On the Concentration of Quantum States in Phase Space

June 16, 1993

R.F. Werner [1,2]

Abstract. Let $E(x)$, for x in a $2d$-dimensional phase space, be an irreducible Weyl system, and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ a convex function with $\Phi(0) = 0$. We discuss the maximum of $\int dx\ \Phi\bigl(|\langle\varphi, E(x)\psi\rangle|^2\bigr)$ with respect to unit vectors φ, ψ. When $\Phi(t) = t^p$ with $1 < p < \infty$ the maximum is attained if and only if φ and ψ are coherent states with respect to the same quadratic form. We show that this statement is not correct for more general convex functions Φ.

[1] *FB Physik, Universität Osnabrück, Postfach 4469, D-4500 Osnabrück, Germany.*
[2] *Electronic mail:* `reinwer@dosuni1.rz.Uni-Osnabrueck.DE`

The study of non-relativistic quantum mechanics in terms of an associated classical phase space is an old subject. Perhaps the most widely known map taking quantum states to functions on phase space is the Wigner transformation [**19**]. However, since the Wigner function of a state need not be integrable, it often represents a "probability" density, in which an infinite positive probability is cancelled by an infinite negative probability to give formally the normalization to unity. These infinities make the Wigner function practically useless in studies in which the statistical interpretation of Quantum Theory is taken seriously, or, on the technical side, whenever norm estimates of states or observables are desired.

There is a well-known alternative to the Wigner function avoiding these difficulties, which is variously known as the "Husimi-function"[**14**], a "phase space observable"[**5,7**], a "Berezin upper/lower symbol"[**13**], or "convolution with a coherent state"[**16**]. One price to pay in all these approaches is that while the Wigner function has an intrinsic characterization in terms of the Weyl operators alone [**10**], these positive distribution functions depend on the choice of a "reference state", usually a coherent state. (In the latter case the frequency ω of the oscillator Hamiltonian $(P^2 + \omega^2 Q^2)/2$ of which the coherent reference state is the ground state is a free parameter).

Apart from this arbitrariness these positive distributions have the unpleasant feature of introducing additional "uncertainty": they have larger variances for both position and momentum than the corresponding quantum state (When it makes sense, the Wigner function reproduces the quantum variances). The least increase in variance is guaranteed by constructing the positive distribution with coherent (minimal uncertainty) states, and this is indeed the only option usually considered in the literature. However, the variance is only a very crude measure of the spread of a probability distribution in phase space, and for many purposes other measures of "concentration in phase space" are relevant. Folklore would suggest that, whatever the notion of concentration, the most concentrated classical densities associated with any quantum state are those of a coherent state, where the positive density is also constructed with a coherent reference state.

In this note we give a positive and a negative result in this direction. The positive result is a strengthening of a result by Lieb [**8**], and makes the above statement precise when the measure of concentration is an $\mathcal{L}^p$ norm, or the Shannon entropy of the classical density. This result, plus an analogy with a well-known inequality for the rearrangement of convolutions had led the present author to conjecture a much more general inequality. The conjecture was presented at the Nottingham Quantum Probability workshop. In the meantime, however, it has turned out that it fails in general: there are measures of concentration of quantum states in phase space which are *not* maximized by coherent states.

We now fix the basic notations and conventions for phase space quantum mechanics. For a more complete exposition the reader is referred to [**16,17**]. By a *phase space* we will mean a $2d < \infty$ dimensional real vector space X with a non-degenerate antisymmetric bilinear form $\sigma : X \times X \to \mathbb{R}$. The connection between quantum mechanics and the classical phase space structure is made by an *irreducible Weyl system*, i.e. an irreducible set of unitary operators $\big(E(x)\big)_{x\in X}$ on a Hilbert space $\mathcal{H}$ such that $t \mapsto E(tx)$ is strongly continuous, and

$$E(x)E(y) = e^{i\sigma(x,y)/2}\; E(x+y) \quad . \tag{1}$$

By von Neumann's Uniqueness Theorem **[9]** such a system is uniquely defined up to unitary equivalence, and we may take, without loss of generality,

$$\begin{aligned} \mathcal{H} &= \mathcal{L}^2(\mathbb{R}^d, d\xi) \\ \sigma(x_1,x_2) &= p_1\cdot q_2 - p_2\cdot q_1 \qquad \text{with} \quad x_i = (p_i,q_i) \in \mathbb{R}^d \times \mathbb{R}^d \\ \big(E(p,q)\psi\big)(\xi) &= \exp(\tfrac{i}{2}p\cdot q + ip\cdot\xi)\; \psi(\xi+q) \quad . \end{aligned} \tag{2}$$

Von Neumann's Theorem also implies that symplectic linear transformations of X are implemented by unitary operators. We will only need the phase space inversion $x \mapsto -x$, given by the unitary operator $(U_-\psi)(\xi) = \psi(-\xi)$. It will be convenient to denote the automorphisms of phase space translation and inversion of classical and quantum observables by the same letter α:

$$\begin{array}{ll} \alpha_x(A) = E(x)AE(x)^* & \qquad (\alpha_x f)(y) = f(y-x) \\ \alpha_-(A) = U_- A U_-^* & \qquad (\alpha_- f)(y) = f(-y) \quad . \end{array} \tag{3}$$

A crucial formula of the theory is the square integrability of the Weyl system in the sense that

$$\int dx |\langle\psi, E(x)\varphi\rangle|^2 = \|\psi\|^2\, \|\varphi\|^2 \quad , \tag{4}$$

where "dx" is the Lebesgue measure on phase space, normalized using Planck's constant $\mathrm{h} = 2\pi$, i.e. $dx = (2\pi)^{-d} dp_1 \cdots dp_d\, dq_1 \cdots dq_d = (2\pi)^{-d} d^d p\, d^d q$.

The correspondence between classical states and observables, given by functions on X, and quantum states and observables, given by operators on $\mathcal{H}$, is best expressed in terms of an extended *convolution* operation "$\bigstar$" such that the convolution of two operators gives a function, and the convolution of an operator and a function is an operator **[16]**. Let $\mathcal{T}(\mathcal{H})$ denote the trace class. Then, for $f, g \in \mathcal{L}^1(X, dx)$, and $A, B \in \mathcal{T}(\mathcal{H})$, we define

$$\begin{aligned} (f\bigstar g)(x) &= \int dy\; f(y)\; g(x-y) \\ f\bigstar A = A\bigstar f &= \int dy\; f(y)\; \alpha_y(A) \\ (A\bigstar B)(x) &= \mathrm{tr}\big(A(\alpha_x\alpha_- B)\big) \quad . \end{aligned} \tag{5}$$

Note that the second an third lines follow from the first by substituting the appropriate action of α_x and α_-, and replacing the integral by the trace if necessary. Note that $A\bigstar B$ is an integrable function by virtue of equation (4)

These definitions make the space $\mathcal{R}^1 = \mathcal{L}^1(X,dx) \oplus \mathcal{T}(\mathcal{H})$ into a $\mathbb{Z}_2$-graded commutative Banach algebra. Its Gel'fand transform is the extension of the Fourier transform on $\mathcal{L}^1(X,dx)$, which is best written in the form

$$\begin{aligned}\big(\mathcal{F}f\big)(x) &= \int dy\ e^{i\sigma(x,y)} f(y)\\ \big(\mathcal{F}T\big)(x) &= \mathrm{tr}\big(E(x)T\big) \quad ,\end{aligned} \tag{6}$$

for $f \in \mathcal{L}^1(X,dx)$, $T \in \mathcal{T}(\mathcal{H})$. The definitions (5) of convolution make sense also when one of the factors is in $\mathcal{R}^1$, and the other is in $\mathcal{R}^\infty := \mathcal{L}^\infty(X,dx) \oplus \mathcal{B}(\mathcal{H})$. As in the classical case we may apply interpolation theory to extend the definitions to suitable spaces $\mathcal{R}^p := \mathcal{L}^p(X,dx) \oplus \mathcal{T}^p(\mathcal{H})$, and Young's inequality holds (for the definition of $\mathcal{T}^p$, see section IX.4 in [**11**]). For the Fourier transform we have the Hausdorff-Young inequality, the Riemann-Lebesgue Lemma, and "twisted" versions of Bochner's Theorem, and the formulas relating products in $\mathcal{R}^1$ and convolutions of Fourier transforms [**16**]. A fundamental result is the extension of Wiener's Approximation Theorem, which states that the phase space translates $\alpha_x(T)$ of a trace class operator $T \in \mathcal{T}(\mathcal{H})$ are norm dense in $\mathcal{T}(\mathcal{H})$ if and only if $\mathcal{F}T$ has no zeros. As a consequence the ideal theory of the Banach algebra $\mathcal{R}^1$, including all questions of harmonic synthesis (relating ideals to the set of points where all Fourier transforms of elements of the ideal vanish) is completely reduced to the classical case. Another useful consequence is a one-to-one correspondence of phase space translation invariant subspaces of $\mathcal{L}^\infty(X,dx)$ and $\mathcal{B}(\mathcal{H})$, respectively [**16,17**]. Under this correspondence the continuous functions vanishing at infinity are associated with the compact operators, the CCR algebra is associated with the almost periodic functions, and so on. One obtains simple proofs for some operator theoretic theorems by using correspondence for "reduction to the classical case". Typical examples are the Riemann- Lebesgue Lemma for the inverse Fourier transform ("$\int dx\ f(x)E(x)$ is a compact operator for $f \in \mathcal{L}^1(X,dx)$"), and the theorem that an operator $A \in \mathcal{B}(\mathcal{H})$, which is strongly continuous for α_x, is in the norm closed subspace generated by the $E(x)$ (i.e. in the CCR-algebra) if and only if the orbit $\{\alpha_x(A) \,|\, x \in X\}$ is norm-precompact.

We have cited these results to give support to the intuition that the operation "$\star$" indeed deserves the name "convolution", and that it is reasonable to expect quantum analogues of results in classical harmonic analysis. On the other hand, the results cited above do not refer to the ordering of $\mathcal{L}^1(X,dx)$ and $\mathcal{T}(\mathcal{H})$. But it is precisely the difference in these order structures which determines the geometry of the respective state spaces, and hence make all the difference between quantum and classical theories. Of course, the convolution of positive objects is positive. In fact, the convolutions (5) may be characterized by their order properties: any normal operator from $\mathcal{L}^\infty(X,dx)$ to $\mathcal{B}(\mathcal{H})$, or in the opposite direction, which takes positive elements into positive elements, and intertwines the actions α_x, is of the form "convolution with a fixed density matrix" (positive trace class operator of trace 1) [**16,18**]. This is the general form of the positive classical distribution functions mentioned in the introduction.

The operator of convolution with a density matrix is "doubly stochastic" in the sense that it preserves $\mathbb{1}$, and maps the trace into the integral, and conversely. As a corollary one has the Berezin-Lieb inequalities [**13,16**]

$$\int dx\, \Phi\Big(f(x)\Big) \geq \mathrm{tr}\, \Phi\Big(f\bigstar T_1\Big) \geq \int dx\, \Phi\Big((T_2\bigstar T_1\bigstar f)(x)\Big) \quad , \tag{7}$$

where Φ is any positive convex function on $\mathbb{R}^+$ with $\Phi(0) = 0$, and T_i are density matrices. This is saying that convolution with T_i produces functions/operators which are less and less concentrated in phase space. If one wants to get sharp estimates of quantum operators $f\bigstar T_1$ from the Berezin-Lieb inequalities, one would like to have the difference between the two sides as small as possible, which requires $T_1\bigstar T_2$ to be concentrated in phase space as sharply as possible. Since $|(T_1\bigstar T_2)(x)| \leq 1$ for all x, δ-function-like concentration is impossible. For studying quantum-classical correspondence we therefore need to learn more about the set of functions

$$\mathcal{D} := \Big\{ T_1\bigstar T_2 \Big| \, T_i \text{ density matrices } \Big\} \subset \mathcal{L}^1(X, dx) \quad . \tag{8}$$

Unfortunately, very little is known about this set.

The aim of the present note is to describe this set better with regard to "concentration in phase space". We can make this notion precise by setting up a theory of "stochastic majorization" in the context of semi-finite W*-algebras with distinguished faithful trace. The basic order relation of such a theory "$A \succ B$", read as "A is more mixed than B", is defined for A, B positive elements, not necessarily in the same W*-algebra. One of the equivalent ways of defining it is that $\mathrm{tr}\, \Phi(A) \leq \mathrm{tr}\, \Phi(B)$, holds for*all* positive convex functions Φ vanishing at 0. This is not quite the same as the Alberti-Uhlmann ordering [**1**], since it does not trivialize in the abelian case, and depends explicitly on the trace chosen in each algebra. The Berezin-Lieb inequalities can be written as $T_2\bigstar T_1\bigstar f \succ T_1\bigstar f \succ f$, and are hence an example of a comparison of elements of different algebras in this ordering. When both $A \succ B$ and $B \succ A$, we say that A and B are *rearrangements* of each other. This is equivalent to saying that they have the same distribution function with respect to the respective traces.

In phase space the natural trace is given by the $\int dx$. Therefore, we would like to find density matrices T_i such that

$$\int dx\, \Phi\Big((T_2\bigstar T_1)(x)\Big) \stackrel{!}{=} \max \quad . \tag{9}$$

A more detailed version of this problem asks for the rearrangements of T_1 and T_2 making such an integral maximal for a given Φ. If we find rearrangements T_1' and T_2' of T_1 and T_2 making this integral maximal for all Φ simultaneously, we have found a smallest element of the set $T_1'\bigstar T_2'$ in the ordering $\succ$.

It may seem unreasonable that such least mixed elements should exist. However, a classic result by F. Riesz [**12**], presented also in the classic book by Hardy, Littlewood, and

Pòlya [**6, Theorem 379**], says that for the classical convolution of functions they do exist: the maximizing T_i' are the symmetrically decreasing rearrangements of the given functions T_i. In order to state a quantum analogue of this result, we have to say what "symmetrically decreasing" means for an operator (we assume $d = 1$ for simplicity). A symmetrically decreasing function is a decreasing function of $h = (p^2 + q^2)/2$. Then we shall call a trace class operator T *symmetrically decreasing*, if it has the same eigenvectors as $H = (P^2 + Q^2)/2$, where P, Q are momentum and position operators in the standard representation (2), and if the eigenvalues of A are decreasing with respect to the eigenvalues of H. It is clear that every positive trace class operator T has a unique decreasing rearrangement $\widehat{T}$ in this sense, computed by forgetting its eigenbasis, and arranging the eigenvalues along the spectrum of H. There is a simple kind of rearrangement inequalities in [**6**] stating that the integral of a product of two functions becomes maximal, when they are rearranged to be monotonic with respect to each other (e.g. both symmetrically decreasing). Such statements carry over trivially to the rearrangement of operators. However, the result of Riesz is much deeper. Its quantum analogue would be the affirmative answer to the following Question, and was conjectured to be true by the author at the QP workshop in Nottingham.

Question 1. *Let $T_1, T_2 \in \mathcal{T}(\mathcal{H})$, or let $T_1 \in \mathcal{T}(\mathcal{H})$ and $T_2 \in \mathcal{L}^1(X, dx)$ be positive, and let $\widehat{T_i}$ denote the symmetrically decreasing rearrangement of T_i. Is it always the case that*

$$T_1 \star T_2 \succ \widehat{T_1} \star \widehat{T_2} \quad ?$$

One can show, using the generating function of the Laguerre polynomials, that the convolution of symmetrically decreasing objects (functions or operators) is symmetrically decreasing. This readily implies (as in [**6**]) that if the above question has an affirmative answer for projection operators T_1, T_2, this holds for general T_i, and also for an arbitrary number $n \geq 2$ of factors.

Of course, the most interesting case is to find least mixed elements of the set $\mathcal{D}$ of probability densities, without constraint on the rearrangement classes. This amounts to maximizing (9) over all density matrices T_1, T_2. It is clear from the convexity of Φ that this maximum will be attained when both density matrices are pure. The only symmetrically decreasing one-dimensional projection is the ground state projection of H, which represents a coherent state. Since pure states are a rearrangement class by itself, we arrive at the following special version of Question 1. Since it makes no reference to the symmetrically decreasing rearrangement of general operators, we can state it in any number d of degrees of freedom.

Question 2. *Let $\varphi, \psi \in \mathcal{H}$ be unit vectors, and let Φ be a positive convex function vanishing at zero. Let χ be a coherent unit vector. Is it true that*

$$\int dx\, \Phi\Big(|\langle\varphi, E(x)\psi\rangle|^2\Big) \leq \int dx\, \Phi\Big(|\langle\chi, E(x)\chi\rangle|^2\Big) = \int_0^\infty dt\ \frac{t^{d-1}}{(d-1)!}\ \Phi\Big(e^{-t}\Big) \quad ?$$

Apart from the classical analogy the main piece of supporting evidence for conjecturing the above Questions to have affirmative answers was the following Theorem, which is a slight extension of a result of Lieb [**8**], who proved it under the additional assumption that one of the two vectors φ, ψ is already coherent.

Theorem. *The answer to Question 2 is "yes", when $\Phi(t) = t^s$ with $1 < s < \infty$.*

Proof : The proof follows closely the arguments in [**8**], who considers only the special case in which φ is already assumed to be Gaussian. The main idea is to utilize the best constants in the Young and Hausdorff-Young inequalities [**2,3**] in $\mathbb{R}^n$, i.e.

$$\|f\bigstar g\|_r \leq \big(C(p)C(q)/C(r)\big)^d \ \|f\|_p \ \|g\|_q$$
$$\text{for} \quad 1 + r^{-1} = p^{-1} + q^{-1} \quad ; \quad 1 \leq p, q \leq \infty$$
$$\left\|\widehat{f}\right\|_s \leq \big(C(s')\,(2\pi)^{1/s}\big)^d \, \|f\|_{s'}$$
$$\text{for} \quad s^{-1} = 1 - (s')^{-1} \quad ; \quad 2 \leq s \leq \infty$$

where $$\ln C(s) = \frac{\ln s}{2s} - \frac{\ln s'}{2s'} \quad .$$

We introduce the functions

$$f_q(\xi) = \overline{\psi(\xi)}\varphi(\xi + q) \quad ,$$

for $\xi, q \in \mathbb{R}^d$. Then by definition (2) of the Weyl operators $E(x)$, the left hand side of the inequality becomes

$$I(s) = \int \frac{d^d q \, d^d p}{(2\pi)^d} \left|\widehat{f}_q(p)\right|^{2s} = \int \frac{d^d q}{(2\pi)^d} \ \left\|\widehat{f}_q\right\|_{2s}^{2s}$$
$$\leq C(s')^{2ds} \ \int d^d q \ \|f_q\|_{s'}^{2s} \quad ,$$

where $(s')^{-1} = 1 - (2s)^{-1}$. On the other hand,

$$\|f_q\|_{s'}^{s'} = \int d\xi \ |\psi(\xi)|^{s'} \ |\varphi(\xi + q)|^{s'} = \big(|\psi|^{s'} \bigstar |\varphi|^{s'}\big)(-q) \quad .$$

Hence, with $r = 2s/s' = 2s - 1$, and $p = q = 2/(1 + r^{-1}) = (2s-1)/s$:

$$I(s) \leq C(s')^{2ds} \ \left\| |\psi|^{s'} \bigstar |\varphi|^{s'} \right\|_r^r$$
$$\leq C(s')^{2ds} \ \big(C(p)^2 (C(r)\big)^{dr} \left\| |\psi|^{s'} \right\|_p^r \left\| |\varphi|^{s'} \right\|_p^r \quad .$$

Since $ps' = 2$, the norms on the right hand side are equal to 1. The logarithm of the constant is $\ln C = -d \ln s$. Since

$$\int_0^\infty dt \ \frac{t^{d-1}}{(d-1)!} \ e^{-ts} = s^{-d} \quad ,$$

this proves the Theorem.

■

This result can be used to obtain bounds on other quantities. On the basis of Lieb's result this was done by Uhlmann [**15**]. Here we get correspondingly stronger results: for example, let N be a measurable set of phase space measure m, and let T_1, T_2 be density matrices. Then

$$\int_N dx\ \big(T_1 \bigstar T_2\big)(x) \leq m\ \exp\Big(\frac{-d}{e} m^{1/d}\Big) \quad , \tag{10}$$

as long as $m \leq e^d$. The proof is immediately reduced to the case of pure states T_i by taking convex combinations. Then the Theorem gives a bound on the p-norm of $T_1 \bigstar T_2$, and we use this together with $\|\chi_N\|_q = m^{1/q}$ for $p^{-1} + q^{-1} = 1$, and optimizing with respect to p. It is interesting to compare this with the bound that an affirmative answer to Question 2 would have guaranteed: to compute this we would only have to take N as a ball of the appropriate radius, and to take T_1, T_2 as coherent. Taking $d = 1$ for simplicity, we would get the improvement

$$\int_N dx\ \big(T_1 \bigstar T_2\big)(x) \leq 1 - e^{-m} \leq m e^{-m/e} \quad . \tag{11}$$

In fact, this improved bound would be equivalent to Question 2.

The first indication that the answer to Question 2 (and *a fortiori* the answer to Question 1) is "no", came from an analysis of the expression in Question 2, when both φ and ψ are close to the same coherent state χ. We briefly indicate the argument without giving the details of the computations. Let us consider differentiable curves $t \mapsto \varphi_t, \psi_t$ of unit vectors, with $\varphi_0 = \psi_0 = \chi$. For our purposes we can write

$$\varphi_t = \chi + t\dot\varphi + \tfrac{1}{2} t^2\ \ddot\varphi + \mathbf{O}(t^3) \quad , \tag{12}$$

and similarly for ψ. The normalization condition then means that $\Re e\langle \chi, \dot\varphi\rangle = 0$, and $\Re e\langle \chi, \ddot\varphi\rangle + \langle \dot\varphi, \dot\varphi\rangle = 0$. Let Φ be a fixed positive convex function with $\Phi(0) = 0$, which we assume to be twice differentiable. Then, setting $\rho_t(x) = |\langle \varphi_t, E(x)\psi_t\rangle|^2$, and $\rho = \rho_0(x)$, our aim is to decide whether $I(t) = \int dx\ \Phi(\rho_t(x))$ has a local maximum at $t = 0$. Hence we expand I to second order in t. The first derivative of I vanishes because χ is an eigenvector of the operator

$$A = \int dx\, \Phi'\big(\rho(x)\big)\ E(x)|\chi\rangle\langle\chi|E(x)^* = \Phi'(\rho) \bigstar P_0 \quad , \tag{13}$$

where $P_0 = |\chi\rangle\langle\chi|$ is the ground state projection of the harmonic oscillator. For the vanishing of the first order it would be sufficient that φ_0 and ψ_0 are both oscillator eigenvectors. Using the equation for the vanishing of the first order, one can eliminate the $\ddot\varphi$ and $\ddot\psi$ from $\ddot I$. Moreover, in many of the terms making up $\ddot I$ the angular integration gives zero. In particular, all terms vanish in which only one derivative appears, or in which all $\dot\varphi, \dot\psi$ appear in the antilinear (resp. linear) arguments of the scalar products. Then the second order becomes

$$\begin{aligned} \ddot I(0) &= -2\Big\{\langle\dot\varphi, B\dot\varphi\rangle + \langle\dot\psi, B\dot\psi\rangle + 2\Re e\langle\dot\psi, C\dot\varphi\rangle\Big\} \quad , \\ \text{where} \qquad B &= \big(\textstyle\int dx\ \rho\, \Phi'(\rho)\big)\mathbb{1} - \big(\Phi'(\rho) + \rho\Phi''(\rho)\big) \bigstar P_0 \quad , \\ \text{and} \qquad C &= -\int dx\, \sqrt{\rho}\, \Phi'(\rho)\ E(x) - \int dx\, \rho\, \Phi''(\rho)\ E(x) P_0 E(x) \quad . \end{aligned} \tag{14}$$

(The omission of a star on the last factor in C is not a misprint). One readily verifies that both B and C are diagonal in the oscillator eigenbasis, and the definiteness of the second derivative becomes equivalent to

$$B_n \geq |C_n| \quad , \tag{15}$$

where B_n and C_n are the n^{th} eigenvalues of B and C. We get

$$\begin{aligned} B_n &= \int_0^\infty dt\; e^{-2t}\Phi''\big(e^{-t}\big)\; \left\{\sum_{k=1}^{n-1} \frac{t^k}{k!}\right\} \\ C_n &= \int_0^\infty dt\; e^{-2t}\Phi''\big(e^{-t}\big)\; \left\{\mathrm{L}_n(t) - \mathrm{L}_{n-1}(t) - (-1)^n\frac{t^n}{n!}\right\} \quad , \end{aligned} \tag{16}$$

where L_n denotes the n^{th} Laguerre polynomial. Here we have performed various partial integrations so that only Φ'' appears rather than Φ and Φ'. Hence the weight in both integrals is an essentially arbitrary positive function. If we insert $\Phi(t) = t^p$, we get

$$B_n = 1 - p^{-(n-1)} \quad \geq \quad p^{-n}(p-1)\big|(p-1)^{n-1} + (-1)^n\big| = |C_n| \quad , \tag{17}$$

which is clear from the Theorem. On the other hand, we can choose for Φ (a smooth approximation of) $\Phi(\rho) = (\rho - \lambda)_+$, so that the integrand has a delta function $\delta(e^{-t} - \lambda)$. Hence it suffices to discuss the condition (15) for the expressions in braces in equation (16), pointwise for each t. Since $\dot\varphi$ and $\dot\psi$ are orthogonal to χ, they do not appear in $\ddot I$. If $\varphi_t = E(tx)\chi$, we get $\dot\varphi$ proportional to the first oscillator function. But the value of the integral is constant under such shifts. Hence along such paths $\ddot I = 0$, and we must have $B_1 = C_1 = 0$. Further we have $B_2 = t = -C_2$. But

$$B_3 = t + \frac{1}{2}t^2 \quad \not\geq \quad \big|-t + t^2\big| = |C_3| \quad , \tag{18}$$

and in fact the inequality (15) fails for all $n \geq 3$ and sufficiently large t. Hence we have proven that the answer to Questions 1 and 2 is "no" in general.

Of course, this proof is rather indirect, and an explicit counterexample would be preferable. We can take the above computation as a guideline, and set

$$\begin{aligned} \varphi &= \cos\gamma\ \chi_0 + \sin\gamma\ \chi_3 \\ \psi &= \cos\gamma\ \chi_0 - \sin\gamma\ \chi_3 \quad , \end{aligned} \tag{19}$$

where χ_n is the n^{th} oscillator eigenvector. Then with $\gamma \approx 0.0606$, and $\lambda \approx 0.0039$, we get

$$\int\! dx\, \Big(|\langle\varphi, E(x)\psi\rangle|^2 - \lambda\Big)_+ \approx 6.11 * 10^{-5} + \int\! dx\, \Big(|\langle\chi_0, E(x)\chi_0\rangle|^2 - \lambda\Big)_+ \quad , \tag{20}$$

so the difference is indeed positive. This computation has to be done with some care, since we are looking for a relatively small difference. It is simplified by observing that for small (resp. large) radial coordinates the density $|\langle\varphi, E(x)\psi\rangle|^2$ is larger (resp. smaller) than λ

for all angles, so that these parts can be integrated analytically. The remaining integral is then done in polar coordinates, where the integration over the angle can also be done analytically, leaving an expression involving an arcsin.

It is clear that this counterexample is not optimal. It would be interesting to maximize the above integral with respect to φ, ψ for any fixed value of λ. It would also be interesting to see whether there is also a counterexample when one of the two vectors is coherent, corresponding to the special case of the Theorem originally treated by Lieb. Another open problem is to determine the best constants of the Young and Hausdorff-Young inequalities for the convolutions (5) in analogy with [**2**]. Are they attained for Gaussians, as in the classical case?

Acknowledgements

The author is indebted to Roland Franzius for an independent verification of the numerical counterexample at the end of the paper. He also acknowledges financial support from the DFG (Bonn).

References

[**1**] P.M. Alberti and A. Uhlmann: *Stochasticity and partial order*, D. Reidel, Dordrecht 1982

[**2**] W. Beckner: "Inequalities in Fourier analysis", *Ann.Math.* **102**(1975) 159–182

[**3**] H.J. Brascamp and E.H. Lieb: "Best constants in Young's inequality, its converse, and its generalization to more than three functions", *Advan.Math.* **20**(1976) 151–173

[**4**] I. Daubechies: "Continuity statements and counterintuitive examples in connection with Weyl quantization", *J.Math.Phys.* **24**(1983) 1453–1461

[**5**] E.B. Davies: *Quantum theory of open systems*, Academic Press, London 1976

[**6**] G. Hardy, J.E. Littlewood, and G. Pólya: *Inequalities*, Cambridge University Press, Cambridge 1934

[**7**] A.S. Holevo: *Probabilistic and statistical aspects of quantum theory*, North Holland, Amsterdam 1982

[**8**] E.H. Lieb: "Proof of an entropy conjecture by Wehrl", *Commun.Math.Phys.* **62**(1978) 35–41

[**9**] J. von Neumann: "Die Eindeutigkeit der Schrödingerschen Operatoren", *Math.Ann.* **104**(1931) 570–578

[**10**] R.F. O'Connell and E.P. Wigner: "Quantum-Mechanical distribution functions: conditions for uniqueness", *Phys.Lett.* **83A** (1981) 145–148

[**11**] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, vol.II: Fourier analysis and self-adjointness*, London: Academic Press, 1975

[**12**] F. Riesz: "Sur une inégalité intégrale", *J.London.Math.Soc.* **5**(1930) 162–168

[**13**] B. Simon: "The classical limit of quantum partition of functions", *Commun.Math.Phys.* **71**(1980) 247–276

[**14**] K. Takahashi: "Wigner and Husimi functions in quantum mechanics", *J.Phys.Soc.Jap.* **55**(1986) 762–779

[**15**] A. Uhlmann: "Remarks on the relation between quantum and classical entropy as proposed by A. Wehrl", *Rep.Math.Phys.* **18**(1980) 177–182

[**16**] R.F. Werner: "Quantum harmonic analysis on phase space", *J.Math.Phys.* **25** (1984) 1404–1411

[**17**] R.F. Werner: "Physical uniformities on the state space of non-relativistic quantum mechanics", *Found.Phys.* **13** (1983) 859–881

[**18**] R.F. Werner: "The classical limit of quantum mechanics", In preparation

[**19**] E.P. Wigner: "On the Quantum Correction for Thermodynamic Equilibrium", *Phys.Rev.* **40** (1932) 749