mu B$ is well defined (note that the spectrum of B is equal to $\{1, \ldots N\}$). Under the above conditions we can rewrite (8) as:

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{\frac{-i\mu}{2}} \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \tag{13}$$

where

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -e^{-r}(\sin\mu B)^{-1}(\cos\mu B \sinh r + A) & -e^{-r}(\sin\mu B)^{-1}(\cos\mu B \sinh r - A) \end{pmatrix} \tag{14}$$

$$\mathbf{P}^{-1} = \begin{pmatrix} -\cos\mu B \sinh r(2A)^{-1} + \frac{1}{2} & -e^{r}\sin\mu B(2A)^{-1} \\ \cos\mu B \sinh r(2A)^{-1} + \frac{1}{2} & e^{r}\sin\mu B(2A)^{-1} \end{pmatrix} \tag{15}$$

and

$$\mathbf{D} = \begin{pmatrix} \cos\mu B \cosh r - A & 0 \\ 0 & \cos\mu B \cosh r + A \end{pmatrix} \equiv \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \tag{16}$$

Therefore

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{\frac{-in\mu}{2}} \mathbf{P}\mathbf{D}^{\mathbf{n}}\mathbf{P}^{-1} \begin{pmatrix} \Phi(0) \\ \Pi(0) \end{pmatrix} \tag{17}$$

where we have used the fact that $\mathbf{D}$ does not depend on t_n. As the next step in exhibiting a positive quantum characteristic exponent for our model let us consider a "rotation" in (Φ, Π) variables. To do so let us define:

$$\Phi^{\epsilon} = \frac{1}{2}[e^{i\epsilon}a + e^{-i\epsilon}a^*] \tag{18}$$

and

$$\Pi^\epsilon = \frac{1}{2i}[e^{i\epsilon}a - e^{-i\epsilon}a^*] \tag{19}$$

Clearly

$$(\Phi^\epsilon)^2 + (\Pi^\epsilon)^2 = a^*a + \frac{1}{2} = (\Phi)^2 + (\Pi)^2. \tag{20}$$

Moreover

$$\begin{pmatrix} \Phi^\epsilon(t_n^+) - \Phi(t_n^+) \\ \Pi^\epsilon(t_n^+) - \Pi(t_n^+) \end{pmatrix} = e^{\frac{-in\mu}{2}} \mathbf{P}\mathbf{D}^{\mathbf{n}}\mathbf{P}^{-1} \begin{pmatrix} \Phi(0)^\epsilon - \Phi(0) \\ \Pi(0)^\epsilon - \Pi(0) \end{pmatrix}. \tag{21}$$

Therefore:

$$\begin{pmatrix} D_\epsilon(\Phi^\epsilon)^{(n)} \\ D_\epsilon(\Pi^\epsilon)^{(n)} \end{pmatrix} = e^{\frac{-in\mu}{2}} \mathbf{P}\mathbf{D}^{\mathbf{n}}\mathbf{P}^{-1} \begin{pmatrix} -\Pi(0) \\ \Phi(0) \end{pmatrix} \tag{22}$$

where $D_\epsilon \Phi^\epsilon = \lim_{\epsilon \to 0} \frac{\Phi^\epsilon - \Phi}{\epsilon}$ $\quad(D_\epsilon \Pi^\epsilon = \lim_{\epsilon \to 0} \frac{\Pi^\epsilon - \Pi}{\epsilon})$.

Remarks:

i) Let us note that $D_\epsilon \Phi^\epsilon$ is well defined as the derivative with respect to the parameter ϵ of the one–parameter family Φ^ϵ of closed operators. Similary, $D_\epsilon \Pi^\epsilon$ is related to Π^ϵ.

ii) The operators Φ^ϵ, Π^ϵ and the phase angle ϵ can be used to characterize squeezed states (cf. [7]).

iii) Let us recall that the basic motivation for the study of characteristic exponents is the problem of stability. Thus, here we study the stability properties of the evolution of Milburn model with respect to the phase angle ϵ.

As the final step of calculating a positive quantum exponent for considered model of the kicked oscilator let us introduce the following cut–off in the (Φ, Π) variables. We replace $\Phi(\Pi)$ by $\Phi_\delta = \int_{-\delta}^{+\delta} \lambda dE_\Phi(\lambda)$ $\quad (\Pi_\delta = \int_{-\delta}^{+\delta} \lambda dE_\Pi(\lambda)$) where $\delta \in R^+$ and $\{E_\Phi(\lambda)\}$ $(\{E_\Pi(\lambda)\})$ stands for the spectral resolution of $\Phi(\Pi)$ respectively.

Consequently, we will consider

$$\begin{pmatrix} D_\epsilon(\Phi_\delta^\epsilon)^{(n)} \\ D_\epsilon(\Pi_\delta^\epsilon)^{(n)} \end{pmatrix} = e^{\frac{-in\mu}{2}}\mathbf{P}\mathbf{D}^{\mathbf{n}}\mathbf{P}^{-1}\begin{pmatrix} -\Pi_\delta(0) \\ \Phi_\delta(0) \end{pmatrix} \tag{23}$$

Let us remarks that (23) strongly converges to (22) as $\delta \to \infty$. Consequently, (23) is well defined approximation of the genuine dynamics.

Then, introducing new variables: $\tilde{\Phi}, \tilde{\Pi}$

$$\begin{pmatrix} \tilde{\Phi}_\delta \\ \tilde{\Pi}_\delta \end{pmatrix} \stackrel{\text{def}}{=} \mathbf{P}^{-1}\begin{pmatrix} \Phi_\delta \\ \Pi_\delta \end{pmatrix} \tag{24}$$

one has (cf. the second equality in (16))

$$\|D_\epsilon \tilde{\Phi}_\delta^\epsilon\| = \|\Lambda_1^n \tilde{\Pi}_\delta(0)\| \tag{25}$$

$$\|D_\epsilon \tilde{\Pi}_\delta^\epsilon\| = \|\Lambda_2^n \tilde{\Phi}_\delta(0)\| \tag{26}$$

Further, let us remark that 0 does not belongs to the spectrum of Λ_2. Hence, 0 is in the resolvent set of Λ_2 and Λ_2 is a bijection. Therefore,

$$\|\Lambda_2^n \tilde{\Phi}_\delta(0)\| = C\|\Lambda_2^n\| \tag{27}$$

where

$$C = \sup_{g:\, g=\Lambda_2^n f} \|\tilde{\Phi}_\delta(0)g\| \quad (\neq 0) \tag{28}$$

On the other hand, the following property of the norm

$$\|A^* A\| = \|A\|^2 \tag{29}$$

implies

$$\|\Lambda_2^{2^k}\| = \|\Lambda_2\|^{2^k} \tag{30}$$

for any $k \in \mathcal{N}$. Consequently

$$\lambda^q(\tilde{\Pi}_\delta) = \lim_{k\to\infty} \frac{1}{2^k} \log \|D_\epsilon \tilde{\Pi}^\epsilon_\delta(t_{2^k})\| \tag{31}$$

$$= \lim_{k\to\infty} \frac{1}{2^k} \log \|\Lambda_2^{2^k}\| = \log \|\Lambda_2\|$$

Now, it is clear that for large enough r (r is the effective constant for kicks) the norm of Λ_2 is larger than 1. Therefore, we can conclude that for some values of μ, r (i.e. χ, τ, κ) the quantum characteristic exponent λ^q for quantum variable $\tilde{\Pi}_\delta$ is strictly positive. To comment on this result let us remark that it is obvious that in polyparametric cases physics as well as mathematics allow numerous combinations of stability in certain directions and irregularity in others. Therefore, we can expect chaotic evolution of our model for some values of χ, τ, κ and regularity for others and we get a confirmation of such behaviour ($\lambda^q(\tilde{\Phi}_\delta)$ can be negative). Moreover, let us recall that the original Milburn model also has such a property. Let us finish with the conclusion that working in the "pure" quantum description of a genuine nonlinear quantum system we are able to speak rigorously about chaotic properties of the quantum evolution without any (semi)classical limits and other (semi)classical approximations leading to semiquantal models.

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