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KMS States for the Weyl Algebra

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Abstract

We consider the possible automorphism groups for the Weyl algebra over **R** resp. T and classify those for which KMS states, unique or not unique, exist.

1 Introduction

In elementary quantum mechanics the canonical equilibrium state is defined by assigning to an observable a the expectation value

$$\omega_\beta(a) = \mathrm{Tr}\ e^{-\beta H} a / \mathrm{Tr}\ e^{-\beta H} \tag{1.1}$$

where H is the Hamiltonian which determines the time development by

$$a_t = e^{itH} a e^{-itH}. \tag{1.2}$$

For (1.1) to make sense it is necessary that H has a purely discrete spectrum bounded from below (for $\beta > 0$). Nevertheless (1.1) has been generalized to quantum field theory where H has a continuous spectrum [HHW]. This generalization works via the following abstraction. $a \to a_t$ is a one parameter group of automorphisms of the algebra $\mathcal{A}$ of observables and an equilibrium state ω_β is characterized by the KMS condition where the time evolution is analytically continued to imaginary times

$$\omega_\beta(ab) = \omega_\beta(b a_{i\beta}). \tag{1.3}$$

Within this framework one can, for example, also define the equilibrium state for an infinite system of free fermions or bosons by the quasifree state over the smeared creation and destruction operators a^*_f, a_g

$$\omega_\beta(a^*_f a_g) = \left\langle g \left| \frac{1}{e^{\beta h} \pm 1} \right| f \right\rangle. \tag{1.4}$$

This defines a state even if the one particle Hamiltonian h has a continuous spectrum. In the case of fermions h need not even be bounded from below [RST].

In this note we propose to investigate to what extent the generalization (1.3) opens new possibilities even in elementary wave mechanics. To get a simple picture we take a restricted algebra $\mathcal{A}$ so that Aut $\mathcal{A}$, the groups of its automorphisms can easily be classified. For $\mathcal{A}$ we take the algebra generated by the exponentials of the position operators x and momentum operators p

$$W(q,k) = e^{i(kx+qp)}, \qquad (q,k) \in \mathbf{R}^2. \tag{1.5}$$

The noncommutativity of x and p is encoded in the Weyl relations (2.1) between these operators and if $\mathcal{A} = W_R$ consists just of the polynomials in $W(q,k)$ then Aut W_R are just the inhomogeneous linear symplectic transformations of (x,p). They are generated by Hamiltonians which are polynomials in (x,p) up to second order.

Instead of infinite space we can also consider particles on a circle. The corresponding algebra is W_T with (1.5) valid for $(q,k) \in \mathbf{R} \times \mathbf{Z}$. Here not all polynomials lead to a time automorphism group.

Our goal is to find out for which automorphisms there are KMS states. We summarize the result in the following Table.

	Automorphisms for W_T	KMS state for W_R	KMS state for W_T
1. $H = p$	yes	no	no
2. $H = x$	yes	no	no
3. $H = \frac{1}{2}(p^2 + \nu^2 x^2)$	yes for $\nu = 0$	yes	yes, nonunique
4. $H = \frac{1}{2}(p^2 - x^2)$	no	no	no
5. $H = \frac{1}{2}(xp + px)$	no	no	no
6. $H = \frac{1}{2}p^2 + gx,\ g \neq 0$	yes	no	no

We can understand the result by the following considerations:

1, 2 give shifts, 3, 4 is a harmonic force (resp. the free motion for $\nu = 0$), 5 is the dilation and 6 the free fall. According to arguments given at the beginning only 3 with $\nu \neq 0$ should be expected to lead to KMS states because for $\nu = 0$ it has a continuous spectrum in $[0, \infty]$ and the other Hamiltonians even from $-\infty$ to ∞. This turns out to be not quite so, the free Hamiltonian p^2 also leads to KMS states Ω_β but the representation given by ω_β is not the standard representation and acts instead in a nonseparabel Hilbert space. This is connected with the infinite volume and disappears when one goes to a finite configuration space. Periodic boundary conditions correspond to considering the subalgebra generated by $W(q, k)$, $k \in \mathbf{Z}$, but now 3 ($\nu \neq 0$) and 4, 5 do not lead to automorphisms. In this case only $H_3(\nu = 0) = H_6(g = 0)$ leads to KMS states but a new phenomenon appears which one knows only from phase transitions in infinite systems. For given β there is a one parameter family of KMS states for the same time evolution. This is related to the fact that on the torus the formal Hamiltonian $\frac{1}{2}p^2$ has a one parameter family of selfadjoint extensions but they all lead to the time evolution $x \to x + pt$, $p \to p$. The different selfadjoint extensions of p correspond to different inequivalent irreducible representations of W_T. In them different values are assigned to the central elements $e^{2\pi i n p} = e^{2\pi i n \gamma}$, $\gamma \in [0, 1)$ and in each of them we can construct a KMS state. These are all the extremal KMS states. This phenomenon is not typical of the thermodynamic limit but appears already for finite matrix algebras. In contradistinction to phase transitions the value γ for the central elements is independent of β.

One can even consider a finite phase space by taking

$$(q, k) \in \left(\frac{1}{g_1}\mathbf{Z}, \frac{1}{2\pi g_2}\mathbf{Z} \right), \qquad g_i \in \mathbf{Z}.$$

According to naive arguments such a phase space has only $g_1 g_2$ cells (of volume $h = 1$) and indeed $\mathcal{A}$ has then $g_1 \cdot g_2$ dimensional irreducible representations. This kind of quantum system has been studied for comparison with classical transformations on T^2 (as phase space) and it appears in the motion on T^2 (as configuration space) with a perpendicular magnetic field. In this case the situation is as for the Hamiltonian 3 and we do have KMS states. However, if there is in addition an electric field we are in the situation 1 and there are no KMS states. Of course, there are always invariant states like the tracial state which corresponds to infinite temperature. In the cases where the KMS states exist it will turn out to be easier to get their explicit form by looking at the consequences of (1.3) than to calculate the traces in (1.1).

Summarizing we can say that the spectral type of the Hamiltonian (discrete or continuous) is not relevant for the existence of KMS states. This might be expected since H is a representation dependent object unless the automorphism is inner. Nevertheless in our examples we found that there are no KMS states in those cases where H was not semibounded. So only in case 3, i.e. for harmonic forces or free time evolution KMS states exist. If we restrict ourselves to subalgebras, then there can exist several KMS states, but one should not talk about phase transition, because the parametrization of the KMS states is independent of the temperature.

2 The Weyl Algebra and Its Automorphisms

If we exponentiate the operators x and p to $W(q,k) = e^{i(kx+qp)}$ these Weyl operators satisfy the multiplication law

$$W(q,k)W(\bar{q},\bar{k}) = e^{i(q\bar{k}-k\bar{q})/2} W(q+\bar{q}, k+\bar{k}), \qquad (q,k) \in \mathbf{R}^2. \tag{2.1}$$

We shall consider the algebra W generated by these operators (i.e. their finite linear combinations), the C^*-algebra $\bar{W}$ (the norm closure of W), and the v. Neumann algebras $\pi(W)''$ (the weak closure in a representation π). Furthermore we shall consider the subalgebras W_T ("a particle on a torus") where $(q,k) \in \mathbf{R} \times \mathbf{Z}$ and W_C ("the rotation algebra") where $(q,k) \in \Theta\mathbf{Z} \times \mathbf{Z}$. W is simple, W_T has the center $\mathcal{Z} = \{W(n,0), n \in 2\pi\mathbf{Z}\}$ and similarly W_C if Θ is rational. If Θ is irrational, W_C is again simple. The automorphisms of W correspond to inhomogeneous linear symplectic transformations of (x,p) and generate $(z = (q,k))$

$$\alpha_{L,S} W(z) \to e^{i\langle L|z\rangle} W(Sz), \tag{2.2}$$

with $S \in SL(2,R)$, L linear, real.

Since $\alpha_{L,S} \circ \alpha_{\bar{L},\bar{S}} = \alpha_{\bar{L}+L\bar{S}, S\circ\bar{S}}$ we see Aut $W = \mathbf{R}^2 \otimes SL(2,R)$. This group is five-dimensional and there are the following one parameter subgroups.

1. Shift in x: $L = (0,t)$, $S = \mathbf{1}$,
2. Shift in p: $L = (t,0)$, $S = \mathbf{1}$,
3. Harmonic oscillator: $L = 0$, $S = \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix}$,
4. Repulsive harmonic force: $L = 0$, $S = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$,
5. Dilation: $L = 0$, $S = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$,
6. Free fall: $L = (gt, \frac{g}{2}t^2)$, $S = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

The groups 1 and 2 (resp. 4 and 5) are conjugate. In $\bar{W}$ there are other inner automorphisms, f.i. the ones generated by $e^{it(W+W^*)}$ but they shall not be considered here. 1 and 2 leave W_T and W_C and 6 leaves W_T invariant. Thus they also give automorphisms of these subalgebras though x is not a globally defined function on the torus. Since $\|W(z) - W(\bar{z})\| = 2$ for $z \neq \bar{z}$ only 1 and 2 are normcontinuous.

3 States

A state ω is a positive linear functional over W with $\omega(\mathbf{1}) = \mathbf{1}$. It gives a representation π_ω of W in a Hilbert space $\mathcal{H}_\omega$ the vectors of which we shall denote by $|q,k\rangle := W(q,k)|0,0\rangle$. ω is determined by

$$f(q,k) := \omega(W(q,k)) = \langle 0,0|W(q,k)|0,0\rangle$$

and the scalar product of two vectors is

$$\langle \bar{q},\bar{k}|q,k\rangle = \langle 0,0|W^*(\bar{q},\bar{k})W(q,k)|0,0\rangle = e^{i(q\bar{k}-k\bar{q})} f(q-\bar{q},k-\bar{k}). \tag{3.1}$$

Thus f defines a state over W iff $|f(q,k)| \leq 1$ and $e^{i(q\bar{k}-k\bar{q})/2} f(q-\bar{q},k-\bar{k})$ is a positive kernel. If f is continuous then $(q,k) \to \pi_\omega(W(q,k))$ is weakly continuous and v. Neumann's uniqueness theorem says that in this case $\pi_\omega(W)$ is isomorphic to the standard representation. W has also a tracial state ω_0 with $f_0(q,k) = \delta_q\delta_k := 1$ for $q = k = 0$ and 0 otherwise. π_{ω_0} is not weakly continuous and it acts in a nonseparable Hilbert space since $\langle \bar{q},\bar{k}|q,k\rangle = \delta_{q-\bar{q}}\delta_{k-\bar{k}}$ (compare also [NT], [AMS]. Thus the $|q,k\rangle$ are a nondenumerable orthogonal basis.

Since W_T has a nontrivial center the uniqueness theorem does no longer hold and there is a one parameter family of irreducible representations π_γ. Physically γ is the Bohm-Aharonov phase and mathematically it gives the c-number assigned to the central elements $\pi_\gamma(W(2\pi n,0)) = e^{2\pi i n\gamma}$. Note that π_γ is generated by ω_γ,

$$\omega_\gamma(W(q,k)) = e^{i\gamma q}\delta_k \tag{3.2}$$

which gives

$$\langle \bar{q},\bar{k}|q,k\rangle = \delta_{k\bar{k}} e^{i(q-\bar{q})(\gamma+k/2)}$$

and thus

$$|q,k\rangle = e^{iq(\gamma+k/2)}|0,k\rangle.$$

Therefore

$$\pi_\gamma(W(2\pi n,0))|q,k\rangle = e^{2\pi i n\gamma}|q,k\rangle.$$

We shall now turn to KMS states. Since the automorphisms are of the form $W(z) \to e^{i\langle z|L_t\rangle}W(S_t z)$ the KMS condition is ($\sigma(z,\bar{z}) = \frac{1}{2}(q\bar{k}-k\bar{q})$)

$$\omega(W(z)W(\bar{z})) = \omega(W(\bar{z})e^{i\langle z|L_{i\beta}\rangle}W(S_{i\beta}z)) \tag{3.3}$$

which means

$$e^{i\sigma(z,\bar{z})}f(z+\bar{z}) = e^{i\sigma(\bar{z},S_{i\beta}z)+i\langle z|L_{i\beta}\rangle}f(S_{i\beta}z+\bar{z}) \quad \forall\, z,\bar{z} \in \mathbf{R}^2. \tag{3.4}$$

For $\bar{z} = -z$ we conclude

$$1 = e^{-i\sigma(z,S_{i\beta}z)+i\langle z|L_{i\beta}\rangle}f((S_{i\beta}-1)z), \qquad \forall\, z \in \mathbf{R}^2. \tag{3.5}$$

For the subgroups 1 and 2 where $S_t = 1$, $L \neq 0$ we see that there are no KMS states.

Case 3:

We have $L = 0$ and

$$S_{i\beta} = \begin{pmatrix} \cosh\beta & i\sinh\beta \\ -i\sinh\beta & \cosh\beta \end{pmatrix}$$

and thus

$$f(z) = \exp\left[i\sigma(z, \frac{1}{S_{i\beta}-1}z)\right]$$

or in components

$$f(q,k) = \exp\left[-\frac{1}{2}(q^2+k^2)\coth\frac{\beta}{2}\right]. \tag{3.6}$$

This satisfies the requirements (3.4), for $\beta > 0$ in addition that $0 \le f \le 1$, for $\beta \to 0$ ($T \to \infty$) it gives the tracial state and for $\beta \to \infty$ ($T \to 0$) the standard harmonic oscillator ground state representation.

Since conjugate automorphisms have conjugate KMS states it remains to consider 4 or 5.

Case 5:

We have $L = 0$,

$$S_{i\beta} = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}$$

and (3.4) gives the condition

$$f(z) = \exp\left[i\sigma(z, \frac{1}{S_{i\beta}-1}z)\right],$$

that is

$$f(q,k) = \exp\left[\frac{i}{2}qk\left(\frac{e^{-i\beta}}{e^{-i\beta}-1} - \frac{e^{i\beta}}{e^{i\beta}-1}\right)\right] = \exp\left[-qk\frac{1}{2}\cot\frac{\beta}{2}\right].$$

Since this does not satisfy $|f(q,k)| \le 1$ there are no KMS states for 4 and 5.

Case 6:

$$S_{i\beta} - 1 = \begin{pmatrix} 0 & i\beta \\ 0 & 0 \end{pmatrix}$$

and

$$L_{i\beta} = \left(i\beta g, -\frac{g}{2}\beta^2\right)$$

$S_{i\beta} - 1$ is not invertible but (3.5) can be used and reads

$$1 = \exp\left[-\beta\frac{k^2}{2} - \frac{g}{2}\beta^2 k - i\beta g q\right] f(i\beta k, 0) \qquad \forall\,(q,k).$$

Thus for $g \neq 0$ there are no KMS states.

Case 6, $g = 0$:

We see that $f(q,0) = \exp[-q^2/2\beta]$ which requires again $\beta > 0$. For $k \neq 0$ we have to go back to (3.4),

$$\exp\left[\frac{i}{2}(q\bar{k} - k\bar{q})\right] f(q+\bar{q}, k+\bar{k}) = \exp\left[\frac{i}{2}(\bar{q}k - \bar{k}(q+i\beta k))\right] f(q+i\beta k+\bar{q}, k+\bar{k})$$

or for $\bar{q} = k = 0$

$$\exp\left[\frac{i}{2}\bar{k}q\right] f(q,\bar{k}) = \exp\left[-\frac{i}{2}\bar{k}q\right] f(q,\bar{k})$$

and thus $f(q,k)=0$ for $k\neq 0$. The KMS state

$$f_\beta(q,k)=\delta_k \exp\left[\frac{-q^2}{2\beta}\right] \tag{3.7}$$

gives again a representation of W which is not strongly continuous in k and thus x does not exist as operator, only e^{ikx}. However, the time evolution $(q,k)\to(q+kt,k)$ is represented strongly continuously and thus possesses a generator H. On the contrary for W_T this representation is strongly continuous (**Z** has discrete topology). In this case the central elements $W(2\pi n,0)$ are not represented by multiples of unity since the vectors $|q,k\rangle$ for different q point in different directions and

$$\pi_\beta(W(2\pi n,0))|\bar q,\bar k\rangle = e^{i\pi(n\bar k)}|\bar q+2\pi n,\bar k\rangle.$$

Thus (3.7) does not give an extremal KMS state but can be decomposed as an integral over extremal KMS states over W_T. The latter are superpositions of the states (3.2)

$$f_{g,\gamma}(q)=e^{i(g+\gamma)q},\qquad g\in\mathbf{Z},\ \gamma\in[0,1),$$

$$f_{\beta,\gamma}(q)=\sum_{g\in\mathbf{Z}} f_{g,\gamma}(q)e^{-\beta(g+\gamma)^2/2}\Big/\sum_{g\in\mathbf{Z}} e^{-\beta(g+\gamma)^2/2}, \tag{3.8}$$

$$f_\beta(q)=\int_0^1 d\gamma\; f_{\beta,\gamma}(q).$$

The states defined by $f_{\beta,\gamma}$ are already KMS states since (3.4) requires

$$f(q)=e^{iqk-\beta k^2/2}f(q+i\beta k)$$

which follows from (3.8) since $k\in\mathbf{Z}$. Furthermore $f_{\beta,\gamma}$ satisfies $f(q+2\pi n)=e^{2\pi i n\gamma}f(q)$. Since

$$\langle q+2\pi n,\bar k|q,k\rangle_\gamma=\delta_{\bar k,k}e^{in\pi(k-2\gamma)}\langle q,k|q,k\rangle_\gamma$$

we see that $\forall\, k$

$$|q+2\pi n,k\rangle = e^{i\pi nk+2\pi i n\gamma}|q,k\rangle$$

and

$$\pi_\gamma(W(2\pi n,0))|q,k\rangle = e^{2\pi i n\gamma}|q,k\rangle,$$

that is the central elements are represented in π_γ by c-numbers. The same still holds in $\pi_{\beta,\gamma}$ but not in π_β. In $\mathcal{H}_{\beta,\gamma}$ we have

$$\langle q+2\pi n,\bar k|q,k\rangle_{\beta,\gamma}=\delta_{\bar k,k}\sum_g e^{-in\pi(k+2\gamma+2g)}e^{-\beta(\gamma+g)^2/2}\Big/\sum_g e^{-\beta(\gamma+g)^2/2}=\delta_{\bar k k}e^{-in\pi(k+2\gamma)}.$$

Thus again

$$\pi_{\beta,\gamma}(W(2\pi n,0))|q,k\rangle = e^{2\pi i n\gamma}|q,k\rangle$$

and $\pi_{\beta,\gamma}(W(2\pi,0)) = e^{2\pi i\gamma}$. Integrating over γ makes $|\langle q+2\pi n,k|q,k\rangle|<1$ and thus $\pi_\beta(W(2\pi n,0))$ is no longer a c-number.

We can conversely show that in this way we have obtained all possible KMS states for the shift on W_T. As already stated the KMS condition (3.4) implies ($\beta = 1$)

$$\omega(W(q,k)) = \exp\left[-i(k-n)q + ikb - \frac{(k-n)}{2}n\right]\omega(W(q+in,k))$$

$\forall\, b, n$ which demands

$$\omega(W(q,k)) = \delta_k f(q)$$

and

$$f(q) = e^{inq - n^2/2} f(q+in).$$

We have to assume that f is analytic and it decreases so that we can pass to the Fourier transform

$$\tilde{f}(p) = \int f(q) e^{iqp} dq = \int_{-\infty}^{\infty} e^{in(q+in)+n^2/2} f(q+in) e^{i(q+in)p+np} dq$$

or

$$\tilde{f}(p) = \exp\left[\frac{n^2}{2} + np\right]\tilde{f}(p+n)$$

Thus we can choose $\tilde{f}(p)$ arbitrary in the interval [0,1) and this determines then $\tilde{f}(p)$ resp. $f(q)$ completely. The choice $\tilde{f}(p) = \delta(p-\gamma)$ gives the extremal KMS state ω_γ, and their convex combinations give all the other choices.

4 Application to the Quantum Hall Current

The Weyl algebra is the appropriate description for the quantum Hall effect, i.e. a particle on a two-dimensional torus in a magnetic field orthogonal to the plane [B,GNT,T]. Then

$$\exp[i\alpha v_x]\exp[i\gamma v_y] = \exp[i\gamma\alpha B]\exp[i\gamma v_y]\exp[i\alpha v_x]$$

and

$$\exp[i\alpha v_x]\exp[inx] = \exp[i\alpha n]\exp[inx]\exp[i\alpha v_x],$$
$$\exp[i\beta v_y]\exp[imy] = \exp[i\beta m]\exp[imy]\exp[i\beta v_y],$$
$$\alpha, \gamma \in \mathbf{R}^2, \qquad (mn, n) \in \mathbf{Z}^2.$$

Introducing new variables

$$\exp[in\bar{x}] = \exp[inx]\exp\left[-i\frac{n}{B}v_y\right]$$
$$\exp[im\bar{y}] = \exp[imy]\exp\left[i\frac{m}{B}v_x\right]$$

the new variables $\bar{x}, \bar{y}$ describe the centers of the Larmor precession. Now

$$\left[e^{i(n\bar{x}+m\bar{y})}, e^{i(\alpha v_x + \beta v_y)}\right] = 0$$

whereas

$$\exp[in\bar{x}]\exp[im\bar{y}] = \exp\left[i\frac{nm}{B}\right]\exp[im\bar{y}]\exp[in\bar{x}].$$

Therefore

$$\mathcal{M} = W(\mathbf{R}^2, B) \otimes W\left(\mathbf{Z}^2, \frac{1}{B}\right).$$

The time evolution of particles in a magnetic field only affects $W(\mathbf{R}^2, B)$ and here corresponds to case 3 with $H = \frac{1}{2}(v_x^2 + v_y^2)$ so that we already found the possible KMS states

$$\omega_\beta(W(\alpha, \beta, n, m) = f(\alpha, \beta)\bar{f}(n, m)$$

with $f(\alpha)$ from (3.6) and $\bar{f}$ can be chosen arbitrarily.

If we add an electric field in the y-direction this corresponds to the Hamiltonian

$$H = \frac{1}{2}(v_x^2 + v_y^2) - \frac{E}{B}v_x + E\bar{y} = \frac{1}{2}\left(\left(v_x - \frac{E}{B}\right)^2 + v_y^2\right) - \frac{E^2}{2B^2} + E\bar{y}$$

which means

$$\tau_t \exp[im\bar{x} + in\bar{y}] = \exp\left[i\frac{E}{B}mt + im\bar{x} + in\bar{y}\right].$$

Therefore in the velocity space the Hamiltonian is only affected by a shift and we can get the same class of KMS states as for case 3 but in the $(\bar{x}, \bar{y})$ space we are in case 1 and know that there does not exist a KMS state. This result should be compared with the observation [GNT] that in the presence on an electric field time evolution rotates between inequivalent irreducible representations (for B rational they exist). Only for $\beta = 0$ exists the invariant tracial state which leads to a type II_1-representation.

The reason why there are no KMS states for $E \neq 0$ is the same why there are no KMS states for the shift on the torus. There the Hamiltonian $H = p$ has eigenvalues $g \in \mathbf{Z}$. Thus the occupancy of the levels has the Boltzmann factor $e^{-\beta g}$ and this does not lead to a (normalized) probability distribution. As mentioned at the beginning in the second quantized theory there is no difficulty for fermions. There the Fermi distribution $\sim 1/(1 + e^{\beta g})$ gives a KMS state. However, it is not locally normal (relative to the vacuum) which means that there is an infinite particle density. This is clear since all levels for $g \ll -\beta^{-1}$ are practically filled. In reality an electric field has only a finite range and then stability of matter guarantees that $H - \mu N$ is positive and this obstacle to KMS states does not exist. These facts are not reproduced by the motion on the torus.

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